Applications of Renormalization to Irrationally Indifferent Complex Dynamics

by

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Abstract

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This thesis comprises of two main results which are proved using renormalization techniques.

For the first result, we show that a quadratic polynomial with a fixed Siegel disc of bounded type rotation number is conformally mateable with the basilica polynomial $f_{\mathbf{B}}(z) := z^2 - 1$.

For the second result, we study sufficiently dissipative complex quadratic Hénon maps with a semi-Siegel fixed point of inverse golden-mean rotation number. It was recently shown in [GaRYa] that the Siegel disks of such maps are bounded by topological circles. We investigate the geometric properties of such curves, and demonstrate that they cannot be C^1-smooth.

To my parents, for their love and devotion.

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Contents

Preface

In the last several decades, renormalization has emerged as a key theme in the field of low dimensional dynamics through a series of seminal works of Douady and Hubbard [DH2], Sullivan [Su], McMullen [Mc1, Mc2], Lyubich [Ly1], Yoccoz [Yo1, Hu], and many others. Loosely speaking, the renormalization of a dynamical system is defined as a rescaled first return map on an appropriately chosen subset of the phase space. Iterating this procedure reveals the small-scale asymptotic behaviour of the dynamics, which is often universal and insensitive to the incidental details of the system.

The renormalization approach has been particularly useful in the study of indifferent dynamical systems, which are often the most challenging cases. The numerous important examples of this include the works of Herman [He], Lanford [La1, La2], Yoccoz [Yo2], Shishikura [Sh1, ISh], and Yampolsky ([Ya2, Ya3]). In this thesis, we consider two different applications of renormalization: the topological modelling of the dynamics of Siegel rational maps, and the analysis of the geometric properties of Siegel disks for dissipative Hénon maps. We present each topic in their own independent self-contained chapter, which we summarize below.

1. Mating the Basilica with a Siegel Disk

In the first chapter, we study the following one-parameter family of quadratic rational maps

$$R_a(z) := \frac{a}{z^2 + 2z} \quad \text{for} \quad a \in \mathbb{C} \setminus \{0\},$$

called the *basilica family*. The characteristic feature of a map R_a is that it has a superattracting two-periodic orbit. The unique (up to an affine change of coordinates) quadratic polynomial that satisfies this property is given by

$$f_{\mathbf{B}}(z) := z^2 - 1 \overset{conj}{\sim} \frac{1}{u^2 + 2u}.$$

For the shape of its filled Julia set, we call $f_{\mathbf{B}}$ the *basilica polynomial*. It follows that for $a \neq 1$, the maps R_a provide examples of non-polynomial dynamical systems.

Analogous to the Mandelbrot set $\mathcal{M}$ for the quadratic polynomials, we can define the non-escape locus $\mathcal{M}_{\mathbf{B}}$ in the parameter space of the basilica family. Comparing the plot of $\mathcal{M}$ shown in Figure 1.1 and the plot of $\mathcal{M}_{\mathbf{B}}$ shown in Figure 1.3, we see that the two sets are structurally very similar. In fact, it is conjectured that the maps in $\mathcal{M}_{\mathbf{B}}$ are realizations of the *matings* of $f_{\mathbf{B}}$ and the quadratic polynomials in $\mathcal{M}$. Loosely speaking, this means that the dynamics of R_a in $\mathcal{M}_{\mathbf{B}}$ is the amalgamation of the dynamics of $f_{\mathbf{B}}$ (from whence it obtains the superattracting two-periodic orbit) with the dynamics of some corresponding quadratic polynomial in $\mathcal{M}$.

In most cases, this conjecture has been verified through the works of Rees, Tan and Shishikura [Re, Tan, Sh2]; Haïssinsky [Ha]; Aspenberg and Yampolsky [AYa]; and Dudko [Du]. The only parameters that are not accounted for, but for which we still expect a positive answer, are the "nice" Siegel parameters contained in the boundary of hyperbolic components that are not too "deep" inside $\mathcal{M}_{\mathbf{B}}$. For our first result, we settle the conjecture for a key subclass of such parameters. Specifically, we show that if R_{a_ν} has a fixed Siegel disk of bounded type rotation number $\nu = e^{2\pi\theta i}$, then R_{a_ν} is a mating with the basilica.

The main ideas of the proof are as follows. First, by using a similar argument as the one found in [YaZ], we prove the existence of a Blaschke product F_ν whose dynamics outside the grand orbit of the unit disc matches that of R_{a_ν}. This Blaschke product F_ν can then be transformed into R_{a_ν} by a quasiconformal surgery replacing the unit disc with a Siegel disc (see Theorem 1.4.5). This proves that the boundary of the Siegel disc for R_{a_ν} is a quasicircle, and that it contains a critical point (see Main Theorem 1A).

Using Main Theorem 1A, we construct chains of iterated preimages of the Siegel disc connected by iterated preimages of the critical point in the dynamical space of R_{a_ν}. These structures are called *bubble rays*, and they play an analogous role to *external rays* for polynomials. Using these bubble rays, we create a dynamical partition for R_{a_ν}. This naturally defines a correspondence between the map R_{a_ν} and the topological model given by the mating of the basilica polynomial $f_{\mathbf{B}}$ and the Siegel quadratic polynomial $f_{\mathbf{S}}$ with rotation number ν. Theorem 1.4.5 then allows us to use a result in the renormalization theory of critical circle maps developed by Yampolsky in [Ya3] called *complex a priori bounds*. Using this estimate, we are able to show that the dynamic partition elements for R_{a_ν} shrink to points. This implies that the correspondence between R_{a_ν} and the mating of $f_{\mathbf{B}}$ and $f_{\mathbf{S}}$ is one-to-one. From this key fact, the rest of the result follows readily.

2. The Siegel Disk of a Dissipative Hénon Map Has Non-Smooth Boundary

In the second chapter, we study the following two-dimensional extension of a one-parameter family of quadratic polynomials

$$H_{c,b}(x,y) = (x^2 + c - by, x) \quad \text{for } c \in \mathbb{C} \text{ and } b \in \mathbb{C} \setminus \{0\}$$

called the *(complex quadratic) Hénon family.* More specifically, we are interested in Hénon maps $H_{\mu_*,\nu} = H_{c_{\mu_*,\nu}, b_{\mu_*,\nu}}$ that has a fixed point $\mathbf{p}_0$ with multipliers $\mu_* = e^{2\pi i \theta_*}$ and $\nu \in \mathbb{D} \setminus \{0\}$, where

$$\theta_* := \frac{\sqrt{5}-1}{2}$$

is the inverse golden-mean.

By a classic theorem of Siegel, there exists a neighborhoods N of $(0,0)$ and $\mathcal{N}$ of $\mathbf{p}_0$, and a biholomorphic change of coordinates

$$\phi : (N, (0,0)) \to (\mathcal{N}, \mathbf{p}_0)$$

such that

$$H_{\mu_*,\nu} \circ \phi = \phi \circ L,$$

where $L(x,y) := (\mu_* x, \nu y)$. This linearizing map can be biholomorphically extended to

$$\phi : (\mathbb{D} \times \mathbb{C}, (0,0)) \to (\mathcal{C}, \mathbf{p}_0)$$

so that the image $\mathcal{C} := \phi(\mathbb{D} \times \mathbb{C})$ is *maximal* (see [MNTU]). We call $\mathcal{C}$ and $\mathcal{D} := \phi(\mathbb{D} \times \{0\})$ the *Siegel cylinder* and the *Siegel disk* of $H_{\mu_*,\nu}$ respectively.

Consider the quadratic polynomial

$$f_{c_*}(x) = x^2 + c_*$$

with a Siegel fixed point x_0 of multiplier μ_*. Let D be its Siegel disk, and let $\psi : \mathbb{D} \to D$ be its biholomoprhic linearizing map. It is well-known that ψ extends quasi-symmetrically to the boundary. Moreover, ∂D contains the critical point of f_{c_*}. Since ∂D is invariant under f_{c_*}, it follows immediately that ∂D cannot be a smooth curve.

For the Hénon map $H_{\mu_*,\nu}$, it was recently shown in [GaRYa] that ϕ restricted to the Siegel disk $\mathcal{D}$ extends homeomorphically, but not C^1-smoothly to the boundary $\partial\mathcal{D}$ (see Theorem 2.1.2). However, this does not imply that $\partial\mathcal{D}$ is itself not a C^1-smooth curve. Moreover, unlike in the one-dimensional case, $\partial\mathcal{D}$ does not contain the critical point for $H_{\mu_*,\nu}$, as no such point exists. Indeed, $H_{\mu_*,\nu}$ is a diffeomorphism with a constant Jacobian equal to $b_{\mu_*,\nu} \neq 0$.

Our proof of non-smoothness relies instead on the renormalization theory developed by Gaidashev and Yampolsky in [GaYa]. Loosely speaking, they showed that high iterates of $H_{\mu_*,\nu}$ restricted to appropriately chosen nested neighborhoods that intersect $\partial\mathcal{D}$ converge to a *universal* degenerate one-dimensional dynamical system with a simple critical point. Geometrically, this means that $\partial\mathcal{D}$ contains a sequence of "near critical" points for higher iterates of $H_{\mu_*,\nu}$. Moreover, the higher the iterate, the more "near critical" such points become. Hence, if $\partial\mathcal{D}$ were C^1-smooth, then by the invariance of $\partial\mathcal{D}$, these "near critical" points would force $\partial\mathcal{D}$ to have corners. Such corners would accumulate to a singularity, which would contradict the smoothness of $\partial\mathcal{D}$.

Chapter 1

Mating the Basilica with a Siegel Disk

1.1 The Definition of Mating

The simplest non-linear examples of holomorphic dynamical systems are given by the quadratic polynomials in $\mathbb{C}$. By an affine change of coordinates, any quadratic polynomial can be uniquely normalized as

$$f_c(z) := z^2 + c \quad \text{for some} \quad c \in \mathbb{C}.$$

This is referred to as the *quadratic family*.

The critical points for f_c are ∞ and 0. Observe that ∞ is a superattracting fixed point for f_c. Let $\mathbf{A}_c^\infty$ be the attracting basin of ∞. It follows from the maximum modulus principle that $\mathbf{A}_c^\infty$ is a connected set. The complement of $\mathbf{A}_c^\infty$ is called the *filled Julia set* K_c. It is known that the boundary of K_c is equal to the Julia set $J_c := J(f_c)$ for f_c (see [M3]).

The non-escape locus in the parameter space for f_c, referred to as the *Mandelbrot set*, is defined as the following compact subset of $\mathbb{C}$:

$$\mathcal{M} := \{c \in \mathbb{C} \mid 0 \notin \mathbf{A}_c^\infty\}.$$

It is known that $\mathcal{M}$ is connected (see [DH1]). Moreover, it is not difficult to prove that J_c is connected (and therefore, $\mathbf{A}_c^\infty$ is simply connected) if and only if $c \in \mathcal{M}$. In fact, if $c \notin \mathcal{M}$, then $J_c = K_c$ is a Cantor set, and the dynamics of f_c restricted to J_c is conjugate to the dyadic shift map (see [M2]). We also define the following subset of the Mandelbrot set:

$$\mathcal{L} := \{c \in \mathcal{M} \mid J_c \text{ is locally connected}\}.$$

It should be noted that $\mathcal{L}$ is a proper subset of $\mathcal{M}$ (for example, if $c \in \mathcal{M}$ is Cremer, then it is known that J_c is not locally connected).

Some of the most celebrated results in holomorphic dynamics are centered on the quadratic family f_c, including those obtained by Douady and Hubbard [DH2], Milnor [M1], Yoccoz [Yo1], and Lyubich [Ly2]. Having been the focal point in the field since the subject first emerged, the dynamics of the quadratic

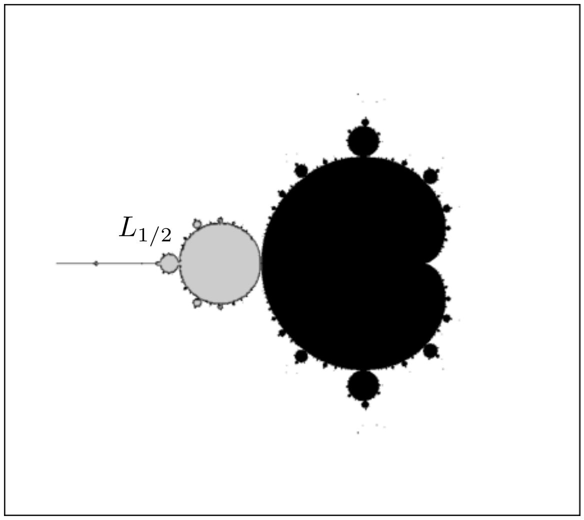

Figure 1.1: The Mandelbrot set $\mathcal{M}$. The 1/2-limb $L_{1/2}$ is highlighted.

family is now almost completely understood. In contrast, obtaining a similarly explicit dynamical description of other families of rational maps remains a wide open area of research. One of the most natural starting point for advancement in this direction is the study of non-polynomial quadratic rational maps. In this section, we describe a construction, originally put forward by Douady and Hubbard (see [Do]), which produces quadratic rational maps by combining the dynamics of two quadratic polynomials.

Suppose $c \in \mathcal{L}$. Since J_c is connected, $\mathbf{A}_c^\infty$ must be a simply connected domain. Let

$$\phi_c : \mathbf{A}_c^\infty \to \mathbb{D}$$

be the unique conformal Riemann mapping such that $\phi_c(\infty) = 0$ and $\phi_c'(\infty) > 0$. It is not difficult to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}_c^\infty & \xrightarrow{f_c} & \mathbf{A}_c^\infty \\ \downarrow{\scriptstyle \phi_c} & & \downarrow{\scriptstyle \phi_c} \\ \mathbb{D} & \xrightarrow{z \mapsto z^2} & \mathbb{D} \end{array}$$

and hence, ϕ_c is the Böttcher uniformization of f_c on $\mathbf{A}_c^\infty$. Moreover, since J_c is locally connected, Carathéodory's theory implies that the inverse of ϕ_c extends continuously to the boundary of $\mathbb{D}$ (see [M3]). If we let

$$\tau_c := \phi_c^{-1}|_{\partial \mathbb{D}},$$

we obtain a continuous parametrization of J_c by the unit circle $\partial \mathbb{D} = \mathbb{R}/\mathbb{Z}$ known as a *Carathéodory loop.* Observe that f_c, when restricted to J_c, acts via τ_c as the angle doubling map:

$$f_c(\tau_c(t)) = \tau_c(2t).$$

Now, suppose $c_1, c_2 \in \mathcal{L}$. Using τ_{c_1} and τ_{c_2}, we can glue the dynamics of f_{c_1} and f_{c_2} together to construct a new dynamical system as follows. First, we construct a new dynamical space $K_{c_1} \vee K_{c_2}$ by gluing the filled Julia sets K_{c_1} and K_{c_2}:

$$K_{c_1} \vee K_{c_2} := (K_{c_1} \sqcup K_{c_2})/\{\tau_{c_1}(t) \sim \tau_{c_2}(-t)\}. \tag{1.1}$$

We refer to the resulting equivalence relation $\sim$ as *ray equivalence*, and denote it by $\sim_{\text{ray}}$. For a point x in K_{c_1} or K_{c_2}, we denote the ray equivalency class of x by $[x]_{\text{ray}}$.

We now define a new map

$$f_{c_1} \vee f_{c_2} : K_{c_1} \vee K_{c_2} \to K_{c_1} \vee K_{c_2},$$

called the *formal mating of* f_{c_1} *and* f_{c_2}, by letting $f_{c_1} \vee f_{c_2} \equiv f_{c_1}$ on K_{c_1} and $f_{c_1} \vee f_{c_2} \equiv f_{c_2}$ on K_{c_2}. Note that the definition of $f_{c_1} \vee f_{c_2}$ is consistent, since on their Julia sets, both f_{c_1} and f_{c_2} act by angle doubling.

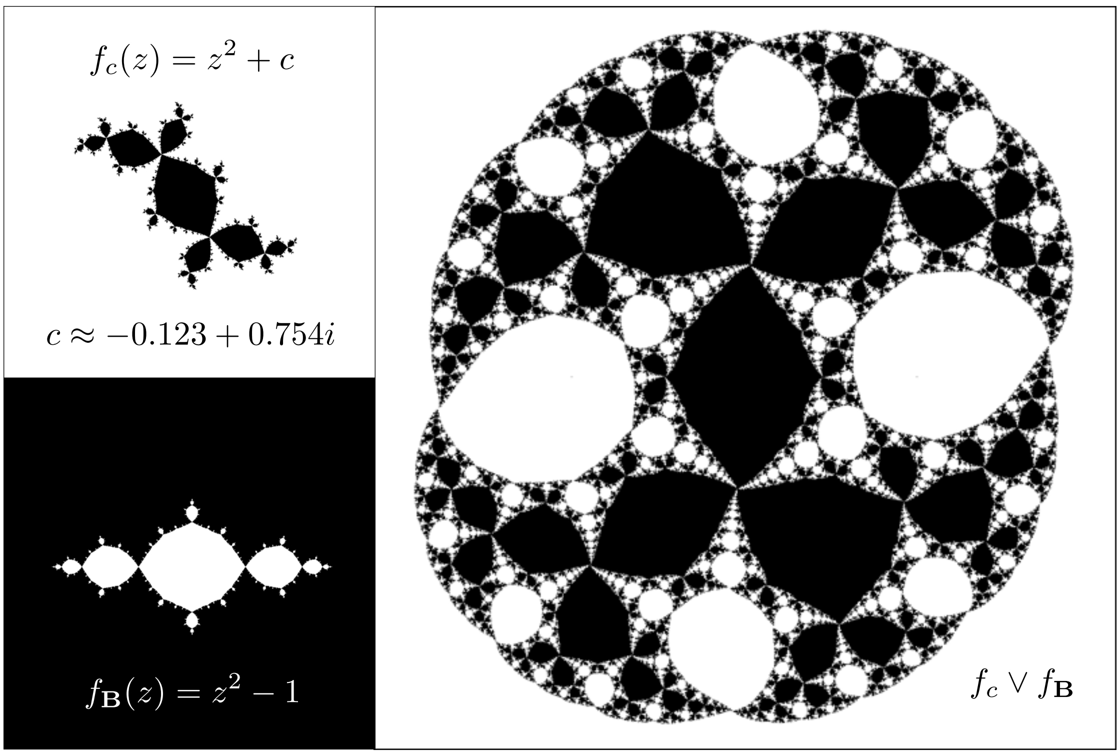

Figure 1.2: The Douady rabbit f_c with $c \approx -0.123 + 0.754i$ mated with the basilica polynomial $f_{\mathbf{B}}$.

If the space $K_{c_1} \vee K_{c_2}$ is homeomorphic to the 2-sphere, then f_{c_1} and f_{c_2} are said to be *topologically mateable*. If, in addition, there exists a quadratic rational map R and a homeomorphism

$$\Lambda : K_{c_1} \vee K_{c_2} \to \hat{\mathbb{C}}$$

such that Λ is conformal on $\mathring{K}_{c_1} \sqcup \mathring{K}_{c_2} \subset K_{c_1} \vee K_{c_2}$, and the following diagram commutes:

$$\begin{tikzcd}
K_{c_1} \vee K_{c_2} \arrow[r, "f_{c_1} \vee f_{c_2}"] \arrow[d, "\Lambda"] & K_{c_1} \vee K_{c_2} \arrow[d, "\Lambda"] \\
\hat{\mathbb{C}} \arrow[r, "R"] & \hat{\mathbb{C}}
\end{tikzcd}$$

then f_{c_1} and f_{c_2} are said to be *conformally mateable*. The quadratic rational map R is called a *conformal mating of* f_{c_1} *and* f_{c_2}. We also say that R *realizes the conformal mating of* f_{c_1} *and* f_{c_2}.

In applications, it is sometimes more useful to work with the following reformulation of the definition of conformal mateability:

Proposition 1.1.1. *Suppose $c_1, c_2 \in \mathcal{L}$. Then f_{c_1} and f_{c_2} are conformally mateable if and only if there exists a pair of continuous maps*

$$\Lambda_1 : K_{c_1} \to \hat{\mathbb{C}} \quad \textit{and} \quad \Lambda_2 : K_{c_2} \to \hat{\mathbb{C}}$$

such that for all $i, j \in \{1, 2\}$ the following three conditions are satisfied:

(i) $\Lambda_i(z) = \Lambda_j(w)$ *if and only if* $z \sim_{ray} w$,

(ii) Λ_i *is conformal on* $\mathring{K}_{c_i}$, *and*

(iii) there exists a rational function R of degree 2 such that the following diagrams commute:

$$\begin{array}{ccc} K_{c_1} & \xrightarrow{f_{c_1}} & K_{c_1} \\ \downarrow{\scriptstyle \Lambda_1} & & \downarrow{\scriptstyle \Lambda_1} \\ \hat{\mathbb{C}} & \xrightarrow{R} & \hat{\mathbb{C}} \end{array} \quad \textit{and} \quad \begin{array}{ccc} K_{c_2} & \xrightarrow{f_{c_2}} & K_{c_2} \\ \downarrow{\scriptstyle \Lambda_2} & & \downarrow{\scriptstyle \Lambda_2} \\ \hat{\mathbb{C}} & \xrightarrow{R} & \hat{\mathbb{C}} \end{array}$$

Proof. Assume that there exists a pair of maps Λ_1 and Λ_2 satisfying (i), (ii) and (iii). Consider the space $K_{c_1} \vee K_{c_2}$ given by (1.1). Define $\Lambda : K_{c_1} \vee K_{c_2} \to \hat{\mathbb{C}}$ by letting $\Lambda|_{K_{c_1}} := \Lambda_1$ and $\Lambda|_{K_{c_2}} := \Lambda_2$. By (i), this definition is consistent. Conformal mateability readily follows from the other two properties of Λ_1 and Λ_2.

Assume that f_{c_1} and f_{c_2} are conformally mateable, and let Λ be the conjugacy between $f_{c_1} \vee f_{c_2}$ and a rational map R. Define

$$\Lambda_1 := \Lambda|_{K_{c_1}} \quad \text{and} \quad \Lambda_2 := \Lambda|_{K_{c_2}}.$$

The properties (i), (ii) and (iii) follow immediately. □

Corollary 1.1.2. *Suppose R is a conformal mating of f_{c_1} and f_{c_2} for some $c_1, c_2 \in \mathcal{L}$. Then R has a locally connected Julia set $J(R)$.*

Proof. Let $\Lambda_1 : K_{c_1} \to \hat{\mathbb{C}}$ and $\Lambda_2 : K_{c_2} \to \hat{\mathbb{C}}$ be as given in Proposition 1.1.1. Note that

$$J(R) = \Lambda_1(J_{c_1}) = \Lambda_2(J_{c_2}).$$

Since the continuous image of a compact locally connected set is locally connected, the result follows. □

Example 1.1.3. For $c \in \mathcal{L}$, the quadratic polynomial f_c is trivially conformally mateable with the squaring map $f_0(z) = z^2$. This follows from choosing Λ_1 and Λ_2 in Proposition 1.1.1 to be the identity map on K_c and the inverse of the Böttcher uniformization of f_c on $\mathbf{A}_c^\infty$ respectively. Note that the conformal mating of f_c and f_0 is realized by f_c itself. The following result shows that with the exception of this trivial case, the mating construction always yields a non-polynomial dynamical system.

Proposition 1.1.4. *Suppose a quadratic polynomial $P : \mathbb{C} \to \mathbb{C}$ is a conformal mating of f_{c_1} and f_{c_2} for some $c_1, c_2 \in \mathcal{L}$. Then either f_{c_1} or f_{c_2} must be equal to the squaring map f_0.*

Proof. Let $J(P)$ and $\mathbf{A}_P^\infty$ denote the Julia set and the attracting basin of infinity for P respectively. We have

$$J(P) = \Lambda_1(J_{c_1}) = \Lambda_2(J_{c_2}).$$

Hence, $\mathbf{A}_P^\infty$ must be contained in either $\Lambda_1(\mathring{K}_{c_1})$ or $\Lambda_2(\mathring{K}_{c_2})$. Assume for concreteness that it is contained in the former. Since $\Lambda_1|_{\mathring{K}_{c_1}}$ is conformal, and

$$f_{c_1}(z) = \Lambda_1^{-1} \circ P \circ \Lambda_1(z) \quad \text{for all} \quad z \in K_{c_1},$$

we see that $\Lambda_1^{-1}(\infty)$ must be a superattracting fixed point for f_{c_1}. The only member in the quadratic family that has a bounded superattracting fixed point is the squaring map f_0. ☐

Example 1.1.5. Consider the formal mating of the basilica polynomial $f_{\mathbf{B}}(z) := f_{-1}(z) = z^2 - 1$ with itself. The glued space $K_{\mathbf{B}} \vee K_{\mathbf{B}}$ consists of infinitely many spheres connected together at discrete nodal points (refer to Section 1.5.1 for the structural properties of $K_{\mathbf{B}}$). Hence, it is not homeomorphic to the 2-sphere. Therefore, $f_{\mathbf{B}}$ is not conformally mateable with itself (since it is not even topologically mateable with itself). This is actually a specific instance of a more general result, which we state below.

Let H_0 be the *principal hyperbolic component* defined as the set of $c \in \mathcal{M}$ for which f_c has an attracting fixed point $z_c \in \mathbb{C}$. It is conformally parametrized by the multiplier of z_c:

$$\lambda : c \mapsto f_c'(z_c)$$

(see e.g. [M2]). Note that λ extends to a homeomorphism between $\overline{H_0}$ and $\overline{\mathbb{D}}$.

A connected component of $\mathcal{M} \setminus \overline{H_0}$ is called a *limb*. It is known (see e.g. [M2]) that the closure of every limb intersects ∂H_0 at a single point. Moreover, the image of this point under λ is a root of unity. Henceforth, the limb growing from the point $\lambda^{-1}(e^{2\pi i p/q})$ for some $p/q \in \mathbb{Q}$ will be denoted by $L_{p/q}$. For example, the parameter value -1 for the basilica polynomial $f_{\mathbf{B}}(z) = z^2 - 1$ is contained in the 1/2-limb $L_{1/2}$.

The following standard observation is due to Douady [Do]:

Proposition 1.1.6. *Suppose c_1 and c_2 are contained in complex conjugate limbs $L_{p/q}$ and $L_{-p/q}$ of the Mandelbrot set $\mathcal{M}$. Then f_{c_1} and f_{c_2} are not topologically mateable.*

Proof. There exists a unique repelling fixed point $\alpha_1 \in K_1$ (resp. $\alpha_2 \in K_2$) such that $K_1 \setminus \{\alpha_1\}$ (resp. $K_2 \setminus \{\alpha_2\}$) is disconnected. Since c_1 and c_2 are contained in complex conjugate limbs, α_1 and α_2 are in the same ray equivalency class. Hence they are glued together to a single point in $K_{c_1} \vee K_{c_2}$. Removing this single point from $K_{c_1} \vee K_{c_2}$ leaves it disconnected, which is impossible if $K_{c_1} \vee K_{c_2}$ is homeomorphic to the 2-sphere. For more details, see [M2]. ☐

1.2 Matings with the Basilica Polynomial

Matings can be particularly useful in describing the dynamics in certain one-parameter families of rational maps. The best studied example of such a family is

$$R_a(z) := \frac{a}{z^2 + 2z} \quad \text{for} \quad a \in \mathbb{C} \setminus \{0\},$$

which is referred to as the *basilica family.*

The critical points for R_a are ∞ and -1. Observe that $\{\infty, 0\}$ is a superattracting 2-periodic orbit for R_a. Let $\mathcal{A}_a^\infty$ be the attracting basin of $\{\infty, 0\}$. The boundary of $\mathcal{A}_a^\infty$ is equal to the Julia set $J(R_a)$.

Proposition 1.2.1. *Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quadratic rational map with a superattracting 2-periodic orbit. Then by a linear change of coordinates, f can be normalized as either:*

(i) R_a for some $a \in \mathbb{C} \setminus \{0\}$, or

(ii) $z \mapsto \frac{1}{z^2}$.

Proof. By a linear change of coordinates, we may assume that f has a superattracting 2-periodic orbit $\{\infty, 0\}$ with a critical point at ∞. Let

$$f(z) = \frac{a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0}.$$

Since $f(\infty) = 0$ and $f(0) = \infty$, we have $a_2 = b_0 = 0$, $b_2 \neq 0$ and $a_0 \neq 0$. If $a_1 \neq 0$, then for r sufficiently large, we have

$$f(re^\theta) \sim \frac{a_1}{b_2 r} e^{-\theta}.$$

This implies that ∞ cannot be a critical point for f by the argument principle. Hence, we must have $a_1 = 0$. These observations yield the following expression for f:

$$f(z) = \frac{a}{z^2 + bz} \qquad \text{with} \qquad a \in \mathbb{C} \setminus \{0\}.$$

If the second critical point for f is equal to 0, then by using a similar argument as above, we see that $b = 0$. In this case, we have

$$f(\lambda z)/\lambda = \frac{1}{z^2},$$

where λ is a cube root of a.

On the other hand, if the second critical point for f is not equal to 0, then we may assume by a linear change of coordinates that it is equal to -1. A straightforward computation shows that $f'(-1) = 0$ if and only if $b = 2$, which means $f = R_a$ as claimed. ☐

Analogously to $\mathcal{M}$, the non-escape locus in the parameter space for R_a is defined as

$$\mathcal{M}_{\mathbf{B}} := \{a \in \mathbb{C} \setminus \{0\} \mid -1 \notin \mathcal{A}_a^\infty\}.$$

We also define the following subset of $\mathcal{M}_{\mathbf{B}}$:

$$\mathcal{L}_{\mathbf{B}} := \{a \in \mathcal{M}_{\mathbf{B}} \mid J(R_a) \text{ is locally connected}\}.$$

The basilica polynomial

$$f_{\mathbf{B}}(z) := z^2 - 1$$

is the only member of the quadratic family that has a superattracting 2-periodic orbit. Let $K_{\mathbf{B}}$ be the filled Julia set for $f_{\mathbf{B}}$. The following result is an analogue of the Böttcher uniformization theorem for the quadratic family. Refer to [AYa] for the proof.

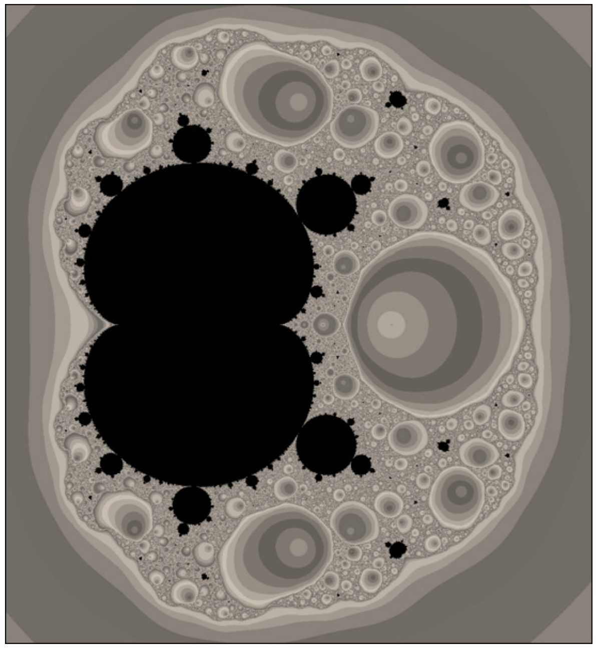

Figure 1.3: The non-escape locus $\mathcal{M}_{\mathbf{B}}$ for R_a (in black). At the center of the largest component of $\mathcal{M}_{\mathbf{B}}$ is the rational map R_1, which realizes the conformal mating of the basilica polynomial $f_{\mathbf{B}}$ with the squaring map f_0. Compare with Figure 1.1. Note that instead of a copy of the 1/2-limb $L_{1/2}$, the main component of $\mathcal{M}_{\mathbf{B}}$ has a second cusp at 0 (see Example 1.1.5).

Proposition 1.2.2. *Suppose $a \in \mathcal{M}_{\mathbf{B}}$. Then there exists a unique conformal map $\psi_a : \mathcal{A}_a^\infty \to \mathring{K}_{\mathbf{B}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}_a^\infty & \xrightarrow{R_a} & \mathcal{A}_a^\infty \\ \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_a} \\ \mathring{K}_{\mathbf{B}} & \xrightarrow{f_{\mathbf{B}}} & \mathring{K}_{\mathbf{B}} \end{array}$$

Moreover, if B is a connected component of $\mathcal{A}_a^\infty$, then ψ_a extends to a homeomorphism between $\overline{B}$ and $\psi_a(\overline{B})$.

Suppose for some $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$, the quadratic polynomials f_c and $f_{\mathbf{B}}$ are conformally mateable. If $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a conformal mating of f_c and $f_{\mathbf{B}}$, then F has a superattracting 2-periodic orbit. By Proposition 1.2.1, F can be normalized as R_a for some $a \in \mathcal{L}_{\mathbf{B}}$.

In view of Proposition 1.2.2, it is natural to ask whether for every $a \in \mathcal{L}_{\mathbf{B}}$, the quadratic rational map R_a is a conformal mating of f_c and $f_{\mathbf{B}}$ for some $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$. It turns out this cannot be true: for some $a \in \mathcal{L}_{\mathbf{B}}$, the map R_a can only be identified as the product more general form of mating called *mating with laminations* between f_c and $f_{\mathbf{B}}$ with $c \notin \mathcal{L}$ (see [Du]). However, the following weaker statement does hold. The proof is completely analogous to the proof of Proposition 1.1.4, so we omit it here.

Proposition 1.2.3. *Suppose R_a is a conformal mating. Then R_a is a conformal mating of f_c and $f_{\mathbf{B}}$ for some $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$.*

The principal motivation in this chapter is to answer the following question:

Motivating Question. *Suppose $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$. Are f_c and $f_{\mathbf{B}}$ conformally mateable? If so, is there a unique member of the basilica family that realizes their conformal mating?*

We now summarize the known results on this topic.

Theorem 1.2.4 (Rees, Tan, Shishikura [Re, Tan, Sh2])**.** *Suppose $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$. If f_c is hyperbolic, then f_c and $f_{\mathbf{B}}$ are conformally mateable. Moreover, their conformal mating is unique up to conjugacy by a Möbius map.*

Theorem 1.2.4 is actually a corollary of a much more general result which states that two post-critically finite quadratic polynomials f_{c_1} and f_{c_2} are (essentially) mateable if and only if c_1 and c_2 do not belong to conjugate limbs of the Mandelbrot set. See [Tan] for more details.

Theorem 1.2.5 (Aspenberg, Yampolsky [AYa])**.** *Suppose $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$. If f_c is at most finitely renormalizable and has no non-repelling periodic orbits, then f_c and $f_{\mathbf{B}}$ are conformally mateable. Moreover, their conformal mating is unique up to conjugacy by a Möbius map.*

Theorem 1.2.6 (Dudko [Du])**.** *Suppose $c \in \mathcal{L} \cap (\mathbb{C} \setminus L_{1/2})$. If f_c is at least 4 times renormalizable, then f_c and $f_{\mathbf{B}}$ are conformally mateable. Moreover, their conformal mating is unique up to conjugacy by a Möbius map.*

Together, Theorem 1.2.4, 1.2.5 and 1.2.6 provide a positive answer to the main question in almost all cases. However, the parameters contained in the boundary of hyperbolic components that are not too "deep" inside the Mandelbrot set are still left unresolved. We discuss these parameters in greater detail in the next section.

1.3 Matings in the Boundary of Hyperbolic Components

Let H be a hyperbolic component of $\mathcal{M} \setminus L_{1/2}$. By Theorem 1.2.4, the quadratic polynomial f_c and the basilica polynomial $f_{\mathbf{B}}$ are conformally mateable for all $c \in H$. Our goal is to determine if this is also true for $c \in \partial H \cap \mathcal{L}$.

Choose a parameter value $c_0 \in H$, and let $a_0 \in \mathcal{M}_{\mathbf{B}}$ be a parameter value such that R_{a_0} is a conformal mating of f_{c_0} and $f_{\mathbf{B}}$. Since R_{a_0} must be hyperbolic, a_0 is contained in some hyperbolic component $H_{\mathbf{B}}$ of $\mathcal{M}_{\mathbf{B}}$.

For all $c \in \overline{H}$, the quadratic polynomial f_c has a non-repelling n-periodic orbit $\mathbf{O}_c := \{f_c^i(z_c)\}_{i=0}^{n-1}$ for some fixed $n \in \mathbb{N}$ (see e.g. [M2]). Likewise, for all $a \in \overline{H_{\mathbf{B}}}$, the quadratic rational map R_a has a non-repelling n-periodic orbit $\mathcal{O}_a := \{R_a^i(w_a)\}_{i=0}^{n-1}$. Define the multiplier maps $\lambda : \overline{H} \to \overline{\mathbb{D}}$ and $\mu : \overline{H_{\mathbf{B}}} \to \overline{\mathbb{D}}$ by:

$$\lambda(c) := (f_c^n)'(z_c) \quad \text{ and } \quad \mu(a) := (R_a^n)'(w_a).$$

It is known that λ and μ are homeomorphisms which are conformal on the interior of their domains (see [M2]).

The following result can be proved using a standard application of quasiconformal surgery (see chapter 4 in [BF]).

Proposition 1.3.1. *Define a homeomorphism $\phi_H : \overline{H} \to \overline{H_{\mathbf{B}}}$ by*

$$\phi_H := \mu^{-1} \circ \lambda.$$

Then for all $c \in H$, the quadratic rational map $R_{\phi_H(c)}$ is a conformal mating of f_c and $f_{\mathbf{B}}$.

Our goal is to extend the statement of Proposition 1.3.1 to the boundary of H where possible.

Consider $c \in \partial H$, and let $a = \phi_H(c) \in \partial H_{\mathbf{B}}$. The multiplier of $\mathbf{O}_c$ and $\mathcal{O}_a$ is equal to $e^{2\pi\theta i}$ for some $\theta \in \mathbb{R}/\mathbb{Z}$. The number θ is referred to as the *rotation number.* If θ is rational, then $\mathbf{O}_c$ and $\mathcal{O}_a$ are parabolic. In this case, an application of trans-quasiconformal surgery due to Haïssinsky implies the following result (see [Ha]).

Theorem 1.3.2. *Suppose that the rotation number θ is rational, so that $\mathbf{O}_c$ and $\mathcal{O}_a$ are parabolic. Then f_c and $f_{\mathbf{B}}$ are conformally mateable, and R_a is the unique member of the basilica family that realizes their conformal mating.*

If θ is irrational, then $\mathbf{O}_c$ is either Siegel or Cremer. In the latter case, it is known that the Julia set J_c for f_c is non-locally connected (see e.g. [M3]). This means that the formal mating of f_c and $f_{\mathbf{B}}$ cannot be defined, and hence, they are not conformally mateable.

For our discussion of the Siegel case, we first recall a classical result of Siegel [S]. An irrational number x is said to be *Diophantine of order κ* if there exists a fixed constant $\epsilon > 0$ such that for all $p/q \in \mathbb{Q}$, the following inequality holds:

$$|x - \frac{p}{q}| \geq \frac{\epsilon}{q^\kappa}.$$

The set of all irrational numbers that are Diophantine of order κ is denoted $\mathcal{D}(\kappa)$. The smallest possible value of κ such that $\mathcal{D}(\kappa)$ is non-empty is 2 (see [M3]).

Theorem 1.3.3 (Siegel [S])**.** *Let $f : U \to V$ be an analytic function. Suppose f has an indifferent periodic orbit $\mathcal{O}$ with an irrational rotation number θ. If $\theta \in \mathcal{D}(\kappa)$ for some $\kappa \geq 2$, then $\mathcal{O}$ is a Siegel orbit.*

There is a classical connection between Diophantine classes and continued fraction approximations (see e.g. [M3]). In particular, if

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

is the continued fraction representation of x, then $x \in \mathcal{D}(2)$ if and only if all the a_i's are uniformly bounded. In view of this, we say that the numbers contained in $\mathcal{D}(2)$ are of *bounded type.* Siegel quadratic polynomials of bounded type are prominently featured in the study of renormalization (see e.g. [P, Mc1, Ya1, Ya3]).

Theorem 1.3.4 (Peterson [P])**.** *Suppose a quadratic polynomial f_c has an indifferent periodic orbit with an irrational rotation number of bounded type. Then f_c has a locally connected Julia set J_c.*

In this chapter, we present a positive answer to the motivating question (stated in Section 1.2) for quadratic polynomials $f_{\mathbf{S}}$ that have an indifferent fixed point with an irrational rotation number of bounded type. Note that by Theorem 1.3.3, the indifferent fixed point is Siegel, and by Theorem 1.3.4, the formal mating of $f_{\mathbf{S}}$ and $f_{\mathbf{B}}$ is well defined.

The solution to the uniqueness part of the main question is elementary.

Proposition 1.3.5. *Suppose $\lambda \in \overline{\mathbb{D}}$. Then there exists a unique $c \in \mathcal{M}$ (resp. $a \in \mathcal{M}_{\mathbf{B}}$) such that f_c (resp. R_a) has a non-repelling fixed point $z_0 \neq \infty$ with multiplier λ.*

Proof. Suppose f_c has a fixed point $z_0 \neq \infty$ with multiplier $\lambda \in \mathbb{C}$. It is easy to check that the value of c is given by

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

Hence, c is uniquely determined.

Likewise, suppose R_a has a fixed point with multiplier $\lambda \in \mathbb{C}$. Then the value of a is given by

$$a = -\frac{8\lambda}{(\lambda - 1)^3}.$$

Hence, a is uniquely determined. □

Our main results are stated below.

Main Theorem 1A. *Suppose $\nu \in \mathbb{R} \setminus \mathbb{Q}$ is of bounded type. Let R_{a_ν} with $a_\nu \in \mathcal{M}_{\mathbf{B}}$ be the unique member of the basilica family that has a Siegel fixed point z_0 with rotation number ν. Let S be the fixed Siegel disc containing z_0. Then S is a quasidisk, and contains the critical point -1 in its boundary.*

Main Theorem 1B. *Suppose $\nu \in \mathbb{R} \setminus \mathbb{Q}$ is of bounded type. Let $f_{\mathbf{S}}$ be the unique member of the quadratic family that has a Siegel fixed point with rotation number ν. Then $f_{\mathbf{S}}$ and $f_{\mathbf{B}}$ are conformally mateable, and R_{a_ν} is the unique member of the basilica family that realizes their conformal mating.*

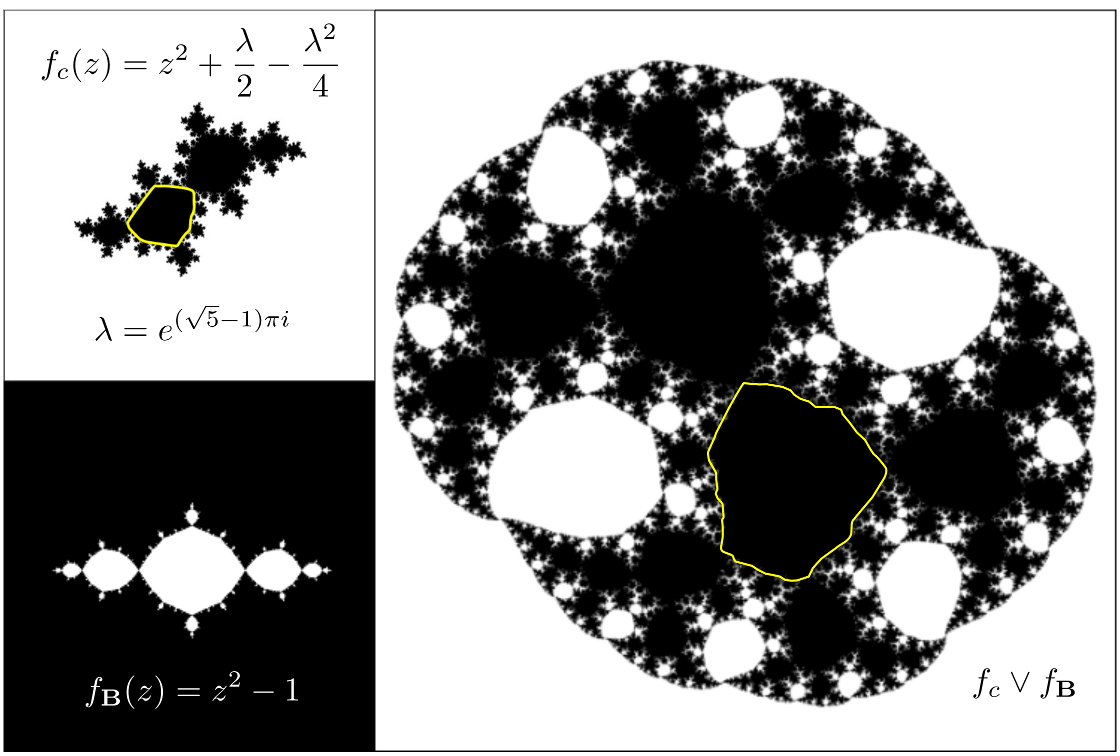

Figure 1.4: The Siegel polynomial f_c with $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$ and $\lambda = e^{(\sqrt{5}-1)\pi i}$ mated with the basilica polynomial $f_{\mathbf{B}}$. The Siegel disc is highlighted.

1.4 The Construction of a Blaschke Product Model

Consider the Blaschke product

$$F_{a,b}(z) := -\frac{1}{e^{i\theta}} \frac{z(z-a)(z-b)}{(1-\bar{a}z)(1-\bar{b}z)},$$

where $ab = re^{i\theta}$ with $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi)$. Note that 0 is a fixed point with multiplier $-r$.

Lemma 1.4.1. *For any value of r and θ, the parameters $a = a(r,\theta)$ and $b = b(r,\theta)$ can be chosen such that $F_{a,b}$ has a double critical point at 1.*

Proof. Let

$$F'_{a,b}(z) = \frac{P(z)}{Q(z)}.$$

Then

$$F''_{a,b}(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}.$$

Thus, the condition

$$F'_{a,b}(1) = F''_{a,b}(1) = 0$$

is equivalent to

$$P(1) = P'(1) = 0.$$

A straightforward computation shows that

$$P(z) = \overline{\kappa}z^4 - 2\overline{\zeta}z^3 + (3 - |\kappa|^2 + |\zeta|^2)z^2 - 2\zeta z + \kappa,$$

where

$$\kappa := ab \quad \text{and} \quad \zeta := a + b.$$

Thus, $F_{a,b}$ has a double critical point at 1 if the following two equations are satisfied:

$$2\kappa - 3\zeta + (3 - |\kappa|^2 + |\zeta|^2) = \overline{\zeta} \tag{1.2}$$

$$3\kappa - 2\zeta + (3 - |\kappa|^2 + |\zeta|^2) = \overline{\kappa}. \tag{1.3}$$

Subtracting (1.2) from (1.3), we see that

$$\kappa - \zeta = \overline{\kappa} - \overline{\zeta}.$$

Substituting $\kappa = x + iy$ and $\zeta = u + iy$ into (1.2), we obtain

$$u^2 - 4u + (2x - x^2 + 3) = 0. \tag{1.4}$$

The equation (1.4) has two solutions: $u = -x + 3$ and $u = x + 1$. The first solution corresponds to the relation

$$\zeta = -\overline{\kappa} + 3.$$

Therefore, by choosing a and b to be the solutions of

$$z^2 + (re^{-i\theta} - 3)z + re^{i\theta} = 0,$$

we ensure that the map $F_{a,b}$ has a double critical point at 1. $\square$

Lemma 1.4.2. *Let $a = a(r, \theta)$ and $b = b(r, \theta)$ satisfy the condition in Lemma 1.4.1. Then for all $r > 1$ sufficiently close to 1, there exists a local holomorphic change of coordinates ϕ at 0 so that the map $G := \phi^{-1} \circ F_{a,b}^2 \circ \phi$ takes the form*

$$G(z) = r^2 z(1 + z^2 + \mathcal{O}(z^3)).$$

Proof. Expanding $F_{a,b}(z)$ as a power series around 0, we have

$$F_{a,b}(z) = -rz + \lambda z^2 + \mathcal{O}(z^3)$$

for some $\lambda = \lambda(r, \theta)$ depending continuously on r and θ. Define

$$\psi_\mu(z) := z + \mu z^2 \quad \text{for} \quad \mu \in \mathbb{C}.$$

A straightforward computation shows that

$$H(z) := \psi_\mu^{-1} \circ F_{a,b} \circ \psi_\mu(z) = -rz + (\lambda + (1 + r)\mu)z^2 + \mathcal{O}(z^3).$$

Thus, by choosing

$$\mu = \frac{-\lambda}{1 + r},$$

we have

$$H(z) = -rz(1 + \nu z^2 + \mathcal{O}(z^3))$$

for some $\nu = \nu(r, \theta)$ depending continuously on r and θ.

Observe that the second iterate of H is equal to

$$H^2(z) = r^2 z(1 + (1 + r^2)\nu z^2 + \mathcal{O}(z^3)).$$

When $r = 1$, the point 0 is a parabolic fixed point of multiplicity 2. This means that $\nu(1, \theta)$ cannot be equal to zero for all $\theta \in [0, 2\pi)$. Hence, for some $\epsilon > 0$ sufficiently small, $\nu(r, \theta)$ is not equal to zero for all $r \in (1, 1 + \epsilon)$ and $\theta \in [0, 2\pi)$. After one more change of coordinates, we arrive at

$$G(z) := \sqrt{(1 + r^2)\nu} \cdot H^2 \left(\frac{z}{\sqrt{(1 + r^2)\nu}} \right) = r^2 z(1 + z^2 + \mathcal{O}(z^3)).$$

$\square$

Lemma 1.4.3. *Let $a = a(r, \theta)$ and $b = b(r, \theta)$ satisfy the condition in Lemma 1.4.1. Then for all $r > 1$ sufficiently close to 1, the Blaschke product $F_{a,b}$ has an attracting 2-periodic orbit near 0.*

Proof. Consider the map $G := \phi^{-1} \circ F_{a,b}^2 \circ \phi$ defined in Lemma 1.4.2. We prove that G has two attracting fixed points near 0.

Observe that G satisfies

$$|G(z)| = r^2|z|(1 + \mathrm{Re}(z^2) + (\text{higher terms}))$$

and

$$\arg(G(z)) = \arg(z) + \mathrm{Im}(z^2) + (\text{higher terms}).$$

Consider the wedge shaped regions

$$V_\epsilon^+ := \{\rho e^{2\pi i t} \in \mathbb{C} \mid 0 \le \rho \le \epsilon, \frac{3}{16} \le t \le \frac{5}{16}\}$$

and

$$V_\epsilon^- := -V_\epsilon^+.$$

It is easily checked that $G(V_\epsilon^+) \subset V_\epsilon^+$ and $G(V_\epsilon^-) \subset V_\epsilon^-$. Since 0 is the only fixed point on the boundary of these regions, and it is repelling, V_ϵ^+ and V_ϵ^- must each contain an attracting fixed point for G. □

Theorem 1.4.4. *Given any angle $\nu \in [0, 2\pi)$, there exists a Blaschke product F_ν that satisfies the following three properties:*

(i) There exists a superattracting 2-periodic orbit $\mathcal{O} = \{\infty, F_\nu(\infty)\}$ with a critical point at ∞.

(ii) The rotation number of the map $F_\nu|_{\partial\mathbb{D}}$ is equal to ν.

(iii) The point 1 is a double critical point.

Proof. The family of Blaschke products $\{F_{a,b}\}$ that satisfy Lemma 1.4.1 and 1.4.3 are continuously parameterized by r and θ. Let $\rho(r, \theta)$ denote the rotation number of the map $F_{a,b}|_{\partial\mathbb{D}}$. In [YaZ], it is proved that $\rho(1, \cdot)$ is not nullhomotopic. By continuity, $\rho(r, \cdot)$ is also not nullhomotopic. Thus, for any angle $\nu \in [0, 2\pi)$, there exists θ such that $\rho(r, \theta) = \nu$.

So far, we have proved the existence of a Blaschke product $F_{a,b}$ that has an attracting 2-periodic orbit near zero, has a double critical point at 1, and whose restriction to $\partial\mathbb{D}$ has rotation number equal to ν. A standard application of quasiconformal surgery turns the attracting 2-periodic orbits of $F_{a,b}$ into superattracting orbits (the surgery must be symmetric with respect to the unit circle to ensure that the resulting map is also a Blaschke product). Then after conjugating by the appropriate Blaschke factor, we obtain the desired map F_ν. □

Theorem 1.4.5. *Suppose ν is irrational and of bounded type. Let F_ν be the Blaschke product constructed in Theorem 1.4.4. Then there exists a quadratic rational function R_ν and quasiconformal maps $\psi : \mathbb{D} \to \mathbb{D}$, and $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\psi(1) = 1$; $\phi(1) = 1$, $\phi(\infty) = \infty$, and $\phi(\psi(0)) = 0$; and*

$$R_\nu(z) = \begin{cases} \phi \circ \psi \circ \mathrm{Rot}_\nu \circ \psi^{-1} \circ \phi^{-1}(z) & : \textit{if } z \in \phi(\mathbb{D}) \\ \phi \circ F_\nu \circ \phi^{-1}(z) & : \textit{if } z \in \hat{\mathbb{C}} \setminus \phi(\mathbb{D}), \end{cases}$$

where Rot_ν denotes rigid rotation by angle ν.

Proof. Since ν is of bounded type, there exists a unique homeomorphism $\psi : (\partial\mathbb{D}, 1) \to (\partial\mathbb{D}, 1)$ such that

$$\psi \circ \mathrm{Rot}_\nu \circ \psi^{-1} = F_\nu|_{\partial\mathbb{D}},$$

and ψ extends to a quasiconformal map on $\mathbb{D}$.

Define

$$g(z) = \begin{cases} \psi \circ \mathrm{Rot}_\nu \circ \psi^{-1}(z) & : \text{if } z \in \mathbb{D} \\ F_\nu(z) & : \text{if } z \in \hat{\mathbb{C}} \setminus \mathbb{D}. \end{cases}$$

By construction, g is continuous.

To obtain a holomorphic map with the same dynamics as g, we define and integrate a new complex structure μ on $\hat{\mathbb{C}}$. Start by defining μ on $\mathbb{D}$ as the pullback of the standard complex structure σ_0 by ψ^{-1}. Next, pull back μ on $\mathbb{D}$ by the iterates of g to define μ on the iterated preimages of $\mathbb{D}$. Finally, extend μ to the rest of $\hat{\mathbb{C}}$ as the standard complex structure σ_0.

Let $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the unique solution of the Beltrami equation

$$\partial_{\overline{z}}\phi(z) = \mu(z)\partial_z\phi(z)$$

such that $\phi(1) = 1$, $\phi(\infty) = \infty$ and $\phi(\psi(0)) = 0$. Then the map

$$R_\nu := \phi \circ g \circ \phi^{-1}$$

gives us the desired quadratic rational function. □

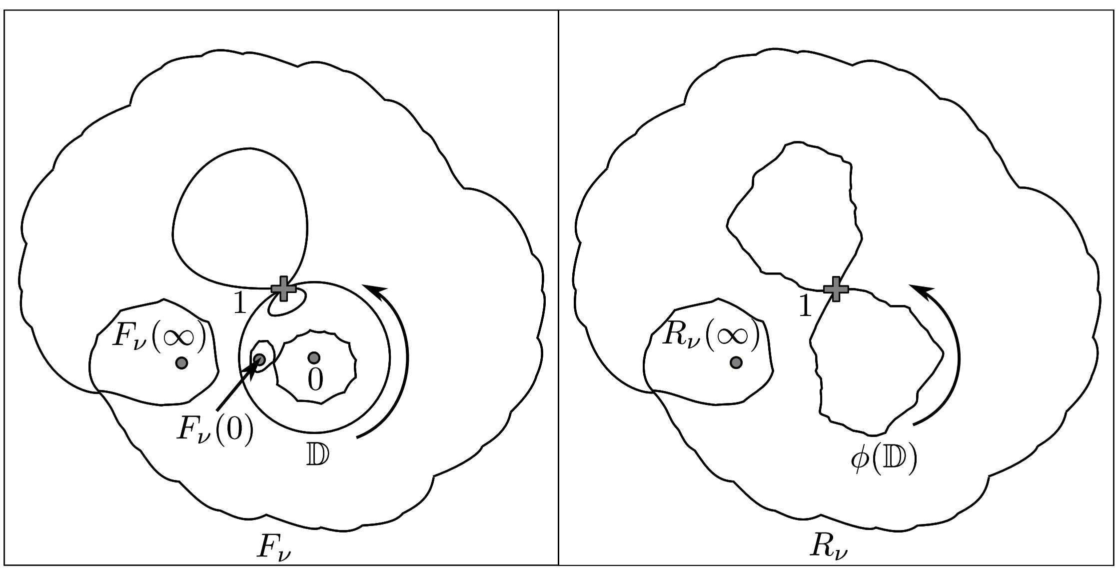

Figure 1.5: An illustration of the quasiconformal surgery in Theorem 1.4.5. The image of $\mathbb{D}$ under the quasiconformal map ϕ is a Siegel disc for R_ν. Also note that the double critical point for F_ν (represented by a cross) becomes a single critical point for R_ν.

Proof of Main Theorem 1A.

Consider the quadratic rational function R_ν constructed in Theorem 1.4.5. Observe that R_ν satisfies the following three properties:

(i) There exists a superattracting 2-periodic orbit $\{\infty, R_\nu(\infty)\}$ with a critical point at ∞.

(ii) The image of $\mathbb{D}$ under the quasiconformal map ϕ is a Siegel disc with rotation number ν.

(iii) The point 1 is a critical point, and is contained in $\partial\phi(\mathbb{D})$.

Clearly, the critical value $R_\nu(\infty)$ is not equal to the critical point 1. The theorem now follows from Proposition 1.2.1. □

1.5 The Construction of Bubble Rays

1.5.1 For the basilica polynomial

Consider the basilica polynomial

$$f_{\mathbf{B}}(z) := z^2 - 1.$$

Note that $f_{\mathbf{B}}$ has a superattracting 2-periodic orbit $\{0, -1\}$, and hence, is hyperbolic. Denote the Julia set and the filled Julia set for $f_{\mathbf{B}}$ by $J_{\mathbf{B}}$ and $K_{\mathbf{B}}$ respectively. The following is a consequence of the hyperbolicity of $f_{\mathbf{B}}$ (see e.g. [M3]).

Proposition 1.5.1. *The Julia set $J_{\mathbf{B}}$ for $f_{\mathbf{B}}$ is locally connected.*

A connected component of $\mathbf{B} := \mathring{K}_{\mathbf{B}}$ is called a *bubble*. Let $\mathbf{B}_0$ be the bubble containing the critical point 0. We have

$$\mathbf{B} = \bigcup_{n=0}^{\infty} f_{\mathbf{B}}^{-n}(\mathbf{B}_0).$$

Let $B \subset \mathbf{B}$ be a bubble. The *generation of* B, denoted by $\text{gen}(B)$, is defined to be the smallest number $n \in \mathbb{N}$ such that $f_{\mathbf{B}}^n(B) = \mathbf{B}_0$. The *center of* B is the unique point $z \in B$ that is mapped to 0 under $f_{\mathbf{B}}^{\text{gen}(B)}$.

Proposition 1.5.2. *There exists a unique repelling fixed point* $\mathbf{b}$ *contained in* $\partial\mathbf{B}_0$*.*

Note that the repelling fixed point $\mathbf{b}$ in Proposition 1.5.2 is the α-fixed point of $f_{\mathbf{B}}$ (see [M2]).

Let $b \in J_{\mathbf{B}}$ be an iterated preimage of $\mathbf{b}$. The *generation of* b, denoted by $\text{gen}(b)$, is defined to be the smallest number $n \in \mathbb{N}$ such that $f_{\mathbf{B}}^n(b) = \mathbf{b}$. Suppose b is contained in the boundary of some bubble B. If the generation of b is the smallest among all iterated preimages of $\mathbf{b}$ that are contained in ∂B, then b is called the *root of* B. It is easy to see that every bubble has a unique root.

Proposition 1.5.3. *Let $b \in J_{\mathbf{B}}$ be an iterated preimage of* $\mathbf{b}$*. Then there are exactly two bubbles B_1 and B_2 in* $\mathbf{B}$ *which contain b in their closures. Moreover, we have*

$$\overline{B_1} \cap \overline{B_2} = \{b\}.$$

Proof. There are exactly two bubbles, $\mathbf{B}_0$ and $f_{\mathbf{B}}(\mathbf{B}_0)$, that contain $\mathbf{b}$ in their closures. Moreover, we have

$$\overline{\mathbf{B}_0} \cap \overline{f_{\mathbf{B}}(\mathbf{B}_0)} = \{\mathbf{b}\}.$$

There exists a neighbourhood N containing b such that N is mapped conformally onto a neighbourhood of $\mathbf{b}$ by $f_{\mathbf{B}}^{\text{gen}(b)}$. The result follows. □

Let $b \in J_{\mathbf{B}}$ be an iterated preimage of $\mathbf{b}$, and let B_1 and B_2 be the two bubbles that contain b in their closures. Suppose $\text{gen}(B_1) < \text{gen}(B_2)$. Then B_1 and B_2 are referred to as the *parent* and the *child at* b respectively. Note that b must be the root of B_2.

Consider a set of bubbles $\{B_i\}_{i=0}^n$ in $\mathbf{B}$, and a set of iterated preimages $\{b_i\}_{i=0}^n$ of $\mathbf{b}$ such that the following properties are satisfied:

(i) $B_0 = \mathbf{B}_0$ and $b_0 = \mathbf{b}$, and

(ii) for $1 \leq i \leq n$, the bubbles B_{i-1} and B_i are the parent and the child at b_i respectively.

The set

$$\mathcal{R}^{\mathbf{B}} := \overline{f_{\mathbf{B}}(\mathbf{B}_0)} \cup (\bigcup_{i=0}^{n} \overline{B_i})$$

is called a *bubble ray for* $f_{\mathbf{B}}$ (the inclusion of $\overline{f_{\mathbf{B}}(\mathbf{B}_0)}$ is to ensure that a bubble ray is mapped to a bubble ray). For conciseness, we use the notation $\mathcal{R}^{\mathbf{B}} \sim \{B_i\}_{i=0}^n$. The bubble ray $\mathcal{R}^{\mathbf{B}}$ is said to be *finite* or *infinite* according to whether $n < \infty$ or $n = \infty$. Lastly, $\{b_i\}_{i=0}^n$ is called the *set of attachment points for* $\mathcal{R}^{\mathbf{B}}$.

Proposition 1.5.4. *If $B \subset \mathbf{B}$ is a bubble, then there exists a unique finite bubble ray $\mathcal{R}^{\mathbf{B}} \sim \{B_i\}_{i=0}^n$ such that $B_n = B$. Consequently, if $\mathcal{R}_1^{\mathbf{B}} \sim \{B_i^1\}_{i=0}^n$ and $\mathcal{R}_2^{\mathbf{B}} \sim \{B_i^2\}_{i=0}^m$ are two bubble rays, then there exists $N \geq 0$ such that $B_i^1 = B_i^2$ for all $i \leq N$, and $B_i^1 \neq B_i^2$ for all $i > N$.*

Proof. We can construct a finite bubble ray ending in B as follows. First, let $\tilde{B}_0 = B$. Next, let $\tilde{b}_0$ be the root of $\tilde{B}_0$, and let $\tilde{B}_1$ be the parent of $\tilde{B}_0$ at $\tilde{b}_0$. Proceeding inductively, we obtain a sequence of bubbles $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \ldots$, and a sequence of roots $\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots$, such that $\tilde{B}_{i+1}$ is the parent of $\tilde{B}_i$ at $\tilde{b}_i$. Since $\operatorname{gen}(\tilde{B}_{i+1})$ is strictly less than $\operatorname{gen}(\tilde{B}_i)$, this sequence must terminate at $\tilde{B}_n = \mathbf{B}_0$ for some $n \geq 0$. Then $\mathcal{R}^{\mathbf{B}} \sim \{\tilde{B}_{n-i}\}_{i=0}^n$ is the desired finite bubble ray. The uniqueness of $\mathcal{R}^{\mathbf{B}}$ follows from the uniqueness of the root of a bubble and Proposition 1.5.3. □

Let $\mathcal{R}^{\mathbf{B}} \sim \{B_i\}_{i=0}^\infty$ be an infinite bubble ray. We say that $\mathcal{R}^{\mathbf{B}}$ *lands at* $z \in J_{\mathbf{B}}$ if the sequence of bubbles $\{B_i\}_{i=0}^\infty$ converges to z in the Hausdorff topology. The following result is a consequence of the hyperbolicity of $f_{\mathbf{B}}$ (see [DH1]).

Proposition 1.5.5. *There exists $0 < s < 1$, and $C > 0$ such that for every bubble $B \subset \mathbf{B}$, we have*

$$\operatorname{diam}(B) < C s^{\operatorname{gen}(B)}.$$

Consequently, every infinite bubble ray for $f_{\mathbf{B}}$ lands.

Denote the attracting basin of infinity for $f_{\mathbf{B}}$ by $\mathbf{A}_{\mathbf{B}}^\infty$. Let

$$\phi_{\mathbf{A}_{\mathbf{B}}^\infty} : \mathbf{A}_{\mathbf{B}}^\infty \to \mathbb{C} \setminus \overline{\mathbb{D}}$$

and

$$\phi_{\mathbf{B}_0} : \mathbf{B}_0 \to \mathbb{D}$$

be the Böttcher uniformization of $f_{\mathbf{B}}$ on $\mathbf{A}_{\mathbf{B}}^\infty$ and $\mathbf{B}_0$ respectively. Using $\phi_{\mathbf{A}_{\mathbf{B}}^\infty}$ and $\phi_{\mathbf{B}_0}$, we can encode the dynamics of bubble rays for $f_{\mathbf{B}}$ in two different ways: via external angles, and via bubble addresses.

Suppose that $\mathcal{R}^{\mathbf{B}}$ is an infinite bubble ray, and let $z \in J_{\mathbf{B}}$ be its landing point. Then there exists a unique external ray

$$\mathcal{R}_{-t}^\infty := \{\arg(\phi_{\mathbf{A}_{\mathbf{B}}^\infty}) = -t\}$$

which lands at z (note that arg stands for the argument of a complex number—e.g. if $w = re^{2\pi i\theta}$, then $\arg(w) = \theta$). The *external angle of* $\mathcal{R}^{\mathbf{B}}$ is defined to be t. Henceforth, the infinite bubble ray with external angle t will be denoted $\mathcal{R}_t^{\mathbf{B}}$.

Let $b \in \partial \mathbf{B}_0$ be an iterated preimage of $\mathbf{b}$. Define

$$\operatorname{adr}(b) := \arg(\phi_{\mathbf{B}_0}(b)).$$

If b' is an interated preimage of $\mathbf{b}$ and $b' \notin \partial \mathbf{B}_0$, then there exists a unique bubble $B \subset \mathbf{B}$ such that B is the parent at b'. In this case, define

$$\operatorname{adr}(b') := \operatorname{adr}(f_{\mathbf{B}}^{\operatorname{gen}(B)}(b')).$$

Let $\mathcal{R}^{\mathbf{B}}$ be a bubble ray and let $\{b_i\}_{i=0}^n$ be the set of attachment points for $\mathcal{R}^{\mathbf{B}}$. The *bubble address of* $\mathcal{R}^{\mathbf{B}}$ is defined to be

$$\operatorname{adr}(\mathcal{R}^{\mathbf{B}}) := (\operatorname{adr}(b_0),\ \operatorname{adr}(b_1),\ \ldots,\ \operatorname{adr}(b_n)),$$

where the tuple is interpreted to be infinite if $\mathcal{R}^{\mathbf{B}}$ is an infinite bubble ray.

If $B \subset \mathbf{B}$ is a bubble, then by Proposition 1.5.4, there exists a unique finite bubble ray $\mathcal{R}^{\mathbf{B}} \sim \{B_i\}_{i=0}^n$ such that $B = B_n$. The *bubble address of* B is defined to be

$$\operatorname{adr}(B) := \operatorname{adr}(\mathcal{R}^{\mathbf{B}}).$$

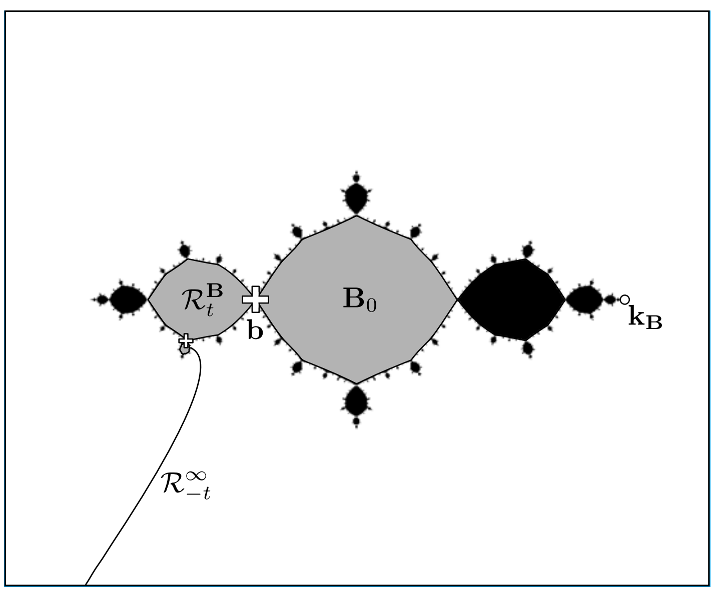

Figure 1.6: The infinite bubble ray $\mathcal{R}_t^{\mathbf{B}}$ with $t \approx 0.354841$ for the basilica polynomial $f_{\mathbf{B}}$. The bubbles contained in $\mathcal{R}_t^{\mathbf{B}}$ are colored in light gray. The white crosses represent the set of attachment points for $\mathcal{R}_t^{\mathbf{B}}$.

1.5.2 For the Siegel polynomial

Suppose $\nu \in \mathbb{R} \setminus \mathbb{Q}$ is of bounded type, and let $f_{\mathbf{S}}$ be the unique member of the quadratic family that has a Siegel fixed point z_0 with rotation number ν. Denote the Siegel disc, the Julia set and the filled Julia set for $f_{\mathbf{S}}$ by $\mathbf{S}_0$, $J_{\mathbf{S}}$ and $K_{\mathbf{S}}$ respectively. By Theorem 1.3.4, $J_{\mathbf{S}}$ is locally connected. A quasiconformal surgery procedure due to Douady, Ghys, Herman, and Shishikura (see e.g. [P]) implies the following:

Theorem 1.5.6. *The Siegel disc $\mathbf{S}_0$ is a quasidisc whose boundary contains the critical point 0.*

A connected component of $\mathbf{S} := \mathring{K}_{\mathbf{S}}$ is called a *bubble*. Note that

$$\mathbf{S} = \bigcup_{n=0}^{\infty} f_{\mathbf{S}}^{-n}(\mathbf{S}_0).$$

Let $S \subset \mathbf{S}$ be a bubble. The *generation of* S, denoted by $\mathrm{gen}(S)$, is defined to be the smallest number $n \in \mathbb{N}$ such that $f_{\mathbf{S}}^n(S) = \mathbf{S}_0$. The *center of* S is the unique point $z \in S$ that is mapped to the Siegel fixed point z_0 by $f_{\mathbf{S}}^{\mathrm{gen}(S)}$.

Let $s \in J_{\mathbf{S}}$ be an iterated preimage of the critical point 0. The *generation of* s, denoted by $\mathrm{gen}(s)$, is defined to be the smallest number $n \in \mathbb{N}$ such that $f_{\mathbf{S}}^n(s) = 0$.

Proposition 1.5.7. *Let $s \in J_{\mathbf{S}}$ be an iterated preimage of the critical point 0. Then there are exactly two bubbles S_1 and S_2 in $\mathbf{S}$ which contain s in their closure. Moreover, we have*

$$\overline{S_1} \cap \overline{S_2} = \{s\}.$$

The construction of a *bubble ray* $\mathcal{R}^{\mathbf{S}}$ for $f_{\mathbf{S}}$ is completely analogous to the construction of a bubble ray $\mathcal{R}^{\mathbf{B}}$ for $f_{\mathbf{B}}$.

Proposition 1.5.8. *If $S \subset \mathbf{S}$ is a bubble, then there exists a unique finite bubble ray $\mathcal{R}^{\mathbf{S}} \sim \{S_i\}_{i=0}^n$ such that $S_n = S$. Consequently, if $\mathcal{R}_1^{\mathbf{S}} \sim \{S_i^1\}_{i=0}^n$ and $\mathcal{R}_2^{\mathbf{S}} \sim \{S_i^2\}_{i=0}^m$ are two bubble rays, then there exists $M \geq 0$ such that $S_i^1 = S_i^2$ for all $i \leq M$, and $S_i^1 \neq S_i^2$ for all $i > M$.*

The following proposition is a consequence of complex a priori bounds due to Yampolsky (see [Ya1]). It is proved in the same way as Proposition 1.8.5.

Proposition 1.5.9. *Every infinite bubble ray $\mathcal{R}^{\mathbf{S}}$ for $f_{\mathbf{S}}$ lands.*

Denote the attracting basin of infinity for $f_{\mathbf{S}}$ by $\mathbf{A}_{\mathbf{S}}^{\infty}$. Let

$$\phi_{\mathbf{A}_{\mathbf{S}}^{\infty}} : \mathbf{A}_{\mathbf{S}}^{\infty} \to \mathbb{C} \setminus \overline{\mathbb{D}}$$

be the Böttcher uniformization of $f_{\mathbf{S}}$ on $\mathbf{A}_{\mathbf{S}}^{\infty}$.

Suppose $\mathcal{R}^{\mathbf{S}}$ is an infinite bubble ray, and let $z \in J_{\mathbf{S}}$ be its landing point. Then there exists a unique external ray

$$\mathcal{R}_t^{\infty} := \{\arg(\phi_{\mathbf{A}_{\mathbf{S}}^{\infty}}) = t\}$$

which lands at z. The *external angle of* $\mathcal{R}^{\mathbf{S}}$ is defined to be t. Henceforth, the infinite bubble ray with external angle t will be denoted $\mathcal{R}_t^{\mathbf{S}}$.

Let $s \in \partial \mathbf{S}_0$ be an iterated preimage of 0. Define

$$\operatorname{adr}(s) := \operatorname{gen}(s).$$

The *bubble address of a bubble* $S \subset \mathbf{S}$ for $f_{\mathbf{S}}$ can now be defined in the same way as its counterpart for $f_{\mathbf{B}}$.

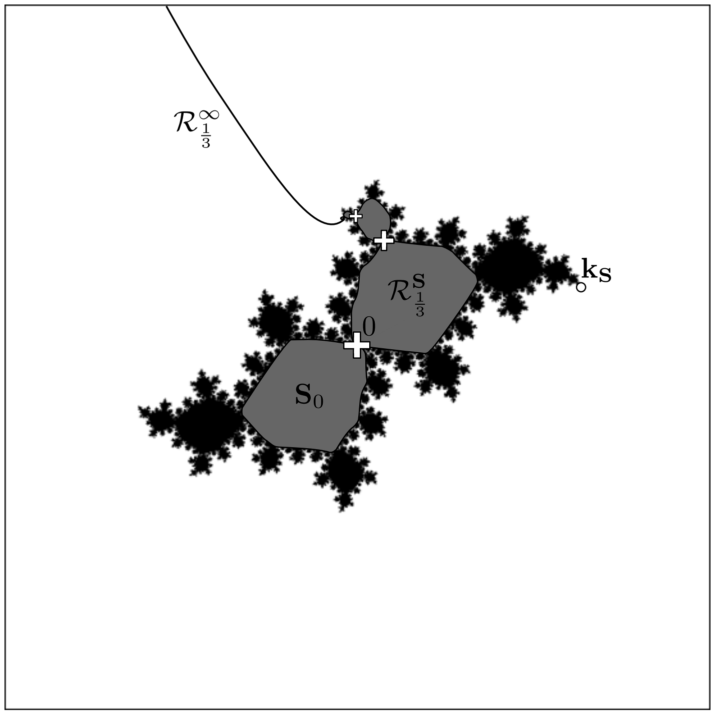

Figure 1.7: The infinite bubble ray $\mathcal{R}^{\mathbf{S}}_{\frac{1}{3}}$ for the Siegel polynomial $f_{\mathbf{S}}$. The bubbles contained in $\mathcal{R}^{\mathbf{S}}_{\frac{1}{3}}$ are colored in dark gray. The white crosses represent the set of attachment points for $\mathcal{R}^{\mathbf{S}}_{\frac{1}{3}}$.

1.5.3 For the candidate mating

Consider the quadratic rational function R_ν constructed in Theorem 1.4.5. Denote the Fatou set and the Julia set for R_ν by $F(R_\nu)$ and $J(R_\nu)$ respectively. A connected component of $F(R_\nu)$ is called a *bubble.*

The critical points for R_ν are ∞ and 1. Recall that $\{\infty, R_\nu(\infty)\}$ is a superattracting 2-periodic orbit, and thus is contained in $F(R_\nu)$. Let $\mathcal{B}_\infty$ be the bubble containing ∞. The set

$$\mathcal{B} := \bigcup_{n=0}^{\infty} R_\nu^{-n}(\mathcal{B}_\infty)$$

is the basin of attraction for $\{\infty, R_\nu(\infty)\}$.

The quadratic rational function R_ν has a Siegel fixed point at 0 with rotation number ν. Denote the Siegel disc for R_ν (the set $\phi(\mathbb{D})$ in Section 1.4) by $\mathcal{S}_0$. As noted in the proof of Main Theorem 1A, the critical point 1 is contained in $\partial\mathcal{S}_0$. Consider the set of iterated preimages of $\mathcal{S}_0$

$$\mathcal{S} := \bigcup_{n=0}^{\infty} R_\nu^{-n}(\mathcal{S}_0),$$

It is easy to see that $F(R_\nu) = \mathcal{B} \cup \mathcal{S}$.

Proposition 1.5.10. *Suppose $U \subset F(R_\nu)$ is a bubble. Then ∂U is locally connected.*

Proof. The result follows immediately from Proposition 1.2.2 and Main Theorem 1A. □

Lemma 1.5.11. *Suppose $X \subset J(R_\nu)$ is a closed, connected, non-recurring set (that is, $R_\nu^n(X) \cap X = \varnothing$ for all $n \in \mathbb{N}$). Then X cannot intersect the boundary of bubbles from both $\mathcal{B}$ and $\mathcal{S}$.*

Proof. Suppose that there exists two bubbles $B \subset \mathcal{B}$ and $S \subset \mathcal{S}$ such that X intersects both ∂B and ∂S. Without loss of generality, we may assume that $B = \mathcal{B}_\infty$ and $S = \mathcal{S}_0$. Observe that $R_\nu^{2n}(X)$ intersects $\partial\mathcal{B}_\infty$ and $\partial\mathcal{S}_0$ for all $n \geq 0$. Likewise, $R_\nu^{2n+1}(X)$ intersects $R_\nu(\mathcal{B}_\infty)$ and $\partial\mathcal{S}_0$ for all $n \geq 0$.

Let $Y := X \cup R_\nu^2(X)$, and consider the set

$$W := \hat{\mathbb{C}} \setminus (\mathcal{B}_\infty \cup \mathcal{S}_0 \cup Y).$$

We claim that if C is a component of W, then C is arcwise connected. Let c be a point in $C \setminus \mathring{C} \subset \partial\mathcal{B}_\infty \cup \partial\mathcal{S}_0$. Since Y is a closed set, there exists a neighbourhood N of c such that $N \cap Y = \varnothing$. By Proposition 1.5.10, it follows that c is arcwise accessible from $N \cap \mathring{C}$. Thus, every point in C is arcwise accessible from $\mathring{C}$. Since C is connected, this implies that C is arcwise connected.

Now, let C' be the component of W that contains $R_\nu(\mathcal{B}_\infty)$. We claim that $\partial\mathcal{S}_0 \cap W$ is not contained in C'. Choose a point x_0 contained in $X \cap \partial\mathcal{S}_0$. Since $\partial\mathcal{S}_0$ is homeomorphic to a circle, we see that $\partial\mathcal{S}_0 \setminus \{x_0, R_\nu^2(x_0)\}$ has exactly two components: γ_1 and γ_2. Choose two points $w_1 \in \gamma_1 \cap W$ and $w_2 \in \gamma_2 \cap W$. If $\partial\mathcal{S}_0 \cap W$ is contained in C', then there exists a simple curve $\Gamma \subset C'$ whose endpoints are w_1 and w_2. The complement of $\mathcal{S}_0 \cup \Gamma$ has exactly two components: one which contains x_0, and one which contains $R_\nu^2(x_0)$. This contradicts the fact that $\mathcal{B}_\infty \cup Y$ is connected.

We conclude that there exists at least one connected component of W that intersects $\partial\mathcal{S}_0$ but does not intersect $\overline{R_\nu(\mathcal{B}_\infty)}$. Denote this component by D. Since X is non-recurring, we have

$$R_\nu^{2n+1}(X) \cap D = \varnothing \quad \text{for all} \quad n \geq 0.$$

However, since the orbit of $R_\nu(x_0)$ under R_ν^2 is dense in $\partial\mathcal{S}_0$, there exists $N \geq 0$ such that

$$R_\nu^{2N+1}(x_0) \in \partial\mathcal{S}_0 \cap D.$$

This is a contradiction. □

Proposition 1.5.12. *Let $B \subset \mathcal{B}$ and $S \subset \mathcal{S}$ be two bubbles. Then $\partial B \cap \partial S = \varnothing$.*

Proof. Suppose that $\partial B \cap \partial S$ contains a point x_0. Since S is an iterated preimage of a Siegel disc, x_0 must be non-recurrent. This contradicts Lemma 1.5.11. □

Proposition 1.5.13. *There exists a unique repelling fixed point β contained in $\partial\mathcal{B}_\infty$.*

Proposition 1.5.14. *Let u be an iterated preimage of β (resp. of 1). Then there are exactly two bubbles U_1 and U_2 in $\mathcal{B}$ (resp. in $\mathcal{S}$) which contain u in their closure. Moreover, we have*

$$\overline{U_1} \cap \overline{U_2} = \{u\}.$$

A *bubble ray* for R_ν can be constructed using bubbles in either $\mathcal{B}$ or $\mathcal{S}$. In the former case, the bubble ray is denoted $\mathcal{R}^{\mathcal{B}}$, and in the latter case, it is denoted $\mathcal{R}^{\mathcal{S}}$. The details of the construction will be omitted as it is very similar to the construction of a bubble ray $\mathcal{R}^{\mathbf{B}}$ for $f_{\mathbf{B}}$ or $\mathcal{R}^{\mathbf{S}}$ for $f_{\mathbf{S}}$.

Proposition 1.5.15. *If $B \subset \mathcal{B}$ is a bubble, then there exists a unique finite bubble ray $\mathcal{R}^{\mathcal{B}} \sim \{B_i\}_{i=0}^n$ such that $B_n = B$. Consequently, if $\mathcal{R}_1^{\mathcal{B}} \sim \{B_i^1\}_{i=0}^n$ and $\mathcal{R}_2^{\mathcal{B}} \sim \{B_i^2\}_{i=0}^m$ are two bubble rays, then there exists $N \geq 0$ such that $B_i^1 = B_i^2$ for all $i \leq N$, and $B_i^1 \neq B_i^2$ for all $i > N$. The analogous statement is also true for bubble rays in $\mathcal{S}$.*

The *bubble address of a bubble $U \subset F(R_\nu)$* for R_ν is defined in the same way as its counterpart for $f_{\mathbf{B}}$ or $f_{\mathbf{S}}$. However, since R_ν is not a polynomial, the external angle of a bubble ray $\mathcal{R}^{\mathcal{B}}$ or $\mathcal{R}^{\mathcal{S}}$ cannot be defined using external rays. To circumvent this problem, we need the following theorem.

Theorem 1.5.16. *There exists a unique conformal map $\Phi_{\mathbf{B}} : \mathbf{B} \to \mathcal{B}$ such that the bubble addresses are preserved, and the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{f_{\mathbf{B}}} & \mathbf{B} \\ \downarrow{\Phi_{\mathbf{B}}} & & \downarrow{\Phi_{\mathbf{B}}} \\ \mathcal{B} & \xrightarrow{R_\nu} & \mathcal{B} \end{array}$$

Likewise, there exists a unique conformal map $\Phi_{\mathbf{S}} : \mathbf{S} \to \mathcal{S}$ such that the bubble addresses are preserved, and the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{f_{\mathbf{S}}} & \mathbf{S} \\ \downarrow{\Phi_{\mathcal{S}}} & & \downarrow{\Phi_{\mathcal{S}}} \\ \mathcal{S} & \xrightarrow{R_\nu} & \mathcal{S} \end{array}$$

Furthermore, if $B \subset \mathbf{B}$ (resp. $S \subset \mathbf{S}$) is a bubble, then $\Phi_{\mathbf{B}}$ (resp. $\Phi_{\mathbf{S}}$) extends to a homeomorphism between $\overline{B}$ and $\overline{\Phi_{\mathbf{B}}(B)}$ (resp. $\overline{S}$ and $\overline{\Phi_{\mathbf{S}}(S)}$).

Proof. For each bubble $B \subset \mathbf{B}$, there exists a unique bubble $B' \subset \mathcal{B}$ such that

$$\operatorname{adr}(B) = \operatorname{adr}(B').$$

Define $\Phi_{\mathbf{B}}|_B$ to be the unique conformal map between B and B' which sends the center and the root of B to the center and the root of B' respectively. Then by construction, $\Phi_{\mathbf{B}}$ conjugates $f_{\mathbf{B}}$ on $\mathbf{B}$ with R_ν on $\mathcal{B}$. Moreover, $\Phi_{\mathbf{B}}$ extends continuously to boundary of bubbles by Proposition 1.5.10.

The map $\Phi_{\mathbf{S}}$ is defined similarly. ☐

Let $\mathcal{R}^{\mathcal{B}} \sim \{B_i\}_{i=0}^\infty$ be an infinite bubble ray for R_ν. The *external angle of* $\mathcal{R}^{\mathcal{B}}$ is defined to be the external angle of the infinite bubble ray $\mathcal{R}^{\mathbf{B}} \sim \{\Phi_{\mathbf{B}}^{-1}(B_i)\}_{i=0}^\infty$ for $f_{\mathbf{B}}$. The external angle of an infinite bubble ray $\mathcal{R}^{\mathcal{S}}$ is defined similarly. Henceforth, the infinite bubble rays for R_ν with external angle t will be denoted by $\mathcal{R}_t^{\mathcal{B}}$ and $\mathcal{R}_t^{\mathcal{S}}$.

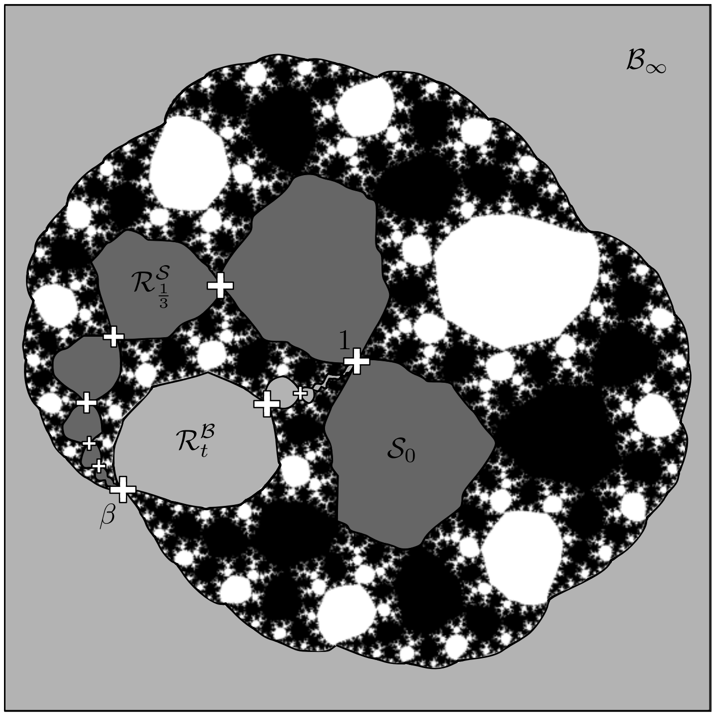

Figure 1.8: The infinite bubble rays $\mathcal{R}^{\mathcal{B}}_t$ with $t \approx 0.354841$ and $\mathcal{R}^{\mathcal{S}}_{\frac{1}{3}}$ for R_ν. The bubbles contained in $\mathcal{R}^{\mathcal{B}}_t$ and $\mathcal{R}^{\mathcal{S}}_{\frac{1}{3}}$ are colored in light gray and dark gray respectively. The white crosses represent the set of attachment points for $\mathcal{R}^{\mathcal{B}}_t$ and $\mathcal{R}^{\mathcal{S}}_{\frac{1}{3}}$. Compare with Figure 1.6 and 1.7.

1.6 The Construction of Puzzle Partitions

1.6.1 For the basilica polynomial

Consider the basilica polynomial $f_{\mathbf{B}}$ discussed in Section 1.5.1. By definition, the infinite bubble ray $\mathcal{R}^{\mathbf{B}}_t$ for $f_{\mathbf{B}}$ with external angle $t \in \mathbb{R}/\mathbb{Z}$ has the same landing point as the external ray $\mathcal{R}^\infty_{-t}$.

Lemma 1.6.1. *Let $\mathcal{R}^{\mathbf{B}}_{t_1}$ and $\mathcal{R}^{\mathbf{B}}_{t_2}$ be two distinct infinite bubble rays for $f_{\mathbf{B}}$, and define*

$$X^{\mathbf{B}}_{t_1,t_2} := \overline{\mathcal{R}^{\mathbf{B}}_{t_1} \cup \mathcal{R}^\infty_{-t_1}} \cup \overline{\mathcal{R}^{\mathbf{B}}_{t_2} \cup \mathcal{R}^\infty_{-t_2}}.$$

Then $\hat{\mathbb{C}} \setminus X^{\mathbf{B}}_{t_1,t_2}$ has exactly two connected components: $C^{\mathbf{B}}_{(t_1,t_2)}$ and $C^{\mathbf{B}}_{(t_2,t_1)}$. If $t \in (t_1,t_2) \subset \mathbb{R}/\mathbb{Z}$, then $\mathcal{R}^\infty_{-t} \subset C^{\mathbf{B}}_{(t_1,t_2)}$. Similarly, if $t \in (t_2,t_1) \subset \mathbb{R}/\mathbb{Z}$, then $\mathcal{R}^\infty_{-t} \subset C^{\mathbf{B}}_{(t_2,t_1)}$.

Proof. First, consider the set

$$\tilde{X}^{\mathbf{B}}_{t_1,t_2} := K_{\mathbf{B}} \cup \overline{\mathcal{R}^\infty_{-t_1}} \cup \overline{\mathcal{R}^\infty_{-t_2}}.$$

Observe that the complement $\hat{\mathbb{C}} \setminus \tilde{X}^{\mathbf{B}}_{t_1,t_2}$ has exactly two connected components: $\tilde{C}^{\mathbf{B}}_{(t_1,t_2)}$ and $\tilde{C}^{\mathbf{B}}_{(t_2,t_1)}$, which are given by

$$\tilde{C}^{\mathbf{B}}_{(t_1,t_2)} = \bigcup_{t \in (t_1,t_2)} \mathcal{R}^\infty_{-t}$$

and

$$\tilde{C}^{\mathbf{B}}_{(t_2,t_1)} = \bigcup_{t \in (t_2,t_1)} \mathcal{R}^\infty_{-t}.$$

Now, let $\mathcal{R}^{\mathbf{B}}_{t_1} \sim \{B^1_i\}_{i=0}^{\infty}$ and $\mathcal{R}^{\mathbf{B}}_{t_2} \sim \{B^2_i\}_{i=0}^{\infty}$, and let $N \geq 0$ be the number given in Proposition 1.5.4. Define

$$\hat{X}^{\mathbf{B}}_{t_1,t_2} := \bigcup_{i=N}^{\infty} \overline{B^1_i} \cup \bigcup_{i=N}^{\infty} \overline{B^2_i} \cup \overline{\mathcal{R}^{\infty}_{-t_1}} \cup \overline{\mathcal{R}^{\infty}_{-t_2}}.$$

Observe that $\hat{X}^{\mathbf{B}}_{t_1,t_2} \subset \tilde{X}^{\mathbf{B}}_{t_1,t_2}$, and that the complement $\hat{\mathbb{C}} \setminus \hat{X}^{\mathbf{B}}_{t_1,t_2}$ also has exactly two connected components. Let $\hat{C}^{\mathbf{B}}_{(t_1,t_2)}$ be the component containing $\tilde{C}^{\mathbf{B}}_{(t_1,t_2)}$, and let $\hat{C}^{\mathbf{B}}_{(t_2,t_1)}$ be the component containing $\tilde{C}^{\mathbf{B}}_{(t_2,t_1)}$.

Let b be the root of the bubble B^1_N, and consider the set

$$Y := X^{\mathbf{B}}_{t_1,t_2} \setminus \hat{X}^{\mathbf{B}}_{t_1,t_2}.$$

If $Y = \varnothing$, then the result is proved. Otherwise, there are three possibilities:

i) $Y = \overline{f_{\mathbf{B}}(\mathbf{B}_0)} \setminus \{b\}$,

ii) $Y = \overline{\mathbf{B}_0} \setminus \{b\}$, or

iii) $Y = \overline{f_{\mathbf{B}}(\mathbf{B}_0)} \cup (\bigcup_{i=0}^{N-1} \overline{B^1_i}) \setminus \{b\}$.

In all three cases, it follows from Proposition 1.5.4 that Y is disjoint from either $\hat{C}^{\mathbf{B}}_{(t_1,t_2)}$ or $\hat{C}^{\mathbf{B}}_{(t_2,t_1)}$. Assume for concreteness that it is disjoint from the former. Then immediately we have $C^{\mathbf{B}}_{(t_1,t_2)} \equiv \hat{C}^{\mathbf{B}}_{(t_1,t_2)}$. Moreover, since Y is simply connected, and its closure intersects $\partial\hat{C}^{\mathbf{B}}_{(t_2,t_1)}$ at only one point (namely, at b), the set $C^{\mathbf{B}}_{(t_2,t_1)} = \hat{C}^{\mathbf{B}}_{(t_2,t_1)} \setminus Y$ must be connected. □

The infinite bubble ray $\mathcal{R}^{\mathbf{B}}_0$ and the external ray $\mathcal{R}^{\infty}_0$ land at the same repelling fixed point $\mathbf{k}_{\mathbf{B}} \in \mathbb{C}$. For $n \in \mathbb{N}$, the *puzzle partition of level* n for $f_{\mathbf{B}}$ is defined as

$$\mathcal{P}^{\mathbf{B}}_n := f^{-n}_{\mathbf{B}}(\overline{\mathcal{R}^{\mathbf{B}}_0 \cup \mathcal{R}^{\infty}_0}) = \bigcup_{i=0}^{2^n-1} \overline{\mathcal{R}^{\mathbf{B}}_{\frac{i}{2^n}} \cup \mathcal{R}^{\infty}_{-\frac{i}{2^n}}}.$$

Note that the puzzle partitions form a nested sequence: $\mathcal{P}^{\mathbf{B}}_1 \subsetneq \mathcal{P}^{\mathbf{B}}_2 \subsetneq \mathcal{P}^{\mathbf{B}}_3 \ldots$.

By Lemma 1.6.1, the complement of the puzzle partition of level n is equal to

$$\hat{\mathbb{C}} \setminus \mathcal{P}^{\mathbf{B}}_n = \bigsqcup_{i=0}^{2^n-1} C^{\mathbf{B}}_{(\frac{i}{2^n},\frac{i+1}{2^n})}.$$

The *puzzle piece of level* n for $f_{\mathbf{B}}$ is defined as

$$P^{\mathbf{B}}_{[\frac{i}{2^n},\frac{i+1}{2^n}]} := \overline{C^{\mathbf{B}}_{(\frac{i}{2^n},\frac{i+1}{2^n})}} \quad \text{for} \quad i \in \{0, \ldots, 2^n - 1\}.$$

The interval $[\frac{i}{2^n}, \frac{i+1}{2^n}] \subset \mathbb{R}/\mathbb{Z}$ is referred to as the *angular span of* $P^{\mathbf{B}}_{[\frac{i}{2^n},\frac{i+1}{2^n}]}$. Note that a puzzle piece of level $n \geq 2$ is mapped homeomorphically onto a puzzle piece of level $n-1$ by $f_{\mathbf{B}}$.

Proposition 1.6.2. *Let $x \in J_{\mathbf{B}}$, and let $n \in \mathbb{N}$. If x is not contained in $\partial\mathcal{P}^{\mathbf{B}}_n$, or there is a unique bubble B contained in $\mathcal{P}^{\mathbf{B}}_n$ such that $x \in \partial B$, then there is a unique puzzle piece of level n that contains x. Otherwise, x is contained in exactly two puzzle pieces of level n.*

Proof. First, suppose x is not contained in $\partial\mathcal{P}^{\mathbf{B}}_n$. Then x is contained in a single connected component of $\hat{\mathbb{C}} \setminus \mathcal{P}^{\mathbf{B}}_n$. The closure of this component is the unique puzzle piece of level n containing x.

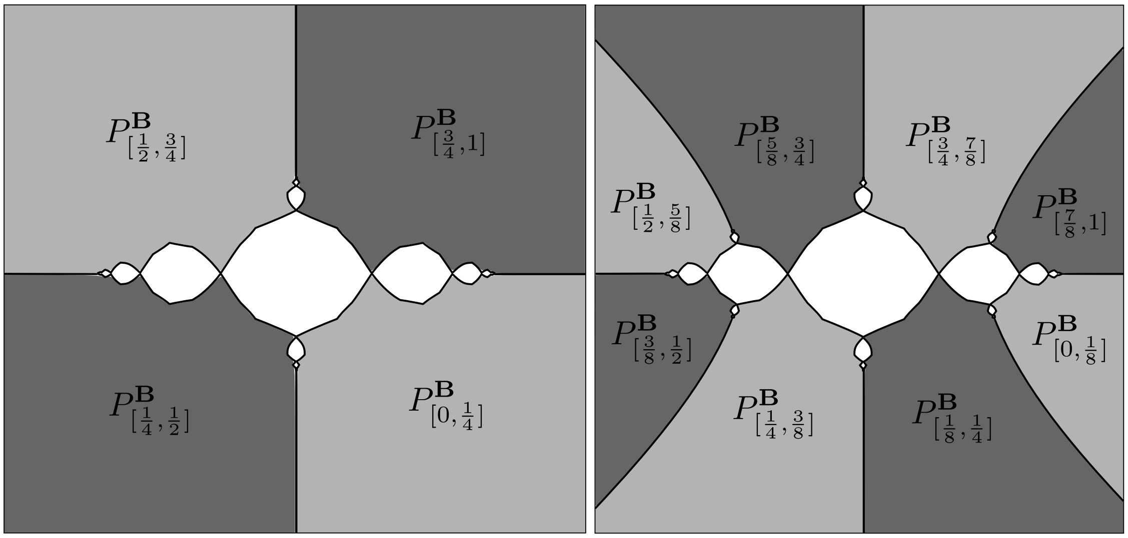

Figure 1.9: The puzzle pieces of level 2 (left) and 3 (right) for $f_{\mathbf{B}}$.

Now, suppose $x \in \partial \mathcal{P}^{\mathbf{B}}_n$. Then there are three possible cases:

i) There is a unique bubble B contained in $\mathcal{P}^{\mathbf{B}}_n$ such that $x \in \partial B$.

ii) There are two bubbles B_1 and B_2 contained in $\mathcal{P}^{\mathbf{B}}_n$ such that $\overline{B_1} \cap \overline{B_2} = \{x\}$.

iii) The point x is an iterated preimage of $\mathbf{k}_{\mathbf{B}}$.

Case i) Since $\mathcal{P}^{\mathbf{B}}_n$ contains finitely many bubble rays whose landing points are all distinct from x, we can choose a sufficiently small disc D centered at x such that $D \cap \mathcal{P}^{\mathbf{B}}_n \subset \overline{B}$. Then $D \cap (\hat{\mathbb{C}} \setminus \mathcal{P}^{\mathbf{B}}_n) = D \cap (\hat{\mathbb{C}} \setminus \overline{B})$ has a single connected component, which must be contained in a unique puzzle piece of level n. The result follows.

Case ii) By a similar reasoning as in Case i), we may choose a sufficiently small disc D centered at x such that $D \cap \mathcal{P}^{\mathbf{B}}_n \subset \overline{B_1 \cup B_2}$. Thus, we see that $D \cap (\hat{\mathbb{C}} \setminus \mathcal{P}^{\mathbf{B}}_n) = D \cap (\hat{\mathbb{C}} \setminus \overline{B_1 \cup B_2})$ has exactly two connected components, say D_1 and D_2. Let $P^{\mathbf{B}}_{[t_1,t_2]}$ be the puzzle piece of level n containing D_1. Then D_2 must be contained in $C^{\mathbf{B}}_{(t_2,t_1)}$, which is disjoint from $P^{\mathbf{B}}_{[t_1,t_2]}$. This implies that D_2 is contained in a puzzle piece distinct from $P^{\mathbf{B}}_{[t_1,t_2]}$. The result follows.

Case iii) Let $t \in \mathbb{Q}/\mathbb{Z}$ be the unique dyadic rational such that the bubble ray $\mathcal{R}^{\mathbf{B}}_t \subset \mathcal{P}^{\mathbf{B}}_n$ lands at x. Then it is easy to see that $P^{\mathbf{B}}_{[t,t+\frac{1}{2^n}]}$ and $P^{\mathbf{B}}_{[t-\frac{1}{2^n},t]}$ are the two puzzle pieces of level n that contain x. $\square$

A *nested puzzle sequence* is a collection of puzzle pieces

$$\Pi^{\mathbf{B}} = \{P^{\mathbf{B}}_{[s_k,t_k]}\}_{k=1}^{\infty}$$

such that $P^{\mathbf{B}}_{[s_{k+1},t_{k+1}]} \subsetneq P^{\mathbf{B}}_{[s_k,t_k]}$ for all $k \geq 1$. Note that this is equivalent to the condition that $[s_{k+1}, t_{k+1}] \subsetneq [s_k, t_k]$. The set

$$L(\Pi^{\mathbf{B}}) := \bigcap_{k=1}^{\infty} P^{\mathbf{B}}_{[s_k,t_k]}$$

is called the *limit of* $\Pi^{\mathbf{B}}$.

Proposition 1.6.3. *Let* $\Pi^{\mathbf{B}} = \{P^{\mathbf{B}}_{[s_k,t_k]}\}_{k=1}^{\infty}$ *be a nested puzzle sequence. Then* $L(\Pi^{\mathbf{B}}) \cap \mathbf{B} = \varnothing$.

Proof. Let $B \subset \mathbf{B}$ be a bubble. Since B is eventually mapped to $\mathbf{B}_0 \subset \mathcal{P}^{\mathbf{B}}_1$ by $f_{\mathbf{B}}$, there exists $N \geq 1$ such that $B \subset \mathcal{P}^{\mathbf{B}}_n$ for all $n \geq N$. This means that B is disjoint from any puzzle piece of level greater than N. Since $P^{\mathbf{B}}_{[s_k,t_k]}$ must be of level at least k, we have $B \cap P^{\mathbf{B}}_{[s_k,t_k]} = \varnothing$ for all $k \geq N$. $\square$

The *external angle* $t \in \mathbb{R}/\mathbb{Z}$ *of* $\Pi^{\mathbf{B}}$ is defined by

$$\{t\} = \bigcap_{k=1}^{\infty} [s_k, t_k].$$

Henceforth, a nested puzzle sequence for $f_{\mathbf{B}}$ with external angle $t \in \mathbb{R}/\mathbb{Z}$ will be denoted by $\Pi^{\mathbf{B}}_t$.

Proposition 1.6.4. *Let* $\Pi^{\mathbf{B}}_t := \{P^{\mathbf{B}}_{[s_k,t_k]}\}_{k=1}^{\infty}$ *be a nested puzzle sequence. Then*

$$L(\Pi^{\mathbf{B}}_t) = \overline{\mathcal{R}^{\infty}_{-t}}.$$

Proof. It follows from Lemma 1.6.1 that $\overline{\mathcal{R}^{\infty}_{-t}} \subset L(\Pi^{\mathbf{B}}_t)$. If $s \neq t$, then for k sufficiently large, we have $s \notin [s_k, t_k]$. This means that $\overline{\mathcal{R}^{\infty}_{-s}}$ is disjoint from $P^{\mathbf{B}}_{[s_k,t_k]}$. The result now follows from Proposition 1.6.3. $\square$

A nested puzzle sequence $\Pi^{\mathbf{B}}_t$ is said to be *maximal* if there is no nested puzzle sequence which contains $\Pi^{\mathbf{B}}_t$ as a proper subset. If two nested puzzle sequences are contained in the same maximal nested puzzle sequence, they are said to be *equivalent*.

Proposition 1.6.5. *Suppose* $\Pi^{\mathbf{B}}_s$ *and* $\Pi^{\mathbf{B}}_t$ *are two equivalent nested puzzle sequences. Then* $s = t$*, and* $L(\Pi^{\mathbf{E}}_s) = L(\Pi^{\mathbf{E}}_t)$.

Proof. Let $\Pi^{\mathbf{B}}_s = \{P^{\mathbf{B}}_{[s_k,t_k]}\}_{k=1}^{\infty}$, and let $\hat{\Pi}^{\mathbf{B}}_u = \{P^{\mathbf{B}}_{[r_k,u_k]}\}_{k=1}^{\infty}$ be the maximal nested puzzle sequence containing $\Pi^{\mathbf{B}}_s$. Since $P^{\mathbf{B}}_{[s_k,t_k]} \subseteq P^{\mathbf{B}}_{[r_k,u_k]}$ for all $k \geq 1$, we have

$$L(\Pi^{\mathbf{B}}_s) \subset L(\hat{\Pi}^{\mathbf{B}}_u).$$

On the other hand, since $\Pi^{\mathbf{B}}_s \subset \hat{\Pi}^{\mathbf{B}}_u$, we have

$$L(\hat{\Pi}^{\mathbf{B}}_u) \subset L(\Pi^{\mathbf{B}}_s).$$

The proof that $s = t$ is similar. $\square$

Proposition 1.6.6. *Let* $x \in J_{\mathbf{B}}$*. If* x *is an iterated preimage of* $\mathbf{b}$ *or* $\mathbf{k_B}$*, then there are exactly two maximal nested puzzle sequences whose limit contains* x*. Otherwise, there is a unique maximal nested puzzle sequence whose limit contains* x*.*

Proof. This is an immediate consequence of Proposition 1.6.2. $\square$

Proposition 1.6.7. *Let* $x \in J_{\mathbf{B}}$*. If* x *is an iterated preimage of* $\mathbf{b}$*, then* x *is biaccessible. Otherwise,* x *is uniaccessible.*

Proof. Suppose $\Pi_t^{\mathbf{B}} = \{P^{\mathbf{B}}_{[s_k,t_k]}\}_{k=1}^{\infty}$ and $\tilde{\Pi}_t^{\mathbf{B}} = \{\tilde{P}^{\mathbf{B}}_{[u_k,v_k]}\}_{k=1}^{\infty}$ are two maximal puzzle sequences whose external angles are both equal to $t \in \mathbb{R}/\mathbb{Z}$. If $\Pi_t^{\mathbf{B}}$ and $\tilde{\Pi}_t^{\mathbf{B}}$ are nonequivalent, then there exists $k \in \mathbb{N}$ such that $(s_k, t_k) \cap (u_k, v_k) = \varnothing$. However, since t is contained in both $[s_k, t_k]$ and $[u_k, v_k]$, we must have $t = t_k = u_k$ or $t = s_k = v_k$. In either case, t must be a dyadic rational.

The result now follows from Proposition 1.6.4 and 1.6.6. □

1.6.2 For the Siegel polynomial

Consider the Siegel polynomial $f_{\mathbf{S}}$ discussed in Section 1.5.2. By definition, the infinite bubble ray $\mathcal{R}_t^{\mathbf{S}}$ for $f_{\mathbf{S}}$ with external angle $t \in \mathbb{R}/\mathbb{Z}$ has the same landing point as the external ray $\mathcal{R}_t^{\infty}$. The following result is a direct analog of Lemma 1.6.1, and can be proved in the same way.

Lemma 1.6.8. *Let $\mathcal{R}_{t_1}^{\mathbf{S}}$ and $\mathcal{R}_{t_2}^{\mathbf{S}}$ be two infinite bubble rays for $f_{\mathbf{S}}$, and define*

$$X^{\mathbf{S}}_{t_1,t_2} := \overline{\mathcal{R}^{\mathbf{S}}_{t_1} \cup \mathcal{R}^{\infty}_{t_1}} \cup \overline{\mathcal{R}^{\mathbf{S}}_{t_2} \cup \mathcal{R}^{\infty}_{t_2}}.$$

Then $\hat{\mathbb{C}} \setminus X^{\mathbf{S}}_{t_1,t_2}$ has exactly two connected components: $C^{\mathbf{S}}_{(t_1,t_2)}$ and $C^{\mathbf{S}}_{(t_2,t_1)}$. If $t \in (t_1, t_2) \subset \mathbb{R}/\mathbb{Z}$, then $\mathcal{R}_t^{\infty} \subset C^{\mathbf{S}}_{(t_1,t_2)}$. Similarly, if $t \in (t_2, t_1) \subset \mathbb{R}/\mathbb{Z}$, then $\mathcal{R}_t^{\infty} \subset C^{\mathbf{S}}_{(t_2,t_1)}$.

The bubble ray $\mathcal{R}_0^{\mathbf{B}}$ and the external ray $\mathcal{R}_0^{\infty}$ both land at the same repelling fixed point $\mathbf{k}_{\mathbf{S}} \in \mathbb{C}$. A *puzzle partition* $\mathcal{P}_n^{\mathbf{S}}$, a *puzzle piece* $P^{\mathbf{S}}_{[t_1,t_2]}$, and a *nested puzzle sequence* $\Pi_t^{\mathbf{S}}$ for $f_{\mathbf{S}}$ are defined in the same way as their counterparts for $f_{\mathbf{B}}$.

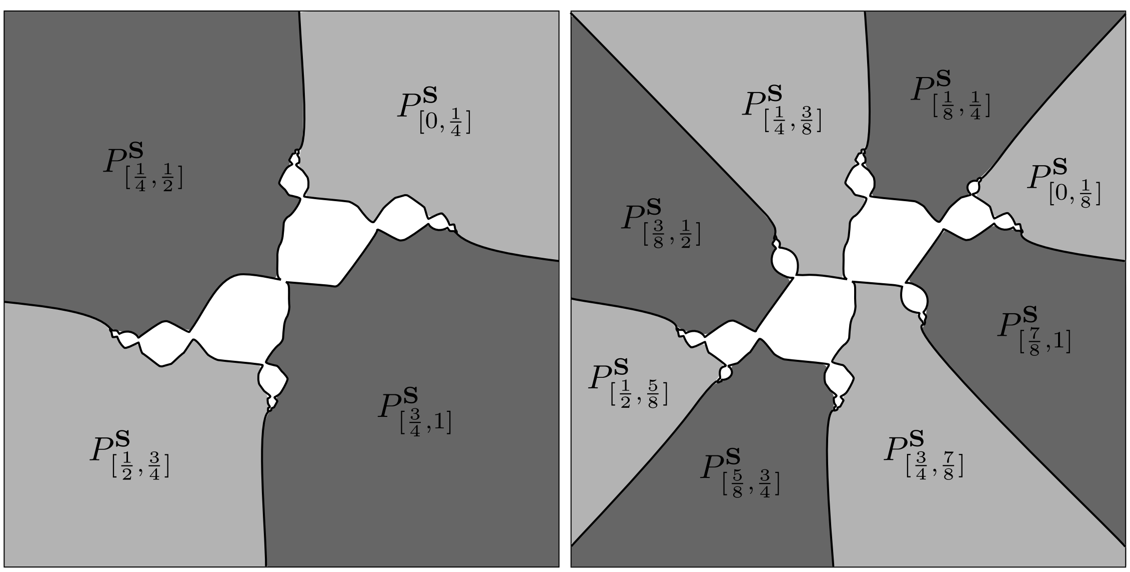

Figure 1.10: The puzzle pieces of level 2 (left) and 3 (right) for $f_{\mathbf{S}}$.

The following four results are analogs of Proposition 1.6.2, 1.6.4, 1.6.6 and 1.6.7. The proofs are identical, and hence, they will be omitted here.

Proposition 1.6.9. *Let $x \in J_{\mathbf{S}}$, and let $n \in \mathbb{N}$. If x is not contained in $\partial\mathcal{P}_n^{\mathbf{S}}$, or there is a unique bubble S contained in $\mathcal{P}_n^{\mathbf{S}}$ such that $x \in \partial S$, then there is a unique puzzle piece of level n that contains x. Otherwise, x is contained in exactly two puzzle pieces of level n.*

Proposition 1.6.10. *Let* $\Pi_t^{\mathbf{S}} := \{P_{[s_k,t_k]}^{\mathbf{S}}\}_{k=1}^{\infty}$ *be a nested puzzle sequence. Then*

$$L(\Pi_t^{\mathbf{S}}) = \overline{\mathcal{R}_t^{\infty}}.$$

Proposition 1.6.11. *Let* $x \in J_{\mathbf{S}}$*. If* x *is an iterated preimage of* 0 *or* $\mathbf{k_S}$*, then there are exactly two maximal nested puzzle sequences whose limit contains* x*. Otherwise, there is a unique maximal nested puzzle sequence whose limit contains* x*.*

Proposition 1.6.12. *Let* $x \in J_{\mathbf{S}}$*. If* x *is an iterated preimage of* 0*, then* x *is biaccessible. Otherwise,* x *is uniaccessible.*

1.6.3 For the candidate mating

Consider the quadratic rational function R_ν constructed in Theorem 1.4.5. The following result is an analog of Lemma 1.6.1 and 1.6.8.

Lemma 1.6.13. *Let* $\mathcal{R}_{t_1}^{\mathcal{B}}$ *and* $\mathcal{R}_{t_2}^{\mathcal{B}}$ *be two infinite bubble rays in* $\mathcal{B}$*, and let* $\mathcal{R}_{s_1}^{\mathcal{S}}$ *and* $\mathcal{R}_{s_2}^{\mathcal{S}}$ *be two infinite bubble rays in* $\mathcal{S}$*. Suppose* $\mathcal{R}_{t_1}^{\mathcal{B}}$ *and* $\mathcal{R}_{s_1}^{\mathcal{S}}$ *land at the same point* x_1*, and* $\mathcal{R}_{t_2}^{\mathcal{B}}$ *and* $\mathcal{R}_{s_2}^{\mathcal{S}}$ *land at the same point* x_2*. Define*

$$X_{s_1,s_2}^{t_1,t_2} := \overline{\mathcal{R}_{t_1}^{\mathcal{B}} \cup \mathcal{R}_{s_1}^{\mathcal{S}}} \cup \overline{\mathcal{R}_{t_2}^{\mathcal{B}} \cup \mathcal{R}_{s_2}^{\mathcal{S}}}.$$

Then $\hat{\mathbb{C}} \setminus X_{s_1,s_2}^{t_1,t_2}$ *has exactly two connected components:* $C_{(s_1,s_2)}^{(t_1,t_2)}$ *and* $C_{(s_2,s_1)}^{(t_2,t_1)}$*, such that*

$$\Phi_{\mathbf{B}}(\mathbf{B} \cap C_{(t_1,t_2)}^{\mathbf{B}}) = \mathcal{B} \cap C_{(s_1,s_2)}^{(t_1,t_2)}, \qquad \Phi_{\mathbf{B}}(\mathbf{B} \cap C_{(t_2,t_1)}^{\mathbf{B}}) = \mathcal{B} \cap C_{(s_2,s_1)}^{(t_2,t_1)},$$

$$\Phi_{\mathbf{S}}(\mathbf{S} \cap C_{(s_1,s_2)}^{\mathbf{S}}) = \mathcal{S} \cap C_{(s_1,s_2)}^{(t_1,t_2)}, \qquad \text{and} \qquad \Phi_{\mathbf{S}}(\mathbf{S} \cap C_{(s_2,s_1)}^{\mathbf{S}}) = \mathcal{S} \cap C_{(s_2,s_1)}^{(t_2,t_1)},$$

where $\Phi_{\mathbf{B}} : \mathbf{B} \to \mathcal{B}$ *and* $\Phi_{\mathbf{S}} : \mathbf{S} \to \mathcal{S}$ *are the maps given in Theorem 1.5.16.*

Proof. Consider the bubble rays $\mathcal{R}_{t_1}^{\mathbf{B}} \sim \{B_i^1\}_{i=0}^{\infty}$, $\mathcal{R}_{t_2}^{\mathbf{B}} \sim \{B_i^2\}_{i=0}^{\infty}$, $\mathcal{R}_{s_1}^{\mathbf{S}} \sim \{S_i^1\}_{i=0}^{\infty}$, and $\mathcal{R}_{s_2}^{\mathbf{S}} \sim \{S_i^2\}_{i=0}^{\infty}$ for $f_{\mathbf{B}}$ and $f_{\mathbf{S}}$. Let $N \geq 0$ and $M \geq 0$ be the numbers given in Proposition 1.5.4 and 1.5.8 respectively. Define

$$Y_{t_1,t_2}^{\mathbf{B}} := \bigcup_{i=N}^{\infty} \overline{B_i^1} \cup \bigcup_{i=N}^{\infty} \overline{B_i^2} \qquad \text{and} \qquad Y_{s_1,s_2}^{\mathbf{S}} := \bigcup_{i=M}^{\infty} \overline{S_i^1} \cup \bigcup_{i=M}^{\infty} \overline{S_i^2}$$

Recall the definition of $\hat{C}_{(t_1,t_2)}^{\mathbf{B}}$ and $\hat{C}_{(t_2,t_1)}^{\mathbf{B}}$ for $f_{\mathbf{B}}$ given in the proof of Lemma 1.6.1. Let $\hat{C}_{(s_1,s_2)}^{\mathbf{S}}$ and $\hat{C}_{(s_2,s_1)}^{\mathbf{S}}$ be the analogous structures for $f_{\mathbf{S}}$. Define

$$\gamma_{(t_1,t_2)}^{\mathbf{B}} := Y_{t_1,t_2}^{\mathbf{B}} \cap \partial \hat{C}_{(t_1,t_2)}^{\mathbf{B}}, \qquad \text{and} \qquad \gamma_{(t_2,t_1)}^{\mathbf{B}} := Y_{t_1,t_2}^{\mathbf{B}} \cap \partial \hat{C}_{(t_2,t_1)}^{\mathbf{B}}.$$

The sets $\gamma_{(s_1,s_2)}^{\mathbf{S}}$ and $\gamma_{(s_2,s_1)}^{\mathbf{S}}$ are defined analogously.

The maps $\Phi_{\mathbf{B}}$ and $\Phi_{\mathbf{S}}$ extend continuously to $Y_{t_1,t_2}^{\mathbf{B}}$ and $Y_{s_1,s_2}^{\mathbf{S}}$. Define

$$\hat{X}_{s_1,s_2}^{t_1,t_2} := \Phi_{\mathbf{B}}(Y_{t_1,t_2}^{\mathbf{B}}) \cup \Phi_{\mathbf{S}}(Y_{s_1,s_2}^{\mathbf{S}}) \cup \{x_1, x_2\}.$$

It follows from Proposition 1.5.12 and 1.5.15 that the complement $\hat{\mathbb{C}} \setminus \hat{X}_{s_1,s_2}^{t_1,t_2}$ has exactly two connected components. Since $\Phi_{\mathbf{B}}$ and $\Phi_{\mathbf{S}}$ are orientation preserving, the boundary of one of these components contains $\Phi_{\mathbf{B}}(\gamma_{(t_1,t_2)}^{\mathbf{B}})$ and $\Phi_{\mathbf{S}}(\gamma_{(s_1,s_2)}^{\mathbf{S}})$, and the boundary of the other contains $\Phi_{\mathbf{B}}(\gamma_{(t_2,t_1)}^{\mathbf{B}})$ and $\Phi_{\mathbf{S}}(\gamma_{(s_2,s_1)}^{\mathbf{S}})$. Denote the former component by $\hat{C}_{(s_1,s_2)}^{(t_1,t_2)}$ and the latter component by $\hat{C}_{(s_2,s_1)}^{(t_2,t_1)}$.

Now, given a bubble $U \subset \mathbf{B}$, let $\mathcal{R}^{\mathbf{B}}_U \sim \{U_i\}_{i=0}^n$ be the unique finite bubble ray such that $U_n = U$. Since $\Phi_{\mathbf{B}}$ extends to a homeomorphism on $\mathcal{R}^{\mathbf{B}}_U \cup \mathcal{R}^{\mathbf{B}}_{t_1} \cup \mathcal{R}^{\mathbf{B}}_{t_2}$, it follows from the above definitions together with Proposition 1.5.12 and 1.5.15 that $\Phi_{\mathbf{B}}(U) \subset \hat{C}^{(t_1,t_2)}_{(s_1,s_2)}$ if $U \subset \hat{C}^{\mathbf{B}}_{(t_1,t_2)}$, and $\Phi_{\mathbf{B}}(U) \subset \hat{C}^{(t_2,t_1)}_{(s_2,s_1)}$ if $U \subset \hat{C}^{\mathbf{B}}_{(t_2,t_1)}$. A completely symmetric argument shows that the analogous statement is true for bubbles in $\mathbf{S}$.

The rest of the proof is similar to that of Lemma 1.6.1, and hence, will be omitted here. □

In order to construct the puzzle partitions for R_ν, we need to prove that every infinite periodic bubble ray lands at a repelling periodic orbit point. This requires the following classical result in holomorphic dynamics (see e.g. [M1]).

Lemma 1.6.14 (Snail's Lemma). *Let $V \subset \mathbb{C}$ be a neighbourhood of 0, and let $f : V \to \mathbb{C}$ be a holomorphic function. Suppose there exists a path $\gamma : [0, \infty) \to V \setminus \{0\}$ which is mapped into itself by f in such a way that $f(\gamma(t)) = \gamma(t+1)$ and γ converges to 0. Then 0 is a fixed point for f, and $f'(0) = 1$ or $|f'(0)| < 1$.*

Proposition 1.6.15. *Let $\mathcal{R}_t = \mathcal{R}^{\mathcal{B}}_t$ or $\mathcal{R}^{\mathcal{S}}_t$ be an infinite bubble ray. If t is rational, then $\mathcal{R}_t$ lands. If t is p-periodic, then $\mathcal{R}_t$ lands at a repelling p-periodic point.*

Proof. Let Ω be the set of cluster points for $\mathcal{R}_t$. Define

$$\Lambda := \Omega \cup \{\infty, R_\nu(\infty)\} \cup \overline{\mathbf{S}_0}.$$

Observe that

$$R^p_\nu : \hat{\mathbb{C}} \setminus R^{-p}_\nu(\Lambda) \to \hat{\mathbb{C}} \setminus \Lambda$$

is a regular 2^p-fold covering of connected hyperbolic spaces. Moreover, since $\Lambda \subsetneq R^{-p}_\nu(\Lambda)$, the inclusion map

$$\iota : \hat{\mathbb{C}} \setminus R^{-p}_\nu(\Lambda) \to \hat{\mathbb{C}} \setminus \Lambda$$

is a strict contraction in the hyperbolic metric. Hence, the map $\iota \circ R^{-p}_\nu$ lifts to the universal cover $\mathbb{D}$ of $\hat{\mathbb{C}} \setminus \Lambda$ to a map

$$\hat{R}^{-p}_\nu : \mathbb{D} \to \mathbb{D}$$

which is also a strict contraction in the hyperbolic metric.

Now, choose a bubble $U \subset \mathcal{R}_t$ such that $\mathrm{gen}(U) > 1$, and let x_0 be a point contained in U. For every $k \geq 0$, there exists a unique point $x_k \in \mathcal{R}_t$ such that $R^{kp}_\nu(x_k) = x_0$. Let $\gamma_0 \subset \mathcal{R}_t$ be a curve from x_0 to x_1, and let γ_k be the unique component of $R^{-kp}_\nu(\gamma_0)$ whose end points are x_k and x_{k+1}.

By the strict contraction property of $\hat{R}^p_\nu$, the hyperbolic lengths of γ_n must go to zero as n goes to infinity. Hence, if $z \in \Omega$, then for any neighbourhood N of z, there exists a smaller neighbourhood $N' \subset N$ such that if $\gamma_n \cap N' \neq \varnothing$, then $\gamma_n \subset N$. In other words, $R^p_\nu(N) \cap N \neq \varnothing$. Since this is true for all neighbourhood of z, the map R^p_ν must fix z.

The set of fixed points for R^p_ν is discrete. Since Ω is connected, this implies that Ω must be equal to the single point set $\{z\}$. By Lemma 1.6.14, we conclude that z is a repelling fixed point.

If t is strictly preperiodic, then $\mathcal{R}_t$ is the preimage of some periodic infinite bubble ray. The result follows. □

Proposition 1.6.16. *The bubble rays $\mathcal{R}^{\mathcal{B}}_0$ and $\mathcal{R}^{\mathcal{S}}_0$ land at the same repelling fixed point $\kappa \in \mathbb{C}$.*

Proof. The quadratic rational map R_ν has exactly three fixed points, two of which must be the Siegel fixed point 0 and the repelling fixed point β. Clearly, a bubble ray cannot land at 0, so it suffices to prove that a fixed bubble ray cannot land at β.

Let D be a sufficiently small disc centered at β such that R_ν is conformal on D. The set $D \cap (\hat{\mathbb{C}} \setminus \overline{\mathcal{B}_\infty \cup R_\nu(\mathcal{B}_\infty)})$ has two connected components D_1 and D_2 such that $D_1 \subset R_\nu(D_2)$ and $D_2 \subset R_\nu(D_1)$. Suppose $\mathcal{R}$ is a bubble ray that lands at β. Then $\mathcal{R}$ must be disjoint from either D_1 or D_2. Hence, $\mathcal{R}$ cannot be fixed. $\square$

Proposition 1.6.17. *Let $t \in \mathbb{R}/\mathbb{Z}$ be a dyadic rational. Then $\mathcal{R}^{\mathcal{B}}_t$ and $\mathcal{R}^{\mathcal{S}}_t$ land at the same iterated preimage of κ.*

Proof. Define $D_n := \{\frac{i}{2^n}\}_{i=0}^{2^n-1} \subset \mathbb{R}/\mathbb{Z}$, and let $t \in D_n$ for some $n \geq 0$. Note that the case $n = 0$ is proved in Proposition 1.6.16. Proceeding inductively, assume that $n > 0$, and that the result is true for the dyadic rationals in D_{n-1}.

If $t \in D_n \setminus D_{n-1}$, then t can be expressed as

$$t = \frac{i}{2^{n-1}} + \frac{1}{2^n} \qquad \text{for some} \qquad i \in \{0, \ldots, 2^{n-1} - 1\}.$$

Observe that t is the unique member of D_n contained in the interval $(\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})$. It follows from Lemma 1.6.13 that $\mathcal{R}^{\mathcal{B}}_t$ is the only member of $\{\mathcal{R}^{\mathcal{B}}_s\}_{s \in D_n}$ whose landing point lies in $C^{(\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})}_{(\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})}$. Likewise, $\mathcal{R}^{\mathcal{S}}_t$ is the only member of $\{\mathcal{R}^{\mathcal{S}}_s\}_{s \in D_n}$ whose landing point lies in $C^{(\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})}_{(\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})}$. By Proposition 1.6.16, $\mathcal{R}^{\mathcal{B}}_t$ and $\mathcal{R}^{\mathcal{S}}_t$ must land at the same point. $\square$

For $n \in \mathbb{N}$, define the *puzzle partition of level* n for R_ν by

$$\mathcal{P}_n := R_\nu^{-n}(\overline{\mathcal{R}^{\mathcal{B}}_0 \cup \mathcal{R}^{\mathcal{S}}_0}) = \bigcup_{i=0}^{2^n-1} \overline{\mathcal{R}^{\mathcal{B}}_{\frac{i}{2^n}} \cup \mathcal{R}^{\mathcal{S}}_{\frac{i}{2^n}}}.$$

By Lemma 1.6.13 and 1.6.17, the complement of the puzzle partition of level n is equal to

$$\hat{\mathbb{C}} \setminus \mathcal{P}_n = \bigsqcup_{i=0}^{2^n-1} C^{(\frac{i}{2^n}, \frac{i+1}{2^n})}_{(\frac{i}{2^n}, \frac{i+1}{2^n})}.$$

A *puzzle piece of level* n for R_ν is defined as

$$P_{[\frac{i}{2^n}, \frac{i+1}{2^n}]} := \overline{C^{(\frac{i}{2^n}, \frac{i+1}{2^n})}_{(\frac{i}{2^n}, \frac{i+1}{2^n})}} \qquad \text{for} \qquad i \in \{0, \ldots, 2^n - 1\}.$$

The interval $[\frac{i}{2^n}, \frac{i+1}{2^n}] \subset \mathbb{R}/\mathbb{Z}$ is referred to as the *angular span of* $P_{[\frac{i}{2^n}, \frac{i+1}{2^n}]}$.

Proposition 1.6.18. *Let $P_{[t_1,t_2]}$ be a puzzle piece with angular span $[t_1, t_2] \subset \mathbb{R}/\mathbb{Z}$. If $\mathcal{R}_t = \mathcal{R}^{\mathcal{B}}_t$ or $\mathcal{R}^{\mathcal{S}}_t$ is an infinite bubble ray with external angle $t \in [t_1, t_2]$, then the accumulation set of $\mathcal{R}_t$ is contained in $P_{[t_1,t_2]}$.*

The following result is an analog of Proposition 1.6.2. The proof is very similar, and hence, it will be omitted here.

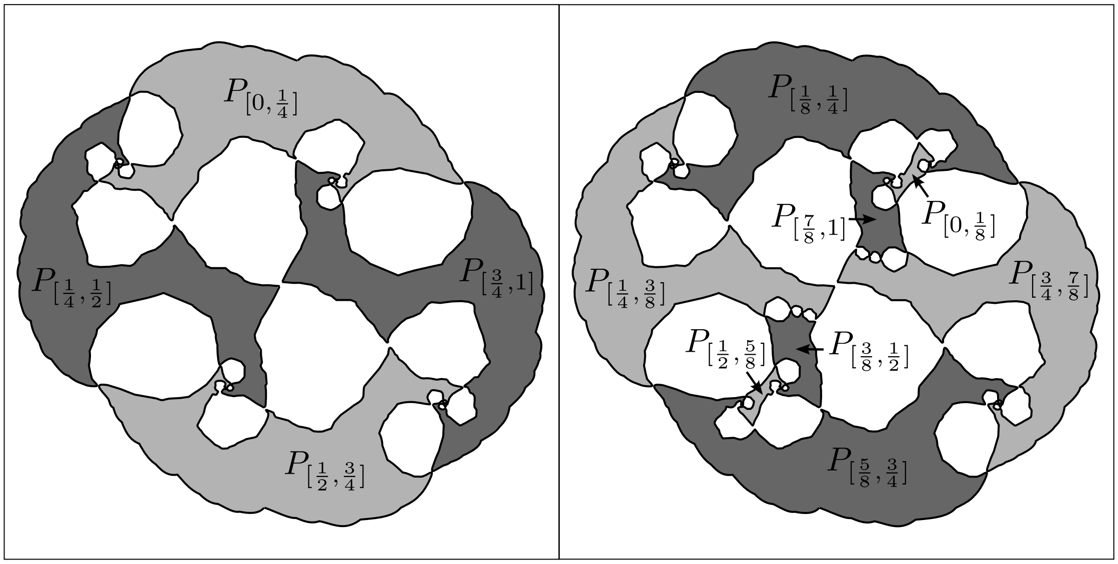

Figure 1.11: The puzzle pieces of level 2 (left) and 3 (right) for R_ν. Compare with Figure 1.9 and 1.10.

Proposition 1.6.19. *Let $x \in J(R_\nu)$, and let $n \in \mathbb{N}$. If x is not contained in $\partial\mathcal{P}_n$, or there is a unique bubble U contained in $\mathcal{P}_n$ such that $x \in \partial U$, then there is a unique puzzle piece of level n that contains x. Otherwise, x is contained in exactly two puzzle pieces of level n.*

A *nested puzzle sequence* Π_t for R_ν is defined in the same way as its counterpart for $f_{\mathbf{B}}$.

Proposition 1.6.20. *Let $x \in J(R_\nu)$. If x is an iterated preimage of κ, β or 1, then there are exactly two maximal nested puzzle sequences whose limit contains x. Otherwise, there is exactly one maximal nested puzzle sequence whose limit contains x.*

Proof. This is an immediate consequence of Proposition 1.6.19. □

Proposition 1.6.21. *Let Π_t be a nested puzzle sequence for R_ν. Its limit $L(\Pi_t)$ cannot intersect the boundary of bubbles from both $\mathcal{B}$ and $\mathcal{S}$.*

Proof. It is easy to see that the limit set of any nested puzzle sequence is closed, connected, and contained in $J(R_\nu)$. Moreover, it must be either pre-periodic or non-recurrent.

Now, suppose that $L(\Pi_t)$ intersects the boundary of bubbles from both $\mathcal{B}$ and $\mathcal{S}$. We may assume that $L(\Pi_t)$ contains a point $x \in \partial\mathcal{S}_0$. Note that the orbit of x is dense in $\partial\mathcal{S}_0$. Hence, if $L(\Pi_t)$ is periodic, then $L(\Pi_t)$ must contain $\partial\mathcal{S}_0$, which is clearly impossible. Therefore, $L(\Pi_t)$ must be non-recurrent. This contradicts Lemma 1.5.11. □

Let Π_t be a nested puzzle sequence. We say that Π_t *shrinks to x* if its limit $L(\Pi_t)$ is equal to $\{x\}$.

Proposition 1.6.22. *Let $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$ be a nested puzzle sequence, and let $\hat{\Pi}_t = \{P_{[r_k,u_k]}\}_{k=1}^{\infty}$ be the unique maximal nested puzzle sequence containing Π_t. Then Π_t shrinks to a point $x \in J(R_\nu)$ if and only if $\hat{\Pi}_t$ does.*

The following result is proved in the next two sections.

Theorem 1.6.23 (the Shrinking Theorem)**.** *Every nested puzzle sequence for R_ν shrinks to a point.*

1.7 A Priori Bounds for Critical Circle Maps

A C^2 homeomorphism $f : S^1 \to S^1$ is called a *critical circle map* if it has a unique critical point $c \in S^1$ of cubic type. Let $\rho = \rho(f)$ be the rotation number of f. In this section, f will be analytic, and ρ will be irrational.

The rotation number ρ can be represented as an infinite continued fraction:

$$\rho = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}.$$

The *nth partial convergent of* ρ is the rational number

$$\frac{p_n}{q_n} = [a_1, \dots, a_n].$$

The sequence of denominators $\{q_n\}_{n=1}^{\infty}$ represent the *closest return times* of the orbit of any point to itself. It satisfies the following inductive relation:

$$q_{n+1} = a_n q_n + q_{n-1}.$$

Let $\mathcal{D}_n \subset S^1$ be the closed arc containing c with end points at $f^{q_n}(c)$ and $f^{q_{n+1}}(c)$. The arc $\mathcal{D}_n$ can be expressed as the union of two closed subarcs A_n and A_{n+1}, where A_n has its end points at c and $f^{q_n}(c)$. The subarc A_n is called the *nth critical arc.* The q_nth iterated preimage of A_n under f is denoted by A_{-n}. The set of closed arcs

$$\mathcal{P}_n^{S^1} = \{A_n, f(A_n), \dots, f^{q_{n+1}-1}(A_n)\} \cup \{A_{n+1}, f(A_{n+1}), \dots, f^{q_n-1}(A_{n+1})\},$$

which are disjoint except at the end points, is a partition of S^1. The collection $\mathcal{P}_n^{S^1}$ is called the *dynamical partition of level n.* The following is an important estimate regarding dynamical partitions due to Swiątek and Herman (see [Sw]):

Theorem 1.7.1 (Real *a priori* bounds)**.** *Let $f : S^1 \to S^1$ be a critical circle map with an irrational rotation number ρ. Then for all n sufficiently large, every pair of adjacent atoms in $\mathcal{P}_n^{S^1}$ have K-commensurate diameters for some universal constant $K > 1$.*

Below, we present an adaptation of complex a priori bounds of [Ya1] (see also [YaZ]) to our setting.

Consider the quadratic rational function R_ν discussed in Section 1.5.3 and 1.6.3. Denote the Siegel disc for R_ν by $\mathcal{S}_0$. By Theorem 1.4.5, there exist a Blaschke product F_ν and a quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$R_\nu(z) = \phi \circ F_\nu \circ \phi^{-1}(z) \quad \text{for all} \quad z \in \hat{\mathbb{C}} \setminus \mathcal{S}_0.$$

Recall that $\{\infty, F_\nu(\infty)\}$ and $\{0, F_\nu(0)\}$ are superattracting 2-periodic orbits for F_ν. Denote the bubble (the connected component of the Fatou set) for F_ν containing 0 and ∞ by $\mathcal{A}_0$ and $\mathcal{A}_\infty$ respectively. By Theorem 1.4.4, the restriction of F_ν to S^1 is a critical circle map.

A *puzzle piece of level n for* F_ν is the image of a puzzle piece of level n for R_ν under ϕ^{-1}. The *nth critical puzzle piece*, denoted P_n^{crit}, is defined inductively as follows:

(i) P_0^{crit} is the puzzle piece of level 1 which contains the first critical arc A_1.

(ii) P_n^{crit} is the puzzle piece which contains the preimage arc A_{-n}, and is mapped homeomorphically onto P_{n-1}^{crit} by $F_\nu^{q_n}$.

Observe that $\Pi_{\text{even}} := \{P_{2n}^{crit}\}_{n=0}^{\infty}$ and $\Pi_{\text{odd}} := \{P_{2n+1}^{crit}\}_{n=0}^{\infty}$ form two disjoint nested puzzle sequences for F_ν at the critical point 1.

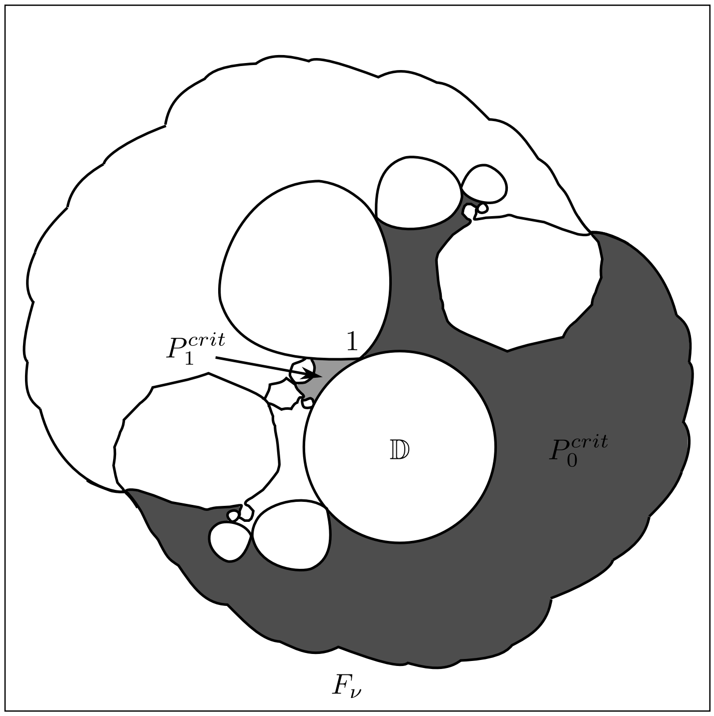

Figure 1.12: The 0th and 1st critical puzzle piece for F_ν.

Lemma 1.7.2. *Let $\mathcal{A}_\infty \cup F_\nu(\mathcal{A}_\infty)$ be the immediate attracting basin of the superattracting 2-periodic orbit $\{\infty, F_\nu(\infty)\}$ for F_ν. Then there exists $N \geq 0$ such that for all $n \geq N$, the nth critical puzzle piece P_n^{crit} is disjoint from the closure of $\mathcal{A}_\infty \cup F_\nu(\mathcal{A}_\infty)$.*

Proof. The result follows immediately from Proposition 1.6.21. □

Theorem 1.7.3. *For all n sufficiently larger than the constant N in Lemma 1.7.2, we have the following inequality:*

$$\frac{\operatorname{diam}(P_n^{crit})}{\operatorname{diam}(A_{-n})} \leq C_1 \sqrt[3]{\frac{\operatorname{diam}(P_{n-1}^{crit})}{\operatorname{diam}(A_{-(n-1)})}} + C_2,$$

where C_1 and C_2 are universal constants.

Proof. Similarly to [YZ], we first lift a suitable inverse branch of F_ν to the universal covering space.

Define the exponential map $\operatorname{Exp} : \mathbb{C} \to \mathbb{C}$ by

$$\operatorname{Exp}(z) := e^{2\pi i z}.$$

Let $I = (\tau - 1, \tau) \subset \mathbb{R}$ be an open interval such that $0 \in I$, and

$$\operatorname{Exp}(\tau) = \operatorname{Exp}(\tau - 1) = F_\nu(1).$$

Let

$$\operatorname{Log} : S^1 \setminus \{F_\nu(1)\} \to I$$

be the inverse of Exp restricted to I. The nth critical interval is defined as

$$I_n := \operatorname{Log}(A_n).$$

Denote the component of $\operatorname{Exp}^{-1}(P_n^{crit})$ intersecting I by $\hat{P}_n^{crit}$.

Define

$$\mathcal{A} := \overline{\mathcal{A}_0 \cup F_\nu(\mathcal{A}_0) \cup \mathcal{A}_\infty \cup F_\nu(\mathcal{A}_\infty)},$$

and let $S \subset \mathbb{C}$ be the universal covering space of $\hat{\mathbb{C}} \setminus \mathcal{A}$ with the covering map $\operatorname{Exp}|_S : S \to \hat{\mathbb{C}} \setminus \mathcal{A}$. For any given interval $J \subset \mathbb{R}$, we denote

$$S_J := (S \setminus \mathbb{R}) \cup J.$$

The restriction of the map F_ν to S^1 is a homeomorphism, and hence, has an inverse. We define a lift $\phi : I \to I$ of $(F_\nu|_{\partial\mathbb{D}})^{-1}$ by

$$\phi(x) := \operatorname{Log} \circ F_\nu^{-1} \circ \operatorname{Exp}(x).$$

Note that ϕ is discontinuous at $\operatorname{Log}(F_\nu^2(1))$, which is mapped to $\tau - 1$ and τ by ϕ. Let $n \in \mathbb{N}$. By the combinatorics of critical circle maps, the kth iterate of ϕ on I_n is continuous for all $1 \le k \le q_n$. By monodromy theorem, ϕ^k extends to a conformal map on S_{I_n}.

For $z \in S_J$, let l_z and r_z be the line segment connecting z to $\tau - 1$ and z to τ respectively. The smaller of the outer angles formed between l_z and $(-\infty, \tau - 1)$, and r_z and $(\tau, +\infty)$ is denoted $\widehat{(z, J)}$.

Denote the hyperbolic distance in S_J by $\operatorname{dist}_{S_J}$. A hyperbolic neighbourhood $\{z \in S_J \mid \operatorname{dist}_{S_J}(z, J)\}$ of J forms an angle $\theta \in (0, \pi)$ with $\mathbb{R}$. Denote this neighbourhood by $G_\theta(J)$. Observe that $G_\theta(J) \subset \{z \in S_J \mid \widehat{(z, J)} > \theta\}$.

For $n \in \mathbb{N}$, define $E_n \subset S^1$ as the open arc containing 1 with end points at $F_\nu^{q_{n+1}}(1)$, and $F_\nu^{q_n - q_{n+1}}(1)$. Observe that E_n contains the critical arcs A_n and A_{n+1}. Define

$$G_\theta^n := G_\theta(\operatorname{Log}(E_n)).$$

Consider the constant N in Lemma 1.7.2. Since $P_N^{crit} \cup P_{N+1}^{crit}$ is disjoint from the closure of $\mathcal{A}$, it is contained in some annulus $E \Subset \hat{\mathbb{C}} \setminus \mathcal{A}$. Let $\breve{S} \Subset S$ be the universal cover of E with the covering map $\operatorname{Exp}|_{\breve{S}}$. Choose θ such that $\hat{P}_{N+2}^{crit} \cup \hat{P}_{N+3}^{crit} \subset G_\theta^{N+1}$. Then we have $\hat{P}_n^{crit} \subset G_\theta^{N+1}$ for all $n \ge N + 3$.

Now, suppose we are given $n \ge N + 3$. Let

$$J_0 := I_n, \quad J_{-1} := \phi(J_0), \quad \ldots, \quad J_{-q_n} := \phi^{q_n}(I_n), \tag{1.5}$$

be the orbit of I_n under ϕ. Given any point $z_0 \in S_{J_0}$, let

$$z_0, \quad z_{-1} := \phi(z_0), \quad \ldots, \quad z_{-q_n} := \phi^{q_n}(z_0), \tag{1.6}$$

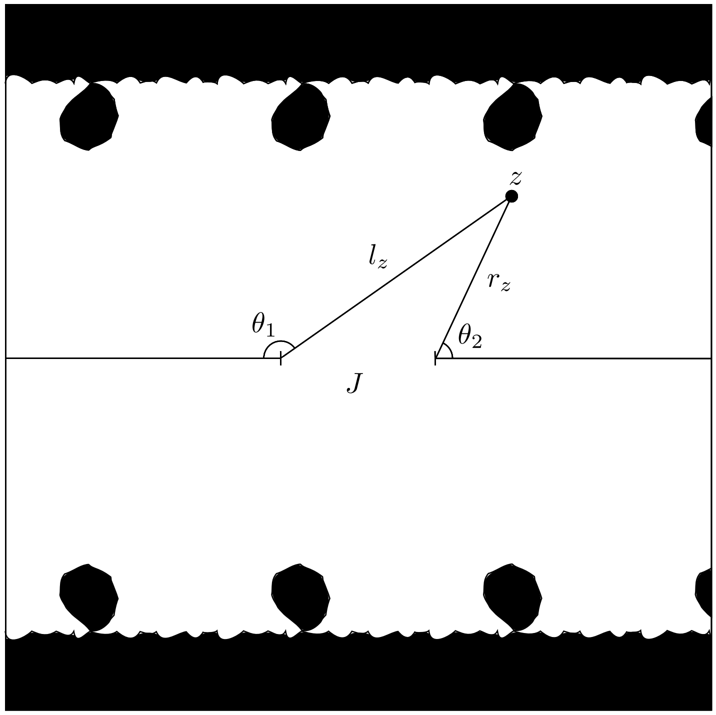

Figure 1.13: Illustration of $\widehat{(z, J)} = \min(\theta_1, \theta_2)$.

be the orbit of z_0 under ϕ.

The following three lemmas are adaptations of lemma 2.1, 4.2 and 4.4 in [Ya1] and lemma 6.1, 6.2 and 6.3 in [YaZ]:

Lemma 1.7.4. *Consider the orbit* (1.6)*. Let $k \leq q_n - 1$. Assume that for some i between* 0 *and k, we have $z_i \in \breve{S}$ and $\widehat{(z_{-i}, J_{-i})} > \epsilon$. Then*

$$\frac{dist(z_{-k}, J_{-k})}{|J_{-k}|} \leq C \frac{dist(z_{-i}, J_{-i})}{|J_{-i}|}$$

for some constant $C = C(\epsilon, \breve{S}) > 0$.

Lemma 1.7.5. *Let J and J' be two consecutive returns of the orbit* (1.5) *of J_0 to I_m for $1 < m < n$, and let ζ and ζ' be the corresponding points of the inverse orbit* (1.6)*. If $\zeta \in G_\theta^m$, then either $\zeta' \in G_\theta^m$ or $\widehat{(\zeta', J')} > \epsilon$ and $dist(\zeta', J') < C|I_m|$, where the constants ϵ and C are independent of m.*

Lemma 1.7.6. *Let J be the last return of the orbit* (1.5) *to the interval I_m preceding the first return to I_{m+1} for $1 \leq m \leq n-1$, and let J' and J'' be the first two returns to I_{m+1}. Let ζ, ζ' and ζ'' be the corresponding points in the inverse orbit* (1.6)*, so that $\zeta' = \phi^{q_m}(\zeta)$ and $\zeta'' = \phi^{q_{m+2}}(\zeta')$. Suppose that $\zeta \in G_\theta^m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon$ and $dist(\zeta'', J'') < C|I_{m+1}|$, or $\zeta'' \in G_\theta^{m+1}$, where the constants ϵ and C are independent of m.*

The interested reader can follow the proofs of Lemma 1.7.4, 1.7.5 and 1.7.6, and the rest of the proof of Theorem 1.7.3 in [YaZ] *mutatis mutandis.* □

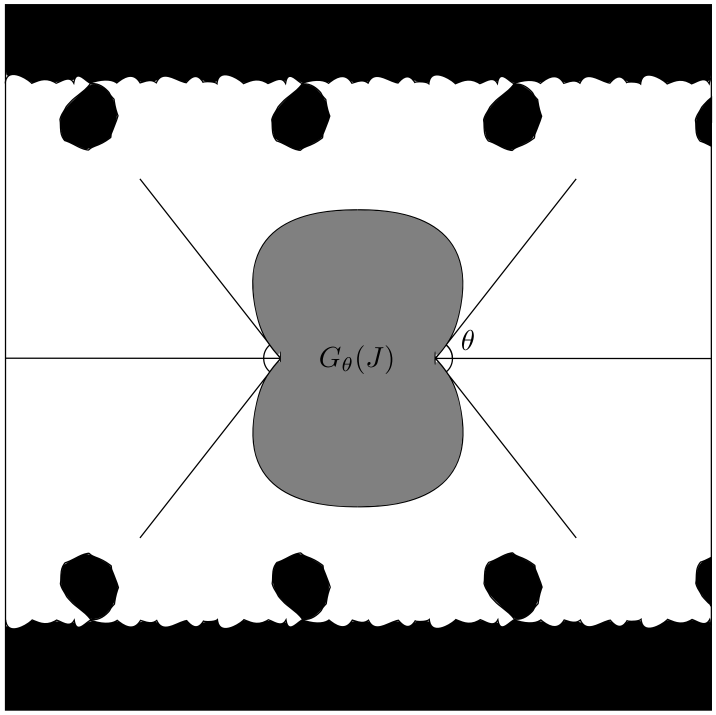

Figure 1.14: Illustration of the hyperbolic neighbourhood $G_\theta(J)$.

Corollary 1.7.7. *For all n sufficiently larger than the constant N in Lemma 1.7.2,* $\text{diam}(P_n^{crit})$ *is K-commensurate to* $\text{diam}(A_{-n})$ *for some universal constant $K \geq 1$. Consequently,* $\text{diam}(P_n^{crit}) \to 0$ *as $n \to \infty$.*

Proof. It suffices to show that any sequence of positive numbers $\{a_n\}_{n=0}^{\infty}$ satisfying the relation

$$a_n \leq C_1 \sqrt[3]{a_{n-1}} + C_2 \quad \text{for all} \quad n \geq 1$$

is bounded.

Consider the sequence $\{b_n\}_{n=0}^{\infty}$ defined inductively by

i) $b_0 = \max(1, a_0)$,

ii) $b_n = C \sqrt[3]{b_{n-1}}$,

where C is chosen so that

$$C\sqrt[3]{k} \geq C_1 \sqrt[3]{k} + C_2 \quad \text{for all} \quad k \geq 1.$$

It is easy to see that $b_n \geq a_n$ for all n.

A straightforward computation shows that

$$b_n = C^{1+\frac{1}{3}+\ldots+\frac{1}{3^{n-1}}} \sqrt[3^{n-1}]{b_0} \xrightarrow{n\to\infty} C^{\frac{3}{2}}.$$

Hence, $\{b_n\}_{n=0}^{\infty}$ and therefore, $\{a_n\}_{n=0}^{\infty}$ are bounded. $\square$

The following result we record for later use:

Lemma 1.7.8. *For all n sufficiently large, the nth critical puzzle piece P_n^{crit} contains a Euclidean disc D_n such that* $\operatorname{diam}(D_n)$ *is K-commensurate to* $\operatorname{diam}(P_n^{crit})$ *for some universal constant $K > 1$.*

Proof. Let D_1 be a disc centered at 1 such that $F_\nu^{q_n}(1) \in \partial D_1$. The map $F_\nu^{q_n}|_{A_n}$ has a well defined inverse branch which extends to D_1. Denote this inverse branch by ψ_n. As a consequence of real a priori bounds, we have the following estimate:

$$\frac{1}{|K_1|} \leq |\psi_n'(1)| \leq |K_1|,$$

where K_1 is some universal constant independent of n.

Observe that the preimage of $\mathbb{D}$ under F_ν consists of two connected components $U_{\text{in}} \subset \mathbb{D}$ and $U_{\text{out}} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Moreover, $\overline{U_{\text{in}}} \cap \overline{U_{\text{out}}} = \{1\}$. It is not difficult to see that ψ_n extends to U_{out}, and that $\psi_n(U_{\text{out}}) \subset P_n^{crit}$.

Now, choose a subdisc $D_2 \subset D_1 \cap U_{\text{out}}$ such that the annulus $A = D_1 \setminus \overline{D_2}$ satisfies the following estimate

$$\frac{1}{|K_2|} \leq \operatorname{mod}(A) \leq |K_2|,$$

for some universal constant K_2 independent of n. By Koebe distortion theorem, ψ_n has uniformly bounded distortion on D_2. Since $\psi_n(D_2) \subset \psi_N(U_{\text{out}}) \subset P_n^{crit}$, the result follows. □

1.8 The Proof of the Shrinking Theorem

We are ready to prove the shrinking theorem stated at the end of Section 1.6. The proof will be split into three propositions.

Proposition 1.8.1. *If Π_t is a nested puzzle sequence such that $L(\Pi_t)$ contains β or κ, then Π_t shrink to a point.*

Proof. We prove the result in the case where $L(\Pi_t)$ contains κ. The proof of the other case is similar.

Since $L(\Pi_t)$ contains κ, it follows that $t = 0$. Observe that $L(\Pi_0)$ is invariant under R_ν. Hence, $L(\Pi_0) \cap \partial\mathcal{S}_0 = \varnothing$.

Let D_r be a disc of radius $r > 0$ centered at κ. Since κ is a repelling fixed point, if r is sufficiently small, then D_r is mapped into itself by an appropriate inverse branch of R_ν. This inverse branch extends to a map $g : N \to N$, where N is a neighbourhood of $L(\Pi_0)$ which is disjoint from $\partial\mathcal{S}_0$ and therefore, the closure of the post critical set for R_ν.

Any set compactly contained within N converges to κ under iteration of R_ν. It follows that $L(\Pi_0) = \{\kappa\}$. □

For the proof of the remaining two propositions, it will be more convenient for us to work with the Blaschke product F_ν rather than R_ν itself. It is clear from the definition that a nested puzzle sequence for R_ν shrinks if and only if the corresponding nested puzzle sequence for F_ν shrinks.

Proposition 1.8.2. *If Π_t is a nested puzzle sequence such that $1 \in L(\Pi_t)$, then Π_t shrink to 1*

Proof. Recall the definition of critical puzzle pieces $\{P_n^{crit}\}_{n=0}^\infty$ for F_ν in Section 1.7. Let $\hat{\Pi}_{\text{even}}$ and $\hat{\Pi}_{\text{odd}}$ be the maximal nested puzzle sequence containing $\{P_{2n}^{crit}\}_{n=0}^\infty$ and $\{P_{2n+1}^{crit}\}_{n=0}^\infty$ respectively. Corollary

1.7.7 and Proposition 1.6.22 imply that $\hat{\Pi}_{\text{even}}$ and $\hat{\Pi}_{\text{odd}}$ both shrink to 1. By Proposition 1.6.20, there is no other maximal nested puzzle sequence at 1. $\square$

For the proof of the final proposition, we need the following lemma.

Lemma 1.8.3. *Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be a rational map of degree $d > 1$. Let $\{(f|_U)^{-n}\}_{n=0}^{\infty}$ be a family of univalent inverse branches of f restricted to a domain U. Suppose $U \cap J(f) \neq \varnothing$. If $V \Subset U$, then*

$$\operatorname{diam}((f|_U)^{-n}(V)) \to 0$$

as $n \to \infty$.

Proposition 1.8.4. *Let w_0 be a point in the Julia set $J(R_\nu)$ which is not an iterated preimage of κ, β or 1. If Π_t is a nested puzzle sequence such that $w_0 \in L(\Pi_t)$, then Π_t shrinks to w_0.*

Proof. Let $z_0 := \phi^{-1}(w_0)$, and consider the forward orbit

$$\mathcal{O} = \{z_n\}_{n=0}^{\infty}$$

of z_0 under F_ν. The proof splits into two cases.

Case 1. Suppose there exists some critical puzzle piece P_M^{crit} such that

$$\mathcal{O} \cap P_M^{crit} = \varnothing.$$

Let z_∞ be an accumulation point of $\mathcal{O}$, and let P^∞ be the puzzle piece of level M containing z_∞. Observe that the orbit of the critical point 1 is dense in $\partial\mathbb{D}$. Hence, P^∞ must be disjoint from $\partial\mathbb{D}$, since otherwise, P^∞ would map into P_M^{crit} by some appropriate inverse branch of F_ν.

Let $U \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ be a neighbourhood of P^∞, and choose a subsequence of orbit points $\{z_{n_k}\}_{k=0}^{\infty}$ from $\mathcal{O}$ such that $z_{n_k} \in P^\infty$. For each k, let

$$g_k : U \to \mathbb{C}$$

be the inverse branch of $F_\nu^{n_k}$ that maps z_{n_k} to z_0. Since P^∞ intersects the Julia set for F_ν, the nested puzzle sequence

$$\Pi := \{g_k(P^\infty)\}_{k=0}^{\infty}$$

must shrink to z_0 by Lemma 1.8.3.

Case 2. Suppose the critical point 1 is an accumulation point of $\mathcal{O}$. Then there exists an increasing sequence of numbers $\{n_k\}_{k=0}^{\infty}$ such that

$$\mathcal{O} \cap P_{n_k}^{crit} \neq \varnothing.$$

Fix k, and let z_{m_k} be the first orbit point that enters the critical puzzle piece $P_{n_k}^{crit}$. Let

$$P^{-n} \subset F_\nu^{-n}(P_{n_k}^{crit})$$

be the nth pull back of $P_{n_k}^{crit}$ along the orbit

$$z_0 \mapsto z_1 \mapsto \ldots \mapsto z_{m_k}. \tag{1.7}$$

Suppose that P^{-n} intersects 1 for some $n > 0$. Then for all $m \leq n$, the puzzle piece P^{-m} must intersect $\partial\mathbb{D}$. Recall that $P^{crit}_{n_k}$ contains the the preimage arc A_{-n_k}. Hence, for every $m \leq n$, the puzzle piece P^{-m} contains the mth preimage of A_{-n_k} under $F_\nu|_{\partial\mathbb{D}}$. By the combinatorics of critical circle maps, it follows that $P^{-q_{n_k}}$ must be the first puzzle piece in the backward orbit $\{P^{-1}, P^{-2}, \ldots, P^{-m_k}\}$ to intersect 1.

Since there are exactly two maximal nested puzzle sequences whose limit contains 1, all puzzle pieces of level $n > n_k + q_{n_k}$ which intersect 1 must be contained in either $P^{crit}_{n_k}$ or $P^{-q_{n_k}}$. Either case would contradict the fact that z_{m_k} is the first orbit point to enter $P^{crit}_{n_k}$. Therefore, P^{-n} does not intersect 1 for all $n \geq q_{n_k}$.

Let $m \leq m_k$ be the last moment when the backward orbit of $P^0 = P^{crit}_{n_k}$ intersect $\partial\mathbb{D}$. By Theorem 1.7.1, Corollary 1.7.7 and combinatorics of critical circle maps, the distance between P^{-m} and $F_\nu(1)$ is commensurate to $\operatorname{diam}(P^{-m})$. Hence, the distance between P^{-m-1} and 1 is commensurate to $\operatorname{diam}(P^{-m-1})$. Therefore, by Theorem 1.7.1 and Koebe distortion theorem, the inverse branch of $F_\nu^{m_k}$ along the orbit (1.7) can be expressed as either

$$F_\nu^{-m_k}|_{P^{crit}_{n_k}} = \eta$$

if $1 \notin P_n$ for all $n > 0$, or

$$F_\nu^{-m_k}|_{P^{crit}_{n_k}} = \zeta_1 \circ Q \circ \zeta_2$$

if $1 \in P^{-q_{n_k}}$, where η, ζ_1 and ζ_2 are conformal maps with bounded distortion, and Q is a branch of the cubic root.

Now, by Lemma 1.7.8, $P^{crit}_{n_k}$ contains a Euclidean disc D_{n_k} such that $\operatorname{diam}(D_{n_k})$ is commensurate to $\operatorname{diam}(P^{crit}_{n_k})$. The above argument implies that the puzzle piece P^{-m_k} must also contain a Euclidean disc D such that $\operatorname{diam}(D)$ is commensurate to $\operatorname{diam}(P^{-m_k})$. Hence, $\operatorname{diam}(P^{-m_k}) \to 0$ as $k \to \infty$, and the nested puzzle sequence

$$\Pi := \{P^{-m_k}\}_{k=0}^{\infty}$$

must shrink to z_0. □

As an application of the shrinking theorem, we prove that every infinite bubble ray for R_ν lands.

Proposition 1.8.5. *Every infinite bubble ray for R_ν lands.*

Proof. Let $\mathcal{R}_t$ be an infinite bubble ray, and let Ω be its accumulation set. If t is a dyadic rational, then $\mathcal{R}_t$ lands at an iterated preimage of κ. Otherwise, there exists a unique nested maximal puzzle sequence $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$ with external angle equal to t. By Proposition 1.6.18, Ω must be contained in $P_{[s_k,t_k]}$ for all $k \geq 1$. The result now follows from the shrinking theorem. □

1.9 The Proof of Conformal Mateability

We are ready to prove that R_ν is a conformal mating of $f_{\mathbf{B}}$ and $f_{\mathbf{S}}$. Recall the maps $\Phi_{\mathbf{B}}$ and $\Phi_{\mathbf{S}}$ in Theorem 1.5.16 defined on the union of the closure of every bubble in $\mathbf{B}$ and $\mathbf{S}$ respectively. Our first task is to continuously extend $\Phi_{\mathbf{B}}$ and $\Phi_{\mathbf{S}}$ to the filled Julia sets $K_{\mathbf{B}} = \overline{\mathbf{B}}$ and $K_{\mathbf{S}} = \overline{\mathbf{S}}$. For brevity, we will limit our discussion to $\Phi_{\mathbf{S}}$. The map $\Phi_{\mathbf{B}}$ can be extended in a completely analogous way.

Let $\tilde{\Phi}_{\mathbf{S}} : J_{\mathbf{S}} \to J(R_\nu)$ be the map defined as follows. For $x \in J_{\mathbf{S}}$, let $\Pi_t^{\mathbf{S}} = \{P^{\mathbf{S}}_{[s_k,t_k]}\}_{k=1}^{\infty}$ be a maximal nested puzzle sequence whose limit contains x. By the shrinking theorem, the corresponding maximal nested puzzle sequence $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$ for R_ν must shrink to a single point, say $y \in J(R_\nu)$. Define $\tilde{\Phi}_{\mathbf{S}}(x) := y$. We claim that $\tilde{\Phi}_{\mathbf{S}}$ is a continuous extension of $\Phi_{\mathbf{S}}$ on $J_{\mathbf{S}}$.

Proposition 1.9.1. *Let $S \subset \mathbf{S}$ be a bubble. If $x \in \partial S$, then $\tilde{\Phi}_{\mathbf{S}}(x) = \Phi_{\mathbf{S}}(x)$.*

Proof. Let $z := \Phi_{\mathbf{S}}(x)$. It is easy to see from the proof of Lemma 1.6.13 that $z \in \overline{\hat{C}^{(s_k,t_k)}_{(s_k,t_k)}}$ for all $k \geq 1$. It follows immediately that $z \in P_{[s_k,t_k]}$ for all $k \geq 1$, and hence, $\{z\} = L(\Pi_t) = \{y\}$. □

Proposition 1.9.2. *The map $\tilde{\Phi}_{\mathbf{S}} : J_{\mathbf{S}} \to J(R_\nu)$ is well defined.*

Proof. Suppose there are two maximal nested puzzle sequences at $x \in J_{\mathbf{S}}$. By Proposition 1.6.11, x is either an iterated preimage of $\mathbf{k_S}$ or 0. The first case follows from Proposition 1.6.18. The second case follows from Proposition 1.9.1. □

Proposition 1.9.3. *Define $\Phi_{\mathbf{S}}(x) := \tilde{\Phi}_{\mathbf{S}}(x)$ for all $x \in J_{\mathbf{S}}$. The extended map $\Phi_{\mathbf{S}} : K_{\mathbf{S}} \to \hat{\mathbb{C}}$ is continuous.*

Proof. It suffices to show that if $\{x_i\}_{i=0}^{\infty} \subset K_{\mathbf{S}}$ is a sequence converging to $x \in J_{\mathbf{S}}$, then the sequence of image points $\{y_i = \Phi_{\mathbf{S}}(x_i)\}_{i=0}^{\infty}$ converges to $y = \Phi_{\mathbf{S}}(x)$. The proof splits into four cases:

i) The point x is an iterated preimage of 0.

ii) There exists a unique bubble $S \subset \mathbf{S}$ such that $x \in \partial S$.

iii) The point x is an iterated preimage of $\mathbf{k_S}$.

iv) Otherwise.

Case i) By Proposition 1.5.7, there exist exactly two bubbles S_1 and S_2 which contain x in their boundary. Moreover, we have $\{x\} = \overline{S_1} \cap \overline{S_2}$. By Proposition 1.9.1, any subsequence of $\{x_i\}_{i=0}^{\infty}$ contained in $\overline{S_1} \cup \overline{S_2}$ is mapped under $\Phi_{\mathbf{S}}$ to a sequence which converges to y. Hence, we may assume that x_i is not contained $\overline{S_1} \cup \overline{S_2}$ for all $i \geq 0$.

By Proposition 1.6.11, there are exactly two maximal nested puzzle sequences $\Pi_t^{\mathbf{S}} = \{P^{\mathbf{S}}_{[s_k,t_k]}\}_{k=1}^{\infty}$ and $\Pi_v^{\mathbf{S}} = \{P^{\mathbf{S}}_{[u_k,v_k]}\}_{k=1}^{\infty}$ whose limit contains x. Let $D_r(x)$ be a disc of radius $r > 0$ centered at x. For every k, we can choose $r_k > 0$ sufficiently small such that $D_{r_k}(x) \cap \mathcal{P}_k^{\mathbf{S}} = D_{r_k}(x) \cap (\overline{S_1} \cup \overline{S_2})$. Let $N_k \geq 0$ be large enough such that $\{x_i\}_{i=N_k}^{\infty}$ is contained in $D_{r_k}(x)$. This implies that $\{x_i\}_{i=N_k}^{\infty} \subset P^{\mathbf{S}}_{[s_k,t_k]} \cup P^{\mathbf{S}}_{[u_k,v_k]}$. It is easy to see that the sequence of image points $\{y_i = \Phi_{\mathbf{S}}(x_i)\}_{i=N_k}^{\infty}$ must be contained $P_{[s_k,t_k]} \cup P_{[u_k,v_k]}$. By Proposition 1.9.2, $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$ and $\Pi_v = \{P_{[u_k,v_k]}\}_{k=1}^{\infty}$ both converge to y, and the result follows.

Case ii) The proof is very similar to Case i), and hence, it will be omitted.

Case iii) Since x is an iterated preimage of $\mathbf{k_S}$, it must be the landing point of some bubble ray $\mathcal{R}_t^{\mathbf{S}}$, where $t \in \mathbb{R}/\mathbb{Z}$ is a dyadic rational. By Proposition 1.6.18, y is the landing point of the corresponding bubble ray $\mathcal{R}_t^{\mathcal{S}}$. Any subsequence of $\{x_i\}_{i=0}^{\infty}$ contained in $\mathcal{R}_t^{\mathbf{S}}$ is mapped under $\Phi_{\mathbf{S}}$ to a sequence in $\mathcal{R}_t^{\mathcal{S}}$ which converges to y. Hence, we may assume that x_i is not contained $\mathcal{R}_t^{\mathbf{S}}$ for all $i \geq 0$.

The remainder of the proof is very similar to Case i), and hence, it will be omitted.

Case iv) By Proposition 1.6.11, there exists a unique maximal nested puzzle sequences $\Pi_t^{\mathbf{S}} = \{P_{[s_k,t_k]}^{\mathbf{S}}\}_{k=1}^{\infty}$ whose limit contains x. Let $D_r(x)$ be a disc of radius $r > 0$ centered at x. Since x is not contained the puzzle partition $\mathcal{P}_n^{\mathbf{S}}$ of any level $n \in \mathbb{N}$, it follows that for every $k \geq 1$, there exists $r_k > 0$ sufficiently small such that $D_r(x) \subset P_{[s_k,t_k]}^{\mathbf{S}}$. Thus, there exists $N_k \geq 0$ such that $\{x_i\}_{i=N_k}^{\infty}$ is contained in $P_{[s_k,t_k]}^{\mathbf{S}}$. It is easy to see that the sequence of image points $\{y_i = \Phi_{\mathbf{S}}(x_i)\}_{i=N_k}^{\infty}$ must be contained in the corresponding puzzle piece $P_{[s_k,t_k]}$ for R_ν. Since the nested puzzle sequence $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$ must shrink to y, the result follows. □

Proposition 1.9.4. *Let $t \in \mathbb{R}/\mathbb{Z}$, and let $x \in J_{\mathbf{B}}$ and $y \in J_{\mathbf{S}}$ be the landing point of the external ray for $f_{\mathbf{B}}$ and $f_{\mathbf{S}}$ with external angle $-t$ and t respectively. Then $\Phi_{\mathbf{B}}(x) = \Phi_{\mathbf{S}}(y)$.*

Proof. Consider the nested puzzle sequences $\Pi_t^{\mathbf{B}} = \{P_{[s_k,t_k]}^{\mathbf{B}}\}_{k=1}^{\infty}$, $\Pi_t^{\mathbf{S}} = \{P_{[s_k,t_k]}^{\mathbf{S}}\}_{k=1}^{\infty}$ and $\Pi_t = \{P_{[s_k,t_k]}\}_{k=1}^{\infty}$. By Proposition 1.6.4 and 1.6.10, we have $L(\Pi_t^{\mathbf{B}}) \cap J_{\mathbf{B}} = \{x\}$ and $L(\Pi_t^{\mathbf{S}}) \cap J_{\mathbf{S}} = \{y\}$. Let z be the point that Π_t shrinks to. By definition, $\Phi_{\mathbf{B}}(x) = z = \Phi_{\mathbf{S}}(y)$. □

Proof of Main Theorem 1B.

We verify the mating criterion given in Proposition 1.1.1. Let $f_{c_1} = f_{\mathbf{B}}$, $f_{c_2} = f_{\mathbf{S}}$, $\Lambda_1 = \Phi_{\mathbf{B}}$, $\Lambda_2 = \Phi_{\mathbf{S}}$, and $R = R_\nu$. Clearly, conditions (ii) and (iii) are satisfied. It remains to check condition (i).

Let $\tau_{\mathbf{B}} : \mathbb{R}/\mathbb{Z} \to J_{\mathbf{B}}$ and $\tau_{\mathbf{S}} : \mathbb{R}/\mathbb{Z} \to J_{\mathbf{S}}$ be the Carathéodory loop for $f_{\mathbf{B}}$ and $f_{\mathbf{S}}$ respectively (refer to Section 1.1 for the definition of Carathéodory loop). Define $\sigma_{\mathbf{B}}(t) := \tau_{\mathbf{B}}(-t)$. By Proposition 1.9.4, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{\sigma_{\mathbf{B}}} & J_{\mathbf{B}} \\ \downarrow{\scriptstyle \tau_{\mathbf{S}}} & & \downarrow{\scriptstyle \Phi_{\mathbf{B}}} \\ J_{\mathbf{S}} & \xrightarrow{\Phi_{\mathbf{S}}} & J(R_\nu) \end{array}$$

It follows that if $z \sim_{ray} w$, then z and w are mapped to the same point under $\Phi_{\mathbf{B}}$ or $\Phi_{\mathbf{S}}$.

To check the converse, it suffices to prove that for $z, w \in J_{\mathbf{S}}$, if $\Phi_{\mathbf{S}}(z) = \Phi_{\mathbf{S}}(w) = x \in J(R_\nu)$, then $z \sim_{ray} w$. First, observe that $\Phi_{\mathbf{S}}$ maps iterated preimages of 0 homeomorphically onto the iterated preimages of 1. Similarly, $\Phi_{\mathbf{S}}$ maps iterated preimages of $\mathbf{k_S}$ homeomorphically onto the iterated preimages of κ. Now, by Proposition 1.6.20, two distinct maximal nested sequences for R_ν shrink to x if and only if x is an iterated preimage of 1, κ or β. If x is an iterated preimage of 1 or κ, then z must be equal to w. If x is an iterated preimage of β, then $z \sim_{ray} w$. □

1.10 Further Thoughts

We finish this chapter with a brief discussion about possible generalizations of our results. The following are the conditions we assumed in our main theorem.

i) The rotation number of the Siegel disk is of bounded type.

ii) The Siegel disk is fixed.

iii) The super-attracting orbit of the hyperbolic polynomial has period 2.

As we explain below, (i) and (ii) are integral to the methods used in our proof, while (iii) can easily be replaced with a more general condition that allows for attracting orbits of any period.

In the proof of the main theorem, we modelled the dynamics of the candidate mating R_ν by a Blaschke product F_ν (see Section 1.4). This allowed us to consider chains of iterated preimages of the Siegel disk joined together at the iterated preimages of the critical point to form bubble rays. Moreover, it gave us a way to adapt Yampolsky's complex a priori bounds to prove that the puzzle pieces that are cut out by these bubble rays shrink to points at the Siegel boundary (see Section 1.7). Without condition (i), the conjugacy between the critical circle map $F_\nu|_{\partial\mathbb{D}}$ and rigid rotation by angle ν cannot extend quasiconformally to $\mathbb{D}$, and hence, we no longer can define the quasiconformal surgery that transforms F_ν to R_ν.

For quadratic polynomials, any result about a fixed Siegel disk tends to generalize to Siegel disks of period greater than one. This is due to the fact that a quadratic polynomial with a periodic Siegel disk can be renormalized to a quadratic-like map with a fixed Siegel disk using external rays. However, no such renormalization technique is known to exist for the basilica family R_a. As a result, we are unable to remove condition (ii).

On the other hand, it is not necessary for us to restrict ourselves to matings of Siegel quadratic polynomials with the basilica polynomial $f_{\mathbf{B}}$. Indeed, the only property of $f_{\mathbf{B}}$ that was used in our proof is the fact that the Fatou components in the filled Julia set for $f_{\mathbf{B}}$ are joined together at discrete points (namely, the iterated preimages of the α-fixed point $\mathbf{b}$). Hence, $f_{\mathbf{B}}$ can be replaced with any hyperbolic quadratic polynomial that satisfies this same property. For example, consider a parameter c which is contained in a satellite component of the main cardioid of the Mandelbrot set $\mathcal{M}$. For such c, the quadratic polynomial f_c is said to be *starlike*. With only a slight adjustment to the construction of the Blaschke product F_ν in Section 1.4 and to the definition of bubble rays in Section 1.5, the argument presented in this chapter can be used to prove the following more general result.

Main Theorem 1B'. *Suppose $\nu \in \mathbb{R} \setminus \mathbb{Q}$ is of bounded type. Let $f_{\mathbf{S}}$ be the unique member of the quadratic family that has a Siegel fixed point with rotation number ν, and let f_c be a starlike polynomial. Then $f_{\mathbf{S}}$ and f_c are conformally mateable.*

Chapter 2

The Siegel Disk of a Dissipative Hénon Map Has Non-Smooth Boundary

2.1 Introduction to Semi-Siegel Hénon Maps

In several complex variables, the archetypical class of examples are given by the following two-dimensional extension of the quadratic family

$$H_{c,b}(x, y) = (f_c(x) - by, x) = (x^2 + c - by, x) \qquad \text{for } c \in \mathbb{C} \text{ and } b \in \mathbb{C} \setminus \{0\}$$

called the *(complex quadratic) Hénon family.*

Since

$$H_{c,b}^{-1}(x, y) = \left(y, \frac{y^2 + c - x}{b}\right),$$

a Hénon map $H_{c,b}$ is a polynomial automorphism of $\mathbb{C}^2$. Moreover, it is easy to see that $H_{c,b}$ has constant Jacobian:

$$\text{Jac}H_{c,b} \equiv b.$$

Note that for $b = 0$, the map $H_{c,b}$ degenerates to the following embedding of f_c:

$$(x, y) \mapsto (f_c(x), x).$$

Hence, the parameter b can be viewed as a measure of how far $H_{c,b}$ is from being a degenerate one-dimensional system. We will always assume that $H_{c,b}$ is a dissipative map (i.e. $|b| < 1$).

As usual, we let $K^\pm$ be the sets of points in $\mathbb{C}^2$ that do not escape to infinity under forward/backward iterations of the Hénon map respectively. Their topological boundaries are $J^\pm = \partial K^\pm$. Let $K = K^+ \cap K^-$ and $J = J^- \cap J^+$. The sets $J^\pm$ and $K^\pm$ are unbounded and connected (see [BS1]), while J and K are compact (see [HOV1]). In analogy to one-dimensional dynamics, the set J is called the *Julia set* of the Hénon map.

A Hénon map $H_{c,b}$ is determined uniquely by the multipliers μ and ν at a fixed point $\mathbf{p}_0$. In particular, we have

$$b = \mu\nu,$$

and

$$c = (1+b)\left(\frac{\mu}{2} + \frac{b}{2\mu}\right) - \left(\frac{\mu}{2} + \frac{b}{2\mu}\right)^2.$$

When convenient, we will write $H_{\mu,\nu}$ instead of $H_{c,b}$ to denote a Hénon map.

A dissipative Hénon map $H_{\mu,\nu}$ has a *semi-Siegel fixed point* $\mathbf{p}_0$ if $\mu = e^{2\pi i\theta}$ for some $\theta \in (0,1) \setminus \mathbb{Q}$, and there exist neighbourhoods N of $(0,0)$ and $\mathcal{N}$ of $\mathbf{p}_0$, and a biholomorphic change of coordinates

$$\phi : (N, (0,0)) \to (\mathcal{N}, \mathbf{p}_0)$$

such that

$$H_{\mu,\nu} \circ \phi = \phi \circ L,$$

where $L(x,y) := (\mu x, \nu y)$. A classic theorem of Siegel states, in particular, that $H_{\mu,\nu}$ is semi-Siegel whenever θ is *Diophantine*. That is, for some constants C and d, we have

$$q_{n+1} < C q_n^d,$$

where p_n/q_n are the continued fraction convergents of θ (see Section 1.3 for a more detailed discussion of Diophantine numbers). In this case, the linearizing map ϕ can be biholomorphically extended to

$$\phi : (\mathbb{D} \times \mathbb{C}, (0,0)) \to (\mathcal{C}, \mathbf{p}_0)$$

so that the image $\mathcal{C} := \phi(\mathbb{D} \times \mathbb{C})$ is *maximal* (see [MNTU]). The set $\mathcal{C}$ is a connected component of the interior of K^+, and its boundary coincides with J^+ (see [BS2]). Let

$$\mathcal{D} := \phi(\mathbb{D} \times \{0\}).$$

Then clearly, $\mathcal{C} = W^s(\mathcal{D})$ and $\mathcal{D} \subset K$. We call $\mathcal{C}$ and $\mathcal{D}$ the *Siegel cylinder* and the *Siegel disk* of the Hénon map respectively.

Remark 2.1.1. The Siegel disk $\mathcal{D}$ must be contained in the center manifold $W^c(\mathbf{p}_0)$ of the semi-Siegel fixed point $\mathbf{p}_0$ (see e.g. [S] for the definition of center manifolds). The center manifold is not unique in general, but all center manifolds of $\mathbf{p}_0$ must coincide on the Siegel disk. This phenomenon is nicely illustrated in [O], Figure 5.

The geometry of Siegel disks in one dimension is a challenging and important topic, studied by numerous authors; including Herman [He], McMullen [Mc3], Petersen [P], Inou and Shishikura [ISh], Yampolsky [Ya3], and others. In the two-dimensional Hénon family, the corresponding problems have been wide open until a very recent work of Gaidashev, Radu, and Yampolsky [GaRYa], who proved:

Theorem 2.1.2 (Gaidashev, Radu, Yampolsky)**.** *Let $\theta_* = (\sqrt{5}-1)/2$ be the inverse golden mean, and let $\mu_* = e^{2\pi i\theta_*}$. Then there exists $\epsilon > 0$ such that if $|\nu| < \epsilon$, then the boundary of the Siegel disk $\mathcal{D}$ of $H_{\mu_*,\nu}$ is a homeomorphic image of the circle. In fact, the linearizing map*

$$\phi : \mathbb{D} \times \{0\} \to \mathcal{D}$$

extends continuously and injectively (but not smoothly) to the boundary.

Theorem 2.1.2 raises a natural question whether the boundary $\partial\mathcal{D}$ can ever lie on a smooth curve. In the present note we answer this in the negative:

Main Theorem 2. *Let $\epsilon > 0$ be as in Theorem 2.1.2 and let $|\nu| < \epsilon$. Then the boundary of the Siegel disk of $H_{\mu_*,\nu}$ is not C^1-smooth.*

2.2 Renormalization of Almost-Commuting Pairs

In this section we give a summary of the relevant statements on renormalization of almost-commuting pairs; we refer the reader to [GaYa] for further details.

2.2.1 One-dimensional renormalization

For a domain $Z \subset \mathbb{C}$, we denote $\mathcal{A}(Z)$ the Banach space of bounded analytic functions $f : Z \to \mathbb{C}$ equipped with the norm

$$\|f\| = \sup_{x\in Z} |f(x)|. \tag{2.1}$$

Denote $\mathcal{A}(Z, W)$ the Banach space of bounded pairs of analytic functions $\zeta = (f, g)$ from domains $Z \subset \mathbb{C}$ and $W \subset \mathbb{C}$ respectively to $\mathbb{C}$ equipped with the norm

$$\|\zeta\| = \frac{1}{2}\left(\|f\| + \|g\|\right). \tag{2.2}$$

Henceforth, we assume that the domains Z and W contain 0.

For a pair $\zeta = (f, g)$, define the *rescaling map* as

$$\Lambda(\zeta) := (s_\zeta^{-1} \circ f \circ s_\zeta, s_\zeta^{-1} \circ g \circ s_\zeta), \tag{2.3}$$

where

$$s_\zeta(x) := \lambda_\zeta x \quad \text{and} \quad \lambda_\zeta := g(0).$$

Definition 2.2.1. We say that $\zeta = (\eta, \xi) \in \mathcal{A}(Z, W)$ is a *critical pair* if

(i) η and ξ have a simple unique critical point at 0, and

(ii) $\xi(0) = 1$.

The space of critical pairs in $\mathcal{A}(Z, W)$ is denoted by $\mathcal{C}(Z, W)$.

Definition 2.2.2. We say that $\zeta = (\eta, \xi) \in \mathcal{A}(Z, W)$ is a *commuting pair* if

$$\eta \circ \xi = \xi \circ \eta.$$

It turns out, requiring strict commutativity is too limiting in the category of analytic functions. Hence, we work with the following less restrictive condition:

Definition 2.2.3. We say that $\zeta = (\eta, \xi) \in \mathcal{C}(Z, W)$ is an *almost commuting pair* (cf. [Bur, Stir]) if

$$\frac{d^i(\eta \circ \xi - \xi \circ \eta)}{dx^i}(0) = 0 \quad \text{for} \quad i = 0, 2.$$

The space of almost commuting pairs in $\mathcal{C}(Z,W)$ is denoted by $\mathcal{B}(Z,W)$.

Proposition 2.2.4 (cf. [GaYa])**.** *The spaces $\mathcal{C}(Z,W)$ and $\mathcal{B}(Z,W)$ have the structure of an immersed Banach submanifold of $\mathcal{A}(Z,W)$ of codimension 3 and 5 respectively.*

Denote

$$c(x) := \bar{x}.$$

Definition 2.2.5. Let $\zeta = (\eta, \xi) \in \mathcal{B}(Z,W)$. The *pre-renormalization* of ζ is defined as:

$$p\mathcal{R}(\zeta) = p\mathcal{R}((\eta,\xi)) := (\eta \circ \xi, \eta).$$

The *renormalization* of ζ is defined as:

$$\mathcal{R}(\zeta) = \mathcal{R}((\eta,\xi)) := \Lambda((c \circ \eta \circ \xi \circ c, c \circ \eta \circ c)).$$

We say that ζ is *renormalizable* if $\mathcal{R}(\zeta) \in \mathcal{B}(Z,W)$.

The following is shown in [GaYa]:

Theorem 2.2.6. *There exist topological disks $\hat{Z} \Supset Z$ and $\hat{W} \Supset W$, and a commuting pair $\zeta_* = (\eta_*, \xi_*) \in \mathcal{B}(Z,W)$ such that the following holds:*

(i) There exists a neighbourhood $\mathcal{N}$ of ζ_* in the submanifold $\mathcal{B}(Z,W)$ such that

$$\mathcal{R} : \mathcal{N} \to \mathcal{B}(\hat{Z}, \hat{W})$$

is an anti-analytic operator.

(ii) The pair ζ_* is the unique fixed point of $\mathcal{R}$ in $\mathcal{N}$.

(iii) The differential $D_{\zeta_*}\mathcal{R}$ is a compact anti-linear operator. The operator

$$L := D_{\zeta_*}\mathcal{R} \circ c$$

has a single, simple eigenvalue with modulus greater than 1. The rest of its spectrum lies inside the open unit disk $\mathbb{D}$ (and hence is compactly contained in $\mathbb{D}$ by the spectral theory of compact operators).

2.2.2 Two-dimensional renormalization

For a domain $\Omega \subset \mathbb{C}^2$, we denote $\mathcal{A}_2(\Omega)$ the Banach space of bounded analytic functions $F : \Omega \to \mathbb{C}^2$ equipped with the norm

$$\|F\| = \sup_{(x,y)\in\Omega} |F(x,y)|. \tag{2.4}$$

Define

$$\|F\|_y := \sup_{(x,y)\in\Omega} |\partial_y F(x,y)|. \tag{2.5}$$

Moreover, for

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

define

$$\|F\|_{\text{diag}} := \sup_{(x,y)\in\Omega} |f_1(x,y) - f_2(x,y)|. \tag{2.6}$$

Denote $\mathcal{A}_2(\Omega, \Gamma)$ the Banach space of bounded pairs of analytic functions $\Sigma = (F, G)$ from domains $\Omega \subset \mathbb{C}^2$ and $\Gamma \subset \mathbb{C}^2$ respectively to $\mathbb{C}^2$ equipped with the norm

$$\|\Sigma\| = \frac{1}{2}\left(\|F\| + \|G\|\right). \tag{2.7}$$

Define

$$\|\Sigma\|_y := \frac{1}{2}\left(\|F\|_y + \|G\|_y\right). \tag{2.8}$$

Moreover,

$$\|\Sigma\|_{\text{diag}} := \frac{1}{2}\left(\|F\|_{\text{diag}} + \|G\|_{\text{diag}}\right). \tag{2.9}$$

Henceforth, we assume that

$$\Omega = Z \times Z \quad \text{and} \quad \Gamma = W \times W,$$

where Z and W are subdomains of $\mathbb{C}$ containing 0. For a function

$$F(x,y) := \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

from Ω or Γ to $\mathbb{C}^2$, we denote

$$\pi_1 F(x) := f_1(x,0) \quad \text{and} \quad \pi_2 F(x) := f_2(x,0).$$

For a pair $\Sigma = (F, G)$, define the *rescaling map* as

$$\Lambda(\Sigma) := (s_\Sigma^{-1} \circ F \circ s_\Sigma, s_\Sigma^{-1} \circ G \circ s_\Sigma), \tag{2.10}$$

where

$$s_\Sigma(x,y) := (\lambda_\Sigma x, \lambda_\Sigma y) \quad \text{and} \quad \lambda_\Sigma := \pi_1 G(0).$$

The following definitions are analogs of Definition 2.2.1, 2.2.2 and 2.2.3.

Definition 2.2.7. For $\kappa \geq 0$, we say that $\Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma)$ is a κ*-critical pair* if

(i) $\pi_1 A$ and $\pi_1 B$ have a simple unique critical point which is contained in a κ-neighbourhood of 0, and

(ii) $\pi_1 B(0) = 1$.

The space of κ-critical pairs in $\mathcal{A}_2(\Omega, \Gamma)$ is denoted by $\mathcal{C}_2(\Omega, \Gamma, \kappa)$.

Definition 2.2.8. We say that $\Sigma = (A, B) \in \mathcal{A}_2(\Omega, \Gamma)$ is a *commuting pair* if

$$A \circ B = B \circ A.$$

Definition 2.2.9. We say that $\Sigma = (A, B) \in \mathcal{C}_2(\Omega, \Gamma, \kappa)$ is an *κ-almost commuting pair* if

$$\left| \frac{d^i \pi_1 [A, B]}{dx^i}(0) \right| := \left| \frac{d^i \pi_1 (A \circ B - B \circ A)}{dx^i}(0) \right| \leq \kappa \quad \text{for} \quad i = 0, 2.$$

The space of κ-almost commuting pairs in $\mathcal{C}_2(\Omega, \Gamma, \kappa)$ is denoted by $\mathcal{B}_2(\Omega, \Gamma, \kappa)$.

Proposition 2.2.10 (cf.[GaYa])**.** *The space $\mathcal{B}_2(\Omega, \Gamma, \kappa)$ has the structure of an immersed Banach submanifold of $\mathcal{A}_2(\Omega, \Gamma)$ of codimension* 1*.*

For $0 < \epsilon, \delta \leq \infty$, let $\mathcal{A}_2(\Omega, \Gamma, \epsilon, \delta)$ be the open subset of $\mathcal{A}_2(\Omega, \Gamma)$ consisting of pairs $\Sigma = (A, B)$ such that the following holds:

1. $\|\Sigma\|_y < \epsilon$, and

2. $\|\Sigma\|_{\text{diag}} < \delta$.

Note that

$$\mathcal{A}_2(\Omega, \Gamma, \infty, \infty) \equiv \mathcal{A}_2(\Omega, \Gamma).$$

We denote

$$\mathcal{C}_2(\Omega, \Gamma, \epsilon, \delta, \kappa) := \mathcal{A}_2(\Omega, \Gamma, \epsilon, \delta) \cap \mathcal{C}_2(\Omega, \Gamma, \kappa), \tag{2.11}$$

and

$$\mathcal{B}_2(\Omega, \Gamma, \epsilon, \delta, \kappa) := \mathcal{A}_2(\Omega, \Gamma, \epsilon, \delta) \cap \mathcal{B}_2(\Omega, \Gamma, \kappa). \tag{2.12}$$

Proposition 2.2.11 (cf. [GaYa])**.** *If ϵ, δ, and κ are sufficiently small, then there exists an analytic projection map $\Pi_{\text{ac}} : C_2(\Omega, \Gamma, \epsilon, \delta, \kappa) \to \mathcal{B}_2(\Omega, \Gamma, \epsilon, \delta, \kappa)$ such that*

$$\Pi_{\text{ac}}|_{\mathcal{B}_2(\Omega, \Gamma, \epsilon, \delta, \kappa)} \equiv \text{ Id}. \tag{2.13}$$

We define an isometric embedding ι of the space $\mathcal{A}(Z)$ to $\mathcal{A}_2(\Omega)$ as follows:

$$\iota(f)(x, y) = \iota(f)(x) := \begin{bmatrix} f(x) \\ f(x) \end{bmatrix}. \tag{2.14}$$

We extend this definition to an isometric embedding of $\mathcal{A}(Z, W)$ into $\mathcal{A}_2(\Omega, \Gamma)$ as follows:

$$\iota((\eta, \xi)) := (\iota(\eta), \iota(\xi)). \tag{2.15}$$

Note that

$$\iota(\mathcal{B}(Z, W)) = \mathcal{B}_2(\Omega, \Gamma, 0, 0, 0).$$

Notation 2.2.12. Let $\mathcal{I}$ be the space of all finite multi-indexes

$$\overline{\omega} = (a_0, \ldots, a_n) \in (\{0\} \cup \mathbb{N})^n \quad \text{for some } n \in \mathbb{N},$$

with the partial ordering relation $\prec$ defined as follows. We have

$$(a_0, a_1, \ldots, a_k, b) \prec (a_0, a_1, \ldots, a_n, a_{n+1})$$

if either $k < n$ and $b \leq a_{k+1}$, or $k = n$ and $b < a_{n+1}$. For a pair $\zeta = (\eta, \xi)$ and a multi-index $\overline{\omega} = (a_0, \ldots, a_n) \in \mathcal{I}$, denote

$$\zeta^{\overline{\omega}} = \phi^{a_n} \circ \ldots \circ \xi^{a_1} \circ \eta^{a_0}$$

where ϕ is either η or ξ, depending on whether n is even or odd. Lastly, define a sequence $\{\overline{\alpha}_0, \overline{\alpha}_1, \ldots\} \subset \mathcal{I}$ such that

$$p\mathcal{R}^n(\zeta) = (\zeta^{\overline{\alpha}_n}, \zeta^{\overline{\alpha}_{n-1}}),$$

where $p\mathcal{R}$ is the pre-renormalization operator defined in Definition 2.2.5.

Lemma 2.2.13. *Let $\tilde{Z} \Subset Z$ and $\tilde{W} \Subset W$ be domains in $\mathbb{C}$. For any four-times 1D renormalizable pair $\zeta_0 = (\eta_0, \xi_0) \in \mathcal{B}(Z, W)$, there exists a neighbourhood $\mathcal{N}(\zeta_0) \subset \mathcal{A}(Z, W)$ of ζ_0 such that if $\zeta = (\eta, \xi) \in \mathcal{N}(\zeta_0)$, then the pair*

$$\mathcal{R}^4(\zeta) := \Lambda(\zeta^{\overline{\alpha}_4}, \zeta^{\overline{\alpha}_3})$$

is a well-defined element of $\mathcal{A}(\tilde{Z}, \tilde{W})$.

Let $\mathcal{D}_2(\Omega, \Gamma, \epsilon) \subset \mathcal{A}_2(\Omega, \Gamma, \epsilon, \infty)$ be the open set consisting of pairs $\Sigma = (A, B)$ such that the following conditions hold.

(i) The pair $\Lambda(\Sigma_1)$ is a well-defined element of $\mathcal{A}_2(\tilde{\Omega}, \tilde{\Gamma})$, where

$$\Sigma_1 = (A_1, B_1) := (A^{-1} \circ \Sigma^{\overline{\alpha}_4} \circ A, A^{-1} \circ \Sigma^{\overline{\alpha}_3} \circ A),$$

and

$$\tilde{\Omega} := (1 - \epsilon)\Omega \quad \text{and} \quad \tilde{\Gamma} := (1 - \epsilon)\Gamma.$$

(ii) The map $\pi_2 B_1$ is conformal on $\pi_1 B^{-1} \circ A_1 \circ A^{-1}(V)$ and $\pi_1 B^{-1} \circ B_1 \circ A^{-1}(V)$, where

$$V := \lambda_{\Sigma_1} Z \cup W \subset \mathbb{C}.$$

We define the renormalization of $\Sigma \in \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ in several steps.

Write

$$\Sigma = (A, B) = \left(\begin{bmatrix} a \\ h \end{bmatrix}, \begin{bmatrix} b \\ g \end{bmatrix} \right),$$

and denote

$$\eta_i(x) := \pi_i A(x) \quad \text{and} \quad \xi_i(x) := \pi_i B(x) \quad , \quad \text{for } i \in \{1, 2\}.$$

Let

$$a_y(x) := a(x, y),$$

and consider the following non-linear changes of coordinates:

$$H(x, y) := \begin{bmatrix} a_y^{-1}(x) \\ y \end{bmatrix} \quad \text{and} \quad V(x, y) := \begin{bmatrix} x \\ \xi_2 \circ \xi_1^{-1} \circ \eta_1^{-1}(y) \end{bmatrix}. \tag{2.16}$$

Observe that

$$A \circ H(x, y) = \begin{bmatrix} a_y \circ a_y^{-1}(x) \\ g(a_y^{-1}(x), y) \end{bmatrix} = \begin{bmatrix} x \\ g(a_y^{-1}(x), y) \end{bmatrix}.$$

Moreover,

$$V^{-1} \circ H^{-1} \circ B = \begin{bmatrix} a_g \circ b \\ \eta_1 \circ \xi_1 \circ \xi_2^{-1} \circ g \end{bmatrix}.$$

Thus, we have

$$\|A \circ H\|_y < O(\epsilon) \quad \text{ and } \quad \|V \circ H \circ B - \iota(\eta_1 \circ \xi_1)\| < O(\epsilon)$$

where defined.

Let

$$A_2 := V^{-1} \circ H^{-1} \circ A_1 \circ H \circ V,$$

and

$$B_2 := V^{-1} \circ H^{-1} \circ B_1 \circ H \circ V.$$

Define the *pre-renormalization* of Σ as

$$p\mathbf{R}(\Sigma) := (A_2, B_2). \tag{2.17}$$

Let

$$\zeta := (\eta_1, \xi_1).$$

From the above inequalities, it follows that

$$\|p\mathbf{R}(\Sigma) - \iota(p\mathcal{R}^4(\zeta))\| < O(\epsilon) \quad \text{ and } \quad \|p\mathbf{R}(\Sigma)\|_y < O(\epsilon^2) \tag{2.18}$$

where defined.

By the argument principle, if ϵ is sufficiently small, then the function $\pi_1 B_1 \circ A_1$ has a simple unique critical point c_a near 0. Set

$$T_a(x, y) := (x + c_a, y), \tag{2.19}$$

Likewise, the function $\pi_1 T_a^{-1} \circ A_1 \circ B_1 \circ T_a$ has a simple unique critical point c_b near 0. Set

$$T_b(x, y) := (x + c_b, y). \tag{2.20}$$

Note that if Σ is a commuting pair (i.e. $A \circ B = B \circ A$), then $T_b \equiv \mathrm{Id}$.

Define the *critical projection* of $p\mathbf{R}(\Sigma)$ as

$$\Pi_{\text{crit}} \circ p\mathbf{R}(\Sigma) = (A_3, B_3) := (T_b^{-1} \circ T_a^{-1} \circ A_2 \circ T_a, T_a^{-1} \circ B_2 \circ T_a \circ T_b). \tag{2.21}$$

Note that

$$0 = \pi_1(B_3 \circ A_3)'(0) = (\pi_1 A_3)'(0) + O(\epsilon^2),$$

and likewise

$$0 = \pi_1(A_3 \circ B_3)'(0) = (\pi_1 B_3)'(0) + O(\epsilon^2).$$

Hence,

$$(\pi_1 A_3)'(0) = O(\epsilon^2) \quad \text{ and } \quad (\pi_1 B_3)'(0) = O(\epsilon^2). \tag{2.22}$$

It follows that there exists a uniform constant $C > 0$ such that the rescaled pair $\Lambda \circ \Pi_{\text{crit}} \circ p\mathbf{R}(\Sigma)$ is contained in $\mathcal{C}_2(\Omega, \Gamma, C\epsilon^2, C\epsilon, C\epsilon^2)$ (recall that this means $\Lambda \circ \Pi_{\text{crit}} \circ p\mathbf{R}(\Sigma)$ is a $C\epsilon^2$-critical pair with $C\epsilon^2$ dependence on y that is $C\epsilon$ away from the diagonal; see (2.11)).

Finally, define the *2D renormalization* of Σ as

$$\mathbf{R}(\Sigma) := \Pi_{\text{ac}} \circ \Lambda \circ \Pi_{\text{crit}} \circ p\mathbf{R}(\Sigma), \tag{2.23}$$

where the projection map Π_{ac} is given in Proposition 2.2.11.

Proposition 2.2.14. *If $\Sigma = (A, B) \in \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ is a commuting pair (i.e. $A \circ B = B \circ A$), then $\mathbf{R}(\Sigma)$ is a conjugate of $(\Sigma^{\overline{\alpha}_4}, \Sigma^{\overline{\alpha}_3})$.*

Theorem 2.2.15. *Let ζ_* be the fixed point of the 1D renormalization given in Theorem 2.1.2. For $\epsilon > 0$, let $\mathcal{N}_\epsilon(\iota(\zeta_*)) \Subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ be a neighbourhood of $\iota(\zeta_*)$ with compact closure. Then there exists a uniform constant $C > 0$ depending on $\mathcal{N}_\epsilon(\iota(\zeta_*))$ such that the 2D renormalization operator*

$$\mathbf{R} : \mathcal{D}_2(\Omega, \Gamma, \epsilon) \to \mathcal{A}_2(\Omega, \Gamma),$$

is a well-defined analytic operator satisfying the following properties:

1. *$\mathbf{R}|_{\mathcal{N}_\epsilon(\iota(\zeta_*))} : \mathcal{N}_\epsilon(\iota(\zeta_*)) \to \mathcal{B}_2(\Omega, \Gamma, C\epsilon^2, C\epsilon, C\epsilon^2)$.*

2. *If $\Sigma = (A, B) \in \mathcal{N}_\epsilon(\iota(\zeta_*))$ and $\zeta := (\pi_1 A, \pi_1 B)$, then*

$$\|\mathbf{R}(\Sigma) - \iota(\mathcal{R}^4(\zeta))\| < C\epsilon.$$

Consequently, if $\mathcal{N}(\zeta_*) \subset \mathcal{B}(Z, W)$ is a neighbourhood of ζ_* such that $\iota(\mathcal{N}(\zeta_*)) \subset \mathcal{N}_\epsilon(\iota(\zeta_*))$, then

$$\mathbf{R} \circ \iota|_{\mathcal{N}(\zeta_*)} \equiv \iota \circ \mathcal{R}^4|_{\mathcal{N}(\zeta_*)}.$$

3. *The pair $\iota(\zeta_*)$ is the unique fixed point of $\mathbf{R}$ in $\mathcal{N}_\epsilon(\iota(\zeta_*))$.*

4. *The differential $D_{\iota(\zeta_*)}\mathbf{R}$ is a compact linear operator whose spectrum coincides with that of $D_{\zeta_*}\mathcal{R}^4$. More precisely, in the spectral decomposition of $D_{\iota(\zeta_*)}\mathbf{R}$, the complement to the tangent space $T_{\iota(\zeta_*)}(\iota(\mathcal{N}(\zeta_*)))$ corresponds to the zero eigenvalue.*

We denote the stable manifold of the fixed point $\iota(\zeta_*)$ for the 2D renormalization operator $\mathbf{R}$ by $W^s(\iota(\zeta_*)) \subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$.

Let $H_{\mu_*,\nu}$ be the Hénon map with a semi-Siegel fixed point $\mathbf{q}$ of multipliers $\mu_* = e^{2\pi i \theta_*}$ and ν, where $\theta_* = (\sqrt{5} - 1)/2$ is the inverse golden mean rotation number, and $|\nu| < \epsilon$. We identify $H_{\mu_*,\nu}$ as a pair in $\mathcal{D}_2(\Omega, \Gamma, \epsilon)$ as follows:

$$\Sigma_{H_{\mu_*,\nu}} := \Lambda(H^2_{\mu_*,\nu}, H_{\mu_*,\nu}). \tag{2.24}$$

The following is shown in [GaRYa]:

Theorem 2.2.16. *The pair $\Sigma_{H_{\mu_*,\nu}}$ is contained in the stable manifold $W^s(\iota(\zeta_*)) \subset \mathcal{D}_2(\Omega, \Gamma, \epsilon)$ of the fixed point $\iota(\zeta_*)$ for the 2D renormalization operator $\mathbf{R}$.*

2.3 The Renormalization Arc

Let

$$\zeta_* = (\eta_*, \xi_*)$$

be the fixed point of the 1D renormalization operator $\mathcal{R}$ given in Theorem 2.1.2. By Theorem 2.2.15, the diagonal embedding $\iota(\zeta_*)$ of ζ_* is a fixed point of the 2D renormalization operator $\mathbf{R}$. Hence, we have

$$\mathbf{R}(\iota(\zeta_*)) = (s_*^{-1} \circ \iota(\zeta)^{\overline{\alpha}_4} \circ s_*, s_*^{-1} \circ \iota(\zeta)^{\overline{\alpha}_3} \circ s_*) = \iota(\zeta_*),$$

where

$$s_*(x, y) := (\lambda_* x, \lambda_* y) \quad , \quad |\lambda_*| < 1.$$

Let $\Sigma = (A, B)$ be a pair contained in the stable manifold $W^s(\iota(\zeta_*))$ of the fixed point $\iota(\zeta_*)$. Assume that Σ is commuting, so that

$$A \circ B = B \circ A.$$

Write

$$\Sigma_n = (A_n, B_n) = \left(\begin{bmatrix} a_n \\ h_n \end{bmatrix}, \begin{bmatrix} b_n \\ g_n \end{bmatrix} \right) := \mathbf{R}^n(\Sigma),$$

and let

$$\eta_n(x) := \pi_1 A_n(x) = a_n(x, 0) \quad \text{and} \quad \xi_n(x) := \pi_1 B_n(x) = b_n(x, 0).$$

By Theorem 2.2.15, we may express

$$A_n = \iota(\eta_n) + E_n \quad \text{and} \quad B_n = \iota(\xi_n) + F_n \tag{2.25}$$

where the error terms E_n and F_n satisfy

$$\|E_n\| < C\epsilon^{2^{n-1}} \quad \text{and} \quad \|F_n\| < C\epsilon^{2^{n-1}}. \tag{2.26}$$

Hence, the sequence of pairs $\{\Sigma_n\}_{n=0}^{\infty}$ converges to $\mathcal{B}_2(\Omega, \Gamma, 0, 0, 0)$ *super-exponentially.*

Denote

$$(a_n)_y(x) := a_n(x, y).$$

Let

$$H_n(x, y) := \begin{bmatrix} (a_n)_y^{-1}(x) \\ y \end{bmatrix} \quad \text{and} \quad V_n(x, y) := \begin{bmatrix} x \\ \pi_2 B_n \circ \xi_n^{-1} \circ \eta_n^{-1}(y) \end{bmatrix}$$

be the non-linear changes of coordinates given in (2.16), let

$$T_n(x, y) := (x + d_n, y),$$

be the translation map given in (2.19), and let

$$s_n(x, y) := (\lambda_n x, \lambda_n y) \quad , \quad |\lambda_n| < 1$$

be the scaling map so that if

$$\phi_n := H_n \circ V_n \circ T_n \circ s_n, \tag{2.27}$$

then by Proposition 2.2.14, we have

$$A_{n+1} = \phi_n^{-1} \circ A_n^{-1} \circ \Sigma_n^{\overline{\alpha}_4} \circ A_n \circ \phi_n$$

and

$$B_{n+1} = \phi_n^{-1} \circ A_n^{-1} \circ \Sigma_n^{\overline{\alpha}_3} \circ A_n \circ \phi_n.$$

Denote

$$\Phi_n^k := \phi_n \circ \phi_{n+1} \circ \ldots \circ \phi_{k-1} \circ \phi_k \quad , \quad \Omega_n^k := \Phi_n^k(\Omega) \quad \text{and} \quad \Gamma_n^k := \Phi_n^k(\Gamma).$$

Define

$$U_n^k := \bigcup_{\overline{\omega} \prec \overline{\alpha}_{k-n}} \Sigma_n^{\overline{\omega}}(\Omega_n^k) \quad \text{and} \quad V_n^k := \bigcup_{\overline{\omega} \prec \overline{\alpha}_{k-n-1}} \Sigma_n^{\overline{\omega}}(\Gamma_n^k).$$

It is not hard to see that $\{U_n^k \cup V_n^k\}_{k=n}^{\infty}$ form a nested sequence. Define the *renormalization arc* of Σ_n as

$$\gamma_n := \bigcap_{k=n}^{\infty} U_n^k \cup V_n^k. \tag{2.28}$$

Proposition 2.3.1. *The renormalization arc γ_n is invariant under the action of Σ_n. Moreover, if*

$$p_n^k := \bigcup_{\overline{\omega} \prec \overline{\alpha}_{k-n}} \Sigma_n^{\overline{\omega}}(\Phi_n^k(\gamma_k \cap \Omega)) \quad \textit{and} \quad q_n^k := \bigcup_{\overline{\omega} \prec \overline{\alpha}_{k-n-1}} \Sigma_n^{\overline{\omega}}(\Phi_n^k(\gamma_k \cap \Gamma)),$$

then

$$\gamma_n = p_n^k \cup q_n^k.$$

Let $\theta_* = (\sqrt{5} - 1)/2$ be the golden mean rotation number, and let

$$I_L := [-\theta_*, 0] \quad \text{and} \quad I_R := [0, 1].$$

Define $L : I_L \to \mathbb{R}$ and $R : I_R \to \mathbb{R}$ as

$$L(t) := t + 1 \quad \text{and} \quad R(t) := t - \theta_*.$$

The pair (R, L) represents rigid rotation of $\mathbb{R}/\mathbb{Z}$ by angle θ_*.

The following is a classical result about the renormalization of 1D pairs.

Proposition 2.3.2. *Suppose $\|\Sigma\|_y = 0$. Then for every $n \geq 0$, there exists a quasi-symmetric homeomorphism between $I_L \cup I_R$ and the renormalization arc γ_n that conjugates the action of $\Sigma_n = (A_n, B_n)$ and the action of (R, L). Moreover, the renormalization arc γ_n contains the unique critical point $c_n = 0$ of η_n.*

The following is shown in [GaRYa].

Theorem 2.3.3. *Let $\Sigma = (A, B)$ be a commuting pair contained in the stable manifold $W^s(\iota(\zeta_*))$ of the 2D renormalization fixed point $\iota(\zeta_*)$. Then for every $n \geq 0$, there exists a homeomorphism between $I_L \cup I_R$ and the renormalization arc γ_n that conjugates the action of $\Sigma_n = (A_n, B_n)$ and the action of (R, L). Moreover, this conjugacy cannot be C^1 smooth.*

Theorem 2.1.2 follows from the above statement and the following:

Theorem 2.3.4 ([GaRYa]). *Suppose*

$$\Sigma = \Sigma_{H_{\mu_*,\nu}},$$

where $\Sigma_{H_{\mu_*,\nu}}$ is the renormalization of the Hénon map given in Theorem 2.2.16. Then the linear rescaling of the renormalization arc $s_0(\gamma_0)$ is contained in the boundary of the Siegel disc $\mathcal{D}$ of $H_{\mu_*,\nu}$. In fact, we have

$$\partial\mathcal{D} = s_0(\gamma_0) \cup H_{\mu_*,\nu} \circ s_0(\gamma_0).$$

Henceforth, we consider the renormalization arc of Σ_n as a continuous curve $\gamma_n = \gamma_n(t)$ parameterized by $I_L \cup I_R$. The components of γ_n are denoted

$$\gamma_n(t) = \begin{bmatrix} \gamma_n^x(t) \\ \gamma_n^y(t) \end{bmatrix}.$$

Lastly, denote the renormalization arc of $\iota(\zeta_*)$ by

$$\gamma_*(t) = \begin{bmatrix} \gamma_*^x(t) \\ \gamma_*^y(t) \end{bmatrix}.$$

The following are consequences of Theorem 2.2.15.

Corollary 2.3.5. *As $n \to \infty$, we have the following convergences (each of which occurs at a geometric rate):*

1. $\eta_n \to \eta_*$,
2. $\lambda_n \to \lambda_*$ *(hence $s_n \to s_*$)*,
3. $\phi_n \to \psi_*$, *where*
$$\psi_*(x,y) = \begin{bmatrix} \eta_*^{-1}(\lambda_* x) \\ \eta_*^{-1}(\lambda_* y) \end{bmatrix}, \qquad \text{and}$$
4. $\gamma_n \to \gamma_*$ *(hence $|\gamma_n^x(0)| \to 0$)*.

2.4 Normality of the Compositions of Microscope Maps

Define

$$\psi_n(x,y) := \begin{bmatrix} \eta_n^{-1}(\lambda_n x) \\ \eta_n^{-1}(\lambda_n y) \end{bmatrix}.$$

For $n \leq k$, denote

$$\Psi_n^k := \psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_{k-1} \circ \psi_k.$$

Let

$$\begin{bmatrix} \sigma_n^k & 0 \\ 0 & \sigma_n^k \end{bmatrix} := (D_{(0,0)}\Psi_n^k)^{-1}.$$

Proposition 2.4.1. *The family $\{\sigma_n^k \Psi_n^k\}_{k=n}^{\infty}$ is normal.*

Proof. By Corollary 2.3.5, there exists a domain $U \subset \mathbb{C}^2$ and a uniform constant $c < 1$ such that for all k sufficiently large, the map ψ_k is well defined on U, and

$$\Omega \cup A_{k+1}(\Omega) \cup \Gamma \cup B_{k+1}(\Gamma) \Subset cU.$$

Thus, by choosing a smaller domain U if necessary, we can assume that ψ_k and hence, Ψ_n^k extends to a strictly larger domain $V \Supset U$. It follows from applying Koébe distortion theorem to the first and second coordinate that $\{\sigma_n^k \Psi_n^k\}_{k=n}^\infty$ is a normal family. □

Proposition 2.4.2. *There exists a uniform constant $M > 0$ such that*

$$||\phi_n - \psi_n|| < M\epsilon^{2^{n-1}}.$$

Proof. The result follows readily from (2.25) and (2.26). □

Proposition 2.4.3. *There exists a uniform constant $K > 0$ such that*

$$\sigma_n^k ||\Phi_n^k - \Psi_n^k|| < K\epsilon^{2^{n-1}}.$$

Proof. By Proposition 2.4.2, we have

$$\phi_{k-1} = \psi_{k-1} + \tilde{E}_{k-1} \qquad \text{and} \qquad \phi_k = \psi_k + E_k,$$

where $||\tilde{E}_{k-1}|| < M\epsilon^{2^{k-2}}$ and $||E_k|| < M\epsilon^{2^{k-1}}$. Observe that

$$\begin{aligned} \phi_{k-1} \circ \phi_k &= \phi_{k-1} \circ (\psi_k + E_k) \\ &= \phi_{k-1} \circ \psi_k + \bar{E}_k \\ &= (\psi_{k-1} + \tilde{E}_{k-1}) \circ \psi_k + \bar{E}_k \\ &= \psi_{k-1} \circ \psi_k + \tilde{E}_{k-1} \circ \psi_k + \bar{E}_k, \end{aligned}$$

where $||\bar{E}_k|| < L\epsilon^{2^{k-1}}$ for some uniform constant $L > 0$ by Corollary 2.3.5. Let

$$E_{k-1} := \tilde{E}_{k-1} + \bar{E}_k \circ \psi_k^{-1}.$$

By Corollary 2.3.5, ψ_k^{-1} is uniformly bounded, and hence, we have

$$||E_{k-1}|| < M\epsilon^{2^{k-2}} + 2L\epsilon^{2^{k-1}} < 2M\epsilon^{2^{k-2}}.$$

Thus, we have

$$\phi_{k-1} \circ \phi_k = \psi_{k-1} \circ \psi_k + E_{k-1} \circ \psi_k.$$

Proceeding by induction, we obtain

$$\Phi_n^k = \Psi_n^k + E_n \circ \psi_{n+1} \circ \ldots \circ \psi_k,$$

where

$$||E_n|| < 2M\epsilon^{2^{n-1}}.$$

By definition, we have

$$\sigma_n^k(\psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_k)'(0) = 1.$$

Factor the scaling constant as

$$\sigma_n^k := \dot{\sigma}_n^k \sigma_{n+1}^k,$$

so that

$$|\dot{\sigma}_n^k \psi_n'(\psi_{n+1} \circ \ldots \circ \psi_k(0))| = 1,$$

and

$$|\sigma_{n+1}^k(\psi_{n+1} \circ \ldots \circ \psi_k)'(0)| = 1.$$

Let

$$M := \sup_{x \in Z} \eta_n'(x).$$

Observe that $\dot{\sigma}_n^k$ is uniformly bounded by $\lambda_n^{-1}M$. Moreover, by Proposition 2.4.1, we have that $\sigma_{n+1}^k(\psi_{n+1} \circ \ldots \circ \psi_k)'$ is also uniformly bounded. Therefore,

$$\begin{aligned}||\sigma_n^k(E_n \circ \psi_{n+1} \ldots \circ \psi_n)'|| &= ||\dot{\sigma}_n^k E_n'(\psi_{n+1} \ldots \circ \psi_n)|| \cdot ||\sigma_{n+1}^k(\psi_{n+1} \ldots \circ \psi_n)'|| \\ &= K||E_n'(\psi_{n+1} \ldots \circ \psi_n)|| \\ &< K\epsilon^{2^{n-1}}\end{aligned}$$

for some universal constant $K > 0$. $\square$

By Proposition 2.4.1 and 2.4.3, we have the following theorem.

Theorem 2.4.4. *The family $\{\sigma_n^k \Phi_n^k\}_{k=n}^{\infty}$ is normal.*

2.5 The Proof of Non-Smoothness.

Let $[t_l, t_r] \subset \mathbb{R}$ be a closed interval, let W be a domain in either $\mathbb{C}$ or $\mathbb{C}^2$, and let $C : [t_l, t_r] \to W$ be a smooth curve. For any $N \subset W$, we define the *angular deviation of C on N* as

$$\partial_\theta(C, N) := \sup_{t,s \in C^{-1}(N)} |\arg(C'(t)) - \arg(C'(s))|, \tag{2.29}$$

where the function arg is defined as

$$\arg(re^{2\pi\theta i}) := 2\pi\theta \tag{2.30}$$

Lemma 2.5.1. *Let $\theta \in \mathbb{R}/\mathbb{Z}$, and let $C_\theta : [0,1] \to \mathbb{C}$ be a smooth curve such that $C_\theta(0) = 0$ and $C_\theta(1) = e^{2\pi\theta i}$. Then for some $t \in [0,1]$, we have*

$$\arg(C_\theta'(t)) = 2\pi\theta.$$

Lemma 2.5.2. *Let*

$$q_2(x) := x^2 \quad \text{and} \quad A_r^R := \{z \in \mathbb{C} \,|\, r < |z| < R\}. \tag{2.31}$$

Suppose $C : [t_l, t_r] \to \mathbb{D}_R$ is a smooth curve such that $|C(t_l)| = |C(t_r)| = R$, and $|C(t_0)| < r$ for some $t_0 \in [t_l, t_r]$. Then for every $\delta > 0$, there exists $M > 0$ such that if $\operatorname{mod}(A_r^R) > M$, then either $\partial_\theta(C, A_r^R)$

or $\partial_\theta(q_2(C), q_2(A_r^R))$ *is greater than* $\pi/3 - \delta$.

Proof. Without loss of generality, assume that $R = 1$, and $C(t_r) = 1$. We prove the case when $r = 0$, so that $C(t_0) = 0$. The general case follows by continuity.

Suppose that $\partial_\theta(C, \mathbb{D}) < \pi/3$. Then by Lemma 2.5.1, we have

$$2\pi/3 < \arg(C(t_l)) < 4\pi/3.$$

This implies that

$$-2\pi/3 < 2\arg(C(t_1)) < 2\pi/3.$$

Hence, by Lemma 2.5.1, we have $\partial_\theta(q_2 \circ C, \mathbb{D}) > \pi/3$. $\square$

Lemma 2.5.3. *Let* $W \subset \mathbb{C}$ *be a domain, and let* $C : [t_l, t_r] \to W^2$ *be a smooth curve given by*

$$C(t) = \begin{bmatrix} C^x(t) \\ C^y(t) \end{bmatrix}.$$

Let $f : W \to f(W)$ *and* $F : W^2 \to F(W^2)$ *be smooth functions such that*

$$F = \iota(f) + E$$

and $\|E\| < \epsilon$. *Suppose*

$$\inf_{x \in U} |f'(x)| > m.$$

Then

$$\|\arg((\iota(f) \circ C^x)') - \arg((F \circ C)')\| < K\epsilon/m$$

for some uniform constant K.

Let $U \subset Z \subset \mathbb{C}$ be a simply-connected domain containing the origin. For all k sufficiently large, the unique critical point c_k of η_k is contained in U. Let $V_k := \eta_k(U)$. Then there exists conformal maps $u_k : \mathbb{D} \to U$ and $v_k : \mathbb{D} \to V_k$ such that the following diagrams commutes:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{u_k} & U \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle \eta_k} \\ \mathbb{D} & \xrightarrow{v_k} & V_k \end{array}$$

By Corollary 2.3.5, we have the following result:

Proposition 2.5.4. *The maps* $u_k : \mathbb{D} \to U$ *and* $v_k : \mathbb{D} \to V_k$ *converge to conformal maps* $u_* : \mathbb{D} \to U$ *and* $v_* : \mathbb{D} \to \eta_*(U)$. *Moreover, the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{u_*} & U \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle \eta_*} \\ \mathbb{D} & \xrightarrow{v_*} & \eta_*(U) \end{array}$$

Proof of Main Theorem 2. By Theorem 2.4.4, the sequence $\{\sigma_0^k \Phi_0^k\}_{k=0}^\infty$ has a converging subsequence. By replacing the sequence by this subsequence if necessary, assume that $\{\sigma_0^k \Phi_0^k\}_{k=0}^\infty$ converges. Consider the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{u_k} & U \\ \Big\downarrow q_2 & & \Big\downarrow \eta_k \\ \mathbb{D} & \xrightarrow{v_k} & V_k \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega & \xrightarrow{\Phi_0^k} & \Omega \\ \Big\downarrow A_k & & \Big\downarrow A_0 \\ A_k(\Omega) & \xrightarrow{\Phi_0^k} & A_0(\Omega) \end{array}.$$

Let $\delta > 0$. Then by Proposition 2.5.4, we can choose $R > 0$ sufficiently small so that if

$$X_k := u_k(\mathbb{D}_R), \quad \text{and} \quad Y_k := v_k(\mathbb{D}_{R^2}),$$

then the following uniform estimates on the angular deviation hold:

1. For any smooth curve $C \subset \mathbb{D}$, we have

$$\partial_\theta(C, \mathbb{D}_R) < \partial_\theta(u_k \circ C, X_k) + \delta \quad \text{and} \quad \partial_\theta(C, \mathbb{D}_{R^2}) < \partial_\theta(v_k \circ C, Y_k) + \delta.$$

2. For any smooth curves $C_1 \subset \Omega$ and $C_2 \subset A_k(\Omega)$, we have

$$\kappa \partial_\theta(C_1, X_k^2) < \partial_\theta(\Phi_0^k \circ C_1, \Phi_0^k(X_k^2))$$

   and

$$\kappa \partial_\theta(C_2, Y_k^2) < \partial_\theta(\Phi_0^k \circ C_2, \Phi_0^k(Y_k^2))$$

   for some uniform constant $\kappa > 0$.

Consider the renormalization arc of Σ_n:

$$\gamma_n(t) = \begin{bmatrix} \gamma_n^x(t) \\ \gamma_n^y(t) \end{bmatrix}.$$

Let

$$\chi_k := u_k^{-1} \circ \gamma_k^x. \tag{2.32}$$

Now, choose $r > 0$ is sufficiently small so that the annulus A_r^R satisfies the condition of Lemma 2.5.2. Next, choose K sufficiently large so that for all $k > K$, we have

$$|\chi_k(0)| < r.$$

Let

$$m_k := \inf_{x \in u_k(A_r^R)} |\eta_k'(x)| > 0.$$

Then m_k is uniformly bounded below by $m_* > 0$. Lastly, denote

$$W_k := v_k \circ q_2(A_r^R) \subset Y_k.$$

Now, suppose towards a contradiction that γ_0, and hence γ_k is smooth for all $k \geq 0$. By the above estimates, we can conclude:

$$\begin{aligned}
\partial_\theta(\gamma_0, \Phi_0^k(X_k^2)) &= \partial_\theta(\Phi_0^k \circ \gamma_k, \Phi_0^k(X_k^2)) \\
&> \kappa \partial_\theta(\gamma_k, X_k^2) \\
&> \kappa \partial_\theta(\gamma_k^x, X_k) \\
&> \kappa \partial_\theta(\chi_k, \mathbb{D}_R) - \kappa\delta \\
&> \kappa \partial_\theta(\chi_k, A_r^R) - \kappa\delta,
\end{aligned}$$

and

$$\begin{aligned}
\partial_\theta(\gamma_0, \Phi_0^k(Y_k^2)) &= \partial_\theta(\Phi_0^k \circ \gamma_k, \Phi_0^k(Y_k^2)) \\
&> \kappa \partial_\theta(\gamma_k, Y_k^2) \\
&> \kappa \partial_\theta(\gamma_k, W_k^2) \\
&= \kappa \partial_\theta(A_k(\gamma_k), W_k^2) \\
&> \kappa \partial_\theta(\iota(\eta_k) \circ \gamma_k^x, W_k^2) - 2\kappa K \epsilon^{2^k}/m_* \qquad (2.33) \\
&> \kappa \partial_\theta(\eta_k \circ \gamma_k^x, W_k) - 2\kappa K \epsilon^{2^k}/m_* \\
&> \kappa \partial_\theta(q_2 \circ \chi_k, q_2(A_r^R)) - \kappa\delta - 2\kappa K \epsilon^{2^k}/m_*.
\end{aligned}$$

where in (2.33), we used Lemma 2.5.3.

By Lemma 2.5.2, either $\partial_\theta(\chi_k, A_r^R)$ or $\partial_\theta(q_2 \circ \chi_k, q_2(A_r^R))$ is greater than $\pi/3 - \delta$. Hence,

$$\max\{\partial_\theta(\gamma_0, \Phi_0^k(X_k^2)), \partial_\theta(\gamma_0, \Phi_0^k(Y_k^2))\} > l$$

for some uniform constant $l > 0$. Since $\Phi_0^k(X_k^2)$ and $\Phi_0^k(Y_k^2)$ both converge to a point in γ_0 as $k \to \infty$, this is a contradiction. □

2.6 Further Thoughts

It is natural to wonder if our results can be extended to rotation numbers other than the inverse-golden mean $\theta_* = (\sqrt{5}-1)/2$. In fact, our proof is quite general and largely independent of which specific rotation number we are considering. The key geometric observation we make is that in the presence of a critical point (or a "near-critical" point for dissipative diffeomorphisms of two variables), there cannot be an invariant smooth curve, since any such curve would have to contain corners. The same argument would apply to, for example, any rotation number of bounded type, since for these rotation numbers, the Siegel boundary of a quadratic polynomial is guaranteed to contain a critical point (see [Do2]).

The real obstruction that prevents us from generalizing our results lies in the fact that the renormalization hyperbolicity theorem in one-dimension (Theorem 2.2.6) has only been established for the inverse golden-mean rotation number. Gaidashev and Yampolsky gave a computer-assisted proof of this result in [GaYa]. From a conceptual point of view, it is expected that the same result should hold for a more general class of rotation numbers. However, in [GaYa], a specific rotation number was used in order to carry out the necessary computations.

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