STABLE HOMOTOPY THEORY

Contents

1. Spectra

Let K^* be a generalized cohomology theory, that is a (sequence of) functor(s) satisfying all Eilenberg-Steenrod axioms except the dimension axiom. For any generalized cohomology theory, there is a corresponding *reduced* theory $\tilde{K}^*$ defined on spaces-with-basepoint.

$$K^*(X, \mathrm{pt}) = \tilde{K}^*(X) + K^*(\mathrm{pt}).$$

All the axioms for K^* can be translated to axioms for $\tilde{K}^*$, one of which is the *suspension isomorphism*:

$$\tilde{K}^{i+1}(SX) = \tilde{K}^i(X).$$

Assume K^* further satisfies the wedge axiom of Milnor and Brown, then K^* is representable. That is, there exists connected E_n with basepoint for each i such that for connected X we have $\tilde{K}^n(X) = [X, E_n]$. Then using the suspension-baseloop adjoint, we see the suspension isomorphism corresponds to a weak equivalence

$$\epsilon'_n : E_n \to \Omega_0 E_{n+1}$$

The sequence of spaces E_n together with the weak equivalences ϵ'_n is an Ω_0-spectrum. A similar discussion for K^* yields (not necessarily connected) F_n representing K^n with weak equivalences $\phi_n : F_n \to \Omega F_{n+1}$. The data of $\{F_n, \phi_n\}$ is that of an Ω-spectrum. We now make a general definition.

Definition 1.1. A spectrum E is a sequence of spaces E_n with basepoint, provided with structure maps, either

$$\epsilon_n : SE_n \to E_{n+1}$$

or

$$\epsilon'_n : E_n \to \Omega E_{n+1}.$$

The two definitions are the same since S and Ω are adjoint. If we choose to work with connected E_n then ϵ'_n automatically maps into Ω_0.

Note that we do not require the structure maps to be weak equivalences. This flexibility allows us to include some important examples. We shall see later that every spectrum is equivalent to an Ω-spectrum.

Definition 1.2. We say E is an Ω-spectrum if ϵ_n or ϵ_n''s are weak equivalences. We say E is an S-spectrum or suspension spectrum if ϵ_n is an equivalence for n sufficiently large.

Example 1.3. (1) (singular cohomology) The Eilenberg-MacLane spectrum $H\mathbb{Z}$ that represents the singular cohomology. The n-th Eilenberg-MacLane space $K(\mathbb{Z}, n)$ is characterized by that $\tilde{H}^n(S^m; \mathbb{Z}) = [S^m, K(\mathbb{Z}, n)]$. It is clear that $K(\mathbb{Z}, n) = \Omega K(\mathbb{Z}, n+1)$ from the path space fibration and the long exact sequence of homotopy groups.

(2) (complex K-theory) $KU_{2n} = \mathbb{Z} \times BU$ and $KU_{2n+1} = U$. By Bott periodicity, there exists weak equivalences $KU_n = \Omega KU_{n+1}$. The spectrum KU represents the complex K-theory. Note that $K^0(X) = [X, \mathbb{Z} \times BU]$ classifies complex vector bundles on X up to stable equivalences. There is a spectrum KO for real K-theory and KSp for quaternionic K-theory.

(3) (Thom spectrum) Let MSO_n be the Thom space of the universal bundle over BSO_n, then there is a natural map $S^1 \wedge MSO_n \to MSO_{n+1}$ induced by the canonical inclusion $SO_n \subset SO_{n+1}$. MSO is called the Thom spectrum for SO (or oriented cobordism). Note that MSO is not an Ω-spectrum.

(4) (suspension spectrum of a space) Let X be a CW-complex, define E_n to be $S^n X$ for $n \geq 0$ and pt otherwise. Then E_n is an S-spectrum, called the suspension spectrum of X.

(5) (sphere spectrum) $\mathbb{S}_n = S^n$ for $n \geq 0$ and pt otherwise, is naturally a spectrum, called the sphere spectrum. It is clear the sphere spectrum is the suspension spectrum of S^0.

(6) (Ω-spectrification) For any spectrum E, we may define a new Ω-spectrum LE by

$$(LE)_n = \lim_k \Omega^k E_{n+k}.$$

It is clear there is a natural map $E \to LE$. It will turn out that $L(-)$ is left adjoint to the forgetful functor from Ω-spectra to spectra.

As homotopy theory studies algebraic invariants of spaces, *stable* homotopy theory studies algebraic invariants of *spectra*, where spaces are replaced by their corresponding suspension spectra.

Even though spaces are used in the definition of spectra, we urge the reader to view spectra as algebraic objects instead of geometric objects. One should constantly compare the category of spectra to the category of chain complexes of abelian groups.

However, there does exist a beautiful (yet often overlooked) geometric discussion which we probably will not delve into. The punch line for the geometric discussion is that every spectrum is like a Thom spectrum and every generalized cohomology theory is like a cobordism theory. We will revisit this geometric point when discussing Anderson duality and universal coefficients theorem.

For the moment, let us simply play with algebra. The most important algebraic invariants for a topological space, without doubts, are homotopy and homology. We shall define homotopy groups for spectra, and we will see that (generalized) homology can be defined using homotopy as well. Further, generalized homology can be computed from singular homology plus a spectral sequence.

Definition 1.4 (homotopy group)**.** Let E be a spectra, for each r we have the following sequence

$$\pi_{n+r} E_n \to \pi_{n+r+1}(SE_n) \to \pi_{n+r+1}(E_{n+1}).$$

Define $\pi_r E := \lim_{n \to \infty} \pi_{n+r} E_n$. It is clear $\pi_r E$ is abelian for all $r \in \mathbb{Z}$.

Example 1.5. (1) If E is an Ω-spectrum then the direct limit is attained. More precisely, the homomorphism $\pi_{n+r}(E_n) \to \pi_{n+r+1}(E_{n+1})$ is an isomorphism for $n + r \geq 1$. This follows from

Theorem 1.6 (Freudenthal suspension)**.** Suppose Y is $(n-1)$-connected, then $S : [X, Y] \to [SX, SY]$ is onto if $\dim X \leq 2n - 1$ and 1-1 if $\dim X < 2n - 1$.

(2) If E is the suspension spectrum of X, then $\pi_r(E) = \lim_n \pi_{n+r}(S^n X)$ is the stable homotopy group of X. This limit is attainted for $n > r + 1$.

(3) By the Pontryagin-Thom argument, we have $\pi_r(MSO) = \Omega_r^{\mathrm{SO}}$ is the cobordism group of r-dimensional oriented manifolds.

(4) (Bott periodicity) Unlike spaces, the homotopy groups for spectra can be non-zero in negative degrees.

$$\pi_n KU = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For KO, $\pi_* KO$ is 8-fold periodic, one full period is

$$\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}.$$

As for KSp, it shares the same homotopy groups with KO, except for a degree shift by 4.

(5) (sphere spectrum) $\pi_*\mathbb{S} = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_{24}, 0, 0, \mathbb{Z}_2, \mathbb{Z}_{240}, \ldots$. All higher homotopy groups of $\mathbb{S}$ are torsion due to Serre. One should treat $\mathbb{S}$ as derived integers.

(6) (Ω-spectrification) Exercise: show that the map $E \to LE$ induces isomorphism on homotopy.

It turns out defining (generalized) homology for a spectrum is much harder than defining homotopy. The best we to achieve that goal is to introduce smash product of two spectra. In this direction, we now introduce many constructions within the category of spectra, including smash product of course.

But before that, we make a dictionary, comparing the category of spectra to the category of chain complexes of abelian groups.

Ch(Ab)	**Spectra**
the ring of integers $\mathbb{Z}$	the sphere spectrum $\mathbb{S}$
abelian group G	Moore spectrum SG
direct sum	wedge sum
tensor product	smash product
hom complex	function spectrum
hom-tensor adjunction	function-smash adjunction
suspension	suspension
mapping cone	mapping cone
exact triangle	cofiber sequence
homology	homotopy
quasi-isomorphism	homotopy equivalence
truncation	truncation
base change spectral sequence	Atiyah-Hirzebruch spectral sequence
localization	localization
completion	completion
derived category	homotopy category
$\cdots$	$\cdots$

More than often, it is not the explicit constructions but the above comparison that is really useful for understanding the category of spectra.

Now let us spell out these constructions, modulo technicalities. Just like when dealing with spaces, it is more convenient to work with a spacial class of spaces–CW-complexes; when dealing with spectra, it is more convenient to work with

Definition 1.7 (CW-spectrum)**.** (1) We say a spectrum E is a CW-spectrum if all E_n's a re CW-complexes and all structure maps are cellular embeddings.

(2) A *subspectrum* A of a CW-spectrum is a CW-spectrum with $A_n \subset E_n$ being a subcomplex for each n.

(3) Let C_n be the set of cells in E_n other than the basepoint, then we get a function $C_n \to C_{n+1}$ by suspension, this function is by definition an injection. Let C be the direct limit $\lim_{n\to\infty} C_n$; an element of C is called a *stable cell* of E. The *stable dimension* of a cell in C_n is its (geometric) dimension minus n. Stable dimension passes to a well-defined $\mathbb{Z}$-valued function on C, which can take negative value.

(4) A subspectrum E' of E is said to be *cofinal* in E is $C' \to C$ is a bijection.

(5) A CW-spectrum is *finite* if it has finitely many stable cells.

Example 1.8. For X a CW-spectrum, let X' be the subspectrum defined by $X'_n = X_n$ for $n \geq 0$ and $X'_n = \text{pt}$ otherwise. Then X' is cofinal in X. For this, there is no real different in considering spectra indexed by $\mathbb{Z}$ or $\mathbb{N}$.

Unless otherwise specified, we will always work with CW-spectra. Recall a theorem of Milnor asserts that the space of maps from a finite CW-complex to any CW-complex is homotopy equivalent to a CW-complex. Thus taking loop space does not leave the category. We now define morphisms.

Definition 1.9 (functions, maps)**.** Assume E is a CW-spectrum and F is an Ω-spectrum.

(1) A *function* f from E to F of degree r is a sequence of maps $f_n : E_n \to F_{n-r}$ compatible with the structure maps, namely the following diagram (strictly, not up to homotopy) commutes:

$$\begin{tikzcd} SE_n \arrow[r, "\epsilon_n"] \arrow[d, "Sf_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ SF_{n-r} \arrow[r, "\phi_{n-r}"] & F_{n-r+1} \end{tikzcd}$$

(2) Take all cofinal $E' \subset E$ and all functions $f' : E' \to F$. Say that two functions $f' : E' \to F$ and $f'' : E'' \to F$ are equivalent if there is a cofinal E''' contained in E' and E'' so that the restrictions of f', f'' to E''' coincide. A *map* from E to F is an equivalence class of functions.

A morphism will be a homotopy class of maps, thus we must define homotopy between maps.

Definition 1.10 (cylinders, homotopy)**.** (1) Let I^+ be the union of the unit interval and a disjoint basepoint. If E is a spectrum, we define the *cylinder spectrum* $Cyl(E)$ to hava terms $I^+ \wedge E_n$ with canonical structure maps. It is clear $Cyl(-)$ is a functor: a map $f : E \to F$ induces a map $Cyl(E) \to Cyl(F)$. Note that we have obvious injection functions:

$$i_0, i_1 : E \to Cyl(E).$$

(2) We say that two maps $f_0, f_1 : E \to F$ are *homotopic* if there is a map $h : Cyl(E) \to F$ such that $f_0 = hi_0$, $f_1 = hi_i$.

Exercise 1.11. Define $Susp(E)$, $Cone(E)$ for spectrum E.

Definition 1.12 (morphism)**.** A *morphism* f from E to F is a homotopy class of maps. Denote the set of degree r morphisms by $[E, F]_r$.

It will follow from a stable version of Freudenthal suspension theorem that $[E, F]_r$ is an abelian group for all r. This allows us to define:

Definition 1.13 (cohomology)**.** Let E, X be spectra, we define the E-cohomology of X to be

$$E^*(X) = [X, E]_{-*}.$$

Lemma 1.14. Let K be a finite CW-complex and identified with its suspension spectrum. Let F be any spectrum. Then

$$[K, F]_r = \lim_{n\to\infty} [S^{n+r}K, F_n].$$

In particular, $[\mathbb{S}, F]_r = \pi_r(F)$.

Proof. Exercise. ■

Theorem 1.15 (stable Freudenthal suspension)**.** $Susp : [X, Y]_* \to [Susp(X), Susp(Y)]_*$ is a 1-1 correspondence.

One can also define morphisms between pairs (X, A) and (Y, B), as well as relative homotopy classes of maps $[X, A; Y, B]$. A version of homotopy extension can be proved:

Lemma 1.16. Suppose that $\pi_*(Y) = 0$, and X, A is a pair of CW-spectra. Then any map $f : A \to Y$ can be extended over X.

Consequently we have:

Theorem 1.17 (Whitehead theorem for spectra)**.** Let $f : E \to F$ be a function such that $f_* : \pi_*E \to \pi_*F$ is an isomorphism, then for any CW-spectrum X,

$$f_* : [X, E]_* \to [X, F]_*$$

is a 1-1 correspondence. Assume further E, F are CW-spectra, then f is an equivalence.

Corollary 1.18. Every CW-spectra is equivalent to an Ω-spectrum.

Proof. $E \to LE$ is an equivalence. ■

Definition 1.19 (Wedge sum/coproduct)**.** Given spectra X_α for $\alpha \in A$, we form $X = \bigvee_\alpha X_\alpha$ by $X_n = \bigvee_\alpha (X_\alpha)_n$ with the obvious structure maps

$$X_n \wedge S^1 = (\bigvee_\alpha (X_\alpha)_n) \wedge S^1 = \bigvee_\alpha (X_\alpha) \wedge S^1 \to \bigvee_\alpha (X_\alpha)_{n+1}.$$

Wedge sum has the property:

$$[\bigvee_\alpha X_\alpha, Y] \cong \prod_\alpha [X_\alpha, Y].$$

Recall finite direct sum and finite direct product are the same. Similarly we have:

Proposition 1.20. Arbitrary products of CW-spectra exists and finite sums (i.e. finite coproducts) are finite products.

Proof. Omitted. ■

Theorem 1.21 (Smash product)**.** For X, Y CW-spectra, there is a CW-spectrum $X \wedge Y$ called smash product of X and Y so that

(1) $X \wedge Y$ is functorial in both X and Y.
(2) $\wedge$ is commutative, associative, and has the sphere spectrum $\mathbb{S}$ as a unit, up to coherent equivalences.
(3) The smash product is distributive over the wedge sum.
(4) Let $X \to Y \to Z$ be a cofibering, then

$$W \wedge X \to W \wedge Y \to W \wedge Z$$

is also a cofibering.

Using smash product, we can define generalized homology for spectra.

Definition 1.22 (homology)**.** Let E, X be spectra, we define the E-homology of X to be

$$E_*(X) = \pi_*(E \wedge X).$$

The last assertion involves an important notion: cofibering.

Definition 1.23 (mapping cone, cofiber sequence)**.** Let $f : X \to Y$ be a morphism between CW-spectra, we may represent it by a function $f' : X' \to Y$ where X' is a cofinal subspectrum of X. Then we can form the mapping cone $Y \cup_{f'} CX$ by $Y_n \cup_{f'_n} CX'_n$ and the canonical structure maps. It is easy to verify that the *equivalence* class of $Y \cup_{f'} CX$ only relies on the morphism f. So we may write $Y \cup_f CX$ for the mapping cone. We have the sequence of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX.$$

This sequence, or anything equivalent to it, is called a *cofiber sequence* or *Puppe sequence*.

Example 1.24. Let X be a CW-spectrum, A a subspectrum. We say A is closed if for every finite subcomplex $K \subset X_n$, $S^m K \subset A_{m+n}$ implies $K \subset A_n$. For the inclusion $i : A \to X$ of closed subspectrum, we can form X/A. the canonical map $X \cup_i CA \to X/A$ is an equivalence by Whitehead's theorem. That is to say,

$$A \to X \to X/A$$

is a cofiber sequence.

Proposition 1.25. For each Z the sequence

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{i^*} [Y \cup_f CX, Z]$$

is exact.

Proof. Same as for CW-complexes. ■

Proposition 1.26. Cofiber sequences can be continued to the right.

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} Susp(X) \xrightarrow{-Susp(f)} Susp(Y)$$

Corollary 1.27 (Key feature)**.** The sequence

$$[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{i_*} [W, Y \cup_f CX]$$

is exact.

Proof. Consider the following diragram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & Y \cup_f CX \arrow[r, "j"] & Susp(X) \arrow[r, "-Susp(f)"] & Susp(Y) \\
W \arrow[r, "1"] & W \arrow[u, "g"] \arrow[r, "i"] & CW \arrow[u, "h"] \arrow[r, "j"] & Susp(W) \arrow[u, "k"] \arrow[r, "-1"] & Susp(W) \arrow[u, "Susp(g)"]
\end{tikzcd}$$

■

This means **cofiber sequences are fiber sequences!**. It follows quickly from definition and the extending-cofiber-sequence argument that cofibering sequences induce long exact sequences in homology and cohomology.

We conclude this lecture with a computational tool for generalized homology and cohomology for finite CW-complexes. Let X be a finite-dimensional CW-complex. The finite assumption can be removed, but then one has to worry about taking limits.

Theorem 1.28 (G.W.Whitehead, Atiyah-Hirzebruch)**.** For each CW-spectrum F there exist spectral sequences

$$H_p(X; \pi_q(F)) \Longrightarrow F_{p+q}(X)$$
$$H^p(X; \pi_{-q}(F)) \Longrightarrow F^{p+q}(X).$$

This spectral sequence is essentially induced by the skeleton filtration of X:

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X.$$

Applying F_* we obtain an Massey exact couple, written in a triangle as follows.

$$\begin{tikzcd}
\sum_p F_*(X_{p-1}) \arrow[rr, "i_*"] & & \sum_p F_*(X_p) \arrow[dl, "j_*"] \\
& \sum_p F_p(X_p, X_{p-1}) \arrow[ul, "\partial"] &
\end{tikzcd}$$

Explicit description of this spectral sequence is possible, we define

$$\begin{aligned}
Z^r_{p,q} &= \ker\{F_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial} F_{p+q-1}(X_{p-1, X_{p-r}})\} \\
&= \operatorname{im}\{F_{p+q}(X_p, X_{p-r}) \xrightarrow{j_*} F_{p+q}(X_p, X_{p-1})\}, \\
B^r_{p,q} &= \operatorname{im}\{F_{p+q+1}(X_{p+r-1}, X_p) \xrightarrow{\partial} F_{p+q}(X_p, X_{p-1})\} \\
&= \ker\{F_{p+q}(X_p, X_{p-1}) \xrightarrow{i_*} F_{p+q-1}(X_{p+r-1}, X_{p-1})\}.
\end{aligned}$$

It is a routine exercise to verify $B^r_{p,q} \subset Z^r_{p,q}$ and ∂ induces a differential d_r on $E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$ whose homology is $E^{r+1}_{p,q}$. The E^∞-page of course is the associated group of $F_m(X)$ with respect to the filtration from the image of

$$F_m(X_p) \to F_m(X).$$

Now we calculate E^1 and E^2-page. From construction,

$$\begin{aligned}
E^1_{p,q} &= F_{p+q}(X_p, X_{p-1}) = \widetilde{F}_{p+q}(X_p/X_{p-1}) \\
&= \widetilde{F}_{p+q}(\bigvee_\alpha S^p) = \sum_\alpha \pi_q(F) = C_p(X; \pi_q(F)).
\end{aligned}$$

We leave it as an exercise for the reader to verify d_1 is the boundary map on cellular chain complex. Therefore, we obtain

$$E^2_{p,q} = H_p(X; \pi_q(F)).$$

The same argument works for cohomology.

Exercise 1.29. Calculate $KU^*(\mathbb{CP}^n)$.

There is an even more general spectral sequence that combines the Atiyah-Hirzebruch spectral sequence and the Serre spectral sequence.

Theorem 1.30. Let $F \to E \to B$ be a Serre fibration with trivial monodromy, and K a CW-spectrum. Then there is a spectral sequence

$$H^p(B; K^q(F)) \Longrightarrow K^{p+q}(E).$$

2. Dualities

We introduce several dualities, concerning spectra, generalized homology and cohomology and manifolds.

- Spanier-Whitehead duality. This is a special case of the hom-tensor adjunction for spectra.
- (Pontryagin-)Anderson duality. This is a generalization of universal coefficients theorem. This is also the analog of Serre-Grothendiec-Verdier duality for coherent sheaves over $Spec\mathbb{Z}$.
- Poincaré duality. This requires discussing products and orientability with respect a given spectrum.

2.1. **Spanier-Whitehead duality.**

Let X be a spectrum, its Spanier-Whitehead dual is supposed to be a spectrum X^* so that $E_*(X^*) = E^{-*}(X)$.

The classical Spanier-Whitehead duality is the Alexander (or sphere) duality.

Theorem 2.1 (Alexander duality)**.** Let K be a compact polyhedron embedded in S^N. Then

$$\widetilde{H}^*(K) \cong \widetilde{H}_{N-*-1}(S^N - K)$$

An interesting application of Alexander duality is when L is a link in S^3. The Spanier-Whitehead dual takes a special form for suspension spectra of compact smooth manifolds. Let M be a closed manifold, then its Spanier-Whitehead dual is the Thom spectrum of its stable normal bundle. This is usually referred to as *Atiyah duality*. Details will be given later.

Form our analogy between spectra and chain complexes, X^* should be nothing but $F(X, \mathbb{S})$, which is analogous to taking the dual cochain complex $C^\bullet = Hom(C_\bullet, \mathbb{Z})$. Therefore, X^* must satisfy the adjunction property that

$$[W \wedge X, \mathbb{S}] = [W, X^*].$$

We can take this as a definition of X^*.

Proposition 2.2. Let X be a CW-spectrum, then the functor $W \mapsto [W \wedge X, \mathbb{S}]_0$ is representable by some CW-spectra X^*.

Proof. One checks the functor satisfies the conditions of Brown representability theorem, which I shall not discuss. ■

We give an explicit construction for the suspension spectrum of a finite CW-complex K. Choose an embedding $K \subset S^n$ and let $L \subset S^n$ be the cell complex which is a deformation retraction of $S^n - K$. We identify L with its suspension spectrum and define $K^* = \Sigma^{-(n-1)}L$. It is clear that the stable homotopy type of $\Sigma^{-n-1}L$ replies only on the homotopy type of $L \simeq S^n - K$. We would like to argue that the stable homotopy type of $\Sigma^{-(n-1)}L$ is independent of choice of embeddings of K into S^n and also only relies on the stable homotopy type of K. First of all, we observe that $S^{n+1} - SK$ deformation retracts to $S^n - K$. So if somebody gives me $X \subset S^n$ and $Y \subset S^m$ with homotopy equivalence $f : S^pX \to S^qY$, then I can consider $S^pX \subset S^{p+n}$ and $S^qY \subset S^{q+m}$ with homotopy type of complements unchanged. Thus I can assume $f : X \to Y$ is a homotopy equivalence with embedding $X \subset S^n$ and $Y \subset S^m$. In this case, consider the join $S^n * S^m = S^{m+n+1}$ where both

X, Y embeds into. Also note that the mapping cylinder M of f embeds into $X * Y \subset S^n * S^m$. In this large sphere we have

$$S^{m+n+1} - X = S^{m+1}(S^n - X),$$
$$S^{m+n+1} - Y = S^{n+1}(S^m - Y),$$

and two maps

$$S^{m+n+1} - X \xleftarrow{f} S^{m+n+1} - M \xrightarrow{g} S^{m+n+1} - Y.$$

Notice that the injections $X \to M \leftarrow Y$ induce isomorphisms of cohomology, and the Alexander duality is natural for inclusions. Therefore f and g induces isomorphisms of homology. Now I can suspend things further to make them all simply connected, so that by Whitehead's theorem we conclude f, g are stable homotopy equivalences. Thus we've proved the assignment $K \mapsto L$ is well-defined, up to stable equivalence, for the suspension spectrum of K. The desuspension is made so that degrees are as expected.

Example 2.3. The sphere spectrum is Spanier-Whitehead self-dual, and the $\mathbb{S}^n$ is Spanier-Whitehead dual to $\mathbb{S}^{-n}$.

Now we examine this construction for closed manifolds to obtain Atiyah duality. Say we have a closed smooth manifold M. Let M_+ be M disjoint union with a base point. Then embed M into a large sphere and thus giving an embedding $M_+ \subset S^n$, with basepoint of M_+ being mapped to, say, the north pole of S^n. Then the compliment $S^n - M_+ = \mathbb{R}^n - M$. Since $\mathbb{R}^n$ is contractible, from the cofiber sequence

$$\mathbb{R}^n - M \to \mathbb{R}^n \to \mathbb{R}^n/(\mathbb{R}^n - M)$$

we conclude $\mathbb{R}^n - M = \Sigma^{-1}(\mathbb{R}^n/\mathbb{R}^n - M) = \Sigma^{-1}Th(N_M)$. Therefore

$$M_+^* = \Sigma^{-(n-1)}(S^n - M_+) = \Sigma^{-n}Th(N_M).$$

Recall that $N_M \oplus T_M = \mathbb{R}^n$, hence $N_M = \mathbb{R}^n - T_M$ and it follows that $Th(N_M) = \Sigma^n Th(-T_M)$. So finally we have

Theorem 2.4 (Atiyah duality)**.** If M is a closed manifold, then $M_+^* = Th(-T_M)$.

Now let us discuss properties of Spanier-Whitehead duality. Assume again X is a finite CW-spectrum. By definition there is a natural isomorphism

$$[W \wedge X, \mathbb{S}]_0 \xleftarrow{T} [W, X^*]_0.$$

Taking $W = X^*$ and $1 : X^* \to X^*$ on the right, we see there is a "evaluation" map

$$e : X^* \wedge X \to \mathbb{S}.$$

Using that T is natural, we see that T carries $f : W \to X^*$ into $W \wedge X \xrightarrow{f \wedge 1} X^* \wedge X \xrightarrow{e} \mathbb{S}$, which yields:

$$T : [W, X^*]_r \to [W \wedge X, \mathbb{S}]_r$$

And by applying the canonical isomorphism suspensions and desuspensions of W shows T is an isomorphism for all r.

Now consider a third spectrum Z, we can make a map

$$[W, Z \wedge X^*]_* \xrightarrow{T} [W \wedge X, Z]_*$$

induced by evaluation.

Proposition 2.5. If W and X are CW-spectra[1], then $T : [W, Z \wedge X^*]_* \to [W \wedge X, Z]_*$ is an isomorphism for any CW-spectrum Z.

Proof. This is easy to show for Z being the suspensions of the sphere spectrum. Then an induction using cofiber sequence and five lemma proves the statement holds for all finite spectrum Z. Finally pass to direct limits from the finite case to complete the proof. ∎

[1] Adams requires both to be finite, but I don't see why finiteness is needed.

Corollary 2.6. For any CW-spectrum E, we have $E_*(X^*) = E^{-*}(X)$[2].

Remark 2.7. If X is the suspension spectrum of a CW-complex, then $E_*(X)$ coincides with the usual E-homology of X. Same for cohomology.

Proof. Take $Z = E$ and $W = \mathbb{S}$ in the previous proposition. ■

Now let us revisit Atiyah duality. Using that $M_+^* = \Sigma^{-n} Th(N_M)$, we have

$$E^{-*}(M) = \widetilde{E}^{-*}(M_+) = \widetilde{E}_*(\Sigma^{-n} Th(N_M)) = \widetilde{E}_{*+n}(Th(N_M)).$$

Suppose there is a Thom isomorphism in E_* for N_M, then we have

$$\widetilde{E}_{*+n}(Th(N_M)) = E_{*+d}(M).$$

Putting these together we have

Theorem 2.8 (Poincaré duality)**.** Assume M^d is orientable for E, then $E_*(M) \cong E^{d-*}(M)$. In particular, if M is orientable then $H_*(M;\mathbb{Z}) \cong H^{d-*}(M;\mathbb{Z})$.

There are versions of Atiyah duality for manifold with boudary which gives the corresponding Poincaré duality as well.

We list some other properties of X^* without proof.

1. if X is finite, then so is X^*.
2. if X is finite, then $(X \wedge Y)^* \simeq X^* \wedge Y^*$.
3. S-dual coverts a cofibering of finite spectra into another cofibering.
4. $X^{**} = X$ if X is finite.

2.2. **Anderson duality.** Let us collect several facts from different fields. First of all, there is the *Pontryagin duality* for abelian topological group which asserts that $G \to \hat{G} = Hom_c(G, \mathbb{R}/\mathbb{Z})$ is reflexive, i.e. the double dual of G is G. This duality has a well-known discrete form for torsion groups, which asserts that $T \to Hom_{\mathbb{Z}}(T; \mathbb{Q}/\mathbb{Z})$[3] is reflexive.

We here describe a even more general version, assuming some compactness. Let A be a finitely generated abelian group, i.e. a coherent sheaf over *Spec*$(\mathbb{Z})$, then the functor

$$A \to Hom(A, \mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$$

is reflexive. We note that $Hom(A, \mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$ should be thought of as a discrete version of $Hom_c(-, \mathbb{R}/\mathbb{Z})$ where continuity is replaced by requiring maps into $\mathbb{Q}/\mathbb{Z}$ can be lifted to its "universal cover" $\mathbb{Q}$. More precisely, we mean the natural evaluation map

$$A \to \left\{ \begin{array}{ccc} Hom(A,\mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ Hom(A,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array} \right\}$$

is an isomorphism[4]. The conceptual way to see this is that $\mathbb{Z}$ is the dualizing sheaf over *Spec*$(\mathbb{Z})$ and $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is an injective resolution of $\mathbb{Z}$. One can also check it by hand, it suffices to verify for $A = \mathbb{Z}$ and $A = \mathbb{Z}_n$ using the structure theorem of finitely generated abelian groups. The above isomorphism is a consequence of the coherent duality over *Spec*$(\mathbb{Z})$.

Turing to topology, there is a well-known isomorphism

$$H^*(X;\mathbb{Q}) \cong Hom(H_*(X;\mathbb{Q}), \mathbb{Q}),$$

and a less known isomorphism

$$H^*(X;\mathbb{Z}_n) \cong Hom(H_*(X;\mathbb{Z}_n), \mathbb{Z}_n).$$

Putting the two together, one can show

[2] Again Adams assumes X is finite.

[3] $Hom(T, \mathbb{Q}/\mathbb{Z})$ is profinite.

[4] This seems to fail if A is not finitely generated, and I recall Yoonjoo gave me a counter-example once. This is somewhat expected, as coherent duality don't(?) usually extend to quasicoherent sheaves.

$$H^*(X;\mathbb{Z}) \xrightarrow{ev} \left\{ \begin{array}{ccc} H_*(X;\mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ H_*(X;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array} \right\}$$

is an isomorphism. This isomorphism was used by Morgan and Sullivan to "glue" L-class and Wu class to a 2-local cohomology class.

We would like to have the same for any generalized cohomology theory. Unfortunately if one simply replace H^* and H_* by E^* and E_* this won't hold, the way to fix this is to introduce certain dual $I_{\mathbb{Z}}E$ of E, called Anderson dual.

Theorem 2.9 (J. Hu)**.** Let E be a spectrum of finite type, and X a finite CW-complex. Then

$$E^*(X) \xrightarrow{ev} \left\{ \begin{array}{ccc} I_{\mathbb{Z}}E_*(X;\mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ I_{\mathbb{Z}}E_*(X;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array} \right\}$$

is an isomorphism.

Remark 2.10. The meaning of this theorem is that E-cohomology classes are completely determined by their evaluations over $I_{\mathbb{Z}}E$-homology classes. The finiteness assumptions are needed for coherent duality to work.

I haven't introduced homology with coeffients yet, but let us analyze what it takes for this theorem to work. We easily observe that if $I_{\mathbb{Z}}E_*(X;\mathbb{Q}/\mathbb{Z}) = Hom(E^*(X),\mathbb{Q}/\mathbb{Z})$ and $I_{\mathbb{Z}}E_*(X;\mathbb{Q}) = Hom(E^*(X),\mathbb{Q})$. Then this theorem follows from the coherent duality over $Spec\mathbb{Z}$.

So we simply *define* $I_{\mathbb{Z}}E$ to fulfill this purpose as follows. Consider the functor $X \mapsto Hom(E^*(X),\mathbb{Q})$. Since $Hom(-;\mathbb{Q})$ is exact, this functor defines a generalized homology theory, whose representing spectrum is denoted by $I_{\mathbb{Q}}E$. Similarly since $Hom(-,\mathbb{Q}/\mathbb{Z})$ is exact, we obtain $I_{\mathbb{Q}/\mathbb{Z}}E$. There is a canonical map

$$I_{\mathbb{Q}}E \to I_{\mathbb{Q}/\mathbb{Z}}E$$

induced by $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. We take $I_{\mathbb{Z}}E$ to be the fiber (i.e. desuspension of cofiber) of this map.

Definition 2.11 (Anderson dual)**.** We call $I_{\mathbb{Z}}E$ the *Anderson dual* of E and $I_{\mathbb{Q}/\mathbb{Z}}E$ the *Brown-Comenetz dual* of E.

Recall that the evaluation diagram above is a coherent way of packing universal coefficient theorem, we can unwrapp it. Indeed, Anderson showed

Theorem 2.12 (universal coefficient theorem)**.** Let E be a spectrum of finite type and X a finite CW-complex, then the following sequence is exact:

$$0 \to Ext\left((I_{\mathbb{Z}}E)_{*-1}(X),\mathbb{Z}\right) \to E^*(X) \to Hom\left((I_{\mathbb{Z}}E)_*(X),\mathbb{Z}\right) \to 0.$$

Example 2.13. The Anderson duals of $H\mathbb{Z}, KU, KO, Tmf$ are $H\mathbb{Z}, KU, \Sigma^4 KO$ and $\Sigma^{21}Tmf$ respectively.

I must now define (co)homology with coefficients, this has to do with *Moore spectrum.* Let G be an abelian group. By axiom of choice[5], there is a short resolution of G by free $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z}^A \xrightarrow{i} \mathbb{Z}^B \to G \to 0$$

Then take a map

$$f : \bigvee_{\alpha\in A} \mathbb{S} \to \bigvee_{\beta\in B} \mathbb{S}$$

so that $\pi_0(f) = i$. We define SG (sometimes denoted by MG) the Moore-spectrum of G to be the cofiber of f. The Moore spectrum for G is characterized, up to equivalence, by the properties

[5] if G is finitely generated, then axiom of choice is not needed.

$H\mathbb{Z}_0(SG) = G$ and all other homology groups vanish. If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups, then

$$SA \to SB \to SC$$

is a cofibering.

Definition 2.14. For any spectrum E we define the corresponding spectrum with coefficients in G to be $EG = E \wedge SG$. We also define $E_*(-;G) = EG_*(-)$ and $E^*(-;G) = EG^*(-)$.

It follows immediately from definition that we have long exact sequence induced from short exact sequence of coefficients. We also have universal coefficient theorem:

Proposition 2.15. There exist exact sequences

$$0 \to E_*(X) \otimes G \to E_*(X;G) \to Tor(E_{*-1}(X), G) \to 0,$$

and (if X is a finite spectrum or G is finitely generated)

$$0 \to E^*(X) \otimes G \to E^*(X;G) \to Tor(E^{*+1}(X), G) \to 0.$$

Proof. Follow your nose. ∎

Corollary 2.16. $i : \mathbb{S} \to H\mathbb{Z}$ induces an equivalence $S\mathbb{Q} \simeq H\mathbb{Q}$. Consequently $\pi_*(X) \otimes \mathbb{Q} \cong H_*(X) \otimes \mathbb{Q}$.

Proof. The above universal coefficient theorem shows $S\mathbb{Q} \to H\mathbb{Q}$ induces isomorphism on homotopy. ∎

Finally we are ready to prove Hu's theorem. As I pointed out, it suffices to check that $I_{\mathbb{Z}}E(X;\mathbb{Q}) = Hom(E^*(X), \mathbb{Q})$ and same for $\mathbb{Q}/\mathbb{Z}$. They follows from the following lemma.

Lemma 2.17. Let E be a spectrum, then $I_{\mathbb{Z}}E \wedge S\mathbb{Q} \simeq I_{\mathbb{Q}}E$ and $I_{\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z} \simeq I_{\mathbb{Q}/\mathbb{Z}}E$.

Proof. [6] Smash the cofiber sequence

$$I_{\mathbb{Z}}E \to I_{\mathbb{Q}}E \to I_{\mathbb{Q}/\mathbb{Z}}E$$

with $S\mathbb{Q}$, we obtain the cofiber sequence

$$I_{\mathbb{Z}}E \wedge S\mathbb{Q} \to I_{\mathbb{Q}}E \wedge S\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}E \wedge S\mathbb{Q}.$$

Since $\pi_*(I_{\mathbb{Q}}E) \otimes \mathbb{Q} = \pi_*(I_{\mathbb{Q}}E)$, we conclude that $I_{\mathbb{Q}}E \wedge S\mathbb{Q} \simeq I_{\mathbb{Q}}E$. Similarly since $\pi_*(I_{\mathbb{Q}/\mathbb{Z}}E) \otimes \mathbb{Q} = 0$, we conclude that $I_{\mathbb{Q}/\mathbb{Z}}E \wedge S\mathbb{Q}$ is contractible. Therefore $I_{\mathbb{Z}}E \wedge S\mathbb{Q} \simeq I_{\mathbb{Q}}E$ is proved.

On the other hand, smashing with $S\mathbb{Q}/\mathbb{Z}$ yields a cofiber sequence

$$I_{\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z} \to I_{\mathbb{Q}}E \wedge S\mathbb{Q}/\mathbb{Z} \to I_{\mathbb{Q}/\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z}.$$

I claim that $I_{\mathbb{Q}}E \wedge S\mathbb{Q}/\mathbb{Z}$ is contractible. This follows from $I_{\mathbb{Q}}E \simeq I_{\mathbb{Q}}E \wedge S\mathbb{Q}$ and the cofiber sequence

$$I_{\mathbb{Q}}E \to I_{\mathbb{Q}}E \wedge S\mathbb{Q} \to I_{\mathbb{Q}}E \wedge S\mathbb{Q}/\mathbb{Z}.$$

It then follows that $I_{\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z} \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z}$. Now consider smashing $S\mathbb{Z} = \mathbb{S} \to S\mathbb{Q} \to S\mathbb{Q}/\mathbb{Z}$ with $I_{\mathbb{Q}/\mathbb{Z}}E$, we can dedude that $I_{\mathbb{Q}/\mathbb{Z}}E \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z}$. Therefore $I_{\mathbb{Z}}E \wedge S\mathbb{Q}/\mathbb{Z} \simeq I_{\mathbb{Q}/\mathbb{Z}}E$ as desired. ∎

We conclude this section with

Proposition 2.18. If E is a spectrum of finite type, then $I_{\mathbb{Z}}I_{\mathbb{Z}}E = E$.

Proof. The idea is to build a natural map $E \to I_{\mathbb{Z}}I_{\mathbb{Z}}E$ and show this is an isomorphism on homotopy using the definition of $I_{\mathbb{Z}}$ and the coherent duality over $Spec\mathbb{Z}$. ∎

Remark 2.19. $I_{\mathbb{Z}} = I_{\mathbb{Z}}\mathbb{S}$ is the dualizing object for spectra.

[6] Here's a short proof: this lemma follows immediately from that $I_{\mathbb{Q}}E$ and $I_{\mathbb{Z}}E$ are rationally equivalent.

2.3. **Poincaré duality.** We have seen that the Poincaré duality for closed smooth manifolds follows from Atiyah duality and Thom isomorphism for the stable normal bundle. In order to discuss Thom isomorphism, we need to talk about products (cup, cap, slant, slash). We would also like to have Poincaré duality for *topological* manifolds, where one does not have tangent/normal bundle. An appropriate notion of orientability has to be used. Once everything is set up, the proof for Poincaré duality proceeds in the usual way. The desired statement is the following:

Theorem 2.20 (Poincaré duality)**.** Let E be a spectrum on which cup product and cap product can be defined. Assume M^d is a topological manifold which is orientable in certain sense with respect to E. Then $E^{d-*}(M) \simeq E_*(M)$.

We point out that, even if M is orientable in the usual sense, it might fail to be orientable for a generalized cohomology theory. Recall that when M is smooth, the orientability should require the normal bundle of M to admit Thom isomorphism in E. If $E = KU$, then a vector bundle has a Thom isomorphism for KU *if and only if* the bundle is $Spin^c$[7]. Therefore, only $Spin^c$ manifolds are orientable for KU.

Now in order to talk about products, we introduce the notion of *ring spectrum*. Roughly speaking, a ring spectrum is a ring object in the category of spectrum.

Definition 2.21 (ring spectrum, module spectrum)**.** A spectrum E is said to be a ring spectrum if it has given maps $\mu : E \wedge E \to E$, $\eta : \mathbb{S} \to E$ (of degree 0) such that certain diagrams commute. A ring spectrum is commutative if further $c\mu = \mu$ where $c : E \wedge E \to E \wedge E$ is switching factors. One can also define module spectrum F by requiring there is a map $\nu : E \wedge F \to F$ subject to certain conditions.

If E is a ring spectrum, then one can define cross product. Informally, this is

$$\widetilde{E}^*(X) \otimes \widetilde{E}^*(Y) = [X, E]_* \otimes [Y, E]_* = [X \wedge Y, E \wedge E]_* \xrightarrow{\mu} [X \wedge Y, E]_* = \widetilde{E}^*(X \wedge Y).$$

Using the plus construction, one can obtain a basepoint-free cup product

$$E^*(X) \otimes E^*(Y) \to E^*(X \times Y).$$

The cross product combined with the diagonal map $X \to X \wedge X$ yields cup product:

$$\widetilde{E}^*(X) \otimes \widetilde{E}^*(X) \to \widetilde{E}^*(X).$$

Similarly one has a basepoint-free version. Cap product can also be defined.

Lemma 2.22. If E is a ring spectrum, then $E^*(X)$ is a graded ring; in particular $\pi_*(E) = E_*(\mathrm{pt})$ is a graded ring. If F is a module spectrum over F, then $F^*(X)$ is a module over $E^*(X)$.

Example 2.23. (1) If R is a commutative ring, then HR the Eilenberg-MacLane spectrum with R-coefficients is a ring spectrum. Note that $H\mathbb{Q}/\mathbb{Z}$ is not a ring spectrum since $\pi_* H\mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ is not a ring.
(2) KU, KO are ring spectra, whose cup products correspond to tensor products over $\mathbb{C}$ and $\mathbb{R}$ respectively. But KSp is not a ring spectrum, KSp is a module spectrum over KO.
(3) The Thom spectrum MSO is a ring spectrum. In particular $\pi_* MSO = \Omega_*^{SO}$ is a graded ring, called the cobordism ring whose multiplication corresponds to Cartisian products of oriented manifolds.
(4) $MU, MSp, MSpin, MSpin^c$ are ring spectra. But $MSpin^h$ is not a ring spectrum.

Finally, orientation.

Definition 2.24 (orientation)**.** Let M be a compact topological manifold without boundary. By an orientation of M over E, we mean a class $\omega \in E^*(M \times M, M \times M - \Delta)$ such that

$$i_x^* \omega \in E^*(x \times M, x \times M - x \times x) \cong E^*(M, M - x)$$

is a generator for each $x \in M$.

---

[7]This is not trivial.

In the case $E = H\mathbb{Z}$ it is clear what generator means, in general this needs extra explanation. The pair $(M, M-x)$ is by excision equivalent to $(\mathbb{R}^n, \mathbb{R}^n - 0)$. We say $\varphi \in F^*(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \widetilde{F}^*(S^n)$ is a $\pi_*(F)$-basis for $F^*(\mathbb{R}^n, \mathbb{R}^n - 0)$.

Remark 2.25. If M is smooth, then using tubular neighborhood theorem and that the normal bundle of Δ_M in $M \times M$ is diffeomorphic to TM, the above orientability can be rephrased as the tangent bundle of M admits a Thom class $U \in E^*(TM, TM - M)$ which restricts to each point $x \in M$ is a generator for $E^*(T_xM, T_xM - x)$.

Remark 2.26. Orientations are usually not unique. For $E = H\mathbb{Z}$, if orientation exists, there are two choices. In general, different choices of orientations are differed by a unit in $\pi_0 E$.

Theorem 2.27 (Poincaré duality)**.** Let M be a compact topological manifold without boundary, oriented over E and F is a module spectrum over E. Then we have an isomorphism

$$F_p(M) \xrightarrow{\simeq} F^{d-p}(M).$$

Proof. The proof proceeds in the same way as for singular cohomology. Firstly, there is a local duality for $\mathbb{R}^n$ and the global duality follows from Mayer-Vietoris sequence and five-lemma. ∎

Example 2.28. (1) All topological manifolds are orientable over $H\mathbb{Z}_2$. Orientability in the usual sense is the same as orientable over $H\mathbb{Z}$.

(2) Conner and Folyd showed stably almost complex manifolds and stably special unitary manifolds are orientable over KU, KO respectively. Later Atiyah, Bott and Shapiro showed orientablity over KU, KO *precisely* means $Spin^c$ and $Spin$ respectively (at least in the smooth case). Their construction of orientation classes involve Clifford algebras and their modules.
(3) Stably almost complex manifolds are orientable over MU. In fact, Quillen noticed MU is the universal theory over which stably almost complex manifolds are orientable.
(4) If M is orientable over the sphere spectrum, then it admits Poincaré duality for any generalized cohomology theory. Question: can we find all such M? Examples of M are spheres S^n.
(5) (Sullivan) PL-manifolds are orientable over KO at odd primes. This orientation is tied up to signature, L-class and Adams Ψ_2 operation, the last of which morally explains why one has to localize at odd primes.

Exercise 2.29. Find an explicit formula for the fundamental class of $\mathbb{CP}^n$ for KU.

We conclude this lecture with the following remark.

Proposition 2.30. Let E be a ring spectrum, then the cohomology Atiyah-Hirzebruch spectral sequence for E is multiplicative.

3. Localization and Adams spectral sequence

Many important geometric-topology questions can be solved by calculating the homotopy groups of some spectra. For example, Thom showed $\Omega_*^{SO} = \pi_*(MSO)$; Kervaire and Milnor showed exotic spheres are related to $\pi_*(\mathbb{S})$. Also, the homotopy groups of a spectrum is required as an input for the Atiyah-Hirzebruch spectral sequence. Therefore, we would love to have a method for calculating $\pi_*(X)$ in general.

Recall that $\pi_*(X) = [\mathbb{S}, X]_{-*}$. We shall address a more general question: how can we calculate $[X, Y]_*$? As we know, it is often easier to calculate (co)homology, so we can ask, how much can we tell about $[X, Y]_*$ if we know about $H_*(X)$ and $H_*(Y)$? One can replace H_* by any generalized homology E_* and ask the same question.

Theorem 3.1 (Adams)**.** Let E be a ring spectrum, under certain conditions on X, Y, E, there is a spectral sequence

$$E_2^{p,*} = Ext^{p,*}_{E_*(E)}(E_*(X), E_*(Y)) \Longrightarrow_p [X, Y]_*^E.$$

Since E is assumed to be a ring spectrum $E_*(E)$ is a coalgebra and $E_*(X), E_*(Y)$ are comodules over $E_*(E)$. $Ext^{p,*}$ is the higher derived Hom^* as usual. The notation $[X, Y]_*^E$ means the localization of $[X, Y]_*$ with respect to E. This should be expected, since from $E_*(X)$ and $E_*(Y)$, no information undetectable by E can be obtained.

3.1. **Localization and completion.** Let E be a spectrum. We would like to *localize* a spectrum with respect to E. Recall one can localize a $\mathbb{Z}$-module with respect to a set of primes by inverting all other primes. And localizing at prime p is the same as tensoring with $\mathbb{Z}_{(p)}$. Further, localization is exact. We will see, in good cases, E-localization also has these properties.

Definition 3.2. A map $f : X \to Y$ is an E-(homology) equivalence if $f_* : E_*X \to E_*Y$ is an isomorphism. We say Z is E-acyclic (or E-trivial) if $E_*Z = 0$. We say W is E-local if for all E-equivalence $f : X \to Y$ the induced map

$$f^* : [Y, W]_* \to [X, W]_*$$

is an isomorphism. We say $f : X \to L_E X$ is a E-localization of X if f is an E-equivalence and $L_E X$ is E-local.

It is an easy exercise to see f is an E-equivalence if and only if the cofiber of f is E-trivial. It is also clear that any two E-localization $X \to L_E X$ and $X \to (L_E X)'$ are equivalence under X. Therefore, it is reasonable to call $L_E X$ *the* localization of X if exists.

Theorem 3.3 (Bousfield)**.** Let $Ho(Sp)$[8] be the homotopy category of CW-spectra. Then each $E \in Ho(Sp)$ gives rise to a E-localization functor $L_E : Ho(Sp) \to Ho(Sp)$ and a natural transformation $\eta : 1 \to L_E$

The techniques Bousfield used to show the existence of localization is applicable in many other situations, but we will not need them.

Proposition 3.4. If E is a ring spectrum and F an E-module, then F is E-local. In particular $E \wedge X$ is E-local.

Proof. Any map $f : Z \to F$ factors as

$$Z \xrightarrow{i \wedge 1} E \wedge Z \xrightarrow{1 \wedge f} E \wedge F \xrightarrow{\mu} F.$$

So if Z is E-acyclic, then $[Z, F] = 0$. This proves F is E-local. ∎

Example 3.5. $E \wedge MG$ is E-local.

Definition 3.6 (E-completion)**.** Let E be a ring spectrum, with $j : I \to \mathbb{S}$ the fiber of $i : \mathbb{S} \to E$. From the inverse system

$$\cdots \to I^{\wedge 3} \xrightarrow{j \wedge 1} I^{\wedge 2} \xrightarrow{j \wedge 1} I \xrightarrow{j} \mathbb{S}$$

we can form the inverse system

$$(\mathbb{S}/I^{\wedge n}) \wedge X.$$

Define the E-nilpotent completion $X^\wedge_E$ to be the direct limit of this direct system, with map $X \to X^\wedge_E$ induced by $\mathbb{S} \to \mathbb{S}/I^{\wedge n}$.

Proposition 3.7. The E-nilpotent completion is always E-local. If E is a finite spectrum, or X and I are connective (i.e. bounded below) and E is of finite type, then the map $X \to X^\wedge_E$ is an E-localization.

Proof. The cofiber sequence $I \to \mathbb{S} \to E$, after smashing with $I^{\wedge(n-1)}$, becomes a cofiber sequence $I^{\wedge n} \to I^{\wedge(n-1)} \to E \wedge I^{\wedge(n-1)}$, and so there are cofiber sequences

$$\mathbb{S}/I^{\wedge n} \wedge X \to \mathbb{S}/I^{\wedge(n-1)} \wedge X \to E \wedge I^{\wedge(n-1)} \wedge X.$$

By induction on n, we find $\mathbb{S}/I^{\wedge n} \wedge X$ is E-local, and so the homotopy (inverse) limit $X^\wedge_E$ is E-local.

Now after smashing with E, the cofiber sequence

$$E \wedge I^{\wedge n} \wedge X \to E \wedge I^{\wedge(n-1)} \wedge X \to E \wedge E \wedge I^{\wedge(n-1)} \wedge X$$

has a retraction of the second map via the multiplication of E, and so the first map is nullhomotopic. Therefore the homotopy limit $\lim E \wedge (I^{\wedge n} \wedge X)$ is trivial, and from the cofiber sequences

$$E \wedge (I^{\wedge n} \wedge X) \to E \wedge X \to E \wedge (\mathbb{S}/I^{\wedge n} \wedge X)$$

[8] Objects are CW-spectra and morphisms are homotopy classes of maps.

we find that $E \wedge X \to \lim E \wedge (\mathbb{S}/I^{\wedge n} \wedge X)$ is an equivalence.

This reduces to proving that the map

$$E \wedge \lim(-) \to \lim(E \wedge -)$$

is an equivalence. This is always true if E is finite or if E is of finite type and the homotopy limits is of connective objects. ■

Example 3.8. For X connective (i.e. bounded below), and $E = H\mathbb{F}_p$, $H\mathbb{F}_p$-localization coincides with p-completion.

The following result allows us to compute $[X, Y]_*^E$, and in particular $\pi_*(L_E Y)$. Let E be a ring spectrum, that $\pi_r(E) = 0$ for $r < 0$ and Y is connective (i.e. bounded below).

Theorem 3.9. (i) Suppose $\pi_0(E)$ is a subring R of the rationals. Then $L_E Y = Y \wedge MR$.
(ii) Suppose $\pi_0(E) = \mathbb{Z}/m$ and $\pi_r(Y)$ is finitely generated for all r. Then $L_E Y = Y \wedge M\mathbb{Z}_m^\wedge$.
(iii) Suppose $\pi_0(E) = \mathbb{Z}/m$ and the identity morphism $1 : Y \to Y$ satisfies $m^e \cdot 1 = 0$. Then $L_E Y = Y$.

Proof. Omitted. ■

It might be surprising at first glance that $\pi_0 E$ determines E-localization. Roughly speaking, one can prove if f is an E-equivalence, then f induces isomorphism on $H(\pi_0 E)$-homology. Conversely, if f yields an isomorphism on $H_*(-; \pi_0 E)$, then by universal coefficient theorem, f induces isomorphism on $H_*(-; \pi_* E)$. Hence by Atiyah-Hirzebruch spectral sequence and Zeeman comparison, we conclude f is an E-equivalence.

This proves, provided E is connected and everything is connective, then E-equivalence is the same as $H(\pi_0 E)$-equivalence. So $[X, Y]_*^E$ relies only on $\pi_0(E)$. In particular, for example, $[X, Y]_*^E$ is the same whether $E = MU_{(p)}$ or $bu_{(p)}$.

However, of course, the $MU_{(p)}$-localization and $bu_{(p)}$-localization are different.

Example 3.10. Let $E = H\mathbb{F}_p$, and X connective of finite type, then $\pi_*(X^\wedge_{H\mathbb{F}_p}) = \pi_*(X)^\wedge_p$ is the p-adic completion of $\pi_*(X)$.

We finish the discussion of Bousfield localization with an arithematic square.

Proposition 3.11. Suppose that E and K are spectra such that $L_K L_E X$ is always trivial. Then, for all X, there is a homotopy pullback diagram

$$\begin{tikzcd} L_{E\vee K} X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_K X \arrow[r] & L_E L_K X. \end{tikzcd}$$

Proof. The objects in the diagram

$$L_K X \to L_E L_K X \leftarrow L_E X$$

are either E-local or K-local, and hence automatically $E \vee K$-local; therefore the homotopy pullback is $E \vee K$-local. It suffices to show

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_K X \arrow[r] & L_E L_K X. \end{tikzcd}$$

becomes a homotopy pullback after smashing with $E \vee K$; which is easy to see. ■

Example 3.12 (Sullivan)**.** For all X, there are homotopy pullback diagrams

$$\begin{tikzcd} X_{(p)} \arrow[r] \arrow[d] & X_{\mathbb{Q}} \arrow[d] \\ X_p^\wedge \arrow[r] & (X_p^\wedge)_{\mathbb{Q}}. \end{tikzcd}$$

This corresponds to the Hensel's principle $0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p \oplus \mathbb{Q} \to \mathbb{Q}_p \to 0$.

3.2. **Base change.** We now address the question of recovering $[X,Y]_*$ from E_*X and E_*Y in a special case: assume Y is a E-module. In this case, denote Y by F, then $[X,Y]_* = [X,F]_* = F^{-*}(X)$. So the question becomes, given an E-module spectrum F and E_*X, can we compute $F^*(X)$? First of all, we must know $\pi_*(F)$. It turns out, under suitable assumptions, knowing π_*F is enough.

Proposition 3.13. Suppose E satisfies certain conditions, and suppose $E_*(X)$ is projective over $\pi_*(E)$. Then for all E-module spectrum F, we have an isomorphism

$$F^*(X) \xrightarrow{\simeq} Hom^*_{\pi_*E}(E_*X, \pi_*F).$$

Remark 3.14. The assumption E_*X is automatically satisfied by $X = \mathbb{S}$. Another example is if $E = H\mathbb{F}_p$, then all $\pi_*H\mathbb{F}_p = \mathbb{F}_p$-modules are projective.

A particularly easy case is when X is E-acyclic, in that case no assumption for E is needed.

Lemma 3.15. Let F be a E-module spectrum. If $E_*X = 0$, then $F_*X = 0$ and $F^*X = 0$.

Proof. $E_*X = 0$ means $E \wedge X$ is contractible. Now any morphism $\mathbb{S} \to F \wedge X$ can be factored as

$$\mathbb{S} \to \mathbb{S} \wedge F \wedge X \to E \wedge F \wedge X \to F \wedge X.$$

Since $E \wedge F \wedge X = F \wedge (E \wedge X)$ is contractible, we see all morphisms $\mathbb{S} \to F \wedge X$ are null-homotopic, hence $F \wedge X$ is contractible. This proves $F_*X = 0$.

Similarly, for any morphism $f : X \to F$ can be factored as

$$X = \mathbb{S} \wedge X \to E \wedge X \xrightarrow{1 \wedge f} E \wedge F \to F.$$

Therefore all morphisms $f : X \to F$ are null-homotopic, and $F^*X = 0$. ■

We note, for any X (not necessarily E-acyclic) and $f : X \to F$, we have a E-module morphism $E \wedge X \xrightarrow{1 \wedge f} E \wedge F \to F$. Applying π_* we get a π_*E-module morphism

$$E_*X \to \pi_*F.$$

So we always get a homomorphism

$$F^*(X) \to Hom_{\pi_*E}(E_*X, \pi_*F).$$

For this lecture, we say X is *perfect* for E if the above homomorphism is an isomorphism for all E-module F. Now we must spell out our assumptions on E, in order for the Proposition to hold.

Assumption. E is a direct limit of finite spectra E_α for which $E_*(DE_\alpha)$ is projective over π_*E and DE_α is perfect for E.

Example 3.16. The assumption is satisfied by the following spectra:

$$\mathbb{S}, H\mathbb{F}_p, MO, MU, MSp, K, KO.$$

Instead of proving the Proposition directly, we prove a more general statement.

Theorem 3.17 (base change spectral sequence)**.** Suppose E satisfies the assumption above, then there is a spectral sequence

$$Ext^{p,*}_{\pi_*E}(E_*X, \pi_*F) \Longrightarrow_p F^*(X)$$

whose edge-homomorphism is the homomorphism

$$F^*(X) \to Hom^*_{\pi_*E}(E_*X, \pi_*F).$$

Proof of Proposition from Theorem. If E_*X is projective over π_*E, then $Ext^{p,*}_{\pi_*E}(E_*X, \pi_*F) = 0$ for $p > 0$. Therefore, the spectral sequence collapses to its edge homomorphism. ■

In order to construct this spectral sequence, we must "resolve" X by spectra with π_*E-projective E-homology. In fact, we will construct a resolution of X of the following form, with $X = X_0$ and with the properties listed below.

$$\begin{tikzcd} X_r \arrow[rr, "x_r"] & & X_{r+1} \arrow[dl] \\ & W_r \arrow[ul, "g_r"] & \end{tikzcd}$$

(i) The triangles are cofiber triangles; more precisely $W_r \to X_r \to X_{r+1}$ is a cofibering and $X_{r+1} \to W_r$ has degree -1.
(ii) For each r, $(x_r)_* : E_*(X_r) \to E_*(X_{r+1})$ is zero.
(iii) For each r, $E_*(X_r)$ is projective over $\pi_* E$.
(iv) For each r, the map $F^*(W_r) \to Hom^*_{\pi_* E}(E_*(W_r), \pi_* F)$ is an isomorphism.

As usual, the most difficult step is in the construction of W_0, the rest will be taking cofibers and inductively applying the same construction. The problem now becomes, for X we need to construct W together with $f : W \to X$ so that the map $X \to cofiber(f)$ induces zero after applying E_*.

Recall E is the direct limit of finite spectra E_α. The injection $E_\alpha \to E$ corresponds to a cohomology class $i_\alpha \in E^0(E_\alpha)$ or to a homology class $g_\alpha \in E_0(DE_\alpha)$.

Lemma 3.18. For any spectrum X and any class $e \in E_p(X)$ there is an E_α and a morphism $f : DE_\alpha \to X$ of degree p such that $e = f_*(g_\alpha)$.

Proof. Take a class $e \in E_p(X)$. Then there is a finite subspectrum $i : X' \subset X$ and a class $e' \in E_p(X')$ such that $i_*(E') = e$. Indeed, $X = \lim X'$ and $E_*(X) = \lim E_*(X')$. So we may interprete e' as a morphism $DX' \to E$ of degree p. By assumption, this morphism factors through some E_α since Dx' is a finite spectrum:

$$DX' \xrightarrow{\varphi} E_\alpha \xrightarrow{i_\alpha} E$$

and $\varphi^* i_\alpha = e'$, considered as an element of $E^{-p}(DX')$. Dualizing back,

$$(D\varphi)_* g_\alpha = e' \in E_p(X').$$

Take f to be $DE_\alpha \xrightarrow{D\varphi} X' \xrightarrow{i'} X$. ■

Lemma 3.19. For any spectrum X there exists a spectrum of the form

$$W = \bigvee_\beta S^{p(\beta)} \wedge DE_{\alpha(\beta)}$$

and a morphism $g : W \to X$ (of degree 0) such that $g_* : E_* W \to E_* X$ is an epimorphism.

Proof. Represent E-homology classes by maps from DE_α into X. ■

Note that since $E_*(DE_\alpha)$ is projective over $\pi_*(E)$, $E_* W$ is automatically projective over $\pi_*(E)$ as well. Moreover, since DE_α is perfect for E, so is W. That is,

$$F^*(W) \xrightarrow{\simeq} Hom^*_{\pi_* E}(E_* W, \pi_* F)$$

for all E-module F.

Proof of Theorem. We can inductively construct X_r and W_r as discussed. Then applying F^* we obtain an exact couple

$$\begin{tikzcd} \bigoplus_r F^*(X_r) \arrow[dr] & & \bigoplus_r F^*(X_{r+1}) \arrow[ll] \\ & \bigoplus_r F^*(W_r) \arrow[ur] & \end{tikzcd}$$

By using mapping cylinder and a telescoping construction, we may assume $X = X_0 \subset X_1 \subset \cdots$. Define $X_\infty = \cup_r X_r = \lim X_r$. The spectral sequence should converge to $F^*(X_\infty, X_0)$. But we note that $E_*(X_r) \to E_*(X_{r+1})$ is zero by construction, so $E_*(X) = \lim E_*(X_r) = 0$. Therefore $F^*(X_\infty) = 0$ and $F^*(X_\infty, X_0) = F^*(X)$.

The first page of the spectral sequence is $E_1^{p,*} = F^*(W_p) = Hom^*_{\pi_* E}(E_* W_p, \pi_* F)$ and the boundary is induced by the boundaries in the projective resolution

$$0 \leftarrow E_* X \leftarrow E_*(W_0) \leftarrow E_*(W_1) \leftarrow \cdots$$

Therefore the second page is $E_2^{p,*} = Ext^{p,*}_{\pi_*E}(E_*X, \pi_*F)$ as claimed. ∎

3.3. **Adams spectral sequence.**

The construction of the spectral sequence is quite easy, but the recognition of the first and second pages are hard. The actual computation, of course, is even more painful.

3.3.1. *The construction.* Let E be a ring spectrum. We construct an exact couple Y_p, W_p inductively. Put $Y_0 = Y$. Suppose we have constructed Y_p we define $W_p = E \wedge Y_p$ and Y_{p+1} the fiber of the map $Y_p \to W_p$. That is, we have a cofibering

$$Y_{p+1} \to Y_p \to W_p \xrightarrow{\partial} Y_{p+1}$$

where ∂ has degree -1.

Therefore inductively we obtain

$$\begin{tikzcd} Y_{p-1} \arrow[dr] & & Y_p \arrow[ll] \arrow[dr] & & Y_{p+1} \arrow[ll] \\ & W_{p-1} \arrow[ur] & & W_p \arrow[ur] & \end{tikzcd}$$

Applying the functor $[X, -]^E_*$, we an exact couple

$$\begin{tikzcd} \bigoplus_p [X, Y_{p+1}]^E_* \arrow[rr] \arrow[dr] & & \bigoplus_p [X, Y_p]^E_* \\ & \bigoplus_p [X, W_p]^E_* \arrow[ur, "\partial"] & \end{tikzcd}$$

So we obtain a spectral sequence which, if convergent, converges to $[X, Y]^E_*$. The (decreasing) filtration on $[X, Y]^E_*$ is given by the images of $[X, Y_p]^E_* \to [X, Y]^E_*$.

Note we have the following cofibering

$$I \to \mathbb{S} \to E.$$

It is not hard to see $Y_p = I^p \wedge Y$ and $W_p = E \wedge I^p \wedge Y$. So the spectral sequence we just obtained is functorial. This is the Adams spectral sequence.

3.3.2. *The first and second page.* In order to recognize the first and second page, we assume E satisfy the assumption for the base change spectral sequence and E_*X is projective over π_*E. We further assume

Assumption. E_*E is a flat right π_*E-module.

Remark 3.20. There are two actions from π_*E on $E \wedge E$. The left one is induced by $E \wedge E \wedge E \xrightarrow{\mu \wedge 1} E \wedge E$ and the right one induced by $1 \wedge \mu$. The two actions, of course, commute.

Lemma 3.21. Suppose E_*E is flat over π_*E (as a right-module). Then for all X, the product map

$$E_*E \otimes_{\pi_*E} E_*X \to E_*(E \wedge X)$$

induced by $(E \wedge E) \wedge (E \wedge X) \xrightarrow{1 \wedge \mu \wedge 1} E \wedge E \wedge X$ is an isomorphism of E_*E-comodules.

Proof. This is obviously true for S^p, and use five lemma and induction this is true for finite spectra. Finally a direct limit argument proves the general case. Note that the flatness is used in the second step as we require $E_*E \otimes_{\pi_*E} -$ to be exact. ∎

Since W_p is E-local, we have $[X, W_p]^E_* = [X, W_p]_* = [X, E \wedge Y_p]_*$. Further, observe that $W_p = E \wedge Y_p$ is an E-module, applying base change we get

$$[X, W_p]_* = Hom^*_{\pi_*E}(E_*X, \pi_*(E \wedge Y_p)) = Hom^*_{\pi_*E}(E_*X, E_*Y_p).$$

By base change (for comodules), we have

$$Hom^*_{E_*E}(E_*X, E_*W_p) = Hom^*_{E_*E}(E_*X, E_*E \otimes_{\pi_*E} E_*Y_p) = Hom^*_{\pi_*E}(E_*X, E_*Y_p).$$

So the first page of the Adams spectral sequence is

$$E_1^{p,*} = [X, W_p]_*^E = Hom^*_{E_*E}(E_*X, E_*Y_p).$$

Now to obtain the second page, recall the differential on the first page is induced by the composition $W_p \to Y_{p+1} \to W_{p+1}$ (of degree -1). From the commutative diagram

$$\begin{tikzcd}
{[X, W_{p-1}]_*} \arrow[r] \arrow[d, "\cong"] & {[X, W_p]_*} \arrow[r] \arrow[d, "\cong"] & {[X, W_{p+1}]_*} \arrow[d, "\cong"] \\
Hom^*_{E_*E}(E_*X, E_*W_{p-1}) \arrow[r] & Hom^*_{E_*E}(E_*X, E_*W_p) \arrow[r] & Hom^*_{E_*E}(E_*X, E_*W_{p+1})
\end{tikzcd}$$

The second page of the Adams spectral sequence is identified with the cohomology of the second row.

Lemma 3.22. E_*W_p is a resolution of E_*Y by extended comodules over E_*E.

Proof. Consider the cofibering $Y_p \to E \wedge Y_p \to Y_{p+1}$ where the second map has degree -1. Smashing this with E we get

$$E \wedge Y_p \underset{1\wedge i}{\overset{\mu\wedge 1}{\leftrightarrows}} E \wedge E \wedge Y_p \to E \wedge Y_{p+1}$$

But $\mu \wedge 1$ is left inverse to $1 \wedge i$, so we have the following short exact sequence, split as a sequence of π_*E-modules.

$$0 \to E_*Y_p \to E_*(W_p) \to E_*(Y_{p+1}) \to 0$$

Hence, the sequence

$$0 \to E_*Y \to E_*W_0 \to E_*W_1 \to E_*W_2 \to \cdots$$

is indeed a resolution of E_*Y. ■

Recall that the usual prescription for computing $Ext_C^{**}(L, M)$ demands a resolution of M be injectives. However, in the case L is projective over R, it will be sufficient to resolve M by relative injectives. More precisely, if L is projective over R and

$$0 \to M \to M_0 \to M_1 \to M_2 \to \cdots$$

is a resolution of M by extended comodules $M_i = C \otimes_R N_i$. Then the cochain complex

$$Hom_C(L, M_\bullet) = Hom_R(L, N_\bullet)$$

computes $Ext_C^{**}(L, M)$ correctly, since $Hom_R(L, -)$ is exact. From here we conclude:

Proposition 3.23. The second page of the Adams spectral sequence is $E_2^{p,*} = Ext^{p,*}_{E_*E}(E_*X, E_*Y)$.

Example 3.24 (classical Adams spectral sequence)**.** Let $E = H\mathbb{F}_p$, then $E_*E = H\mathbb{F}_{p*}H\mathbb{F}_p = A_*$ is the dual mod p Steenrod algebra. Take $X = \mathbb{S}$, then $E_*X = \mathbb{F}_p$, the Adams spectral sequences now reads:

$$E_2^{s,*} = Ext^{s,*}_{A_*}(\mathbb{F}_p, H_*(Y; \mathbb{F}_p)) \Longrightarrow \pi_*(Y)^\wedge_p.$$

Assume Y is a finite spectra, then $Ext^{s,*}_{A_*}(\mathbb{F}_p, H_*(Y; \mathbb{F}_p)) = Ext^{s,*}_A(H^*(Y; \mathbb{F}_p), \mathbb{F}_p)$. In particular, we have

$$E_2^{s,*} = Ext^{s,*}_A(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(\mathbb{S})^\wedge_p.$$

Example 3.25 (Thom's work on unoriented cobordism)**.** Take $Y = MO$ in the previous example. Thom showed $H^*(MO; \mathbb{F}_2)$ is a free A-module using Thom isomorphism and Wu-formula and Serre's thesis. (A modern simply proof of this fact employs the Milnor-Moore theorem.) It follows that the Adams spectral sequence collapses on the second page, and $\pi_*(MO)^\wedge_2$ as $\mathbb{F}_2$-module is spanned by an A-basis of $H^*(MO; \mathbb{F}_2)$.

Example 3.26 (Adams-Novikov spectral sequence)**.** Let $E = BP$ the Brown-Peterson spectrum. Take $X = Y = \mathbb{S}$. We get

$$E_2 = Ext_{BP_*BP}(BP_*, BP_*) \Longrightarrow \pi_*(\mathbb{S})_{(p)}$$

where $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ is the universal ring for p-local, p-typical formal group laws.

3.3.3. *Adams-Margolis trick.* The coalgebra A_* involved in the classical Adams spectral sequence is infinite dimensional, so the computation of the second page of the Adams spectral sequence is already very hard. We introduce a trick for reducing the problem to a *finite dimensional* coalgebra.

Let ku be the connected KU, so that $\pi_*(ku) = \pi_{\geq 0}KU$. Let us try to compute $\pi_*(ku \wedge X)$ using the classical Adams spectral sequence. Then

$$E_2 = Ext_{A_*}(\mathbb{F}_p, H_*(ku \wedge X; \mathbb{F}_p)).$$

For the moment, assume X is a finite spectrum, therefore

$$E_2 = Ext_A(H^*(ku \wedge X; \mathbb{F}_p), \mathbb{F}_p).$$

By Künneth formula, we have $H^*(ku \wedge X; \mathbb{F}_p) = H^*(ku; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(X; \mathbb{F}_p)$. For simplicity of notation, we write H^* for $H^*(-; \mathbb{F}_p)$. The action of A on $H^*(ku) \otimes H^*(X)$ is through $\Delta : A \to A \otimes A$.

Lemma 3.27. We have

$$H^*(ku; \mathbb{F}_2) = A/A(Sq^1 + ASq^{01})$$

where $Sq^{01} = Sq^1Sq^2 + Sq^2Sq^1$ dual to ξ_2; and

$$H^*(ku; \mathbb{F}_p) = \sum_1^{p-1} A/(AQ_0 + AQ_1)$$

for p odd. In fact, $ku_{(p)} = \sum_1^{p-1} BP\langle 1\rangle$.

We note that as A-module, $H^*(ku; \mathbb{F}_2) = A \otimes_B \mathbb{F}_2$ where B is the exterior subalgebra of A generated by Sq^1 and Sq^{01}. Therefore, by applying change-of-rings, we have a spectral sequence

$$Ext_B(H^*(X; \mathbb{F}_2); \mathbb{F}_2) \Longrightarrow \pi_*(ku \wedge X)\hat{_2}$$

One can remove the remove the finiteness assumption on X if we use comodule language.

Proposition 3.28. Let X be a bounded-below spectrum, then we have a spectral sequence

$$E_2 = Ext_{B_*}(\mathbb{F}_2, H_*(X; \mathbb{F}_2)) \Longrightarrow \pi_*(bu \wedge X)\hat{_2}.$$

A similar result holds, replacing ku by ko and B by the subalgebra generated by Sq^1 and Sq^2 respectively.

4. Complex cobordism ring

The goal of this lecture is to determine the structure of complex cobordism ring. The first part is to determine its group structure–we show it is torsion-free. Two proofs are presented: one is due to Milnor, by applying Adams spectral sequnece; the other is due to Bouncristiano and Hacon, by a surgery argument. The second part is to determine its rings structure following Quillen.

4.1. **Milnor's theorem on complex cobordism.** Following Milnor, we base our calculation of π_*MU on the Adams spectral sequence.

Proposition 4.1. $H^*(MU; \mathbb{F}_p)$ is a free module over $A/(Q_0)$, where (Q_0) is the two-sided ideal generated by the Bockstein Q_0.

This proposition follows from the following two lemmas.

Lemma 4.2. The A-module map $A/(Q_0) \to H^*(MU; \mathbb{F}_p)$ induced by the Thom class is monic.

Proof. Using splitting principle, we write the Thom class $U = x_1x_2\ldots$. For any admissible P^I, its action on the Thom class can be computed and reduces to $P^i x^{p^s} = x^{p^s}, x^{p^{s+1}}$ or 0 as $i = 0, p^s$ or any other. ■

Lemma 4.3. Let A be a connected Hopf algebra over a field $\mathbb{F}$. Let M be a connected coalgebra over $\mathbb{F}$ with counit $1 \in M_0$ and a left module over A such that the diagonal map is a map of A-module. Suppose $\nu : A \to M : a \mapsto a \cdot 1$ is a monomorphism. Then M is a free left A-module.

Proof. Let $A_+ \subset A$ be the augmentation ideal of A. Let $\pi : M \to N = M/A_+M$ be the projection. Let $f : N \to M$ be any vector space splitting and define $\phi : A \otimes N \to M : a \otimes n \mapsto af(n)$. We show ϕ is an isomorphism of A-modules.

(i)ϕ is epic. Suppose ϕ is epic in degrees $< k$. Then for $c \operatorname{im} M_k$, consider $c - \phi(1 \otimes \pi c)$. This element by induction is in the image of ϕ and therefore c is in the image of ϕ.

(ii)ϕ is monic. Consider

$$A \otimes N \xrightarrow{1\otimes f} A \otimes M \to M \xrightarrow{\Delta} M \otimes M \xrightarrow{1\otimes\pi} M \otimes N.$$

The injectivity of ν implies the injectivity of the above composition and therefore ϕ being the composition of the first two maps is monic. ∎

Proposition 4.4 (Milnor)**.** Suppose Y is a connective spectrum of finite type. If $H^*(Y;\mathbb{F}_p)$ is a free $A/(Q_0)$-module with even dimensional generators, then π_*Y contains no p-torsion.

The idea of the proof is to apply the Adams spectral sequence

$$Ext_A^{*,*}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \Longrightarrow [X,Y]^{\wedge}_{p,*}$$

in which X is chosen to be the Spanier-Whitehead dual of the Moore spectrum $M\mathbb{F}_p$. First let us analyze $[X,Y]_*$ when $X = DM\mathbb{F}_p$.

$$[DM\mathbb{F}_p, Y]_{-*} = Y^*(DM\mathbb{F}_p) = Y_*(M\mathbb{F}_p) = \pi_*(Y \wedge M\mathbb{F}_p) = \pi_*(Y;\mathbb{F}_p).$$

By universal coefficient theorem, we have

$$0 \to \pi_*(Y) \otimes \mathbb{F}_p \to \pi_*(Y;\mathbb{F}_p) \to Tor(\pi_{*-1}(Y), \mathbb{F}_p) \to 0.$$

Therefore if π_*Y contains p-torsion, then $[X,Y]_m$ must be nontrivial for two consecutive values of m. On the other hand, assuming $H^*(Y;\mathbb{F}_p)$ is free over $A/(Q_0)$ with even generators, we will see $[X,Y]_m$ is zero for m even.

Second, note that $H^*(DM\mathbb{F}_p;\mathbb{F}_p) = H_*(M\mathbb{F}_p;\mathbb{F}_p)$. Then by universal coefficient theorem, the only non-zero groups are

$$H^0(DM\mathbb{F}_p;\mathbb{F}_p) = \mathbb{F}_p \quad H^1(DM\mathbb{F}_p;\mathbb{F}_p) = \mathbb{F}_p$$

Further, $Q_0 : H^0 \to H^1$ is an isomorphism. That is to say, $H^*(X;\mathbb{F}_p)$ is the exterior algebra generated by Q_0. Finally to compute $Ext_A^{*,*}(H^*(Y;\mathbb{F}_p), H^*(X,\mathbb{F}_p))$, we need to resolve $A/(Q_0)$ by free A-modules.

Prelimiaries concerning Steenrod algebra.

Let $\mathscr{R}$ be the set of sequences of integers $(r_0, r_1, r_2, \dots)$ such that $r_i \geq 0$ and $r_i = 0$ for almost all i. If $R = (r_0, r_1, r_2, \dots)$, let $\dim R = \sum 2r_i(p^i - 1)$ and $l(R) = \sum r_i$. Let V_s be the graded free abelian group generated by $R \in \mathscr{R}$ such that $l(R) = s$.

Milnor defined elements Q_i and $\mathscr{P}^R$ in the mod p Steenrod algebra A for $i = 0, 1, 2, \dots$ and $R \in \mathscr{R}$ and proved the following facts about them.

If $U, V \in \mathscr{R}$, $U - V \in \mathscr{R}$ is defined if $u_i \geq v_i$ and is equal to $(u_1 - v_1, u_2 - v_2, \dots)$. Δ_j denotes the sequence with 1 in the j^{th} place and zeros elsewhere.

(i) $\dim Q_i = 2p^i - 1$, $\dim \mathscr{P}^R = \dim R$. Note $Q_0 = \beta$ is the Bockstein.
(ii) $\{Q_i\}$ is the basis for a Grassmann subalgebra, A_o of A, i.e. $Q_iQ_j = 0$ if and only if $i = j$ and $Q_iQ_j + Q_jQ_i = 0$.
(iii) A is a free right A_o-module and $\{\mathscr{P}^R\}$ is a $\mathbb{F}_p$-basis for $A/(Q_0) = A/(A\beta A)$.
(iv) $(Q_0) = AQ_0 + AQ_1 + AQ_2 + \cdots$.

Let $M_s = A \otimes V_s$ and let $d_s : M_s \to M_{s-1}$ be the A-homomorphism of degree +1 given by

$$d_s(1 \otimes R) = \sum Q_j \otimes (R - \Delta_j).$$

Let $\alpha : M_0 \to A/(Q_0)$ be given by

$$\alpha(1 \otimes (0, 0, \dots)) = 1.$$

Lemma 4.5. The following is exact:

$$\to M_s \xrightarrow{d_s} M_{s-1} \to \cdots \to M_0 \xrightarrow{\alpha} A/(Q_0) \to 0.$$

Proof. It is well-known (cf. Cartan) the following is a A_o-free acyclic resolution of $\mathbb{F}_p$:

$$\to A_o \otimes V_s \xrightarrow{d_s}\to A_o \otimes V_{s-1} \to \cdots \to A_o \otimes V_0 \to \mathbb{F}_p.$$

Note

$$A \otimes_{A_o} \mathbb{F}_p = A/\sum AQ_i = A/(Q_0).$$

Applying $A\otimes_{A_o}$ to the above exact sequence yields the desired sequence. But A is free over A_o and hence $A\otimes_{A_o}$ is exact. ■

Remark 4.6. In order for the d_s to be of degree 0, we should add s to the dimension of each element in M_s.

With this, we can prove:

Corollary 4.7. $Ext_A^{*,*}(A/(Q_0), H^*(X;\mathbb{F}_p))$ is concentrated in odd degrees.

Proof. $Hom_A^t(M_s, H^*(X;\mathbb{F}_p)) = Hom_{A_o}^t(A_o \otimes V_s, H^*(X;\mathbb{F}_p))$ has a basis consisting of:

- for each R of dimension t, the homomorphism h_R that carries $1\otimes R$ to 1 and zero elsewhere;
- for each R of dimension $t+1$, the homomorphism h'_R that takes $1 \otimes R$ to Q_0 and zero elsewhere.

The differential, induced by d_s, takes h_R to $h'_{R+\Delta_0}$ and h'_R to 0. Hence $Ext_A^{*,*}$ has as basis the set of elements h'_R with total $r_0 = 0$, which has total dimension $t-s$ equals to $\sum 2r_i(p^i-1)-1$. ■

Proof of Proposition. Since $H^*(Y;\mathbb{F}_p)$ is assumed to be free over $A/(Q_0)$ with even degree generators, then $Ext_A^{*,*}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$ is concentrated in odd degrees. Therefore, by Adams spectral sequence the p-completion of $[X,Y]_*$ is concentrated in odd degrees and so $\pi_*(Y)$ contains no p-torsion. ■

Theorem 4.8. π_*MU is torsion free and the Hurewicz homomorphism $\pi_*MU \to H_*MU$ is a monomorphsim.

4.2. **The geometry of Chern numbers, following Buoncristiano and Hacon.** We assume the reader is familiar with surgery theory. We aim to inductively prove that if M^n has all Chern numbers zero, then $\{M\} = 0$. The strategy is to show, assuming the statement is proved in dimensions$< n$, that if M^n is such a manifold then $\{M, t_M\} = \{N, c\}$ in $\Omega_n^U(BU)$ where $t_M : M \to BU$ defines the weakly-complex structure on M and c is a constant map. For each prime $p \geq 2$, this leads to a relation in $\Omega_*^U(B\mathbb{Z}/p)$, whence p divides $\{M\}$ in Ω_n^U. It then follows that $\{M\} = 0$.

To begin with, let $f : M \to X$ be a map from M into X, a finite dimensional Grassmannian. Assume f maps into the k-skeleton X^k of X. We can perturb f so that it is transversal to all the k-cells of X. Fix a prior a cellular structure of X, say by Schubert cells. Assume e is a Schubert k-cell of X and $\hat{e}$ its center. Then $V(e) = f^{-1}(\hat{e})$ is a $(n-k)$-dimensional normally framed submanifold of M and locally near $V(e)$, M is diffeomorphic to $V(e) \times e$ with f behaves like a projection onto e.

Now let f be induced by tangent bundle of M and assume all Chern numbers of M vanish. Then it follows all Chern numbers of $V(e)$ vanish, indeed $f_*[V(e)] = f_*[M] \cap z$ where z is the dual cohomology class of the Schubert cell e. Then by induction hypothesis f can be cobordant to a map $g : N \to X^{k-1}$. Then make g to be transverse to all $(k-1)$-cells of X^{k-1}. We observe that the preimages of centers of $(k-1)$-cells are cobordant to normally framed submanifolds in M by making F transverse to all $(k-1)$-cells. Therefore we conclude all Chern numbers of preimages by g of $(k-1)$-cells in N are zero. And we can proceed to cobord g into X^{k-2} and so forth. So we have $\{M, t_M\} = \{N, c\}$ in $\Omega_n^U(BU)$ where c is a constant map. Geometrically, we have built a cobordism $F : W \to X$ and the bundle $\xi = F^*(universal)$ over W has the property that $\xi|_M = \tau_M \oplus \mathbb{R}^a$ and $\xi|_N$ is trivial.

Before we move on to the key construction in Buoncristiano and Hacon's geometric argument, let us observe the above surgery process applies to many other occasions, such as unoriented manifold and BO, weakly symplectic manifold and BSp. We will illustrate their idea by a warm-up exercise–prove Thom's theorem that if all Stiefel-Whitney numbers of M vanish, then M bounds an unoriented manifold.

Again, we construct an unoriented cobordism $F : W \to BO$ with $F|_M = t_M$ and $F|_N$ is the constant map. Since W is $(n+1)$-dimensional, we can assume F maps into $BO(n+1)$, thus $\xi|_M = \tau_m \oplus \mathbb{R}$ and $\xi|_N = \mathbb{R}^{n+1}$. The sphere bundle $S(\xi)$ of ξ is a $2n$-dimensional manifold with boundary $S(\tau_M \oplus \mathbb{R})$ and the trivial S^n-bundle over N. Next, we identify, in a neighborhood of M, $M \subset \tau_M \oplus \mathbb{R}$ with $\Delta M \subset M \times M \times \mathbb{R}$. The antipodal map on $\tau_M \oplus \mathbb{R}$ is transferred into the action $(x, y, t) \mapsto (y, x, -t)$ on $M \times M \times \mathbb{R}$. Then cut out a tubular neighborhood of ΔM and glue $M \times M \times [-1, 1]$ to $S(\xi)$ along the identified common boundary component $S(\tau_M \oplus \mathbb{R})$. This way, we obtain a manifold Q whose boundary is the disjoint union of the trivial S^n-bundle over N and $M \times M \times \{\pm 1\}$.

Now observe that Q carries a natural free $\mathbb{Z}/2$-action, therefore $Q' = Q/\mathbb{Z}/2$ is a smooth manifold (with boundary) with a map $Q' \to \mathbb{RP}^q$ classifying the double cover $Q \to Q'$. Restricting the map $Q' \to \mathbb{RP}^q$ to its boundary, we see, on the N-side, we have $N \times \mathbb{RP}^n \to \mathbb{RP}^n \subset \mathbb{RP}^q$; and on the other side we have the constant map. Finally, choose a $\mathbb{RP}^{q-n}$ in $\mathbb{RP}^q$ transversal to $\mathbb{RP}^n$ and misses the image of constant map, then the transverse preimage of $\mathbb{RP}^{q-n}$ is a submanifold of Q' with boundary N. This proves N is a boundary and therefore M is a boundary.

The passage from the unoriented case to the complex case is like moving from the orthogonal group to the unitary group, where $\mathbb{Z}/2 = O(1)$ should be replaced by $S^1 = U(1)$. So in principle, one should replace $\mathbb{RP}^q$ in the previous argument by $\mathbb{CP}^q$ and $M \times M = M^{\mathbb{Z}/2}$ by $LM = M^{S^1}$. However, since LM no longer is finite dimensional, to avoid this technical difficulty, we approximate LM by $M^{\mathbb{Z}/n} = M \times \cdots \times M$; and as usual it suffices to consider $n = p$ for all p prime. Then $\mathbb{RP}^q$ should be replaced by Lens spaces approximating $B\mathbb{Z}/p$.

We now walk through the construction in the stably almost complex case. To begin with, assume $\xi|_M = \tau_M \oplus \mathbb{R}^{2a-n}$ and $\xi|_N = \mathbb{C}^a$. We construct, from a complex vector bundle, a fiber bundle by Lens spaces as follows. First of all, for a complex vector space $\mathbb{C}^a$, we consider the hyperplane $H(\mathbb{C}^a)$ in the p-fold direct sum $\mathbb{C}^a \oplus \cdots \oplus \mathbb{C}^a$ defined by $\sum v_i = 0$. Then H is a $(p-1)a$-dimensional complex vector space carrying a $\mathbb{Z}/p$-action. This action restricted to the unit sphere S_H of H is free, thus the quotient is a Lens spaces $L(\mathbb{C}^a)$ of real dimension $2(p-1)a - 1$. This construction, applied fiberwise, yields a bundle $L(\mathbb{C}^a) \to L(\xi) \to W$ and $L(\xi)$ admits a p-fold covering $S_H(\xi)$.

Similar to the real case, we can identify $M \subset H(\xi|_M)$ with $\Delta M \subset M^p \times H(\mathbb{R}^{2a-n})$ with the $\mathbb{Z}/p$-action on $M^p \times (\mathbb{R}^{2a-n})^p$ given by the canonical cyclic permutation. Then again, we cut out the corresponding neighborhood and glue $M^p \times D_H(\mathbb{R}^{2a-n})$ to $S_H(\xi)$ to obtain a manifold Q with boundary $N \times S_H(\mathbb{C}^a)$ and $M^p \times S_H(\mathbb{R}^{2a-n})$. Note Q carries a free $\mathbb{Z}/p$-action whose quotient is a manifold Q' with boundary. Then the map $Q' \to L(\mathbb{C}^q) \subset B\mathbb{Z}/p$ classifying $Q \to Q'$ has the property that, when restricted to the N-side, it is the composition of the projection $N \times L(\mathbb{C}^a) \to L(\mathbb{C}^a)$ together with an inclusion; whilst when restricted to the M-side, it is factors through $L(\mathbb{R}^{2a-n}) \to L(\mathbb{C}^q)$. Since $\dim L(\mathbb{R}^{2a-n}) < \dim L(\mathbb{C}^a)$, we see the image of the M-side can be made to avoid the $\mathbb{Z}/p$-Poincaré dual, say P, of $L(\mathbb{C}^a)$ in $L(\mathbb{C}^q)$. Therefore, the transverse preimage of P in Q' is a $\mathbb{Z}/p$-manifold with boundary N. This means, $\{N\}$ reduced mod p is zero. We must confess that we haven't checked that all the constructions respect weakly complex structures, I leave it to the reader; once this is done, we can conclude p divides $\{N\}$ in Ω^U_*. Since p can be arbitrary and Ω^U_n is finitely generated, we conclude $\{N\} = 0$ and thus $\{M\} = 0$.

Theorem 4.9. Ω^U_* is torsion-free.

Proof. Suppose M is a torsion, then all Chern numbers of M, after multiplied by some positive integer, are zero, and thus all Chern numbers of M are zero. Therefore M is a boundary. ■

4.3. **The Lazard ring.**

Definition 4.10. A (1-dim commutative) formal group law f over a commutative ring R is a power series $f(x, y) \in R[[x, y]]$ satisfying

(1) (asscociativity) $f(x, f(y, z)) = f(f(x, y), z)$;
(2) (unit) $f(x, 0) = x = f(0, x)$;
(3) (inverse) there exists $g(x) \in R[[x]]$ such that $f(x, g(x)) = f(g(x), x) = x$;
(4) (commutativity) $f(x, y) = f(y, x)$.

It is an easy exercise to show the existence of g follows from the other axioms and

$$f(x,y) = x + y + \sum a_{i,j} x^i y^j.$$

The covariant functor $R \mapsto FGL(R)$ is representable by the Lazard ring $L = \mathbb{Z}[a_{i,j}]$/relations. That is, $FGL(R) = Hom_{\text{Rings}}(L, R)$. The Lazard ring is naturally graded, with $\deg x = \deg y = \deg f = -2$.

Example 4.11. (1) (additive group) $\mathbb{G}_a(x,y) = x + y$ can be defined over any ring.
(2) (multiplicative group) $\mathbb{G}_m(x,y) = x + y + xy$.
(3) (group laws isomorphic to additive group law) Suppose $g(x) = x + \sum b_i x^i$ then g^{-1} exists and

$$g(g^{-1}(x) + g^{-1}(y))$$

is a formal group law isomorphic to the additive group law. It is also clear every formal group law isomorphic to $\mathbb{G}_a$ has this form. Therefore, the functor $R \mapsto \mathbb{G}_a(R)$ classifying group laws isomorphic to the additive group law is representable by $\mathbb{Z}[b_1, b_2, \ldots]$. There is a canonical graded ring homomorphism

$$\phi : L \to \mathbb{Z}[b_1, b_2, \ldots].$$

sending $a_{i,j}$ to the coefficient of $x^i y^j$ in $g(g^{-1}(x) + g^{-1}(y))$.

Proposition 4.12. If $\mathbb{Q} \subset R$ then every formal group law over R is isomorphic to $\mathbb{G}_a$. In particular, $\phi \otimes \mathbb{Q} : L \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, \ldots]$.

Proof. Define the logarithmic differential $\omega(x)$ to be

$$\omega(x) = dx / \frac{d}{dy}|_{y=0} f(x,y).$$

It is easy to see all coefficients of ω are in R. Now since $\mathbb{Q} \subset R$, we can integrate ω to a power series $\log(x)$, which satisfies

$$\log(f(x,y)) = \log(x) + \log(y).$$

The series $\log(x) = \log^f(x)$ is called the logarithm of f. ■

Theorem 4.13 (Lazard)**.** The Lazard ring is a polynomial ring over $\mathbb{Z}$ with generators of dimension $2, 4, 6, \ldots$. Further, the morphism ϕ is monomorphic and

$$Q_{2n}(\phi) : \mathbb{Z} \to \mathbb{Z}$$

is multiplication by p if $n+1$ is a power of p and multiplication by 1 otherwise.

Let E be any complex orientable ring spectrum, that is complex vector bundles admit Thom classes in E yielding Thom isomorphisms. This is equivalent, by splitting principle, to saying complex line bundles are orientable for E. Suppose we have chosen a Thom class for $\mathcal{O}(1)$:

$$x = c_1(\mathcal{O}(1)) \in E^2(\mathbb{CP}^\infty).$$

Then one can show $E^*(\mathbb{CP}^\infty) = E^*(\text{pt})[[x]]$ using that $\mathbb{CP}^\infty$ is the Thom space of $\mathcal{O}(1)$ over $\mathbb{CP}^\infty$.

The map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induced by $\mathcal{O}(1) \otimes \mathcal{O}(1)$ yields a formal group law $f^E(x,y)$ over $E^*(\text{pt})$. This formal group law reflects how the first Chern class of the tensor of two line bundles can be written in terms of Chern classes of factors.

Therefore, there is a natural ring homomorphism $L \to \pi_* E$ classifying f^E. We note that f^E relies on the choice of $x = c_1(\mathcal{O}(1))$. Two different choices are differed by a change of coordinate. For $E = MU$, there is a tautological choice: $\Omega^2(\mathbb{CP}^n) \cong \Omega_{2n-2}(\mathbb{CP}^n)$ is represented by $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. This way, we have a natural map

$$\theta : L \to \pi_* MU.$$

Theorem 4.14 (Quillen)**.** θ is an isomorphism.

Proof. Myshcenko showed the logarithm of f^{MU} is given by

$$\log^{MU}(x) = \sum_{n\geq 0} \mathbb{CP}^n \frac{x^{n+1}}{n+1}.$$

If $\log^L = \sum p_n \frac{x^{n+1}}{n+1}$ is the logarithm of f^L, then θ sends p_n to $\mathbb{CP}^n$. Since $\pi_* MU \otimes \mathbb{Q}$ is generated by $\mathbb{CP}^n$ and p_n generates $L \otimes \mathbb{Q}$, we conclude $\theta \otimes \mathbb{Q}$ is an isomorphism. Then since both L and $\pi_* MU$ are torsion free, we see θ is injective.

It remains to show θ is surjective. From the identity

$$H(x,y) = \sum [H_{ij}] x^i y^j = f^{MU}(x,y)(\sum \mathbb{CP}^n x^n)(\sum \mathbb{CP}^m y^m),$$

we deduce the Milnor hypersurfaces H_{ij} are in the image of θ. But according to Milnor, they generate $\pi_* MU$. ■