Real Quadratic Rational Maps

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Any real rational map \( f \) can be studied in two different ways:

As a piecewise monotone map from the circle \( \mathbb{P}^1(\mathbb{R}) \) to itself, \( f \) can be studied by very elementary methods.

But \( f \) always extends to a rational map \( f_{\mathbb{C}} \) from the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \) to itself, so that all of the tools of holomorphic dynamics are also available.

This talk will study the very special case of

Real Quadratic Maps, with Real Critical Points.
The Moduli Space $M = V / G$

Let $V$ be the smooth manifold consisting of all real quadratic maps

$$f(x) = \frac{ax^2 + bx + c}{a'x^2 + b'x + c'}$$

with real critical points. Let $G$ be the group of orientation preserving fractional linear transformations

$$L(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \quad \text{with} \quad \alpha \delta - \beta \gamma > 0,$$

acting on $V$ by conjugation $f \mapsto L \circ f \circ L^{-1}$.

Then we can form the quotient space $M = V / G$. 
Assertion: $M$ is a Topological Cylinder

This is a picture of part of the universal covering space of $M$.

The white regions correspond to maps such that both critical orbits converge to strongly attracting orbits of low period;

while the black points correspond to maps for which at least one critical orbit does not converge to an attracting periodic orbit.
Main Lemma. Every map in $V$ is conjugate under $G$ to a unique map which satisfies three conditions:

1. $f$ has the form

$$f(x) = \frac{Ax^2 + B}{Cx^2 + D},$$

(Proof: Put the critical points at zero and infinity.)

2. $AD - BC > 0$,

(Conjugate by $L(x) = -1/x$ if necessary.)


(Conjugate by $L(x) = kx$. The required equation is satisfied for one and only one $k > 0$.)

The resulting map $f$ is then uniquely determined!

(But $A, B, C, D$ are only unique up to multiplication by a common non-zero constant.)
The Invariant Interval \( f(\hat{\mathbb{R}}) \).

If we think of \( f \) as a map from the circle \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) to itself, then:

The image \( f(\hat{\mathbb{R}}) \) is a closed interval bounded by the two critical values.

Evidently all of the interesting dynamics is concentrated in this interval.

**Theorem 1.** *For any closed interval \( I \) contained in the circle \( \hat{\mathbb{R}} \) there is one and only one map \( f \) in canonical form for which \( f(\hat{\mathbb{R}}) = I \).*

Thus the moduli space \( M \) is homeomorphic to the set of all intervals \( I \subset \hat{\mathbb{R}} \).
Proof: (Identifying $\hat{\mathbb{R}}$ with $\mathbb{R}/\mathbb{Z}$)

For any map $f(x) = \frac{A x^2 + B}{C x^2 + D}$, the two critical values are

$$v_0 = f(0) = \frac{B}{D} \quad \text{and} \quad v_\infty = f(\infty) = \frac{A}{C}.$$ 

If we normalize the four coefficients $A, B, C, D$ so that $A^2 + C^2 = B^2 + D^2 = 1$, then we can set

$$A = \sin(\pi t_\infty), \quad C = \cos(\pi t_\infty); \quad B = \sin(\pi t_0), \quad D = \cos(\pi t_0).$$

Thus $v_\infty = \tan(\pi t_\infty)$ and $v_0 = \tan(\pi t_0)$.

By definition, $t_\infty$ and $t_0$ are the critical value angles.

More generally, any angle $t \in \mathbb{R}/\mathbb{Z}$ corresponds to a unique point $\tan(\pi t) \in \hat{\mathbb{R}}$. 
**Theorem 2.** \( M \) is diffeomorphic to \( \mathbb{R}/2\mathbb{Z} \times (0, 1) \).

**Proof.** We have two distinct critical value angles

\[ t_\infty, \ t_0 \in \mathbb{R}/\mathbb{Z}, \]

Lift to points \( \hat{t}_\infty, \hat{t}_0 \in \mathbb{R} \) so that \( \hat{t}_0 < \hat{t}_\infty < \hat{t}_0 + 1 \).

Then the corresponding point of \( M \) is uniquely determined by the two numbers

\[ \Sigma = \hat{t}_\infty + \hat{t}_0 \pmod{2\mathbb{Z}}, \quad \text{and} \quad \Delta = \hat{t}_\infty - \hat{t}_0 \in (0, 1). \]

Here \( \Delta \) is precisely the length of the interval \( f(\hat{\mathbb{R}}) \), lifted to \( \mathbb{R}/\mathbb{Z} \).
Two Pictures of $M$.

Red Lines: Polynomials (with critical fixed point).
Blue lines: “co-polynomials”. (One critical point maps to the other.)
The Six Regions in Moduli Space

\[ \begin{array}{cccc}
\theta = 0 & \theta = 0.5 & \eta = -0.5 & \eta = 0 \\
A = 0, C = 1 & A = 1, C = 0 & B = -1, D = 0 & B = 0, D = 1 \\
\end{array} \]
More about $M$.

**Chebyshev curve:** with $f(\text{critical value}) = \text{fixed point}$ with $\mu > 1$. 
Every complex quadratic rational map either:
(1) belongs to the connectedness locus,
   \( \Leftrightarrow \) connected Julia set, or
(2) belongs to the shift locus,
   \( \Leftrightarrow \) totally disconnected Julia set.

For our real maps \( f \), there is a further distinction:

real shift locus
\( \Leftrightarrow \ J(f_{\mathbb{C}}) \subset \hat{\mathbb{R}}, \)

imaginary shift locus
\( \Leftrightarrow \ J(f_{\mathbb{C}}) \cap \hat{\mathbb{R}} = \emptyset. \)
Critically finite maps fall into three types:

**Hyperbolic.**

*(A quadratic map is hyperbolic if both critical orbits converge to attracting periodic orbits.)*

**Half-Hyperbolic** if only one critical orbit converges to an attracting periodic orbit.

**Totally Non-Hyperbolic** if no critical orbit converges to an attracting periodic orbit.

Every hyperbolic and critically finite $f$ is the center point of a hyperbolic component in the connectedness locus.

*But if $f$ is totally non-hyperbolic then $J(f_C)$ is the entire Riemann sphere.*
Theorem: Any critically finite point in $M$ is uniquely determined by its combinatorics.
By definition, a **bone** in $M$ is a maximal smooth curve on which one of the two critical points is periodic.

In the unimodal region:

Theorem of Filom + Yan Gao: Each bone in the unimodal region is a smooth curve from polynomial to co-polynomial; and each locus of constant topological entropy is connected.
Misiurewicz showed that a map with constant \(|\text{slope}| = s > 1\) has topological entropy \(h = \log(s)\).

Here \(s = (\sqrt{5} + 1)/2\).

**Theorem.** To every critically finite co-polynomial there corresponds a critically finite polynomial.

(Conversely to almost every critically finite polynomial there corresponds a critically finite co-polynomial.)
The Filom-Pilgrim family of maps.

Given $0 < p/q < 1$ consider the combinatorics $(m_0, m_1, \cdots, m_{q-1})$ with $m_k \equiv k + p \pmod{q}$.

Here is the PL model for $p/q = 2/5$.

Since the longest edge maps onto the entire interval, we get

$$s^4 - s^3 - s^2 - s - 1 = 0.$$ 

Thus the topological entropy $\log(s_q)$ depends only on $q$. 
Theorem (Filom and Pilgrim).

In the $++-+$ region, loci of constant topological entropy can have arbitrarily many connected components.

Step 1: Each hyperbolic component $H(p/q)$ contains a curve leading from the center point to the ideal point $(\Sigma, \Delta) = (-.5, 1)$.

Furthermore these curves depend monotonically on $p/q$. 
For each $q \geq 3$ there is a curve $C_q$ of constant entropy $\log(s_q)$ which extends from the left ideal point $(-.5, 1)$ through $f_{1/q}$ to the right and ideal point $(+.5, 1)$. (Compare Slide 14.)

The curves $C_q$ divide the square region $-.5 \leq \Sigma \leq .5$ into disjoint connected open sets $U_3, U_4, U_5, \cdots$.

If $q > n(n - 1)$ is prime, there is a $p$ so that $f_{p/q} \in U_n$. 
Key Lemma. As \( f \) tends to the ideal point \((-0.5, 1)\) within the hyperbolic component \( H(p/q) \), the multiplier of the fixed point of \( f_c \) in the upper half plane tends to \( e^{2\pi i p/q} \).
If we are given two of the fixed point multipliers for a quadratic rational map, then the third is given by

\[ \mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1 \mu_2}. \]

Now suppose that \( \mu_1 = r e^{i\theta} \) and \( \mu_2 = r e^{-i\theta} \). Then

\[ \mu_3 = \frac{2 - 2r \cos(\theta)}{1 - r^2}. \]

If the map has real critical points then we must have \( r \geq 1 \).

If \( \mu_3 \to -\infty \), it follows easily that \( r \to 1 \). \( \Box \)
This is a hypothetical picture of what we would get if we replace the ideal point \((-0.5, 1)\) by an entire vertical interval of angles \(0 \leq \theta \leq 1/2\).

This completes the outlined proof of non-monotonicity.

For further details see Filom and Pilgrim’s paper.
Examples of the Thurston Pullback Map.

Wittner ($(5, 6, 1, 0, 2, 3)$) Filom-Pilgrim $3/7 : (3, 4, 5, 6, 0, 1, 2))$

Weakly Obstructed ($(3, 4, 3, 2, 1, 0)$) Str. Obstructed ($(3, 5, 4, 0, 1, 2)$)

Str. Obstructed ($(2, 3, 4, 6, 4, 0, 1)$) Exceptional ($(1, 3, 4, 3, 1, 0)$)


In preparation:
A more detailed manuscript, plus an interactive web site.