TRANSCENDENTAL JULIA SETS WITH FRACTIONAL PACKING DIMENSION

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Abstract. We show how to construct a family of transcendental entire functions whose Julia sets have packing dimension in $(1, 2)$. These are the first examples where the computed packing dimension is not 1 or 2. Our construction will allow us further show that the set of packing dimensions attained is dense in $(1, 2)$, and that the Hausdorff dimension of the Julia sets can be made arbitrarily close to corresponding packing dimension.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental (non-polynomial) entire function. We denote the *nth iterate* of f by f^n. We define the *Fatou set*, $\mathcal{F}(f)$, to be the set of all points so that $\{f^n\}_{n=1}^{\infty}$ locally forms a normal family. Thus the Fatou set is the "stable" set for the dynamics of f. We define the *Julia set*, $\mathcal{J}(f)$, to be the complement of the Fatou set. This is the set where the dynamics of f are chaotic and there is strong sensitivity to initial conditions. A primary aim of complex dynamics is to study the geometric and topological properties of the Julia set. We refer the reader to [CG93] and [Sch10] and for an introduction to complex dynamics the rational and transcendental setting, respectively.

In this paper we prove the following:

Theorem 1.1. *There exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that the packing dimension of $\mathcal{J}(f) \in (1, 2)$.*

Actually, our techniques generate a family of entire functions, and we actually have the following stronger result:

Theorem 1.2. *The set of packing dimensions attained is dense in $(1, 2)$. In particular, let $s \in (1, 2)$ and $\epsilon_0 > 0$ be given. Then there exists a transcendental entire f so that*

$$s - \epsilon_0 \leq \mathrm{Hdim}(\mathcal{J}(f)) \leq \mathrm{Pdim}(\mathcal{J}(f)) \leq s + \epsilon_0.$$

In [Bak75], Baker proved that the Julia set of a transcendental entire function must always contain a compact connected set, and it follows immediately that the Hausdorff dimension of the Julia set must always be ≥ 1. In [Mis81], Misiurewicz showed that the Julia set of e^z was the entire complex plane, and in [McM87] McMullen showed that the Julia sets of the exponential and sine families of entire functions always have Hausdorff dimension 2, but need not be all of $\mathbb{C}$. These examples can also have positive or zero area measure. Reducing the dimension of the Julia set is therefore the difficult task in the transcendental setting, and in [Sta91], Stallard constructed examples in the Eremenko-Lyubich class that had Hausdorff dimension arbitrarily close to 1, and refined this result further in [Sta97] and [Sta00] to include all values in $(1, 2]$. Moreover, in [Sta96], Stallard showed that in the Eremenko-Lyubich class the Hausdorff dimension must be strictly greater than 1. Recently, in [Bis17], Bishop constructed a transcendental entire function with Julia set having Hausdorff dimension 1. This example demonstrates that all values of Hausdorff dimension in $[1, 2]$ can be achieved.

Less is known about the packing dimension in the transcendental setting. In [RS05], Rippon and Stallard show that if f belongs to the Eremenko-Lyubich class (so that the set of singular values of f is compact), then the packing dimension of the Julia set is 2. Bishop's computed the packing dimension of the Julia set of his example above to be 1. Our result is the first of its kind where the computed packing dimension is strictly between 1 and 2. Packing dimension and other various dimensions relevant to the paper are defined in Section 3.

We would like to point out how our construction differs from the constructions cited above. Since Stallard's examples belong to the Eremenko-Lyubich class, the packing dimension of those Julia sets must be 2, even though the Hausdorff dimension can attain any value in $(1, 2)$. The dynamical behavior of our examples is also much different; our functions have multiply connected Fatou components which are impossible in the Eremenko-Lyubich class. Stallard uses a family of functions defined via a Cauchy integral, whereas we use an infinite product construction similar to Bishop. Our example differs from Bishop's because not all of the Fatou components will be multiply connected. Instead of basing our construction off of a polynomial whose Julia set is a Cantor set, we base ours off of a polynomial whose Julia set is a quasicircle. It will follow that the entire function we construct also has Fatou components which get mapped onto an attracting basin whose boundary is this quasicircle. The multiply connected components of the Julia set will accumulate onto the boundary of these quasicircles, so that they are no longer close to being round everywhere. Bishop's techniques need to be extended to cover this new type of component.

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2. Outline of the Proof

We will construct a function $f : \mathbb{C} \to \mathbb{C}$ depending on parameters $N \in \mathbb{N}$, $R \in \mathbb{R}$, and c in the main cardiod of the Mandelbrot set. Define $f_0(z) = z^2+c$. The function f will be an infinite product of the form

$$f(z) = f_0^N(z) \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right) = f_0^N(z)(1+\epsilon(z)). \tag{2.1}$$

Here, $n_k = 2^{N+k-1}$, and the sequence $\{R_k\}$ grows superexponentially and is defined inductively starting from a large initial parameter R. The choices are made so that near the origin, the infinite product can be made uniformly close to the constant function 1. We write the infinite product as $(1+\epsilon(z))$ to emphasize this fact, where $\epsilon(z)$ is a holomorphic function uniformly close to the 0 function in a large neighborhood of the origin. Our construction mirrors Bishop's construction in [Bis17], with a couple key changes, the most important being the iterated polynomial z^2+c has a quasicircle as its Julia set with exactly one attracting basin containing the origin, rather than a Cantor set. As previously mentioned, this will affect the computation of the packing dimension, since the quasicircle has dimension larger than 1 and introduces geometry not present in Bishop's example.

First, in section 5, we will show that f does indeed define an entire function. Given any $s \in (1,2)$, we will choose c so that $\mathrm{Hdim}(J(f_0)) = s$. In a neighborhood of the origin, f is a polynomial-like mapping which is close to f_0^N. We will use an argument analogous to the straightening lemma to show that the Julia set of f, viewed as a polynomial-like mapping, is a quasicircle with dimension as close as we would like to s. It will follow that the Julia set of the entire function f will have Hausdorff dimension close to s as well.

In the next sections we prove several estimates on the growth of the sequence $\{R_k\}$, and decompose the plane into alternating annuli A_n and B_n, where the modulus of A_n is fixed and contains the circle $|z| = |R_k|$ and the modulus of B_n increases as $n \to \infty$. We will show that f looks like a power function z^m on B_n, that $f(B_n) \subset B_n$, and that if a point ever lands in B_n, it diverges locally uniformly to ∞ under f. Therefore, all the interesting dynamical behavior happens in the annuli A_k. We will show that $A_n \subset f(A_{n-1})$, and that all the zeros and critical points of f and the Julia set are inside the A_n's. To accomplish this, we will show (in a way that can be made precise) that f is approximately equal to the kth term in the infinite product on A_k.

From here, we will be able to prove that the Fatou set contains two types of connected components. The first type are the components which are subsets of the escaping set. These components will be infinitely connected wandering domains, and the boundary of such components will be bounded by C^1 closed curves. These curves will accumulate on the outer boundary of each component. Moreover, the regions bounded by these curves will eventually map conformally onto some infinitely connected Fatou component which surrounds the origin, so that the holes in these domains contain conformal copies of each other. The second type of Fatou component comes from the connected component containing the critical point 0 of $f(z)$. This component is an attracting basin, and its inverse images form "trapdoors" in the sense that if z is inside of one of the inverse images, z will eventually land in the basin containing 0 and remain there for the rest of its iteration. The Julia set will contain the boundaries of each of these two types of components. This is not the entire Julia set though. Since f has a multiply connected Fatou component, the work of Dominguez ([Dom97]) implies that the Julia set will also contain points that do not lie on the boundaries of either of these two types of components. These so-called "buried" points in the Julia set either remain bounded, are in the bungee set, or escape, but they do not belong to the fast escaping set. In sections 10 through 13, we will perform a detailed analysis on the dimension of the set of these points. We will show that the dimension of this set is lower bounded by the dimension of the boundary of the basin of attraction containing 0. While the dimension could possibly be larger than the dimension of the boundary of the basin of attraction, we show that we can make this difference in dimension arbitrarily small.

To upper bound the packing dimension, we will follow [Bis17] and study the critical exponent of a Whitney decomposition of the complement of the Julia set of f in a bounded region. Since the Julia set of f will have zero area, it turns out that this critical exponent coincides with the packing dimension, and we will show that this exponent is at most the dimension of the buried points. The key idea in this part of the proof is to iterate small Fatou components to components of unit size where we can calculate the critical exponent of a Whitney decomposition directly. The tradeoff is that this conformal rescaling procedure results in various corrective factors that we must now control independently of which small Fatou component we chose. We do this with a combination of all the technical work done earlier in the paper.

3. Hausdorff, Packing, and Minkowski Dimension

Given a set $A \subset \mathbb{C}$, we define its *α-Hausdorff measure* to be the quantity

$$H^\alpha(A) := \lim_{\delta\to 0} H^\alpha_\delta(A) := \lim_{\delta\to 0}\left(\inf\left\{\sum_{i=1}^\infty \mathrm{diam}(U_i)^\alpha \,:\, A \subset \cup_{i=1}^\infty U_i,\, \mathrm{diam}(U_i) < \delta\right\}\right).$$

The infimum is taken over all countable covers $\{U_i\}$ of A. One can check that if $H^t(A) < 0$, then $H^s(A) = 0$ for all $s > t$, and similarly, if $H^t(A) > 0$, then

$H^s(A) = \infty$ for all $s < t$. It follows that the *Hausdorff dimension*

$$\mathrm{Hdim}(A) := \sup\{t : H^t(A) = \infty\} = \inf\{t : H^t(A) = 0\}$$

is uniquely defined. The Hausdorff measure of a set is unchanged if we restrict the coverings to be open, closed, or even convex. It will change if we restrict our coverings to be by open balls, but only by a bounded amount, since every open set U is contained in a ball around $z \in U$ of twice its diameter. It follows that the Hausdorff dimension does not change if we restrict our coverings to be by open balls, and similar arguments follow for other restricted coverings, such as squares or dyadic squares.

Given a compact set $K \subset \mathbb{C}$, define $N(K, \epsilon)$ to be the minimal number of open balls of radius ϵ needed to cover K. Since K is compact, this number exists and is finite. We define the *upper Minkowski dimension* of K to be

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log(N(K,\epsilon))}{\log(1/\epsilon)} = \sup\{s \geq 0 : \limsup_{\epsilon \to 0} N(K,\epsilon)\epsilon^s = 0\}.$$

Equivalently, one may consider coverings of K by squares of a fixed edge length, and since the diameters of squares and balls are comparable, this would not change the definitions above. For this reason, the upper dimension is often called the *upper box counting dimension*, although we will favor the former notation.

In this paper, we will investigate the Minkowski and packing dimension of *unbounded* Julia sets, so strictly speaking, the definition above does not make sense. We can instead consider the *local upper Minkowski* dimension of the Julia set, which is the upper and lower Minkowski dimension of the Julia set intersected with an open neighborhood of finite diameter. In [RS05], Rippon and Stallard show that the local upper Minkowski dimension of the Julia set of an entire function is constant and coincides with its packing dimension (defined below), except perhaps in a neighborhood of 1 point (a point with finite backward orbit; there is at most 1 by the Picard theorem). Our example will not have an exceptional value of this kind, so their result further implies that the packing dimension and local Minkowski dimension are the same, no matter where we measure the local Minkowski dimension. In light of this, we will abuse notation and refer to the local upper Minkowski dimension of $\mathcal{J}(f)$ by $\overline{\mathrm{Mdim}}(\mathcal{J}(f))$; the neighborhood we are using will always be made clear.

One reason to use the packing dimension is that the upper Minkowski dimension has poor measure theoretic qualities. In particular, one can check for the set $K = \{0\} \cup \{1/n : n = 1, 2, \dots\}$ has Minkowski dimension 1/2, but Hausdorff dimension 0 since it is countable. This means that Minkowski dimension does not satisfy

$$\overline{\mathrm{Mdim}}(\cup A_i) = \sup \overline{\mathrm{Mdim}}(A_i).$$

To fix this issue, we define the *packing dimension* of K to be

$$\mathrm{Pdim}(K) = \inf \left\{ \sup_i \{\overline{\mathrm{Mdim}}\, K_i : K = \cup K_i\} \right\}.$$

Here, the infimum is taken over all partitions of K into countably many subsets K_i. With this definition we see that the packing dimension of the countable set above is 0. The packing dimension can also be defined in terms of α-packing measures, see Section 2.7 in [BP17]. In practice, it is difficult to compute packing dimension using either of these definitions. We will instead use the strategy described below.

For $n \in \mathbb{Z}$, let D_n denote the *nth* generation of dyadic intervals

$$I = [j2^{-n}, (j+1)2^{-n}], \quad j \in \mathbb{Z}.$$

We call $Q \subset \mathbb{R}^d$ a *dyadic cube* if it is the product of d dyadic intervals inside the same D_n. Dyadic cubes satisfy the following simple but useful properties:

(1) The length is $l(Q) = 2^{-n}$, and its diameter is $|Q| = \sqrt{d} \cdot l(Q)$.
(2) Each dyadic cube is contained inside a unique cube $\tilde{Q}$ with the property that $|\tilde{Q}| = 2|Q|$.
(3) Given two dyadic cubes Q_1 and Q_2, either Q_1 or Q_2 have disjoint interiors, or one is contained in the other.

If $\Omega \subset \mathbb{R}^d$ is open, each $x \in \Omega$ belongs to a dyadic cube $Q \subset \Omega$ with the additional property that $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega)$. By property (3), each point is contained in a cube of maximal possible diameter. It follows that the cubes $\{Q_j\}$ form a *Whitney decomposition* of Ω, that is, a collection of cubes that are have disjoint interior, cover Ω, and satisfy the estimate

$$\frac{1}{C} \operatorname{dist}(Q_j, \partial\Omega) \leq |Q_j| \leq C \operatorname{dist}(Q_j, \partial\Omega). \tag{3.1}$$

More generally, we may consider Whitney decompositions of Ω with subsets that are not dyadic cubes, but are instead sets with disjoint interior whose closures cover Ω and satisfy the same estimate above. Whitney decompositions are also conformally invariant in the following sense: if $f : \Omega \to f(\Omega)$ is a conformal map, and Q is a cube in some Whitney decomposition of Ω, then $f(\Omega)$ is covered by at M cubes in a Whitney decomposition of $f(\Omega)$, where M only depends on the constants in (3.1) and not on the conformal mapping (see [GM05], p. 21).

Whitney decompositions allow us to define for any compact $K \subset \mathbb{R}^d$ the *critical exponent*

$$\alpha = \alpha(K) = \inf\{\alpha : \sum |Q|^\alpha < \infty\}$$

where the sum is taken over all cubes in a Whitney decomposition within some bounded distance of K. The critical exponent is well-defined; it does not depend on which Whitney decomposition we choose for the complement of K. The key point is that given two Whitney decompositions of a domain Ω, and a cube inside one of the decompositions, it can be covered by a finite number of cubes in the other collection, and this finite number depends only on the constants in (3.1) and the dimension d. The following is Lemma 2.6.1 in [BP17], which relates the critical exponent of the complement of a set to its upper Minkowski dimension.

Lemma 3.1. *For any compact $K \subset \mathbb{R}^d$, $\alpha(K) \leq \overline{\operatorname{Mdim}}(K)$. If K has zero Lebesgue measure then we have equality.*

To summarize this section, for our application, we have

$$\operatorname{Hdim}(\mathcal{J}(f)) \leq \operatorname{Pdim}(\mathcal{J}(f)) = \overline{\operatorname{Mdim}}(\mathcal{J}(f)).$$

Therefore, to obtain Julia sets with packing dimension in $(1, 2)$, our approach will be to show that $\operatorname{Hdim}(\mathcal{J}(f)) > 1$ and $\overline{\operatorname{Mdim}}(\mathcal{J}(f)) < 2$, for which we will use the lemma above. A complete discussion of these various dimensions, including proofs, can be found in [BP17].

4. Quasiconformal and Polynomial Like Mappings

Douady and Hubbard introduced polynomial-like mappings in [DH85]. In this section we outline and review the properties we will need in this paper.

First, recall that a continuous mapping $f : U \to V$ between two topological spaces is called *proper* if the inverse image of every compact $K \subset V$ is compact. A *degree d polynomial-like map* is a triple (f, U, V), where $f : U \to V$ is a proper holomorphic mapping of degree d, and U and V are homeomorphic to disks with U is relatively compact in V. We define the *filled Julia set* of f by

$$K_f := \bigcap_{n \geq 0} f^{-n}(U).$$

This is precisely the set of points that remain in U for all iterates of f. The Julia set of f is defined to be the boundary ∂K_f, and we denote it by $\mathcal{J}_f$.

Many of the classical propositions in the dynamics of polynomials are true for polynomial-like mappings. For example, suppose that $B \subset K_f$ is the immediate basin of attraction for some attracting fixed point $p \in B$. The same proof in the case of rational functions that B contains a critical point of f applies equally well to the polynomial-like case. Hyperbolicity also makes sense in the setting of polynomial-like maps. We say $f : U \to V$ is *hyperbolic* if there exists a Riemannian metric on J_f and $a > 1$ so that if $z \in J_f$ we have $||D_z f(v)||_{f(z)} \geq a||v||_z$ where $v \in T_z J_f$ is a tangent vector. With the same proof as the case of rational mappings, this definition is equivalent to the definitions that $|(f^N)'| > 1$ on J for some N, or that every critical point is attracted to an attracting fixed point or cycle. In our applications, the polynomial-like map f will come as the restriction of an entire function, and it will be important that we distinguish between f being hyperbolic as a polynomial-like map, versus f being hyperbolic as a transcendental entire function (which our example cannot be, since it has an unbounded set of critical points. See [RGS17].)

The usefulness of polynomial-like mappings comes from the straightening lemma (Theorem 1, p. 296 in [DH85]), which gives a quasiconformal conjugacy between the polynomial-like map and some polynomial of the same degree. To lower bound the Hausdorff dimension of the Julia set of our examples, we will construct a quasiconformal conjugacy in a way similar to the straightening lemma. If U and V are planar domains, we call an orientation preserving homeomorphism $\varphi : U \to V$ *K-quasiconformal* if φ has locally square integrable distributional derivatives which satisfy

$$|\varphi_{\bar{z}}(z)| \leq k|\varphi_z(z)|$$

for all $z \in U$ for $k = (K-1)/(K+1) < 1$. Given a quasiconformal mapping, we define its *dilatation*

$$\mu(z) = \frac{\varphi_{\bar{z}}(z)}{\varphi_z(z)}.$$

The definition says that the dilatation is bounded above by some number less than 1. We will need the following fact about quasiconformal mappings. The proof can be found in Section IV.5.6 of [LV73].

Lemma 4.1 (The Good Approximation Lemma). *Suppose that $\varphi_n : U \to V$ is a sequence of quasiconformal mappings. Suppose that $\varphi_n \to \varphi$ uniformly on compact* *sets, and the corresponding dilatation μ_n of φ_n converges pointwise almost everywhere to some limit almost everywhere. Then φ is quasiconformal with dilatation $\mu = \lim \mu_n$.*

5. Defining the Function

In this section, we specify the parameters defining f and show that it is an entire function. Along the way we will prove some basic estimates regarding the parameters $\{R_k\}$ and n_k defining f that will be useful later in the paper.

Recall that the main cardioid of the Mandelbrot set is the region consisting of all parameters $c = \mu/2(1 - \mu/2)$, where $\mu \in \mathbb{D}$. If c is a parameter in the main cardioid, the Julia set of $z^2 + c$ is a quasicircle with an attracting fixed point in its interior. As mentioned in the introduction, the results of Shishikura and Sullivan imply that for each $s \in (1, 2)$, we may choose c in the main cardioid so that $\mathrm{Hdim}(\mathcal{J}(F_0)) = \mathrm{Pdim}(\mathcal{J}(F_0(z)) = s$ (see [Shi98] p.232 and [Sul83] p.742, along with Theorem 7.6.7 in [PU10]).

Having chosen such a c, recall that we defined $f_0(z) = z^2 + c$, and $f_0^N(z)$ denotes the Nth iterate of f_0. Since f_0^N is a degree 2^N polynomial there exists some $R > 0$ so that if $|z| \geq R$ we have

$$\frac{1}{2} \leq \left| \frac{f_0^N(z)}{z^{2^N}} \right| \leq 2. \tag{5.1}$$

We will always assume R is big enough so that (5.1) holds.

Next given some integer $N > 0$ define a sequence of integers for $k = 0, 1, 2 \ldots$

$$n_k := 2^{N+k-1}.$$

Note that when $k \neq 0$, $n_k \geq 2^N$ and for all k we have $2n_k = n_{k+1}$. Given the R above, define

$$R_1 = 2R.$$

We will construct our infinite product as a sequence of partial products inductively as follows. Given R as above we can define

$$F_1(z) := \left(1 - \frac{1}{2}\left(\frac{z}{R_1}\right)^{n_1}\right),$$

$$f_1(z) := f_0^N(z) \cdot F_1(z),$$

$$R_2 := M(f_1, 2R_1) = \max\{|f_1(z)| : |z| = 2R_1\}.$$

Next, assume that f_{k-1}, F_{k-1} and R_k have all been defined. From there, we define

$$F_k(z) := \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right),$$

$$f_k(z) := f_0^n(z) \prod_{j=1}^{k} F_j(z),$$

$$R_{k+1} := M(f_k, 2R_k) = \max\{|f_k(z)| : |z| = 2R_k\}.$$

With these starting parameters, we want to begin by looking at functions of the form

$$f(z) = \lim_{k\to\infty} f_k(z) = \lim_{k\to\infty} f_0^n(z) \prod_{j=1}^{k} F_j(z). \tag{5.2}$$

Viewing this as a formal infinite product, our first step will be to show that f is indeed an entire function on $\mathbb{C}$.

Lemma 5.1 (The Growth Rate of n_k)**.** *For all $k = 1, 2, \dots,$ we have*

(1) $n_k = 2n_{k-1}$ *for all k, and* $n_k \geq 2^N$ *for all* $k \geq 1$.
(2) $2^N + \sum_{j=1}^{k} n_j = n_{k+1}$.

Proof. The first claim immediate from the definitions. For the second claim we compute

$$\begin{aligned}\sum_{j=1}^{k} n_j &= (2^N + \cdots + 2^{N+k-1}) = 2^{N-1}(2 + \cdots + 2^k) \\ &= 2^{N-1}(2^{k+1} - 2) = 2^{N+k} - 2^N \\ &= n_{k+1} - 2^N.\end{aligned}$$

□

Corollary 5.2. *For all $k \geq 1$,* $\deg(f_k) = 2\deg(F_k)$

Proof. Apply Lemma 5.1 to the equality $\deg(f_k) = 2^N + \sum_{j=1}^{k} n_j$. □

We can deduce the following growth rate estimates for R_k.

Lemma 5.3 (The Growth Rate of R_k)**.** *If $k \geq 1$ then*

$$R_{k+1} \geq 2^{n_k} R_k^{2^N + n_{k-1}} \geq 2^N R_k^{2^N}.$$

Proof. If R is big enough so that (5.1) holds then

$$\begin{aligned} R_2 &:= \max_{|z|=2R_1} |f_0^N(z)| \cdot \left| \left(1 - \frac{1}{2}\frac{z^{n_1}}{R_1^{n_1}}\right) \right| \\ &\geq \frac{1}{2}|2R_1|^{2^N} \cdot \max_{|z|=2R_1} \left| \left(1 - \frac{1}{2}\frac{z^{n_1}}{R_1^{n_1}}\right) \right| \\ &\geq 2^{2^N - 1} R_1^{2^N} \cdot (2^{n_1 - 1} - 1) \\ &\geq 2^{2^N} R_1^{2^N} \geq 4R_1^2. \end{aligned}$$

This is the base case for an induction. Suppose that for some k, we have

$$R_j \geq 2^{n_{j-1}} R_j^{2^N + n_{j-1}} \geq 2^{2^N} R_j^{2^N} \geq 4R_j^2$$

for all $j \leq k$. This induction hypothesis implies that $R_k \geq R_j^2$ for all $j \leq k-1$. Therefore,

$$\begin{aligned}
R_{k+1} \quad &:= \quad \max_{|z|=2R_k} |f_0^N(z)| \cdot \prod_{j=1}^{k} \left| \left(1 - \frac{1}{2}\frac{z^{n_j}}{R_j^{n_j}}\right) \right| \\
&\geq \quad 2^{2^N-1} R_k^{2^N} \prod_{j=1}^{k} \left| 2^{n_j-1} \frac{R_k^{n_j}}{R_j^{n_j}} - 1 \right| \\
&\geq \quad 2^{2^N-1} R_k^{2^N} (2^{n_k-1} - 1) \prod_{j=1}^{k-1} \left| 2^{n_j-1} R_k^{n_{j-1}} - 1 \right| \\
&\geq \quad 2^{2^N - 2(k+1) + \sum_{j=1}^{k} n_j} R_k^{2^N + \sum_{j=1}^{k-1} n_{j-1}} \\
&\geq \quad 2^{2^N - 2(k+1) + n_{k+1}} R_k^{2^N + n_{k-1}} \\
&\geq \quad 2^{2^N + n_k} R_k^{2^N + n_{k-1}}.
\end{aligned}$$

□

The lemma above also contains the following simpler inequalities that will often be sufficient for our purposes. We note them below.

Corollary 5.4 (Other Useful Inequalities)**.** *For $k \geq 1$ we have*

$$R_{k+1} \geq 4R_k^2.$$

For $k \geq 1$ we have:

$$R_{k+1} \geq (2R)^{2^{kN}}.$$

Proof. The first inequality is contained in the proof of Lemma 5.3. The second inequality follows from Lemma 5.3 by induction and the fact that $R_1 = 2R$. Indeed, the base case is clear by definition, and for the larger cases we have

$$R_{k+1} \geq (R_k)^{2^N} \geq (2R)^{2^N \cdot 2^{N(k-1)}} \geq (2R)^{2^{N(k-1)+N}}.$$

□

Now we show that the infinite product we are interested in converges on $\mathbb{C}$.

Corollary 5.5. *The infinite product*

$$f(z) = \lim_{k \to \infty} f_k(z) = \prod_{k=0}^{\infty} F_k(z)$$

converges uniformly on compact subsets of $\mathbb{C}$. In particular, $f(z)$ is a transcendental entire function.

Proof. To check that the infinite product converges, we check that the associated sum

$$\sum_{k=0}^{\infty} |1 - F_k(z)|$$

converges uniformly on compact sets. It suffices to show that f converges uniformly on every closed ball $\{|z| \leq s\}$. If we choose j so that $R_j > 2s$, then we know that for all $k > j$

$$|1 - F_k(z)| = \frac{1}{2}\left|\frac{z}{R_k}\right|^{n_k} \leq \frac{s^{n_k}}{R_k^{n_k}} = O(2^{-n_k}) = O(2^{-2^k}).$$

In this way, the series above is summable, so the series converges uniformly on $\{|z| \leq s\}$. So f defines an entire function, and it is certainly not a polynomial. □

We conclude this section by recording two useful lemmas regarding the growth rate of R. The first lemma will help us study f near $|z| = R_k$ in section 7. The proof follows from Theorem 5.3, and we refer the reader to sections 6 and 8 of [Bis17] for the details of the proof.

Lemma 5.6. *Suppose that $\{R_k\}$ has been defined as in this section, and $m \geq 1$. Then*

$$\prod_{j=1}^{k-1}\left(1 + \left(\frac{R_j}{R_k}\right)^m\right) = 1 + O(R_k^{-1/2}). \tag{5.3}$$

$$\prod_{j=k+1}^{\infty}\left(1 + \frac{R_k}{R_j}\right) = 1 + O(R_k^{-1}). \tag{5.4}$$

Finally, if $|z| \leq 4R_k$, we have

$$\prod_{j=k+1}^{\infty} F_j(z) = 1 + O(R_k^{-1}). \tag{5.5}$$

This second lemma is used in the proof of Lemma 11.1

Lemma 5.7. *Let $N_k = n_1 \cdots n_k$, and let $\alpha > 0$. Then for any $R > 1$,*

$$\sum_{k=1}^{\infty} 2^k N_k R_k^{-\alpha} < \infty$$

Moreover, by choosing R sufficiently large, the sum can be made arbitrarily small.

6. The Hausdorff Dimension Changes by a Small Amount

Recall that we denote $f_0(z) = z^2 + c$. Since the Julia sets of f_0 and f_0^N are the same, they have the same Hausdorff dimension. We can also view the function f as f_0^N perturbed by the infinite product we constructed in the previous section. The parameters R have been chosen so that in a neighborhood around the origin the infinite product defining f is uniformly close to 1. To emphasize this, we write

$$f(z) = f_0^N(z) \cdot (1 + \epsilon(z))$$

where $1 + \epsilon(z)$ coverges locally uniformly to the constant function 1 as $R \to \infty$.

In this section, we will view f and f_0^N as polynomial-like mappings. The filled Julia set K_f, and hence the Julia set J_f of the polynomial-like mapping f will be a subset of the Julia set of f viewed as an entire function. Since f_0^N is a polynomial, the filled Julia set $K_{f_0^N}$ and the Julia set $J_{f_0^N}$ coincide with its usual Julia set and filled Julia set. We will denote the basin of attraction of f_0^N by $B_{f_0^N}$, so that

$K_{f_0^N} = J_{f_0^N} \cup B_{f_0^N}$. We use similar notation for f. Recall that we chose c so that $\mathrm{Hdim}(J_{f_0^N}) = s$. Our goal is to show that since f and f_0^N differ only by a perturbation close to the identity, $\mathrm{Hdim}(J_f) = t > 1$ where $|s - t|$ is as small as we would like. It will follow that the Julia set of the entire function f, $\mathcal{J}(f)$, is at least t, since it will contain J_f.

The following lemma is obvious.

Lemma 6.1. *For sufficiently large R, there exists r and V relatively compact in $B(0, 1/4R)$ so that*

$$f : B(0, r) \to V$$

is a degree 2^N polynomial-like mapping.

Next observe that

$$|f'(z)| \geq |(f_0^N)'(z)|(1 + \epsilon(z))| - |\epsilon'(z)||f_0^N(z)|.$$

Note that on $B(0, r)$, $|f_0^N(z)|$ is bounded and $\epsilon(z) + 1$ can be chosen as close to 1 as we like. By choosing r and R large enough and applying the Cauchy estimates, $|\epsilon'(z)|$ may be chosen as close to zero as we would like. Hence we may choose R and r so that $\sup_{B(0,r)} |f'(z) - (f_0^N)'(z)|$ is as small as we would like. We are now ready to state the main theorem of this section.

Theorem 6.2. *Let $s = \mathrm{Hdim}(\mathcal{J}(f_0))$ and $\epsilon > 0$ be given. Then R and r may be chosen so that the Julia set of the polynomial-like mapping f is a quasicircle such that $|\mathrm{Hdim}(J_f) - s| < \epsilon$.*

Proof. f_0^N is hyperbolic. It follows that there exists some topological annulus A containing the Julia set of f_0^N so that $|(f_0^N)'(z)| \geq \alpha > 1$ on A. By our remark above, we may arrange for $|f'(z)| \geq \alpha' > 1$ on A as well. Since the dynamics are expanding, A is relatively compact in $f_0(A)$ and $f(A)$, which are also topological annuli containing A.

Since f is uniformly close to f_0, the boundary of $f(A)$ and $f_0(A)$ are close to each other in the following sense. If $L(\theta)$ is a ray through the origin with angle θ, then the angles formed by the intersection of $L(\theta)$ and $f(A)$ and $f_0^N(A)$ are close. It follows that there exists a quasiconformal mapping

$$\varphi_0 : \overline{f_0(A) \setminus A} \to \overline{f(A) \setminus A}.$$

Moreover, φ_0 can be chosen to have the following properties:

(1) $\varphi_0 = \mathrm{id}$ on ∂A.
(2) φ_0 is $(1 + \delta)$-quasiconformal, where $\delta \to 0$ and $R \to \infty$.
(3) $f(\varphi_0(z)) = \varphi_0(f_0^N(z))$ on ∂A.

We now set up the following notation. We call $A_n = f_0^{-n}(A)$, and we call $B_n = f^{-n}(A)$. Thus A_n and B_n each form nested sequences of topological annuli.

Next, notice that $U_1 = A \setminus A_1$ and $V_1 = A \setminus B_1$ are each the disjoint union of topological annuli. We call the components the outer and inner annuli, corresponding to which component has larger diameter. We denote these annuli by U_o^1 and V_o^1 for the outer annuli, and U_i^1 and V_i^1 for the inner annuli. We can continue this procedure inductively, obtaining a sequence annuli $U_n = A_n \setminus A_{n+1} = U_o^n \cup U_i^n$ and $V_n = B_n \setminus B_{n+1} = V_o^n \cup V_i^n$. These annuli have the property that the outer boundary of U_o^n coincides with the inner boundary of U_o^{n-1} for all n, and the inner annuli have the property that the outer boundary of U_i^n coincides with the inner boundary of U_i^{n+1}. Similar statements are true for V_o^n and V_i^n. Our goal is, for each n, to construct a quasiconformal $\varphi_n : f_0(A) \to f(A)$ satisfying

(1) φ_n is conformal from A_n to B_n,
(2) $f(\varphi_n(z)) = \varphi_n(f_0^N(z))$ on the components U_i^k, U_0^k, V_i^k, V_0^k for $k = 1, \dots, n$.
(3) $\varphi_{n-1} = \varphi_n$ on the components U_i^k, U_0^k, V_i^k, V_0^k for $k = 1, \dots, n-1$.
(4) φ_n is $(1+\delta)$-quasiconformal, with the same δ as above.

Let us first construct φ_1. Note that $\varphi_0 \circ f_0^N : U_1 \to f(A) \backslash A$ and $f : V_1 \to f(A) \backslash A$ are each 2^N to 1 continuous mappings, hence there exists a lift $\varphi_1 : U_1 \to V_1$ that satisfies

$$\varphi_0 \circ f_0^N = f \circ \varphi_1.$$

Since f and f_0^N are are 2^N to 1 and locally conformal mappings, φ_1 is $(1+\delta)$-quasiconformal. Next, since $\mathrm{id} \circ f_0^N : A_1 \to A$ and $f : B_1 \to A$ are 2^N to 1, there is a lift $\Phi : A_1 \to B_1$ so that

$$\mathrm{id} \circ f_0^N = f \circ \Phi.$$

Note that since each of f and f_0^N are locally conformal and each $2^N - 1$, Φ must be a conformal mapping. We define

$$\varphi_1 = \begin{cases} \varphi_0(z) & z \in f_0(A) \setminus A, \\ \varphi_1(z) & z \in U_1, \\ \Phi(z) & z \in A_1. \end{cases}$$

We should check that φ_1 is continuous. Indeed, if z is on the boundary of U_1 and $f_0(A) \setminus A$ then $f(\varphi_1(z)) = \varphi_0(f_0(z)) = f(\varphi_0(z))$ by the construction of φ_0. So we choose the lift φ_1 (there are 2^N possible choices) so that $\varphi_1 = \varphi_0$ on this common boundary. If z is on the boundary of U_1 and A_1, similarly we argue that

$$f(\Phi(z)) = f_0(z) = \varphi_0(f_0(z)) = f(\varphi_1(z)).$$

So we choose the lift so that $\Phi(z) = \varphi_1(z)$ on their common boundary of definition.

We may continue this procedure inductively, obtaining lifts $\varphi_n : U_n \to V_n$ and $\Phi_n : A_n \to B_n$ that satisfy

$$f(\varphi_n(z)) = \varphi_{n-1}(f_0(z)) \quad \text{on } U_n\text{, and,}$$
$$f(\Phi_n(z)) = \Phi_{n-1}(f_0(z)) \quad \text{on } A_n.$$

Just as before, we may choose the lifts so that these maps agree on their common boundaries. Φ_n is conformal on A_n and φ_n is $(1+\delta)$-quasiconformal on U_n. We define

$$\varphi_n = \begin{cases} \varphi_{n-1}(z) & z \in (f_0(A) \setminus A) \cup \bigcup_{i=1}^{n-1} U_i, \\ \varphi_n(z) & z \in U_n, \\ \Phi_n(z) & z \in A_n. \end{cases}$$

By construction, φ_n evidently satisfies the four properties outlined above.

The result is a sequence of $(1+\delta)$-quasiconformal mappings $\varphi_n : f_0(A) \to f(A)$. The dilatation μ_n converges almost everywhere, since it is eventually constant on each U_n. The φ_n family converges uniformly on compact sets as well. Indeed, since φ_n conjugates the dynamics, it sends the fixed points of f to fixed points of f_0^N. By perhaps taking a subsequence, since there are only finitely many repelling fixed points for f_0^N, we can assume that $\varphi_n(z) = w$ for some repelling fixed point z of f_0^N and w of f for all n. The sequence $\{\varphi_n\}$ is therefore sequentially compact, every sequence has a convergent subsequence. We obtain a limit $\varphi(z)$ which conjugates the dynamics and maps the Julia set of the polynomial f_0^N onto the Julia set of the polynomial-like map f. By the Lemma 4.1, the limit φ is $(1+\delta)$-quasiconformal. Since φ is α-Hölder with α close to one, the dimension of J_f changes only by a small amount (see [Ahl66], p.30). □

The following corollaries are immediate from the above proof.

Corollary 6.3. *The Julia set of the polynomial-like mapping f is a quasicircle with dimension in $(1,2)$ arbitrarily close to $\dim(\mathcal{J}(z^2+c))$.*

Proof. The Julia set of f is the quasiconformal image of a quasicircle. The rest is just a restatement of Theorem 6.2. □

Corollary 6.4. *f is a hyperbolic polynomial-like mapping.*

Proof. We showed f was expanding on $\mathcal{J}(f)$ in the proof above. □

To conclude this section, we review some notation we will use for the rest of the paper. We now know $f : B(0,r) \to V$ viewed as a polynomial-like mapping has a quasicircle Julia set and one attracting basin. We will let K_f denote the filled Julia set of the polynomial-like mapping f, and write $K_f = J_f \cup B_f$, where J_f is the Julia set and B_f is the attracting basin.

7. The Local Behavior of f

We now move on to analyzing the function f away from the origin. The purpose of this section is to show that f behaves like simpler functions (the functions F_j defined in Section 5 and simple power functins) on suitably defined regions of $\mathbb{C}$. Recall that

$$f(z) := f_0^N(z) \cdot \prod_{j=1}^{\infty} F_j(z).$$

To be more specific, we will show, quantitatively, that we can decompose $\mathbb{C} \setminus B(0, R_1/4)$ into regions where f looks approximately like F_j. The observations and estimates here are vital for understanding to precise dynamical behavior of f.

We define

$$H_m(z) = z^m(2 - z^m).$$

A description of the conformal mapping behavior of H_m can be found in Section 9 of [Bis17]. For our purposes, we will need to consider the components of $\mathbb{C}\setminus\{|H_m(z)| = 1\}$. This set has $m+2$ connected components, one unbounded, one containing the origin, and m petals which we denote by Ω_m^p. $H_m : \Omega_m^p \to \mathbb{D}$ is a conformal mapping, and $\operatorname{diam}(\Omega_m^p) \lesssim 1/m$.

Next, we decompose $\mathbb{C} \setminus B(0, R_1/4)$ into annuli as follows.

$$A_k := \left\{ z : \frac{1}{4} R_k \leq |z| \leq 4R_k \right\}, \qquad B_k := \left\{ z : 4R_k \leq |z| \leq \frac{1}{4} R_{k+1} \right\},$$

$$V_k := \left\{ z : \frac{3}{2} R_k \leq |z| \leq \frac{5}{2} R_k \right\}, \qquad U_k := \left\{ z : \frac{5}{4} R_k \leq |z| \leq 3R_k \right\}.$$

Note that V_k is compactly contained inside of U_k.

Lemma 7.1. *With H_m defined above, we have*

$$F_k(z) = \frac{1}{2} \left(\frac{R_k}{z} \right)^{n_k} H_{n_k}\left(\frac{z}{R_k} \right).$$

Proof. This is a simple computation:

$$\begin{aligned}
\frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} H_{n_k}\left(\frac{z}{R_k}\right) &= \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k}\left(\frac{z}{R_k}\right)^{n_k}\left(2-\left(\frac{z}{R_k}\right)^{n_k}\right) \\
&= \frac{1}{2}\left(2-\left(\frac{z}{R_k}\right)^{n_k}\right) \\
&= F_k(z).
\end{aligned}$$

□

Lemma 7.2. *If $z \in A_k$, there is a constant C_k so that*

$$f(z) = C_k H_{n_k}\left(\frac{z}{R_k}\right)(1+O(R_k^{-1})).$$

For $k \geq 2$, the constant C_k is given by the formula

$$C_k = (-1)^{k-1}2^{-k)} R_k^{n_k}\prod_{j=1}^{k-1} R_j^{-n_j}.$$

For $k = 1$ the constant is given by

$$C_1 = \frac{1}{2}R_1^{n_1}.$$

Proof. Write

$$f(z) = f_0^N(z)\cdot\left(\prod_{j=1}^{k-1}F_j(z)\right)\cdot F_k(z)\cdot\left(\prod_{j=k+1}^{\infty}F_j(z)\right)$$

The tail end of the infinite product is easy to estimate; it satisfies the hypotheses of Lemma 5.3, so we obtain

$$\prod_{j=k+1}^{\infty}F_j(z) = 1+O(R_k^{-1}).$$

Next we use Lemma part (2) of 5.1 to compute

$$\begin{aligned}
f_0^N(z)\cdot\left(\prod_{j=1}^{k-1}F_j(z)\right)\cdot F_k(z) &= \left(z^{-2^N}f_0^N(z)\right)\left(\prod_{j=1}^{k-1}z^{-n_j}F_j(z)\right)(z^{n_k}F_k(z)) \\
&= I\cdot II\cdot III.
\end{aligned}$$

We show how to estimate each term. Note first that $z \in A_k$, $|z| > R$, so that by (5.1),

$$I = z^{-2^N}f_0^N(z) = 1+O(R_k^{-1}).$$

Similarly, we can check that for $j = 1, \ldots, k-1$ that

$$z^{-n_j}F_j(z) = z^{-n_j}\left(1-\frac{1}{2}\left(\frac{z}{R_k}\right)^{n_j}\right) = \frac{-1}{2}R_j^{-n_j}\left(1-2\left(\frac{R_j}{z}\right)^{n_j}\right).$$

Since $z \in A_k$, $|z|$ is comparable to R_k. Therefore

$$z^{-n_j}F_j(z) = \frac{-1}{2}R_j^{-n_j}\left(1+O\left(\left(\frac{R_j}{R_k}\right)^{n_j}\right)\right).$$

Finally, observe that since $R > 4$ we have $n_k = 2^{N+k-1} \leq 2^{N-1} R^{2^k/2} \leq R_k^{1/2}$. It follows from Lemma 5.3 that when $z \in A_k$ we have

$$\begin{aligned} II &= \prod_{j=1}^{k-1} \frac{-1}{2} R_j^{-n_j} \left(1 - 2\left(\frac{R_j}{R_k}\right)^{n_j}\right) \\ &= \left(\prod_{j=1}^{k-1} \frac{-1}{2} R_j^{-n_j} (1 + O(R_k^{-1}))\right) \\ &= \left(\prod_{j=1}^{k-1} \frac{-1}{2} R_j^{-n_j}\right) (1 + O(R_k^{-1})). \end{aligned}$$

Finally, by Lemma 7.1 we have

$$III = z^{n_k} F_k(z) = \frac{1}{2} R_k^{n_k} H_{n_k}\left(\frac{z}{R_k}\right).$$

Combining all these observations yields the desired formula, along with the formulas for C_k. □

Lemma 7.3. *Let $R > 8$, $N > 3$, and $k \geq 2$. Then $|C_k| \geq 1/2^k |R_k|^{n_{k-1}} \geq 8|R_k|$. When $k = 1$, then $|C_1| = 1/2|R_1|^{n_1} \geq 8|R_1|$.*

Proof. The $k = 1$ case is obvious. For $k \geq 2$, we can compute, using the fact that $|R_j| \leq \sqrt{|R_k|}$, that

$$\begin{aligned} |C_k| &= \frac{1}{2^k} |R_k|^{n_k} \prod_{j=1}^{k-1} |R_j|^{-n_j} \geq \frac{1}{2^k} |R_k|^{n_k} \prod_{j=1}^{k-1} |R_k|^{-n_j/2} \\ &= \frac{1}{2^k} |R_k|^{n_k - n_{k1}} = \frac{1}{2^k} |R_k|^{n_{k-1}}. \end{aligned}$$

So in this case we see that

$$|C_k| \geq \frac{1}{2^k} |R_k|^{n_{k-1}} > 8|R_k|.$$

□

Lemma 7.4. *For all k, $|C_{k+1}| \geq |C_k| \geq 1$.*

Proof. We compute using the fact that $|R_{k+1}| \geq 2|R_k|$

$$\frac{|C_{k+1}|}{|C_k|} = \frac{1}{2} \frac{|R_{k+1}|^{n_{k+1}}}{|R_k|^{n_{k+1}}} \geq 2^{n_{k+1}-1} \geq 1.$$

□

The next lemma says that f looks like a power function on B_k.

Lemma 7.5. *For $z \in B_k$, we have*

$$f(z) = -C_k \left(\frac{z}{R_k}\right)^{2n_k} (1 + O(R_k^{-1})) \cdot (1 + O(4^{-n_{k+1}})) \cdot (1 + O(4^{-n_k})).$$

Proof. Following the proof of Lemma 7.2, we have

$$f(z) = C_k H_{n_k}\left(\frac{z}{R_k}\right) F_{k+1}(z)(1 + O(R_k^{-1})).$$

We have kept the F_{k+1} term, since $|z| \geq R_k$. However, $|z| \in B_k$, $4|R_k| \leq |z| \leq 1/4|R_{k+1}|$, so that,

$$H_{n_k}\left(\frac{z}{R_k}\right) = \left(\frac{z}{R_k}\right)^{n_k}\left(2 - \left(\frac{z}{R_k}\right)^{n_k}\right) \tag{7.1}$$

$$= \left(\frac{z}{R_k}\right)^{2n_k}\left(2\left(\frac{R_k}{z}\right)^{n_k} - 1\right) \tag{7.2}$$

$$= -\left(\frac{z}{R_k}\right)^{2n_k}\left(1 + O(4^{-n_k})\right). \tag{7.3}$$

A similar computation yields

$$F_{k+1}(z) = \left(1 - \frac{1}{2}\left(\frac{z}{R_{k+1}}\right)^{n_{k+1}}\right) = (1 + O(4^{-n_{k+1}})).$$

□

The following useful corollary is immediate.

Corollary 7.6. *f is never zero on B_k.*

The next proofs follow similar lines of reasoning, and are identical to the proofs in Section 10 of [Bis17]. We record them below.

Lemma 7.7. *For all $k \geq 1$, and for z satisfying $5/4|R_k| \leq |z| \leq 4|R_k|$, we have*

$$f(z) = C_k\left(\frac{z}{R_k}\right)^{2n_k}\left(1 + O\left(\left(\frac{4}{5}\right)^{n_k}\right)\right)(1 + O(R_k^{-1})).$$

Lemma 7.8. *For $k \geq 1$, and $\frac{1}{4}|R_k| \leq |z| \leq \frac{4}{5}|R_k|$, we have*

$$f(z) = 2C_k\left(\frac{z}{R_k}\right)^{n_k} \cdot \left(1 + O\left(\frac{4}{5}\right)^{n_k}\right)(1 + O(R_k^{-1})).$$

Lemma 7.9. *On U_k, we have*

$$f(z) = C_k\left(\frac{z}{R_k}\right)^{2n_k}(1 + h_k(z)).$$

Where $h_k(z)$ is holomorphic on U_k with

$$|h_k(z)| = O\left(\left(\frac{4}{5}\right)^{n_k} + |R_k^{-1}|\right).$$

Corollary 7.10. *$f'(z)$ is non-zero on V_k.*

8. The Mapping Behavior on Annuli

We record the following useful corollary from the results of the previous section.

Corollary 8.1. *For all $k \geq 1$ we have*

$$\frac{1}{8} \leq \frac{|R_{k+1}|}{|C_k| \cdot 2^{2n_k}} \leq 8.$$

Proof. By the first part of the proof of Lemma 7.2 we have for $z \in A_k$ that

$$f(z) = f_k(z)(1 + O(R_k^{-1}).$$

Recall that f_k was the kth partial product of the infinite product defining f. If we further assume $|z| = 2R_k$, Lemma 7.7 applies and

$$\max_{|z|=2R_k} |f_k(z)| = \max_{|z|=2|R_k|} |C_k| \cdot \left(\frac{2|R_k|}{|R_k|}\right)^{2n_k} |(1 + O(4/5)^{n_k})| \cdot |(1 + O(R_k^{-1}))|.$$

By Lemma 2.6, and perhaps choosing N and R larger,

$$\frac{1}{8}|R_{k+1}| \leq |C_k| 2^{2n_k} \leq 8|R_{k+1}|.$$

□

With Corollary 8.1 and the estimates of the previous section, the following estimates follow in the exact same way they do in Sections 11 and 12 of [Bis17].

Theorem 8.2. *For sufficiently large N and R, we have that $A_{k+1} \subset f(A_k)$ and $f(B_k) \subset B_{k+1}$.*

Corollary 8.3. *Each set B_k is in the Fatou set of f.*

Proof. Since B_k maps into B_{k+1} we know that the iterates tend to infinity uniformly. □

We conclude this section by showing that the inner and outer boundary components of the Fatou components which surround the origin are C^1. We will see in the next section this easily implies that all the boundary Fatou components of all the Fatou components are C^1-smooth. The proof of the following lemma can be found in section 18 of [Bis17].

Lemma 8.4. *Suppose h is a holomorphic function on $A = \{z : 1 < |z| < 4\}$ and suppose that $|h|$ is bounded by ϵ on A. Let $H(z) = z^m(1 + h(z))$. For any fixed θ the segment $S(\theta) = \{re^{i\theta} : 3/2 \leq r \leq 5/2\}$ is mapped by H to a curve that makes angle at most $O(\epsilon/m)$ with any radial ray it meets.*

Recall that the annulus $V_k \subset A_k$ gets mapped into A_{k+1} by f. It follows from the lemma that $W = f^{-1}(V_{k+1}) \subset V_k$ is a topological annulus, and the boundary components of W are smooth curves that are at most ϵ_k away from being round circles. From here, with the additional help of Lemma 7.7, we can also deduce that the width of W is approximately $R_k/2n_k$.

Theorem 8.5. *The inner and outer boundary components of the Fatou components surrounding the origin are C^1 smooth.*

Proof. Fix some $k \geq 1$ and define

$$\Gamma_{k,n} = \left\{ z \in A_k : f^j(z) \in A_{k+j}, j = 1, \ldots, n \right\}.$$

Since A_{k+n} is a round annulus, it has a natural foliation of closed circles. $\Gamma_{k,n}$ has an induced foliation of closed analytic curves by pulling back these circles in A_{k+n} via f.

Let $L = L(\theta)$ be a ray through the origin. Then the curves in $\Gamma_{k,n}$ and $\Gamma_{k,n+1}$ intersect this ray with some angle which depends on the particular curve we choose in each family. We let ϕ_n and ϕ_{n+1} denote the supremum of these angles. By the lemma above, we have

$$|\phi_n - \phi_{n+1}| = O(\epsilon_n).$$

By our observation before the proof, the topological annuli $\Gamma_{k,n}$ contract uniformly. Therefore the intersection

$$\bigcap_{n=1}^{\infty} \Gamma_{k,n} = \Gamma_k$$

is some closed set with no interior. We claim Γ is actually C^1 smooth. It is sufficient to show that for each $z \in \Gamma$, there is a vector v_z based at z tangent to Γ. This follows from the summability of the sequence $\{\epsilon_k\}$. The limit Γ_k makes at most angle $O(\sum \epsilon_n)$ with the circular arcs that foliate $V_k \subset A_k$. □

9. Partitioning the Julia And Fatou Set

In this section, we describe a system for cataloging the various components of the Julia set and the Fatou set, along with a description of each type of component. We will also discuss the conformal mapping properties of f, which will be vital to our analysis in the upcoming sections.

Consider the trajectory of a point z with $|z| < 1/4R_1$, assuming that $z \notin K_f$, the filled Julia set of the polynomial-like mapping f. There exists a natural number N so that $|f^N(z)| > R_1$, so z is iterated into some annulus A_1 or B_1. If z is iterated into B_1, the point stays in future B_k's for all subsequent iterates. It is also possible that z is iterated into A_1 but is still iterated into B_k for some k. As discussed earlier, such a z is always in the Fatou set.

The more interesting case is when the orbit of z is disjoint from B_k for all k. There are two subcases to consider. For the first case, define

$$C = \cup_{n=0}^{\infty} f^{-n}(K_f).$$

C is the set of all points that eventually map into the filled Julia set of the polynomial-like mapping f. If z belongs to C, the orbit of z is eventually contained in K_f. If z is in the Julia set, the orbit will eventually be contained in J_f, and if z is in the Fatou set, the orbit will eventually be contained inside of the attracting basin B_f.

The other case is that $z \notin C$. It may not be true that the orbit of z is eventually contained inside of the A_k's for $k \geq 1$. Indeed, the zeros of f are contained in the annular regions A_k, so it is possible (and often happens!) that points can be iterated from A_k to A_j for $j < k$. It is possible that points land back in the region between the inner boundary of A_1 and K_f. It will be useful to label this region by pulling back the annulus A_1. First define

$$A_0 = \{|z| \leq 1/4R_1 : f(z) \in A_1\}.$$

$$A_{-k} = \{|z| \le 1/4R_1 \,:\, f^N(z) \in |z| \le 1/4R_1,\, N = 1, 2, \dots, k-1,\, f^k(z) \in A_1\}.$$

After $k+1$ steps, f maps A_{-k} into A_1. Therefore, we may describe this final case of orbits as the set of all z whose orbit is contained in

$$A := \bigcup_{k \in \mathbb{Z}} (A_k \setminus C).$$

From the definition of A, it is apparent that $f^{-1}(A) = A$.

The next few lemmas describe the dynamical behavior of points in A.

Lemma 9.1. *Let W be a connected component of $f^{-1}(A_k)$ for $k \in \mathbb{Z}$. Then we have the following possibilities.*

1. *$W \subset A_{k-1}$.*
2. *$W \subset A_j$ for $j \ge k$. All possible j occur.*

Proof. By Theorem 8.2, we know that case 1 can occur. If $z \in A_j$ for $j \le k-2$, then $|f(z)| \le 1/4R_k$, since $f(B_j) \subset B_{j+1}$ for all j. The fact that all possible $j \ge 1$ occurs follows from the fact that all the zeros of f are in the annuli A_j for $j \ge 1$, and the fact that f is continuous. □

The cases above are very different. In the first case listed in the lemma, $f|_W : W \to A_k$ is a $2n_k$ to 1 covering map. This follows from Lemma 7.2. If $W \subset A_j$ for $j > k-1$, then $f|_W : W \to A_k$ is a conformal mapping. To prove this, we need to show W doesn't contain any critical points. This follows from the following series of observations. First recall that $\Omega^p_{n_k}$ is a "petal" region where $H_{n_k}(z) = z^{n_k}(2 - z^{n_k})$ restricts to be a conformal mapping onto the disk. The following is Lemma 13.1 in [Bis17]:

Lemma 9.2. *We have $(\mathcal{J}(f) \cap A_k) \subset V_k \cup (R_k \cdot \Omega^p_{n_k})$ for all k.*

The next important lemma says that the diameter of portion of the Julia set inside the petal $R_k \cdot \Omega^p_{n_k}$ is much smaller than the diameter of the petal. This lemma is deduced on page 35 of [Bis17] as a consequence of the proof of the above lemma.

Lemma 9.3. *The diameter of the portion of the Julia set contained in the petal $R_k \cdot \Omega^p_{n_k}$ is bounded by $R_k^{-2}/2^{n_k}$*

The final lemma characterizes the dynamics of the critical points of f. Recall that B_f is the basin of attraction for the polynomial-like mapping f. The following is Lemma 14.2 in [Bis17].

Lemma 9.4. *All critical points of f are either contained in B_f or A_k for some k. If z is a critical point contained in A_k, then $f(z) \in B_k$ and the distance from z to the Julia set is at worst comparable to R_k/m_k. In both cases, z is in the Fatou set.*

We now know enough to show that if W is a component of $f^{-1}(A_k)$ inside of A_j for $j \ge k$, then $f : W \to A_k$ is a conformal mapping with uniformly bounded conformal distortion. We make this sentence precise. First recall the following consequence of the Koebe distortion theorem

Lemma 9.5. *Fix $r < 1$ and let $B = B(0, r) \subset \mathbb{D}$ be an open ball and let $K \subset B$ be a compact set. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is conformal. Then there exists a constant $C = C_r$, independent of f, so that*

$$\frac{1}{C_r}\frac{\operatorname{diam}(f(K))}{\operatorname{diam}(f(B))} \le \frac{\operatorname{diam}(K)}{\operatorname{diam}(B)} \le C_r \frac{\operatorname{diam}(f(K))}{\operatorname{diam}(f(B))}.$$

It is easy to deduce this form the the Koebe distortion theorem; for example, one can use [GM05] Theorem 4.5 p. 22. We stress that the constant C_r is independent of f and only depends on the conformal modulus of the annulus $\mathbb{D} \setminus B$. As r tends to 1, more distortion is possible.

It is easy to see that we need not apply Lemma 9.5 to the unit disk $\mathbb{D}$. Indeed, we may replace $\mathbb{D}$ by any simply connected domain and B by any simply connected domain compactly contained inside of it. Then the constants of the theorem depend only on the conformal modulus of the annulus between these two domains. When we say bounded conformal distortion, we mean a bound on the conformal modulus, and hence a bound on the constants in Lemma 9.5.

Lemma 9.6. *The only components W of $f^{-1}(A_k)$ for that are in A_j for $k \leq j$ are in the petals $R_j \cdot \Omega^p_{n_j}$. Moreover, there exists a ball B so that W is compactly contained in $\frac{1}{2}B$ and so that $f|_{2B}$ is conformal.*

In other words, f maps W to A_k with uniformly bounded conformal distortion.

Proof. A proof of the first sentence is found in Lemma 16.2 of [Bis17]. To prove the second fact, recall that any critical point $z \in A_k$ has distance approximately R_k/n_k from the Julia set of f. W contains the Julia set inside of the petal $R_j \cdot \Omega^p_{n_j}$ and has diameter at most approximately R_j^{-2}/n_j. Therefore, there exists a ball B of unit size so that $2B$ does not contain any critical points, and $1/2B$ contains W. □

Note that the proof above can easily be modified so that if R is initially chosen sufficiently large, we can choose B to make the conformal distortion as close to 1 as we would like. In this case, we just choose B to be a ball of twice the diameter of W.

Now we turn to a systematic labeling of the Fatou components of f. for $k \geq 1$, define Ω_k to be the Fatou component containing B_{k-1}. Here, we interpret $B_0 = f^{-1}(B_1)$. For $k \leq 0$, define $\Omega_k = (f^{-k+1})^{-1}(\Omega_1)$. Fix Ω_j. By Lemma 9.6, there are copies of Ω_k's with small conformal distortion inside of the holes in Ω_j. To be precise, there exists Fatou components Ω in the holes of Ω_j so that Ω eventually iterates onto some Ω_j and f has bounded conformal distortion. Since all these components inside of Ω_j map conformally onto some other Ω_k, this picture is repeated when we zoom into a components of Ω_k-type, again with uniform bounded distortion. The following definition catalogs precisely which Ω_j that Ω maps conformally onto

Definition 9.7. We call a Fatou component ω of *Ω_k-type* if there exists m so that $f^m : \omega \to \Omega_k$ is a conformal mapping. That is, ω maps conformally onto Ω_k.

Such an m is unique, since f^{m+1} will be an n to 1 mapping, where n depends on Ω_k.

At the end of the next section we will have shown that every Fatou component of f is either a preimage of the basin of attraction B_f, or of Ω_k type for some integer k. Moreover, we will have shown that inside a fixed Ω_j, the polynomial hulls of the components of Ω_k-type form a nested sequence of compact sets with diameters tending to zero. The intersection of this sequence of compact sets forms a Cantor-like subset (we say Cantor like since it is totally disconnected and has no isolated points; it is not closed since it accumulates on the boundary of Ω_j).

We would now like to describe the Julia set of f and how it relates to the set of points A. By our discussion above, there are two types of Fatou components of f. They are the components in C, or the Ω_k's and the other domains of Ω_k type. The boundaries of these components are therefore in the Julia set. By Theorem 8.5, the boundaries of the components Ω_k are C^1 smooth and approximately circles. The proof of Theorem 8.5 also proves these boundary components are in the escaping set of f. By Lemma 9.6, and Lemma 9.5, if ω_k is a component of Ω_k type for $k \geq 1$, its inner and outer boundary components are in the escaping set, are C^1 and are approximately circles as well. For the components of C, the boundary eventually maps onto the boundary of K_f. There is a third portion of the Julia set, and it corresponds to the set of all points which move from A_k to A_j for $j \leq k$ infinitely often. If $z \in A_k$ and $f(z) \in A_j$ for $j \leq k$, we will say that *z moves backwards*. Therefore, the points that move backwards infinitely often are precisely the points in the Cantor-like subset described above.

Given the above the discussion, we have the following.

Lemma 9.8. *The Julia set can be decomposed into three pieces*

1. *The Cantor type set of points Y that move backwards in the A_k's infinitely often.*
2. *The C^1 components that escape to ∞.*
3. *Preimages of the Julia set of the quadratic-like map f*

Moreover, if $s = \mathrm{Hdim}(J_f)$, the dimension of the Julia set of the polynomial-like mapping f, $\mathrm{Hdim}(\mathcal{J}(f)) \geq s$.

We conclude the section by proving a lemma describing the nice geometry of the round Fatou components Ω_k, $k \geq 1$. The proof follows from some basic calculations using the fact that f looks like a power mapping on some portions of Ω_k, along with using Lemma 9.6.

Lemma 9.9 (The Shape of Round Fatou components)**.** *Choose some Fatou component Ω_k, for $k \geq 1$. Define $d_j = 2(n_k + \cdots + n_{j-1})$ for $j > k$. Then Ω_k has the following geometric properties*

1. *The inner and outer boundary components of Ω_k are C^1 curves arbitrarily close to round circles.*
2. *For all $j \geq k$, there are $n_j \cdot 2^{d_j}$ many boundary components of Ω_k which lie distance approximately $R_k \cdot 2^{-d_j}$ from the outer boundary component of Ω_k. The boundary components are approximately distance $R_k n_j^{-1} 2^{-d_j}$ apart from each other and lie on an approximately round circle. Their diameter is $O(R_j^{-1})$. All boundary components of Ω_k arise in this manner for some j; we say that such a component is in the jth level of Ω_k.*
3. *All of the boundary components of Ω_k are approximately round circles.*

10. The s-Sum of the Fatou Components: Preliminaries

In the next few sections, our goal to prove the following technical theorem, which we need to show that $\mathrm{Pdim}(\mathcal{J}(f)) < 2$. Recall that $s = \mathrm{Hdim}(J_f)$.

Theorem 10.1. *Let $\epsilon > 0$ be given. Then*

$$\sum_{k=1}^{\infty} \sum_{\omega_k \subset A_1} \mathrm{diam}(\omega_k)^{s+\epsilon} < \infty,$$

where the sum is taken over all Fatou components $\omega_k \subset A_1$ of Ω_k-type for $k \geq 1$.

The proof of Theorem 10.1 will be carried out in Section 13 after we prove some preliminary lemmas. For convenience, we will call the sum in Theorem 10.1 an $(s+\epsilon)$-sum of the components of Ω_k-type. Recall that the components of Ω_k type inside of A_1 are the Fatou components which eventually iterate conformally onto Ω_k. Roughly put, if we zoom into a hole of the component Ω_1, we will see a component of Ω_k type.

To prove Theorem 10.1, we will construct a "self-improving" covering of the Cantor-like set of points Y which move backwards in the A_k's infinitely often. By this we mean that if at some stage the covering contains a component of Ω_k type, the new stage will cover all the holes of that component and be a subset of the previous stage. We will show that the $(s+\epsilon)$-sum of the new covered components can be compared to the diameter of the previous component in such a way that the $(s+\epsilon)$-sum of the diameters of the components at all stages is summable. The following corollary of such a construction will be immediate.

Corollary 10.2. *Let $\epsilon > 0$. Then we may define f in a way that depends on ϵ so that* $\mathrm{Hdim}(Y) \leq s + \epsilon$.

In this section, we will describe this covering before moving on to some technical lemmas we will need in the next sections. The overall idea is that we wait for the first time that a point in $Y \cap A_1$ moves backwards in the A_k's, and refine the covering accordingly. Such a refinement corresponds to zooming into the new layer of holes in the components of Ω_k-type.

Define $W_1^0 = A_1$. For each $z \in A_1 \cap Y$, by definition, there is a first n so that $f^n(z) \in A_k$ for $k \leq n$. $f^{-n}(A_k)$ has several components in A_1, all of which are topological annuli. z is a member of one of these components, which we denote by W_k^n. The collection of all such W_k^n refines the covering of $Y \cap A_1$. We continue to refine the covering by the dynamics as follows. If $z \in W_k^n$, then there is a first q so that $f^{n+q}(z) \in A_j$ for $j \leq k+q$. We replace W_k^n by the component $W_j^{n+q} \subset (f^{n+q})^{-1}(A_j)$ containing z, which is clearly contained inside of W_k^n.

It will be useful to make the following modification to the procedure above. Instead of considering all $j \leq k+q$, we will just consider the $j = k+q-1$ case. To do this, we replace W_{k+q-1}^{n+q} by the polynomial hull of W_{k+q-1}^{n+q}, which we denote as $\hat{W}_{k+q-1}^{n+q}$. Note that these sets have the same diameter, and they contain all components in the cover of the form W_j^{n+q}, $j < k+q$ inside of it. Finally, note that $k+q \geq 0$, since we are waiting for the first time the component moves backwards, and f maps A_k with $k \leq 0$ onto A_{k+1} by definition. In other words, the points in the negatively indexed A_k never move backwards.

11. The s-Sum of the Fatou Components: Refining for $k \geq 1$

In this section, we show that the refining the covering in the previous section results in a decreased $(s+\epsilon)$-sum compared to the previous stage, but only in the case for Fatou components far away from C, the collection of inverse images of K_f. We will deal with Fatou components close to C in the next section.

First we need an easy estimate comparing the diameter of W_k^n to the diameter of the hole inside of it.

Lemma 11.1. *R may be chosen so that, for all $k \geq 1$, $\alpha \geq s$, we have*

$$\operatorname{diam}(W_{k-1}^n)^\alpha \leq \frac{1}{4} \operatorname{diam}(W_k^n)^\alpha.$$

Proof. There is exactly one component of the form W_{k-1}^n contained inside of the hull $\hat{W}_k^n$. By the Kobe distortion theorem

$$\frac{\operatorname{diam} W_{k-1}^n}{\operatorname{diam} W_k^n} \leq C \frac{\operatorname{diam} f^n(W_{k-1}^n)}{\operatorname{diam} f^n(W_k^n)} = C \frac{\operatorname{diam} A_{k-1}}{\operatorname{diam} A_k} \leq C \frac{1}{R_0}.$$

Recall that $R_0 = \operatorname{diam}(A_0) = \operatorname{diam}(f^{-1}(A_1))$. By choosing R large enough, we have the desired result. □

The next lemma is more complicated. We show that at any stage, when we refine a component W_k^n, we can control the diameters of the refined components in terms of the diameter of W_k^n.

Lemma 11.2. *R may be chosen so that, for all $k \geq 1$, $\alpha \geq s$*

$$\sum_{q \geq 1} \sum_{W_{k+q-1}^{n+q}} \operatorname{diam}(W_{k+q-1}^{n+q})^\alpha \leq \frac{1}{4} \operatorname{diam}(W_k^n)^\alpha$$

Proof. First we need to count how many new components of the type W_j^{n+q} we get for each q. First note that by definition of our covering we have the following chain

$$\begin{array}{rcl}
W_{k+q-1}^{n+q} \subset W_k^n & \subset & A_1 \\
f^n(W_{k+q-1}^{n+q}) \subset f^n(W_k^n) & \subset & A_k \\
f^{n+1}(W_{k+q-1}^{n+q}) & \subset & A_{k+1} \\
\vdots & & \vdots \\
f^{n+q-1}(W_{k+q-1}^{n+q}) & \subset & A_{k+q-1} \\
f^{n+q}(W_{k+q-1}^{n+q}) & \subset & A_{k+q-1}.
\end{array}$$

Indeed, we are choosing to cover with the hull of the component that goes back in the $j = n + q - 1$ case, and since each time we choose q so it was the first time this happened, we have this exact sequence of mappings. f acts like a covering map in each of these individual situations. There are two possibilities.

(1) $f : A_k \to A_{k+1}$ and $k \geq 0$. In this case, the mapping is $2n_k$ to 1.
(2) $f : A_k \to A_{k+1}$ and $k < 0$. In this case, f is a polynomial-like mapping and is 2^N to 1.

It follows from taking preimages according to the definition of the covering that the number of components W_j^{n+q} inside of W_k^n is less than

$$2^q n_k \cdots n_{k+q-2} \leq 2^q N_{k+q-2}.$$

Here, $N_k = n_1 \cdot \cdots \cdot n_k$.

Finally, we can use the last petal map, Lemma 9.3, and two applications of the Koebe distortion theorem to conclude that

$$\operatorname{diam}(W_{k+q-1}^{n+q}) \leq \frac{1}{R_{k+q-1}} \operatorname{diam}(W_k^n).$$

So for each q, the contribution to the sum is bounded above by

$$O\left(2^q N_{k+q-2} R_{k+q-1}^{-1}\right) \cdot \operatorname{diam}(W_k^n)$$

By Lemma 5.7, choosing R sufficiently large makes the big-oh term as small as we would like, so the result follows. $\square$

12. The s-Sum of the Fatou Components: Refining for $k \leq 0$

Having dealt with the refinement of all components of Ω_k-type for $k > 0$, we turn to analyzing the refinement with $k \leq 1$. As $k \to -\infty$, the Fatou components Ω_k are no longer approximate circles but annuli shaped like the fractal J_f. To estimate the $(s+\epsilon)$-sum when we refine the covering in these components, we will decompose them into pieces that map conformally onto Ω_1. The Koebe distortion theorem will allow us to compare the $(s+\epsilon)$ sum of the refinement restricted to one of these pieces to the $(s+\epsilon)$-sum of the refinement of a component of Ω_1-type, with a corrective factor given by the diameter of the piece raised to the $(s+\epsilon)$ power. To control these corrective factors, we will show that they form a Whitney decomposition of the complement of J_f, and use Lemma 3.1 and the fact that $\operatorname{Hdim}(J_f) = s$ to obtain a convergent sum. The end result is the following:

Lemma 12.1. *Fix $\epsilon_0 > 0$. Let $W_1^n \in f^{-n}(A_1)$ be an element of the covering of Y. Let $W_j^n \in f^{-n}(A_j)$ be the components in the covering contained inside of $\hat{W}_1^n$. There exists R so that*

$$\sum_{j=1}^{-\infty} \sum_{W_{j+q-1}^{n+q} \subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \leq \frac{1}{4} \sum_{W_{j+q-1}^{n+q} \subset W_1^n} \operatorname{diam}(W_q^{n+q})^{s+\epsilon_0}.$$

Proof. For each Ω_k, $k \leq 0$, f is a 2^N to 1 mapping, so each Ω_k can be decomposed into $2^{N(-k+1)}$ components each of which maps conformally onto Ω_1. This breaks Ω_k into $2^{N(-k+1)}$ quadrilateral pieces $\mathcal{Q} = \{Q_i^k\}_{i=i}^{2^{N(-k+1)}}$. All of the Q_i^k all have holes, and we would like to use these to build a Whitney decomposition of J_f. We denote the filled in components as usual by $\hat{\mathcal{Q}} = \{\hat{Q}_i^k\}_{i=i}^{2^{N(-k+1)}}$. Furthermore, we choose to define Q_i^k by the dynamics of f. To be precise, for each i and k, we may choose $f(Q_i^k) = Q_{i'}^{k+1}$ for some i'. To accomplish this, it suffices to choose an appropriate decomposition of Ω_0, and then define the decomposition of Ω_k for $k < 0$ by inverse images.

We claim $\hat{\mathcal{Q}}$ forms a Whitney decomposition of the complement of the Julia set J_f of the polynomial-like map f. Indeed, there exists a quasiconformal mapping φ from the complement of K_f to the complement of the disk $\mathbb{D}$ that conjugates the dynamics of the polynomial like map f with z^{2^N}. Under this conjugacy, the cubes in $\hat{\mathcal{Q}}$ map to approximate hyperbolic squares that form a Whitney decomposition of the complement of $\mathbb{D}$ invariant under the dynamics of z^{2^N}. It follows that $\hat{\mathcal{Q}}$ is a Whitney decomposition by applying φ^{-1} since the hyperbolic squares are.

Let us return back to W_1^n. Since $\hat{\mathcal{Q}}$ forms a Whitney decomposition of the complement of J_f, $f^{-M}(\hat{\mathcal{Q}})$ forms a Whitney decomposition of the component $f^{-M}(K_f)$ contained inside of $\hat{W}_1^n$. If $W_{j+q-1}^{n+q} \subset W_j^n$, it is contained inside of some quadrilateral $Q = f^{-M}(\hat{Q})$, and there exists $W_{q-1}^{n+q} \subset W_1^n$ so that, by the Lemma

9.5

$$\frac{\operatorname{diam}(W_{j+q-1}^{n+q})}{\operatorname{diam}(Q)} \leq C\frac{\operatorname{diam}(W_{q-1}^{n+q})}{\operatorname{diam}(W_1^n)}.$$

Applying this for all $W_{j+q-1}^{n+q} \subset Q$, we obtain

$$\sum_{W_{j+q-1}^{n+q}\subset Q} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \leq C\frac{\operatorname{diam}(Q)^{s+\epsilon_0}}{\operatorname{diam}(W_1^n)^{s+\epsilon_0}} \sum_{W_{j+q-1}^{n+q}\subset W_1^n} \operatorname{diam}(W_q^{n+q})^{s+\epsilon_0}.$$

Next, if we sum over all the pieces $Q \in f^{-M}(\hat{\mathcal{Q}})$ that make up the decomposition of each Fatou component, we get

$$\sum_{j=1}^{-\infty} \sum_{W_{j+q-1}^{n+q}\subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \leq C\cdot\frac{\sum_{Q\in f^{-M}(\hat{\mathcal{Q}})}\operatorname{diam}(Q)^{s+\epsilon_0}}{\operatorname{diam}(W_1^n)^{s+\epsilon_0}} \sum_{W_{j+q-1}^{n+q}\subset W_1^n} \operatorname{diam}(W_q^{n+q})^{s+\epsilon_0}.$$

Since $f^{-M}(\hat{\mathcal{Q}})$ is a Whitney decomposition of the complement of $f^{-M}(K_f)$, the sum converges by Theorem 3.1, and the sum is comparable to $\operatorname{diam}(W_0^n)$. By Lemma 11.1, and by choosing R to be sufficiently large we have

$$\sum_{j=1}^{-\infty} \sum_{W_{j+q-1}^{n+q}\subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \leq \frac{1}{4} \sum_{W_{j+q-1}^{n+q}\subset W_1^n} \operatorname{diam}(W_q^{n+q})^{s+\epsilon_0}.$$

□

13. The s-Sum of the Fatou Components: Conclusions

By combining the two technical lemmas above, we can now prove the Theorem 10.1. The rest of the section is dedicated to some simple corollaries of the proof below and a discussion of the geometry of the Fatou and Julia sets relevant to the next section.

Proof of Theorem 10.1 *and Corollary* 10.2. It is sufficient to prove that the diameters of all components W_k^n for $k \geq 1$ converge. Indeed, each component ω_k can be associated to an element W_k^n of the covering of Y since the outer boundary component of ω_k is contained in a unique $\hat{W}_k^n$, so that they have comparable diameters.

Suppose we are at the mth stage of the refinement procedure described in Section 10. Let S denote all the collection of sets $\hat{W}_k^n$ obtained at this stage, and S' denote all the components obtained by refining S. All of the components in S' contained in $\hat{W}_k^n$ are contained in $W_j^n \subset \hat{W}_k^n$ for $j \leq k$. We estimate:

$$
\begin{aligned}
\sum_{j=k}^{-\infty} \sum_{W_{j+q-1}^{n+q} \subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} &\leq \frac{1}{4} \sum_{W_{j+q-1}^{n+q} \subset W_1^n} \operatorname{diam}(W_q^{n+q})^{s+\epsilon_0} \\
&= \sum_{j=1}^{k} \sum_{W_{j+q-1}^{n+q} \subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \\
&\quad + \sum_{j=1}^{-\infty} \sum_{W_{j+q-1}^{n+q} \subset W_j^n} \operatorname{diam}(W_{j+q-1}^{n+q})^{s+\epsilon_0} \\
&\leq \frac{1}{4} \sum_{j=1}^{k} \operatorname{diam}(W_j^n) + \frac{1}{4} \operatorname{diam}(W_1^n) \\
&\leq \frac{1}{8} \operatorname{diam}(W_k^n).
\end{aligned}
$$

It follows that the diameters of each refinement decay geometrically, so that the sum of all components of the covering is finite. This proves Theorem 10.1. Since the tail of a convergent series tends to 0, it follows that the sum of the diameters of the mth refinement tends to 0 as m tends to infinity. Since the mth refinement covers Y, it follows that $H^{s+\epsilon_0}(Y) = 0$, so that $\operatorname{Hdim}(Y) \leq s + \epsilon_0$. □

Corollary 13.1. *$J(f)$ has zero Lebesgue measure.*

Proof. The Julia set is the disjoint union of the the set of points that move backwards infinitely often, countably many C^1 curves, and countably many quasicircles with dimension strictly less than 2. All of these components have zero measure. □

14. The Packing Dimension of $J(f)$ is < 2

In this section, we prove that the packing dimension of $J(f)$ can be taken to be arbitrarily close to the Hausdorff dimension, and is therefore less than 2. to accomplish this, we combine the techniques of the previous sections with the techniques used in [Bis17] that resulted in an estimate of packing dimension being 1. Roughly put, we will decompose the compliment of the Julia set into three regions, and estimate the local upper Minkowski dimension using Theorem 3.1. The regions take the three following general forms, the first two being the most straightforward. We have inverse images of the basin B_f, and we also multiply connected Fatou components which are "far away" from the inverse images of B_f in the sense that these components are almost circular. These regions correspond to the components of Ω_k-type (see Definition 9.7) for sufficiently large k. When k is much less than zero, components of Ω_k we see a third type of region. These components are far from circular, since their boundary curves accumulate onto the fractal boundary of the appropriate inverse image of ∂B_f.

The following result will be useful in applying Theorem 3.1. It follows from the results of Sullivan in [Sul83] (see Theorems 3 and 4).

Theorem 14.1. *Let $f : U \to V$ be a hyperbolic polynomial-like map. Then we have* $\operatorname{Pdim}(\partial K_f) = \operatorname{Hdim}(\partial K_f) = \overline{\operatorname{Mdim}}(\partial K_f)$.

In particular, the polynomial-like map $f(z)$ is hyperbolic since its critical point tends to an attracting fixed point, so its packing dimension and upper Minkowski dimension equal the Hausdorff dimension s as well.

In order to apply Theorem 3.1, we need to decompose the Fatou components into simpler pieces. We collect the following lemmas proved in section 20 of [Bis17]. The first lemma will allow us to break the infinitely connected Fatou components into simpler, annular regions.

Lemma 14.2. *Let Ω be a bounded open set containing disjoint open subsets $\{\Omega_j\}$ so that $\Omega \setminus \cup_j \Omega_j$ has zero Lebesgue measure. Then for $t \in [1,2]$ we have*

$$\sum_{Q \in W(\Omega)} \operatorname{diam}(Q)^t \leq \sum_j \sum_{Q \in W(\Omega_j)} \operatorname{diam}(Q)^t$$

where $W(\Omega)$ represents a Whitney decomposition of Ω.

The topological annuli we obtain are close to round annuli if the Fatou component is far enough from the Julia set. This makes calculating the s-Whitney sum easy, according to the next two lemmas.

Lemma 14.3. *If $f : \Omega_1 \to \Omega_2$ is biLipschitz, then for any $t \in (0,2]$, we have*

$$\sum_{Q \in W(\Omega_1)} \operatorname{diam}(Q)^t \cong \sum_{Q \in W(\Omega_2)} \operatorname{diam}(Q)^t.$$

Given a Whitney decomposition W, the s-Whitney sum is the sum of the diamters of the elements of W raised to the s power.

Lemma 14.4. *Let $A(r, r+\delta)$ be a round annulus and $t \geq 1$. Then the t-Whitney sum is*

$$O\left(\frac{1}{(t-1)}\delta_j^{t-1} r^t\right).$$

Recall in Section 9 that we labeled Fatou components which wind around 0 as Ω_k, corresponding to which B_{k-1} they contained. As a corollary to the above work, we have the following basic estimate.

Theorem 14.5. *Let $W(\omega_k)$ be a Whitney decomposition for a component ω_k of Ω_k-type, $k \geq 1$, and let $t \geq s$. Then*

$$\sum_{Q \in W(\omega_k)} \operatorname{diam}(Q)^t = O\left(\frac{1}{(t-1)} \operatorname{diam}(\omega_k)^t\right).$$

Proof. By Lemma 9.6, it sufficies to just consider the Fatou components Ω_k which wind around the origin. The layers of holes in the Fatou set Ω_k lie on Jordan curves that can be chosen arbitrarily close to circles. Connect each of these layers with such a curve, decomposing Ω_k into approximately round annuli. By Lemma 14.2, it suffices to estimate the s-sum of the Whitney decomposition of each of these annuli. But since the Jordan curves may be chosen close to circles, the annuli are biLipschitz equivalent to round annuli $A(r, r+\delta)$, where r is the diameter of Ω_k, with biLipshitz constant independent of the topological annulus. Therefore, by Lemmas 14.3 and 14.4, we have.

$$\sum_{Q \in W(\Omega_k)} = O\left(\frac{1}{(t-1)} \sum_{\delta_j} \operatorname{diam}(\Omega_k)^t\right) = O\left(\frac{1}{(t-1)} \operatorname{diam}(\Omega_k)^s\right).$$

□

Note that the above "necklacing" construction where we decompose the Fatou components into approximately parallel annuli works for any Ω_k with $k \leq 1$ by pulling back the construction under f. Let $\Omega \subset A_1$ be any Fatou component. If Ω is of Ω_k type for some k, the construction can also be pulled back to Ω via f. Hence we have decomposed all Fatou components into topological annuli. Those components that are of Ω_k type for $k \geq 1$ are roughly circular annuli with C^1 boundary components. However, when $k << 0$, the components of Ω_k type will be very close to a copy of the fractal Julia set of the quadratic like map and will approximate the Julia set. The length of these boundary components tends to infinity.

Let $W(A_1)$ be a Whitney decomposition of the necklaced Fatou component Ω_1 together with the necklaced Fatou components contained inside A_1. Since the critical exponent of a set is independent of the Whitney decomposition, we assume that $W(A_1)$ is taken to be the dyadic Whitney decomposition.

Theorem 14.6. *The $(s+\epsilon)$-sum of the Whitney decomposition $W(A_1)$ above converges for any $\epsilon > \epsilon_0$, where s the Hausdorff dimension of ∂K_f.*

Proof. The Fatou components have three types. Those of Ω_k type for $k \geq 1$, Ω_k type for $k < 1$, and those that get iterated to B_f. Therefore,

$$\sum_{Q \subset W(\Omega_1)} \mathrm{diam}(Q)^{s+\epsilon} = I + II + III.$$

Here, I represents the cubes in the components of Ω_k-type for $k \geq 1$, II represents cubes in components of Ω_k-type for $k < 1$, and III represents cubes that get iterated into B_f. We estimate each infinite sum separately.

We have already taken care of I above using Theorem 14.5. Indeed, by Theorem 10.1, the $(s+\epsilon)$-sum of the components of Ω_k-type converges for any $\epsilon > \epsilon_0$.

Next we estimate III. ∂K_f is the Julia set of a hyperbolic polynomial-like mapping. It follows from Theorem 14.1 that the upper Minkowski and Hausdorff dimensions of ∂K_f are both s. All other components which map to ∂K_f do so conformally, and are contained inside of a component ω of Ω_1 type. This means that the dimensions of the boundary of the copies of all the K_f's are all s. It also implies that the copies of K_f map to K_f with bounded distortion. Since the conformal image of a Whitney decomposition is a Whitney decomposition in the range of the conformal mapping, and since the image cubes have bounded finite overlap with a fixed dydadic Whitney decomposition $W(K_f)$ of K_f, we have for a given copy K'_f of the basin of attraction that

$$\sum_{Q \in W(A_1) \cap K'_f} \mathrm{diam}(Q)^{s+\epsilon} \leq C \, \mathrm{diam}(\omega)^{s+\epsilon} \sum_{Q \in W(K_f)} \mathrm{diam}(Q)^{s+\epsilon}.$$

Here ω is the Fatou component of Ω_1-type containing K'_f. Summing over all such components, we have

$$III \leq C \sum_{\omega \subset A_1} \mathrm{diam}(\omega)^{s+\epsilon_0} \cdot \sum_{Q \in W(B_f)} \mathrm{diam}(Q)^{s+\epsilon_0}.$$

Therefore III converges because of Theorem 10.1 and Theorem 3.1.

Lastly we show II is finite. We apply a technique similar to the one used in the proof of Theorem 10.1. Every component of Ω_k-type for $k < 1$ is contained in the polynomial hull of a unique Fatou component of Ω_1-type. We fix such a component and call it ω. We further refine $W(A_1)$ (via Lemma 14.2) by taking our necklaced components and slicing them into pieces which map conformally onto Ω_1. These pieces are chosen to be the same collection $\hat{\mathcal{Q}} = \{\hat{Q}_i^k\}$ from Theorem 10.1, except sliced by the necklacing construction. We pull back this refinement to all other Fatou components being summed over via the conformal mapping f.

Now choose some cube S in the refinement of $W(A_1)$ contained in ω. Then we have

$$\operatorname{diam}(S)^{s+\epsilon} \leq C \operatorname{diam}(\omega)^{s+\epsilon} \operatorname{diam}(f^M(S))^{s+\epsilon}.$$

$f^M(S) \subset Q$ for some $Q \in \hat{\mathcal{Q}}$, therefore

$$\operatorname{diam}(f^M(S))^{s+\epsilon} \leq C \operatorname{diam}(Q)^{s+\epsilon} \cdot \operatorname{diam}(f^{M+k}(S))^{s+\epsilon}.$$

Combining these estimates we have

$$\operatorname{diam}(S)^{s+\epsilon} \leq C \operatorname{diam}(\omega)^{s+\epsilon} \operatorname{diam}(Q)^{s+\epsilon} \operatorname{diam}(f^{M+k}(S))^{s+\epsilon}.$$

Next, we sum over all $S \subset f^{-M}(Q)$. This yields

$$\sum_{S \subset Q} \operatorname{diam}(S)^{s+\epsilon} \leq C \operatorname{diam}(\omega)^{s+\epsilon} \operatorname{diam}(Q)^{s+\epsilon} \sum_{S \subset Q} \operatorname{diam}(f^{M+k}(S))^{s+\epsilon}.$$

Similarly to the previous case, by the Koebe distortion theorem, the sum on the right hand side is comparable to a fixed $(s+\epsilon)$-sum of some Whitney decomposition of a necklaced Ω_1. The sum therefore converges. Next, we sum over all Q in the collection $f^{-M}(\hat{\mathcal{Q}})$ contained in $\hat{\omega}$. Therefore,

$$\sum_{S \subset \hat{\omega}} \operatorname{diam}(S)^{s+\epsilon} \leq C \operatorname{diam}(\omega)^{s+\epsilon} \sum_{Q \in \hat{\mathcal{Q}}} \operatorname{diam}(Q)^{s+\epsilon}.$$

The sum on the right hand side converges by Theorem 3.1. Finally, we sum of all possible components ω of Ω_1-type to see that

$$I \leq C \sum_{\omega \subset A_1} \operatorname{diam}(\omega)^{s+\epsilon}.$$

This sum converges by Theorem 10.1. It follows that III, converges, proving the theorem. □

Corollary 14.7. *The upper Minkowski dimension, and hence the packing dimension of $\mathcal{J}(f) \cap A_1$ is at most $s + \epsilon_0$.*

Proof. The above argument shows that the critical exponent for the Whitney decomposition of $\mathcal{J}(f) \cap A_1$ is less than or equal to $s+\epsilon_0$. Since $\mathcal{J}(f)$ has zero Lebesgue measure, Theorem 3.1 says that the upper Minkowski dimension of $\mathcal{J}(f) \cap A_1$ is also less than or equal to $s + \epsilon_0$. By [RS05], since f has no exceptional values, the local upper Minkowski dimension is constant and coincides with the packing dimension, so that $\operatorname{Pdim}(J(f)) \leq s + \epsilon_0$. □

15. Further Questions

There are two obvious ways to strengthen Theorem 1.1.

Question 15.1. *Can the packing dimension of the Julia sets be taken to be equal to the Hausdorff dimension?*

Our techniques do not quite get us this result. In the proof of Lemma 12.1, we consider an $(s+\epsilon)$-sum of a Whitney decomposition with critical exponent given by s. As ϵ tends to zero, this sum tends to infinity. Therefore, the parameter R tends to ∞. Recall that the Julia set breaks up into inverse images of ∂B_f, C_1 curves, and the Cantor type set of buried points. Since ∂B_f has dimension s, and the curves have dimension 1, it means that any increase in the dimension must come from the buried points. Since the dynamics are expanding on this set, it is believable that that the packing and Hausdorff dimensions of this set coincide. Since the packing dimension of ∂B_f and the packing dimension of the C^1 curves equal their respective Hausdorff dimensions, a proof that the same fact is true for the buried points would prove that packing dimension is equal to the Hausdorff dimension.

Question 15.2. *Are all packing dimensions possible?*

Again, our techniques only show that a dense set of packing dimensions are possible. We believe that either by adding a parameter λ to f or varying the parameter c in the main cardiod continuously, the dimension should vary continuously, much like in the case of the polynomial family $z^2 + c$.

Question 15.3. *Is it possible to generate good computer images of multiply connected Fatou components?*

Our example and Bishop's dimension 1 example each have dynamics that can be explicitly described. Despite this, due to the nature of the large parameters that define the functions we consider and the lack of a good test for whether a point is in the Julia set, it remains difficult to generate good computer images of multiply connected Fatou components. Perhaps by experimenting with more slowly growing parameters and using some theory (our result suggests it may be helpful to count how many times a point travels backwards), we could generate some interesting figures.

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