On the non-existence of almost complex manifolds with sum of Betti numbers 3

Jiahao Hu

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Abstract

In this note we prove that there does not exist an almost complex manifold whose sum of Betti numbers is 3 in complex dimension greater or equal to 3. Michael and Aleks have already proven that such a manifold does not exist except possibly for dimension being a power of 2. We manage to rule out power of 2 as well.

Thoughout this note, we assume X is a compact almost complex $8k$-manifold whose sum of betti numer is 3. Let σ, e, χ^0 be the signature, Euler characteristic and Todd genus of X respectively. Evidently $e = 3$ and by a theorem of Hirzerbruch, $\sigma \equiv e$ modulo 4 for X, so $\sigma = -1$. χ^0 is an integer.

For such a manifold X, the only non-zero rational cohomology groups are $H^0(X;\mathbb{Q})$, $H^{4k}(X;\mathbb{Q})$, and $H^{8k}(X;\mathbb{Q}) \simeq \mathbb{Q}$. So the only possibly non-zero rational Pontryagin classes are p_k and p_{2k}, the only possibly non-zero rational Chern classes are c_{2k} and c_{4k}. Since $p_k = (-1)^k 2c_{2k}$, $p_{2k} = c_{2k}^2 + 2c_{4k}$ and $c_{4k} = e = 3$, the only 'free' characteristic class is c_{2k}.

Our argument then proceed as following. First we apply signature theorem and write it purely in terms of Chern classes to determine c_{2k}^2. Then we can compute χ^0. Finally we show that $0 < |\chi^0| < 1$, but this violates the integrality of χ^0.

By signature theorem,

$$\sigma = h_{2k}p_{2k} + h_{k,k}p_k^2$$

plug in $p_k = (-1)^k 2c_{2k}$, $p_{2k} = c_{2k}^2 + 2c_{4k}$ and use that

$$h_{k,k} = \frac{1}{2}(h_k^2 - h_{2k})$$

we get

$$\sigma = (2h_k^2 - h_{2k})c_{2k}^2 + 2h_{2k}c_{4k} \tag{1}$$

where $h_m = 2^{2m}(2^{2m-1} - 1)\frac{B_{2m}}{(2m)!}$. Here B_{2m} is the $2m$-th Bernoulli number without sign.

$$B_2 = \frac{1}{6}, B_3 = 0, B_4 = \frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \dots$$

Here's a list of first several coefficients h_m and $h_{m,m}$

$$h_1 = \frac{1}{3}, h_2 = \frac{7}{45}, h_3 = \frac{62}{945}, h_4 = \frac{381}{14175}$$

$$h_{1,1} = -\frac{1}{45}, h_{2,2} = -\frac{19}{14175}, h_{3,3} = -\frac{40247}{638512875}$$

Similarly we can compute Todd genus by

$$\chi^0 = t_{4k}c_{4k} + t_{2k,2k}c_{2k}^2$$

and use $t_{2k,2k} = \frac{1}{2}(t_{2k}^2 - t_{4k})$ we get

$$\chi^0 = t_{4k}c_{4k} + \frac{1}{2}(t_{2k}^2 - t_{4k})c_{2k}^2 \tag{2}$$

where $t_{2m} = (-1)^{m+1}\frac{B_{2m}}{(2m)!}$.

Here's a list of first several coefficients t_m and $t_{m,m}$

$$t_1 = \frac{1}{2}, t_2 = \frac{1}{12}, t_3 = 0, t_4 = -\frac{1}{720}, t_5 = 0, t_6 = \frac{2}{60480}, t_7 = 0, t_8 = -\frac{3}{3628800}$$

$$t_{1,1} = \frac{1}{12}, t_{2,2} = \frac{3}{720}, t_{3,3} = -\frac{1}{60480}, t_{4,4} = \frac{5}{3628800}$$

Recall we actually only need to prove the case where k is a power of 2, we can further assume k is even. Hence have relations

$$h_k = -2^{2k}(2^{2k-1} - 1)t_{2k}$$

$$h_{2k} = -2^{4k}(2^{4k-1} - 1)t_{4k}$$

Plug these into (1) and solve for c_{2k}^2 we get

$$c_{2k}^2 = \frac{\sigma + 2^{4k+1}(2^{4k-1} - 1)t_{4k}e}{2^{4k}[(2^{4k-1} - 1)(t_{2k}^2 + t_{4k}) + (3 - 2^{2k+1})t_{2k}^2]} \tag{3}$$

Plug (3) into (2) we have

$$\chi^0 = \frac{2^{4k+1}(2^{4k} - 2^{2k+1} + 1)e \cdot t_{4k}t_{2k}^2 + (t_{2k}^2 - t_{4k})\sigma}{2^{4k+1}[(2^{4k-1} - 1)(t_{2k}^2 + t_{4k}) + (3 - 2^{2k+1})t_{2k}^2]}$$

Let $r_k := t_{4k}/t_{2k}^2$ then

$$\chi^0 = \frac{2^{4k+1}(2^{4k} - 2^{2k+1} + 1)e \cdot t_{4k} + (1 - r_k)\sigma}{2^{4k+1}[(2^{4k-1} - 1)(1 + r_k) + (3 - 2^{2k+1})]} \tag{4}$$

Now we claim $0 < |\chi^0| < 1$. Before we get into tedious estimations, let's first consider what happens when k is big enough. Note that $e = 3, \sigma = -1$ and since $B_{2m} \sim \frac{(2m)!}{2^{2m-1}\pi^{2m}}$ when m is big enough

$$t_{2k} \sim -\frac{1}{2^{2k-1}\pi^{2k}}, t_{4k} \sim -\frac{1}{2^{4k-1}\pi^{4k}}, r_k \sim -\frac{1}{2}$$

we conclude that $\chi^0 \to 0$ as $k \to \infty$, but χ^0 is always negative. So this shows χ^0 is not an integer for big k. As for small k, $\chi^0 = -\frac{3}{39040}$ when $k = 2$.

Sadly fun's over, it's that time when we have to pretend we are analysts and perform a hardcore estimation. Notice that $B_{2m} = \frac{(2m)!\zeta(2m)}{2^{2m-1}\pi^{2m}}$ where $\zeta(2m) = \sum_{n=1}^{\infty}\frac{1}{n^{2m}}(< 2)$, it is easy to show $\zeta(4k) \leq \zeta(2k)^2$, so $r_k \geq -\frac{1}{2}$. And $|t_{4k}| < \frac{1}{2^{4k-2}\pi^{4k}}$.

Let

$$N := 3 \cdot 2^{4k+1}(2^{4k} - 2^{2k+1} + 1) \cdot |t_{4k}| + (1 - r_k)$$
$$D := 2^{4k+1}[(2^{4k-1} - 1)(1 + r_k) + (3 - 2^{2k+1})]$$

then using the above estimations

$$0 < N < \frac{3}{2} + 24\frac{(2^{2k} - 1)^2}{\pi^{4k}} < 26$$

and

$$D > 2^{6k+2}(2^{2k-3} - 1) > 2^{6k+2}$$

So $0 < |\chi^0| = N/D < 1$. Q.E.D.