MAT 514 Summer II 2019, MIDTERM
3:30-4:30, August 1 in Earth and Space 181

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THIS EXAM IS WORTH 50 POINTS. QUESTIONS 1-40 ARE WORTH 1 POINT EACH. CHOOSE AND ANSWER 2 OF 41-44. EACH IS WORTH 5 POINTS. NO BOOKS, NOTES OR CALCULATORS ARE ALLOWED.

1-10 TRUE/FALSE: Write T (for true) or F (for false) in each box.

(1) F $(1+2i) - 3(2-i) = -5 - i$

(2) T $\frac{2-i}{1+2i} = \cancel{i} = -i$

(3) F $(4+3i)^{3} = 2.$ (5)

(4) T $1/i = -i$

(5) F $-16 = 4e^{i\pi}$

(6) F $e^{2019\pi i} = 1$

(7) T $\mathrm{Re}(\frac{1+z}{1-z}) = \frac{1-|z|^2}{|1-z|^2}$

(8) T $\mathrm{Log}(i) = \pi/2$

(9) F $\sin(1) = \frac{e^{i}-e^{-1}}{2}$

(10) T $\cancel{z + \bar{z} = 2i\mathrm{Im}(z)}$ $\mathrm{Arg}(-2) = -\pi$

11-15 Place the letter of the corresponding point in the box. The same letter might be used more than once.

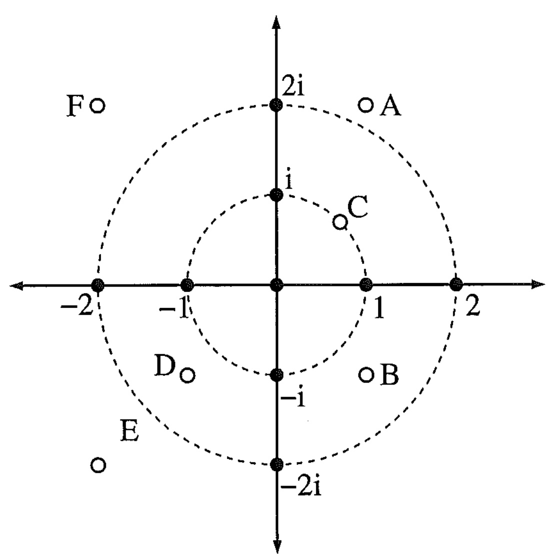
(11) C $|z| = 1$

(12) B $\mathrm{Re}(z) = 1, \mathrm{Im}(z) < 0.$

(13) C $z^2 = i$

(14) A $z = F + 3$

(15) E $z = 2 \cdot D.$



16-20 Match each function with its definition. Assume $z = x + iy$.

$iz = ix - y$.

(16) $\boxed{E}$ $\cosh(z)$

(17) $\boxed{F}$ $\exp(iz)$

(18) $\boxed{B}$ $\cos(z)$

(19) $\boxed{C}$ $\tan(z)$

(20) $\boxed{N}$ 2^z

A. $\frac{1}{2i}(e^{iz} - e^{-iz})$

B. $\frac{1}{2}(e^{iz} + e^{-iz})$

C. $(-i)\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

D. $\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$

E. $\frac{1}{2}(e^{z} + e^{-z})$

F. $e^{-y}(\cos x + i \sin x)$

G. $e^{x}(\cos x - i \sin x)$

H. $e^{x}\cos(y)$

I. $e^{x}\cos(y) + ie^{x}\sin(y)$

J. $e^{z \log i}$

K. $\frac{1}{2}(e^{z} - e^{-z})$

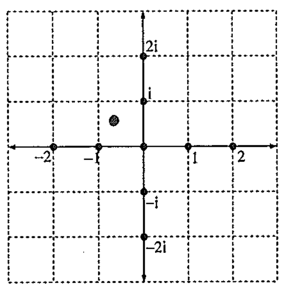
L. $\frac{1}{2}\log\frac{1+z}{1-z}$

M. $e^{i \log z}$

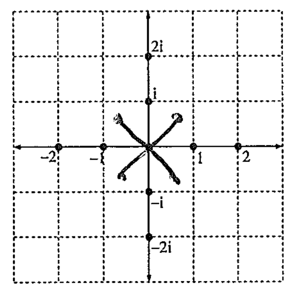
N. none of the above

21-25 Draw the following points or regions as accurately as you can. Assume $z = x + iy$.

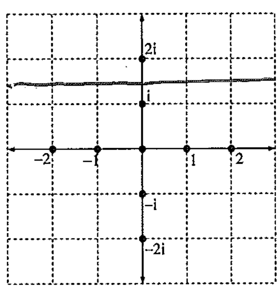
(21) Draw the point $z = e^{i\frac{3\pi}{4}}$.



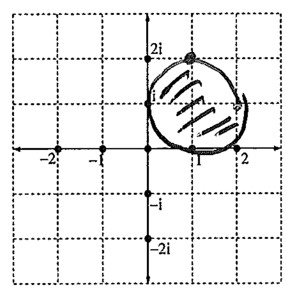
(22) Draw all solutions of $z^4 = -1$.



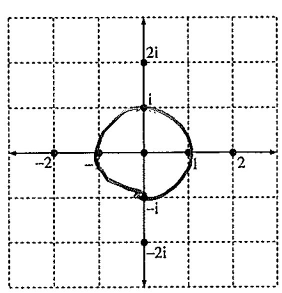
(23) Draw the points $|z - 2i| = |z - i|$.



(24) Draw the region $|z - (1 + i)| \leq 1$.



(25) Draw the set of points so that $\bar{z} = 1/z$



$\bar{z} = \frac{1}{z}$ iff $|z|^2 = 1$

26-35

21-30 TRUE/FALSE: Write T (for true) or F (for false) in each box.

(26) T The function $\sin(z)$ is entire.

(27) T For any $\theta \in \mathbb{R}$, $\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$

(28) F The path $\gamma(t) = e^{2it}$, $t \in [0, 2\pi]$ is a simple, closed path.

(29) T If $c \neq 0$ and $z \neq -d/c$, the derivative of $f(z) = \frac{az+b}{cz+d}$ is $f'(z) = \frac{ad-bc}{(cz+d)^2}$.

(30) F If $f(z) = \cosh(z)$, then $f'(z) = -\sinh(z)$.

(31) F A Mobius transformation sends circles to other lines and circles, but never sends lines to lines.

(32) F There exists an entire function f that only takes real values and is non-constant.

$z = x + iy$

(33) T If $z \in \mathbb{C}$, then $-\bar{z}$ is the reflection of z with respect to the imaginary axis.

(34) T Suppose $f = u + iv$. If the partial derivatives of u and v exist in some open disk $D(z_0, r)$ and are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0, then f is differentiable at z_0 .

(35) T An isolated point of G can never be an interior point of G.

36-40: Give a precise statement of each definition or result.

(36) State the triangle inequality.

If $z, w \in \mathbb{C}$

$$|z+w| \leq |z| + |w|.$$

(37) Define "closed path."

A path $\gamma : [a,b]$ is closed if

$$\gamma(a) = \gamma(b).$$

(38) Suppose f is differentiable at z_0. State the Cauchy-Riemann equations at z_0.

If $f = u + iv$:

$$u_x(z_0) = v_y(z_0)$$

$$u_y(z_0) = -v_x(z_0)$$

(39) For a real number θ, define the function $e^{i\theta}$

$$e^{i\theta} := \cos\theta + i\sin\theta$$

(40) Define "$z \in G$ is an interior point."

$z \in G$ is an interior point if there exists $r > 0$ so that $D(z, r) \subset G$ (the disk centered at z of radius r is contained inside G).

41-44: Answer two of the following questions. Mark clearly which questions you are answering. A third question may be attempted for 3 extra credit points - clearly mark which question is for extra credit.

(41) Let z and w be distinct complex numbers. Show that $|z+w|^2 - |z-w|^2 = 4Re(z\bar{w})$.

(42) At what points $z \in \mathbb{C}$ is the function $f(z) = z^2 - (\bar{z})^2$ differentiable? Is there a region where f is holomorphic? Justify your answer by citing any relevant result or definition from the course.

(43) Find a piecewise smooth, simple, closed parameterization of the semicircle of radius 2 centered at 0 (that is, the center of the corresponding circle is 0), oriented counterclockwise, beginning at $z = -2$. (Note: the semicircle includes a line segment on the real axis).

(44) Describe and sketch the image of the rectangle $R = \{z = x+iy : x \in [0,1], \theta \in [0,\pi]\}$ under the exponential function $\exp(z)$.

(41) $|z+w|^2 - |z-w|^2 = (z+w)(\bar{z}+\bar{w}) - (z-w)(\bar{z}-\bar{w})$

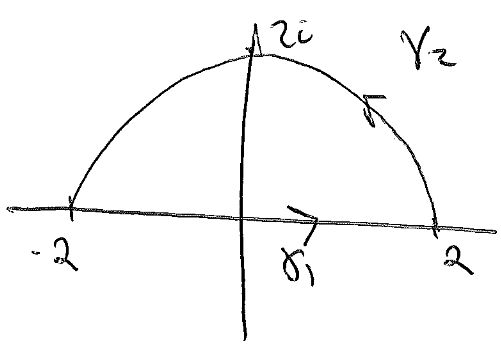
$= |z|^2 + |w|^2 + z\bar{w} + w\bar{z} - |z|^2 - |w|^2 + z\bar{w} + \bar{z}w.$

$= 2z\bar{w} + 2\bar{z}w$

$= 2z\bar{w} + 2\overline{z\bar{w}}$

$= 4\,Re(z\bar{w}).$

(43)



$\gamma_1(t) = -2 + t(2-(-2)) \quad t\in[0,1]$

$= -2 + 4t$

$\gamma_2(t) = 2e^{i\pi(t-1)} \quad t\in[1,2]$

$\gamma(t) = \begin{cases} \gamma_1(t) & t\in[0,1] \\ \gamma_2(t) & t\in[1,2] \end{cases}$

(42) $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\bar{z}^2 = (x-iy)^2 = x^2 - y^2 - 2ixy$

$z^2 - (\bar{z})^2 = 4ixy$.

So $u(z) = 0$

$v(z) = 4xy$

$u_x = 0 \quad v_x = 4y$

$u_y = 0 \quad v_y = 4x$.

$u_x = v_y$ only if $x=0$

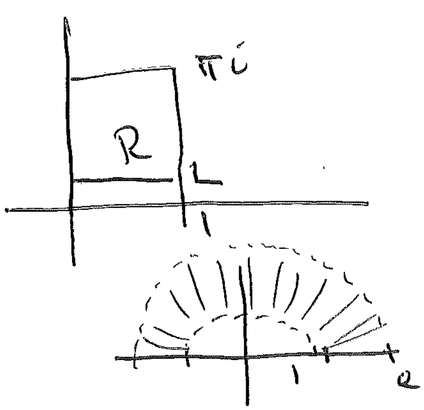
$u_y = -v_x$ only if $y=0$

So f is not differentiable on $\mathbb{C} \setminus \{0\}$.

Since u_x, u_y, v_x, v_y are continuous on $\mathbb{C}$, f <u>is</u> differentiable at 0.

f is holomorphic nowhere: not differentiable on a region.

(43)



$\exp(z) = e^x \cdot (\cos y + i \sin y)$.

Horizontal lines

$L = t + iy_0 \quad t \in [0,1] \quad y_0 \in [0,\pi]$

$\exp(L) = e^t \cdot (\cos(y_0) + i \sin y_0)$

↪ a half annulus / ring.

Additional Page 2

(1) $1+2i-6+3i = -5+5i$

(2) $\dfrac{2-i}{1+2i} = \dfrac{(2-i)(1-2i)}{(1+2i)(1-2i)} = \dfrac{2-2-5i}{5} = -i.$

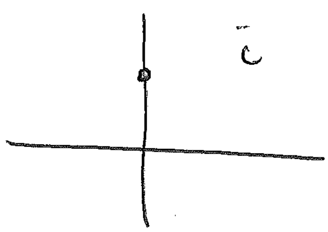
(3) Can't be, consider the modulus.

(5) $|-16| = 16 \qquad |4e^{i\pi}| = 4$

(6) $e^{2019\pi i} = -1$ since 2019 is odd.

(7) $\dfrac{1+z}{1-z} \cdot \dfrac{1-\bar{z}}{1-\bar{z}} = \dfrac{1-|z|^2+z-\bar{z}}{|1-z|^2} = \dfrac{1-|z|^2+2i\,\mathrm{Im}(z)}{|1-z|^2}.$

(8)



$\ln|i| = 0$

$\mathrm{Arg}(i) = \dfrac{\pi}{2}.$

(9) $\sin(1) = \dfrac{e^{i}-e^{-i}}{2i}$

(10) $\mathrm{Re}(z) = \frac{1}{2}(z+\bar{z})$, not $\mathrm{Im}(z)$.

(11) - (15) all clear. (16) - (20) def'ns.

Additional Page 3

(26) $\sin(z) = \frac{1}{2i}\left(\underbrace{\exp(iz)}_{\text{entire}} - \underbrace{\exp(-iz)}_{\text{entire.}}\right)$

(27) De Moivre's Theorem.

(28) $\gamma(t)$ wraps around $C(0,1)$ twice, and hence is not simple.

(29) $f'(z) = \dfrac{a(cz+d) - c(az+b)}{(cz+d)^2} = \dfrac{ad-bc}{(cz+d)^2}$

(30) Derivative of $\cosh(z) = \sinh(z)$.

(31) $f(z) = z+1$ sends lines to lines.

(32) False. C-R eqns

(33) $z = x+iy$ $\quad -\bar{z} = -(x-iy) = -x+iy$

(34) Converse to C-R eqns.

(35) True. See defn.