1. Complex orientable theory

Let $h$ be a multiplicative cohomology theory, i.e. $h$ is a generalized cohomology theory and has a cup product. In particular, $h^*(pt)$ is a ring.

**Definition 1.1.** We say $h$ is complex orientable if there is an isomorphism

$$h^*(\mathbb{C}P^\infty) \simeq h^*(pt)[[t]].$$

This isomorphism is called a complex orientation of $h$. Or equivalently the map induced by inclusion $h^2(\mathbb{C}P^\infty) \to h^2(\mathbb{C}P^1)$ is surjective.

Note that $h$ might have more than one complex orientations. Once the orientation is fixed, we say $h$ is complex oriented.

**Exercise 1.2.** Show that $h$ is complex orientable if and only if for any complex vector bundle $\xi$ over base $X$ we have Thom isomorphism $h^*(\xi) \simeq h^*(X)$.

**Example 1.3.** Ordinary cohomology (with any coefficient) is complex orientable.

2. Formal group law

**Definition 2.1.** Let $R$ be a commutative ring, a formal group law over $R$ is a formal power series $f(u, v) \in R[[u, v]]$ such that

1. $f(u, 0) = u = f(0, u)$
2. $f(u, f(v, w)) = f(f(u, v), w)$
3. $f(u, v) = f(v, u)$

Let $E$ be a complex orientable cohomology theory, then the isomorphism $E(\mathbb{C}P^\infty) \simeq E^*(pt)[[t]]$ permits us to define Chern class by pull-back of $t$ combining splitting principle. Then $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induced by $O = \pi_1^1 O(1) \otimes \pi_2^2 O(1)$ gives a formal group law

$$c_1^E(O) = f(u, c_1^E(\pi_1^1 O(1)), v = c_1^E(\pi_2^2 O(1))) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*[[u, v]]$$

(last isomorphism by Atiyah-Hirzebruch spectral sequence) over the coefficient ring.

For example, $c_1(L \otimes L') = c_1(L) + c_1(L')$ for ordinary Chern class, hence the formal group law is the additive group law $\mathbb{G}_a$ over $\mathbb{Z}$.

3. $K$-theory is complex orientable

It suffices to show $K$-theory admits Thom isomorphism for complex vector bundles.

Let $\xi \to X$ be a $U(n)$-bundle, let $M(\xi)$ be the Thom space of $\xi$, we are to construct a map $K(X) \to \tilde{K}(M(\xi))$ which is an isomorphism, analogous to the Thom isomorphism $H^*(X) \to \hat{H}^*(M(\xi))$. So similarly, we start by defining a Thom class $T(\xi) \in \tilde{K}(M(\xi))$. 

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3.1. Thom class in $K$-theory.

- Thom class $T(\xi)$ is a relative class in $K(D(\xi), \partial D(\xi))$.
- The exterior algebra of $\xi$, $\wedge(\xi) = \wedge^{ev}(\xi) \oplus \wedge^{od}(\xi)$.
- Pull back $\xi$ to a bundle $\xi'$ over $D(\xi)$, then $\wedge(\xi') = \wedge^{ev}(\xi') \oplus \wedge^{od}(\xi')$.
- Moreover, there is a map $\phi : \wedge^{od}(\xi) \to \wedge^{ev}(\xi')$.

Let $(x, v, Y) \in (X, D^{2n}, \wedge^{od}(\xi)) \mapsto (x, v, (v \wedge (v^*)^*) Y) \in (X, D^{2n}, \wedge^{ev}(\xi))$.

Notice that $\phi$ is an isomorphism away from zero section of $D(\xi)$.

Thus, we define $T(\xi) := (\wedge^{ev}(\xi'), \wedge^{od}(\xi'), \phi) \in K(D(\xi), \partial D(\xi))$.

3.2. Thom class in $KO$-theory. Let $\xi \to X$ be an $SU(n)$-bundle

- If $n \equiv 0 \pmod{4}$, then $\wedge \xi = R(\xi) \oplus R_{-}(\xi)$ and $R(\xi) = R^{ev}(\xi) \oplus R^{od}(\xi)$. Pull back to $D(\xi)$ get $t(\xi) = (R^{ev}(\xi'), R^{od}(\xi'), \phi) \in KO(D(\xi), \partial D(\xi))$

- If $n \equiv 2 \pmod{4}$, then $s(\xi) = (\wedge^{ev}(\xi'), \wedge^{od}(\xi'), \phi) \in K\tilde{S}P(M(\xi))$

3.3. Thom isomorphism. To prove the Thom isomorphism in $K$-theory, we use the following theorem of Dold.

**Theorem 3.1.** Suppose $h^*$ is a multiplicative cohomology theory. Let $\xi$ be an $O(n)$-bundle over a finite CW complex $X$. Let $t \in h^n(D(\xi), \partial D(\xi))$ be such that inclusion $i : (D_2^0, \partial D_2^0) \to (D(\xi), \partial D(\xi))$, where $D_2^0$ is the cell over $x \in X$, has $h^n(D_2^0, \partial D_2^0)$ a free $h^*(pt)$-module with generator $i^*(t)$. Then there is an isomorphism

$$h^k(X) \cong h^{k+n}(D(\xi), \partial D(\xi)), a \mapsto \pi^* a \cdot t$$

**Sketch of Proof.** The proof is basically the same as the proof of standard Thom isomorphism, which is an induction on cell and an application of five lemma. $\Box$

Applying this theorem, one only needs to check that $i^*T(\xi)$ is the generator of $K(D_2^{2n}, \partial D_2^{2n}) = K(S^{2n})$, and is the generator of the free $K^*(pt)$-module $K^*(S^{2n})$. By Bott periodicity, splitting principle and the fact that $M(\xi \oplus \eta) = M(\xi) \wedge M(\eta)$, one suffices to check $n = 1$. We then explicitly compute the Thom class of the universal line bundle.

**Proposition 3.2.** The tautological $U(1)$-bundle $\rho_{n-1}$ over $\mathbb{C}P^{n-1}$ has Thom space $\mathbb{C}P^n$, and $T(\rho_{n-1}) = 1 - \rho_n \in K(\mathbb{C}P^n)$.

**Proof.** First of all, the projection

$$\mathbb{C}P^n - [0, 0, \ldots, 1] \to \mathbb{C}P^{n-1}, [z_0, \ldots, z_{n-1}, z_n] \mapsto [z_0, \ldots, z_{n-1}]$$

is the tautological $\rho_{n-1}$ (one can see this by explicitly writing down the bundle transition function), so $M(\rho_{n-1}) \cong \mathbb{C}P^n$. From the view of Thom isomorphism, we have $H^*(M(\rho_{n-1})) = \wedge(u, ux)/(u^2 - ux) = \wedge(u)$ where $u$ is the Thom class, this agrees with the cohomology of $\mathbb{C}P^n$.

This is a combination of several facts.

- For any $U(1)$-bundle $\xi$, $M(\xi)$ is canonically isomorphic to $E(\xi) \circ U(1)/U(1)$, where $E(\xi) \circ U(1)$ is the join of $E(\xi)$ and $U(1)$.
- $S^{2n-1} = U(1) \circ \cdots \circ U(1)$, $\mathbb{C}P^{n-1} = S^{2n-1}/U(1) = U(1) \circ \cdots \circ U(1)/U(1)$. 


• \( M(\rho_{n-1}) = E(\rho_{n-1}) \circ U(1)/U(1). \)

Recall that
\[
T(\rho_{n-1}) = (\wedge^e(\rho_{n-1}), \wedge^o(\rho_{n-1}), \phi) \in K(D(\rho_{n-1}), \partial D(\rho_{n-1}))
\]
and notice that since \( \rho_{n-1} \) is a line bundle, we have
\[
\wedge^e(\rho_{n-1}) = \wedge^0 = \text{trivial bundle},
\]
and
\[
\wedge^o(\rho_{n-1}) = \wedge^1 = \rho_{n-1}.
\]
Thus
\[
T(\rho_{n-1}) = (\varepsilon, [\pi^*\rho]_{n-1}) = 1 - \pi^*\rho_{n-1}. \]
We claim that \( \pi^*\rho_{n-1} \) on \( M(\rho_{n-1}) \) is the tautological bundle \( \rho_n \), indeed one easily sees this by looking at bundle transition function.

\[\square\]

**Remark 3.3.** For \( n = 1 \), \( 1 - \rho_1 \in \tilde{K}(\mathbb{C}P^1) = \tilde{K}(S^2) \) is exactly the generator.

**Corollary 3.4** (Thom isomorphism in \( K \)-theory). For any \( U(n) \)-bundle \( \pi : \xi \to X \), we have an isomorphism \( K(X) \simeq K(D(\xi), \partial D(\xi)) = \tilde{K}(M(\xi)), \eta \mapsto \pi^*\eta \otimes T(\xi) \).

Similarly we have Thom isomorphisms for \( SU(4k) \) and \( SU(4k+2) \) bundles

- \( KO(X) \simeq \tilde{KO}(M(\xi)) \) for \( SU(4k) \)-bundle
- \( KO(X) \simeq \tilde{KS}(M(\xi)) \) for \( SU(4k+2) \)-bundle

4. Complex cobordism theory is complex orientable

This is purely tautologous.

### 4.1. Computation of universal formal group law on \( \Omega_+^U(pt) \).

Suppose
\[
F^\Omega(u, v) = \sum a_{rs}u^rv^s,
\]
then since \( \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \) is the limit of Segre map \( \mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{n+m} \), the pull-back of a hyperplane in \( \mathbb{C}P^{n+m} \) is Milnor manifold \( H_{nm} \),
so we have
\[
[H_{nm}] = \sum_{r=0}^n \sum_{s=0}^m a_{rs}[\mathbb{C}P^{n-r}][\mathbb{C}P^{m-s}].
\]
Therefore one has
\[
H(u, v) = \sum H_{nm}u^nv^m = F^\Omega(u, v)CP(u)CP(v).
\]