Points on the Circle: from Pappus to Thurston

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A P R I L  5, 2018

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The Circle: \( \hat{\mathbb{R}} \cong \mathbb{P}^1(\mathbb{R}) \)

Our “circle” will be the real projective line, \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \).
Geometrically: Projecting from $p$, the point $q$ on the blue line maps to the point $r$ on the red line.

Algebraically: A map from $\mathbb{R}$ to $\mathbb{R}$ is fractional linear if it has the form

$$x \mapsto \frac{ax + b}{cx + d} \quad \text{with} \quad ad - bc \neq 0.$$  

Essential Property: The action of the group of fractional linear transformations on $\mathbb{R}$ is three point simply transitive.
Pappus of Alexandria (4th century).

Pappus defined a numerical invariant, computed from the distances between four points on a line; and proved that it is invariant under projective transformations.
Cross-Ratio: Four Points on the Projective Line.

Definition (non-standard): For \( a, b, c, d \in \mathbb{R} \), let

\[
\text{cr}(a, b, c, d) = \text{cr}
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
= \frac{(a - b)(c - d)}{(a - c)(b - d)} \in \hat{\mathbb{R}}.
\]

= \text{product of row differences} \over \text{product of column differences}

Restriction: At least 3 of the 4 variables must be distinct.

There is a unique continuous extension to the case \( a, b, c, d \in \hat{\mathbb{R}} \). Then:

\[
\text{cr}(a, b, c, d) = 0 \iff a = b \text{ or } c = d,
\]

\[
\text{cr}(a, b, c, d) = \infty \iff a = c \text{ or } b = d,
\]

\[
\text{cr}(a, b, c, d) = 1 \iff a = d \text{ or } b = c,
\]

\[
\text{cr}(a, b, c, d) \in \hat{\mathbb{R}} \setminus \{0, 1, \infty\} \iff a, b, c, d \text{ all distinct}.
\]

\[
\text{cr}(1, \infty, 0, x) = x \text{ for all } x.
\]
Many Points on $\hat{\mathbb{R}}$.

Definition. The moduli space $\mathcal{M}_{0,n}(\mathbb{R}) = \mathcal{M}_n$ is the space of equivalence classes of $n$-tuples $(p_1, \ldots, p_n)$ of distinct points of $\hat{\mathbb{R}}$ modulo the action of the group of fractional linear transformations.

Thus $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ represent the same point of $\mathcal{M}_n$ if and only if there is a fractional linear transformation $g$ such that

$$g(p_j) = q_j$$

for every $j$. 
Embedding $\mathcal{M}_n$ into a product of many circles.

**Easy Lemma.** The $n$-tuples $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ represent the same point of $\mathcal{M}_n$ if and only if:

$$\text{cr}(p_h, p_i, p_j, p_k) = \text{cr}(q_h, q_i, q_j, q_k)$$

for every $1 \leq h < i < j < k \leq n$.

Thus we can embed $\mathcal{M}_n$ into the $\binom{n}{4}$-fold product of circles

$$\hat{\mathbb{R}}(\binom{n}{4}) = \prod_{0 \leq h < i < j < k \leq n} \hat{\mathbb{R}}$$

by sending the equivalence class of $(p_1, \cdots, p_n)$ into the $\binom{n}{4}$-tuple of cross-ratios $\text{cr}(p_h, p_i, p_j, p_k)$, where

$$1 \leq h < i < j < k \leq n.$$
A Non-Standard Definition of \( \overline{\mathcal{M}}_n \)

**Theorem (McDuff and Salamon).** The closure \( \overline{\mathcal{M}}_n \) of \( \mathcal{M}_n \) within the torus

\[
\hat{\mathbb{R}}^\left(\binom{n}{4}\right) = \prod_{0 \leq h < i < j < k \leq n} \hat{\mathbb{R}}
\]

is a smooth, compact, real-algebraic manifold of dimension \( n - 3 \).

**Intuitive Proof that \( \overline{\mathcal{M}}_n \) is a real-algebraic set.**

If the \( p_j \) are all distinct, we can put \( p_1, p_2, \ldots, p_3 \) at \( 1, \infty, 0 \), so that

\[
\text{cr}(p_1, p_2, p_3, p_k) = p_k \quad \text{for all} \quad k.
\]

Thus \( p_4, p_5, \ldots, p_n \) are \( n - 3 \) independent variables, and determine all of the \( \binom{n}{4} \) coordinate cross-ratios.

Clearing denominators, we get a set of \( \binom{n}{4} - 3 \) defining polynomial equations.
The Simplest Cases $n = 3, 4$.

By definition, $\mathcal{M}_3 = \overline{\mathcal{M}}_3$ is a single point.

The subset $\mathcal{M}_4 \subset \mathbb{R}^{(4)} = \mathbb{R}$ is clearly just

$$\hat{\mathbb{R}} \setminus \{0, 1, \infty\} = \mathbb{R} \setminus \{0, 1\};$$

and its closure within $\hat{\mathbb{R}}$ is the entire circle: $\overline{\mathcal{M}}_4 \cong \hat{\mathbb{R}}$

We should think of $\overline{\mathcal{M}}_4$ as a cell complex with three vertices and three edges:
\( \mathcal{M}_5 \) is a “hyperbolic dodecahedron”, covered by twelve right angled hyperbolic pentagons.

The interiors of the twelve pentagons are the twelve connected components of \( \mathcal{M}_5 \).
Both have isometry group of order 120:
\[ \mathfrak{A}_5 \oplus (\mathbb{Z}/2) ; \quad \mathfrak{S}_5 \]
In both cases, each face has an “opposite” face:

In the hyperbolic case, \( A \leftrightarrow B, \quad j \longleftrightarrow j + 5 \pmod{10} \).

Euler Characteristic:
\[ \chi = 12 - 30 + 20 = 2, \quad \chi = 12 - 30 + 15 = -3. \]

Hyperbolic case \( \implies \overline{M}_5 \) is non-orientable;
with no fixed point free involution.
Why Twelve Pentagons in $\overline{M}_5$?
Each top dimensional cell in $M_n$ corresponds to one of the 
$$\frac{(n-1)!}{2}$$
different ways of arranging the labels $1,2,3,\ldots,n$ in cyclic order (up to orientation) around the circle.

Thus $\overline{M}_5$ has $4!/2 = 12$ two-cells.

Within $\overline{M}_5$ there are five different ways that two neighboring points can cross over each other to pass to a different face; hence five edges to each 2-cell.
The Embedding \( \varphi_{I,J} : \overline{M}_{r+1} \times \overline{M}_{s+1} \rightarrow \overline{M}_n \). 13.

Let

\[ \{1, 2, \ldots, n\} = I \cup J \]

be a partition into a set \( I \) with \( r \geq 2 \) elements, and a disjoint set \( J \) with \( s \geq 2 \) elements, where \( r + s = n \).

The image of \( \varphi_{I,J} \) is a union of codimension one faces; and every codimension one face of \( \overline{M}_n \) is included in the image of \( \varphi_{I,J} \) for just one partition \( \{I, J\} \) of \( \{1, 2, \ldots, n\} \).
Iterating this construction.

Mumford Stability Condition:

Each circle must have at least three distinguished points.
Cross-ratios and the Image of $\varphi_{I,J}$.

For each $x \in \overline{M}_n$ and each list of distinct numbers $h, i, j, k$ in \{1, 2, \ldots, n\}, define the limiting cross-ratio

$$\text{cr}_x(h, i, j, k) \in \hat{\mathbb{R}}$$

to be the limit, for any sequence of points $x^n \in M_n$ converging to $x$, of the cross-ratios $\text{cr}(p^n_h, p^n_i, p^n_j, p^n_k)$, where each $(p^n_1, \ldots, p^n_n) \in \hat{\mathbb{R}}^n$ is a representative for the class $x^n \in M_n$.

**Assertion.** The point $x \in \overline{M}_n$ belongs to the image,

$$x \in \varphi_{I,J}(\overline{M}_{r+1} \times \overline{M}_{s+1}) \subset \overline{M}_n,$$

if and only if

$$\text{cr}_x(i, i', j, j') = 0$$

for every $i, i' \in I$ and every $j, j' \in J$. 
Example: $\overline{M}_5$.

The are $\binom{5}{2} = 10$ partitions of $\{1, 2, 3, 4, 5\}$ into subsets of order two and three. Hence there are ten embeddings

$$\overline{M}_3 \times \overline{M}_4 \cong \hat{\mathbb{R}} \hookrightarrow \overline{M}_5.$$ 

These correspond to ten closed geodesics, each made up of three edges.

Thus there are $10 \times 3 = 30$ edges in $\overline{M}_5$.

Each of these geodesics also contains three vertices, Here each vertex is counted twice since it belongs to two different geodesics, so there are $10 \times 3/2 = 15$ vertices.

Thus verifying that $\chi = 12 - 30 + 15 = -3$. 
Example: $\overline{M}_6$

The are $\binom{6}{2} = 15$ partitions of $\{1, 2, 3, 4, 5, 6\}$ into subsets $I$, $J$ of order two and four. Hence there are fifteen embeddings

$$\overline{M}_3 \times \overline{M}_5 \cong \overline{M}_5 \hookrightarrow \overline{M}_6 ;$$

where each copy of $\overline{M}_5$ is made up of twelve pentagons.

Similarly there are $\binom{6}{3}/2 = 10$ partitions into two subsets of order three, yielding ten embeddings of the torus

$$\overline{M}_4 \times \overline{M}_4 \hookrightarrow \overline{M}_6 .$$

Each copy of the torus is made up of $3 \times 3 = 9$ squares.

(Thus the 2-skeleton of $\overline{M}_6$ consists of

$15 \times 12 = 180$ pentagons, plus $10 \times 9 = 90$ squares.)

According to Thurston, every smooth closed 3-manifold can be cut along embedded 2-spheres, tori, and/or Klein bottles into pieces, each of which has a locally homogeneous geometry.
Theorem. If we cut $\overline{M}_6$ open along its ten embedded tori, then the remainder can be given the structure of a complete hyperbolic manifold of finite volume with twenty infinite cusps.

Corollary: The fundamental group $\pi_1(\overline{M}_6)$ maps onto a free group on ten generators.

But $\pi_1(\overline{M}_6)$ also contains free abelian groups $\mathbb{Z} \oplus \mathbb{Z}$. 

Jaco-Shalen-Johannson Decomposition of $\overline{M}_6$. 18.
Proof Outline.
Each 3-cell in $\overline{M}_6$ is bounded by 6 pentagons & 3 squares.

(Take the union of two tetrahedra with a face in common, and chop off three of the corners.)

We want 60 copies of this 3-cell to fit together to form a smooth manifold.

Thus we need all dihedral angles to be $90^\circ$!
In Hyperbolic 3-Space, choose three orthogonal lines of length \( \ell \) starting at the point A. Then their convex closure is a tetrahedron with dihedral angle 90° along three of the edges. We need the dihedral angles along edges between B, C and D to be 45°. For \( \ell \) finite, these angles are always > 45°. But as \( \ell \to \infty \) these dihedral angles will tend to 45°. Two copies yield a model 3-cell; but only by collapsing the three squares to points, and pushing them out to the sphere at infinity.

Thus \( \overline{M}_6 \) with the 10 tori (or 90 squares) removed is a hyperbolic manifold tiled by 120 ideal tetrahedra.
Concluding Remark: The Associahedron.

55 years ago Stasheff, while studying associativity for spaces with a continuous product operation, invented a sequence of objects $A_n$ which we call *associahedra*.

![Diagram of associahedra $A_3$ and $A_4$]

The vertices of $A_n$ correspond to the many ways of making sense of an $n$-fold non-associative product.

**Theorem.** *Each top-dimensional cell of $\overline{M}_n$ is isomorphic as a cell complex to $A_{n-1}$.*
The top cells of $\overline{M}_n$ are Associahedra: Proof Idea. 22.
Some References

The associahedron:

Thurston geometrization conjecture:
  W. Thurston, *3-dimensional manifolds, Kleinian groups* · · · , 1982.

Construction and smoothness of $\overline{M}_n(\mathbb{C})$:

Topology of $\overline{M}_n(\mathbb{C})$:

Proof of Thurston’s conjecture:

Topology of $\overline{M}_n(\mathbb{R})$:

Construction of $\overline{M}_n(\mathbb{C})$ using cross-ratios:
  D. McDuff and D. Salamon, “J-holomorphic curves · · · (Appendix D),” 2012.

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