MAT 514 Summer II 2019, SAMPLE FINAL 1,
Actual midterm is 3:30-4:30, August 1st in Earth and Space 181

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THIS EXAM IS WORTH 50 POINTS. EACH QUESTION IS WORTH 1 POINT, EXCEPT FOR THE QUESTIONS 31-35, IS WORTH 1 POINT. QUESTIONS 31-35 ARE WORTH TWO POINTS EACH. NO BOOKS, NOTES OR CALCULATORS ARE ALLOWED.

1-5 Calculate the order of the zero or the pole at the given point. Write 0 if the function does not have a zero at the given point

(1) $\boxed{1}$ $f(z) = \exp(z) - 1 - z^2/2$ when $z = 0$

(2) $\boxed{0}$ $f(z) = 1 + z + z^2 + \ldots$ when $z = 0$.

(3) $\boxed{3}$ $f(z) = \frac{1}{\sin(z) - z}$ when $z = 0$.

(4) $\boxed{5}$ $f(z) = \sum_{k=5}^{\infty} c_k (z - z_0)^k$ when $z = z_0$.

(5) $\boxed{2}$ $f(z) = \sin(z) - 1$ at $z = \pi/2$.

6-10: Calculate the residue of each function at the given point.

(6) $\boxed{0}$ $f(z) = \exp(z)$ when $z = z_0 \in \mathbb{C}$.

(7) $\boxed{-1}$ $f(z) = \ldots \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \ldots$ when $z = 0$.

(8) $\boxed{-1/2}$ $f(z) = \frac{1}{z(z^2+1)}$ when $z = -i$.

(9) $\boxed{0}$ $f(z) = \frac{1}{z^2}$ when $z = 0$.

(10) $\boxed{1}$ $f(z) = \frac{1}{\sin(z)}$ when $z = 0$.

11-15 Match each function with its power series expansion.

(11) B $\frac{1}{(1-z)^2}$

(12) D $z\sin(z)$

(13) L $\exp(z^2)$

(14) G $\frac{1}{4-z}$

(15) J $\frac{\sin(z)}{z}$

A. $\sum_{k=0}^{\infty} z^k$

B. $\sum_{k=1}^{\infty} kz^{k-1}$

C. $\sum_{k=1}^{\infty} kz^k$

D. $\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+2}}{(2k+1)!}$

E. $\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k)!}$

F. $\sum_{k=0}^{\infty} 4^{-k} z^k$

G. $\sum_{k=0}^{\infty} 4^{-k-1} z^k$

H. $\sum_{k=0}^{\infty} 2^{-k-1} z^k$

I. $\sum_{k=0}^{\infty} 2^{-k} z^k$

J. $\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k+1)!}$

K. $\sum_{k=0}^{\infty} \frac{z^{(k^2)}}{k!}$

L. $\sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$

M. $\sum_{k=0}^{\infty} \frac{z^k}{k!}$

N. none of the above

16-20: Identify the radius of convergence of the following power series

(16) 1 $\sum_{k=0}^{\infty} z^k$

(17) 1 $\sum_{k=1}^{\infty} kz^{k-1}$

(18) ∞ $\sum_{k=0}^{\infty} \frac{(z+2)^k}{k^k}$

(19) $\frac{1}{6}$ $\sum_{k=0}^{\infty} 6^k(z-1)^k$

(20) 1 $\sum_{k=0}^{\infty} k^2(z+1)^k$

21-30 TRUE/FALSE: Write T (for true) or F (for false) in each box.

(21) F Cauchy's theorem implies that if γ is a smooth simple closed curve, and f is holomorphic on a domain U containing γ, then $\int_\gamma f = 0$.

Interior of γ must be in U

(22) F The function $f(z) = 1/z$ has an antiderivative on the domain $\mathbb{C} \setminus \{0\}$

It does not. $\int_{C(0,1)} \frac{1}{z}\,dz = 2\pi i$

(23) T If U is a simply connected region. and f is a holomorphic function, f has an antiderivative on U.

(24) T A power series centered at z_0 with radius of convergence R is holomorphic on $D(z_0, R)$.

(25) F If $\sum_{k=0}^{\infty} c_k z^k$ has radius of convergence 1, the series $\sum_{k=0}^{\infty} c_k$ always converges.

Consider $\sum z^k$ $\sum_{k=0}^{\infty} 1$ diverges

(26) F The function $\exp(1/z)$ has a pole of infinite order at $z = 0$.

Essential Singularity.

(27) T A function is holomorphic if and only if it is analytic.

(28) F When integrating a function f along a smooth path γ, the orientation of γ does not effect the result.

$\int_{-\gamma} f = -\int_\gamma f$

(29) F If $f(z)$ is not an entire function, and γ is a smooth, simple, closed path, then $\int_\gamma f$ can never be 0.

$\frac{1}{z}$ not entire. ~~but~~ $\int_{C(1,1/2)} \frac{1}{z} = 0$ by Cauchy's Thm.

(30) T Suppose $f(z)$ has a pole of order two at z_0. Then the residue of $f(z)$ at $z = z_0$ is $\lim_{z\to z_0} \frac{d}{dz}\left((z-z_0)^2 f(z)\right)$

Formula.

31-35: Give a precise statement of each definition or result. Each question is worth two points

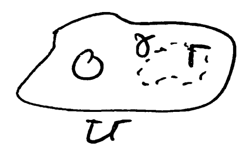
(31) Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path, and let f be continuous. Define "integral of f with respect to γ.

$$\int_\gamma f = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

(32) State Cauchy's Theorem. Include all hypotheses.

Let U be a region, $f: U \to \mathbb{C}$ be holomorphic. Let γ be a smooth, simple closed path whose interior is inside of U. Then

$$\int_\gamma f = 0$$



(33) Let $f : U \to \mathbb{C}$ be a continuous function. Define "F is the antiderivative of f"

$F: U \to \mathbb{C}$ is an antiderivative of f is

- F is holomorphic on U
- $F'(z) = f(z)$

(34) Define "power series centered at $z_0 = -1$".

For $c_0, c_1, c_2 \ldots \in \mathbb{C}$

$$\sum_{k=0}^{\infty} c_k (z+1)^k$$

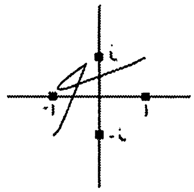
(35) State the Cauchy Integral formula. Include all hypotheses

Let U be a region, $f: U \to \mathbb{C}$ be holomorphic. Let γ be a smooth simple closed path in U. Suppose $w \in U$ is also in the interior of γ. Then

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz$$

36-45: Calculate the following integrals. Unless otherwise stated, assume the path is oriented counter clockwise (i.e, positively oriented.)

(36) $\boxed{0}$ $\int_\gamma z\,dz.$

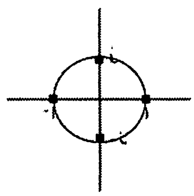


Antiderivative is $\frac{1}{2}z^2$

$\int_\gamma z\,dz = \frac{z^2}{2}\Big|_{-i-1}^{i+1}$

$= \frac{1}{2}(2i - 2i) = 0$

(37) $\boxed{0}$ $\int_\gamma (\bar{z})^2\,dz.$



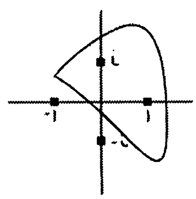
$\gamma(t) = e^{it}$ $\gamma'(t) = ie^{it}$ $t \in [0, 2\pi]$

$f(z) = (\bar{z})^2$, $f(\gamma(t)) = (e^{-it})^2 = e^{-2it}$

$\int_\gamma (\bar{z})^2 = \int_0^{2\pi} e^{-2it} \cdot ie^{it}$

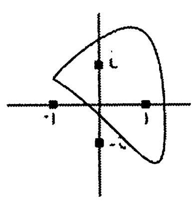
$= i\int_0^{2\pi} e^{-it}\,dt = 0.$

(38) $\boxed{0}$ $\int_\gamma \cos(z)\,dz.$



Cauchy's Thm applies

(39) $\boxed{2\pi i \cos 1}$ $\int_\gamma \frac{\cos(z)}{z-1}\,dz.$



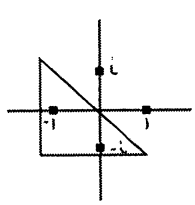
Cauchy Integral Formula

$f(z) = \cos z$ $\boxed{2\pi i \cdot \cos 1} = \int_\gamma \frac{\cos z}{z-1}\,dz$

$w = 1$

Answer.

(40) $\boxed{-\pi i}$ $\int_\gamma \frac{1}{(z-i)(z+1)}\,dz.$



Cauchy Integral Formula

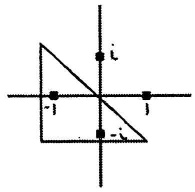
$f(z) = \frac{1}{z-i}$

$2\pi i \cdot f(-1)$

$w = -1$

$= -\pi i$

(41) $\boxed{0}$ $\int_\gamma \frac{1}{(z+i)(z+1)}\,dz$.



Residue Thm

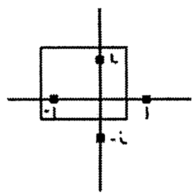
$\mathrm{Res}_{z=-i}(f) = \frac{1}{-i+1}$

$\mathrm{Res}_{z=-1}(f) = \frac{1}{-1+i}$

Sum of Residues is 0

(42) $\boxed{}$ $\int_\gamma \frac{1}{z(z+i)(z+1)}\,dz$.

See side.



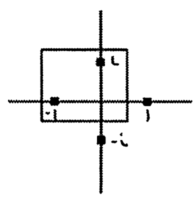
Residue Thm

$\mathrm{Res}_{z=0} = \frac{1}{(0+i)(0+1)} = \frac{1}{i} = -i$

$\mathrm{Res}_{z=-1} = \frac{1}{(-1)\cdot(-1+i)} = \frac{1}{1-i}$

~~Res~~ $\boxed{\text{Answer} = \left(2\pi i\cdot\left(-i + \frac{1}{1-i}\right)\right)}$

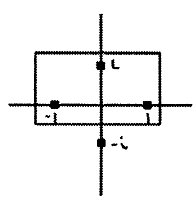
(43) $\boxed{0}$ $\int_\gamma \frac{1}{z^3}\,dz$.



Antiderivative is $-\frac{1}{2}\frac{1}{z^2}$

γ is closed

(44) $\boxed{\pi i}$ $\int_\gamma \sum_{k=-\infty}^{0} \frac{1}{(-k+1)!} z^k\,dz$.

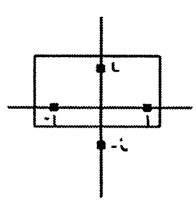


Residue Thm

$c_{-1} = \frac{1}{2!} = \frac{1}{2}$.

$2\pi i \cdot \frac{1}{2} = \pi i$

(45) $\boxed{-\pi i}$ $\int_\gamma \frac{\cos(z)-1}{z^3}\,dz$.



Residue Thm

$\frac{\cos z - 1}{z^3} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right) - 1}{z^3}$

$= -\frac{1}{2}\cdot\frac{1}{z} + \frac{1}{4!}z - \cdots$

c_{-1} $\quad 2\pi i \cdot \frac{-1}{2} = -\pi i$

Additional Page 1

(1) $\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}\right)-1-\frac{z^2}{2} = z + \frac{z^3}{3!} + \cdots$

power is 1.

(2) $f(0)=1$

(3) $\sin(z)-z = \left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)-z$

$= -\frac{z^{3}}{3!}+\frac{z^5}{5!} - \cdots$ order 3

(4) $f(z)= c_5(z-z_0)^{5}+ \cdots$

order 5

(5) $f(\pi/2) = 1-1 = 0$

$f'(\pi/2) = \cos \pi/2 = 0$

$f''(\pi/2) = -\sin \pi/2 = -1 \neq 0.$ Order is 2.

(6) $f(z)$ is entire $\Rightarrow$ all residues are 0.

(7) $c_1 = -1.$

(8) $\lim_{z\to -i} \frac{(z+i)}{z(z+i)(z-i)} = \frac{1}{-i\cdot(-2i)} = -\frac{1}{2}$

Use $-i$ is a pole of order 1

(9) $c_{-1} = 0$

(10) Order of the pole is 1.

$$\lim_{z \to 0} \frac{z}{\sin(z)} = 1$$

(11) $\frac{1}{(1-z)^2} = \frac{d}{dz} \frac{1}{1-z} = \frac{d}{dz} \sum_{k=0}^{\infty} z^k$

(12) $\sin(z)$ p.s, but w/ extra power of z.

(13) $\exp(z^2) = \sum \frac{1}{k!} (z^2)^k = \sum \frac{1}{k!} z^{2k}$

(14) $\frac{1}{4-z} = \frac{1}{4} \cdot \frac{1}{1-\frac{z}{4}} = \frac{1}{4} \cdot \sum_{k=0}^{\infty} (\omega)^4$

$\omega = \frac{z}{4}$

$= \frac{1}{4} \cdot \sum_{k=0}^{\infty} 4^{-k} z^k$

$= \sum_{k=0}^{\infty} 4^{-k-1} z^k.$

(15) Similar to 12.

Additional Page 3

(16) p.s for $\frac{1}{1-z}$.

(17) Derivative has same radius of convergence.

(18) Root test

$$\lim_{k\to\infty} \left(\frac{1}{k^k}\right)^{1/k} = \lim_{k\to\infty} \frac{1}{k} = 0 = \frac{1}{R}$$

$$R = \infty$$

(19) Root Test

$$\lim_{k\to\infty} (6^k)^{1/k} = 6 = \frac{1}{R}$$

$$R = 1/6.$$

(20) Ratio Test

$$\lim_{k\to\infty} \frac{(k+1)^2}{k^2} = \lim_{k\to\infty} \frac{k^2+2k+1}{k^2} = \lim_{k\to\infty} 1 + \frac{2}{k} + \frac{1}{k^2}$$

$$= 1$$