Curvature and Résidu Itératif

John Milnor

Stony Brook University

(work with Araceli Bonifant)

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Example: The Rounded Mandelbrot Set

Connectedness locus for the family of maps

\[ g_\lambda(z) = z^2 + \lambda z \]
Two Fixed Point Invariants.

Consider an isolated fixed point \( z_0 = f(z_0) \) of a holomorphic map \( f : \mathbb{C} \to \mathbb{C} \).

One basic invariant is the **multiplier** \( \lambda = f'(z_0) \).

Another is the **holomorphic index**

\[
\text{ind}(f, z_0) = \frac{1}{2\pi i} \oint_{z_0} \frac{dz}{z - f(z)}.
\]

For a fixed point with \( \lambda \neq 1 \), it is not hard to check that

\[
\text{ind}(f, z_0) = \frac{1}{1 - \lambda}.
\]

If \( \lambda = 1 \), then for any small \( \epsilon \neq 0 \), the one local fixed point for \( f \) will split into \( n \) nearby fixed points \( z_1, \ldots, z_n \) for \( f + \epsilon \), where \( n \geq 2 \) is called the **fixed point multiplicity**.

Furthermore:

\[
\lambda_j = (f + \epsilon)'(z_j) \neq 1.
\]

**Assertion**: \( \text{ind}(f, z_0) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \text{ind}(f + \epsilon, z_j) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \frac{1}{1 - \lambda_j} \).
Résidu Itératif (Jean Écalle, 1976).

Definition. If \( \lambda = 1 \), the difference

\[
\text{résit}(f, z_0) = \frac{n}{2} - \text{ind}(f, z_0)
\]

is called the résidu itératif.

Theorem. For any integer \( k \geq 1 \):

\[
\text{résit}(f^{\circ k}, z_0) = \frac{1}{k} \text{résit}(f, z_0).
\] (1)

Proof. For \( \epsilon \approx 0 \) the fixed point with multiplier one for \( f \) splits into \( n \) fixed points for \( f + \epsilon \) with multipliers \( \lambda_1, \ldots, \lambda_n \approx 1 \).

Therefore

\[
\text{résit}(f^{\circ k}, z_0) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \left( \frac{1}{2} - \frac{1}{1 - \lambda_j^k} \right).
\]

Lemma. \( \left( \frac{1}{2} - \frac{1}{1 - \lambda^k} \right) = \frac{1}{k} \left( \frac{1}{2} - \frac{1}{1 - \lambda} \right) + o(1) \) as \( \lambda \to 1 \).

Equation (1) then follows easily. \( \Box \)
Extended definition (Buff and Epstein, 2002).

The résidu itératif can be defined at any parabolic fixed point, so that

\[ \text{résit}(f^\circ k, z_0) = \text{résit}(f, z_0)/k. \]

If \( \lambda_0 = f'(z_0) \) is a \( p \)-th root of unity, simply define:

\[ \text{résit}(f, z_0) := p \cdot \text{résit}(f^\circ p, z_0), \]

using the Ecalle definition on the right.

We want to relate the résidu itératif to curvature in parameter space.

In the family \( \{ z \mapsto z^2 + \lambda z \} \), each root of unity \( \lambda_0 = e^{2\pi i q/p} \) is a common boundary point for the main hyperbolic component \( H \), and for a satellite component \( S(q/p) \).

**Theorem.** The real part \( \Re(\text{résit}(g_{\lambda_0}, 0)) \) is equal to the average of the two curvatures:

\[ K(\partial H, \lambda_0) = +1 \quad \text{and} \quad K(\partial S(q/p), \lambda_0). \]
Examples:

<table>
<thead>
<tr>
<th>$q/p$</th>
<th>$\text{résit}(g_{\exp(2\pi i q/p)})$</th>
<th>$K_S$</th>
<th>$K_S/p^2$</th>
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<tr>
<td>1/2</td>
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<td>4.5</td>
<td>1.125</td>
</tr>
</tbody>
</table>

Here

\[
\Re(\text{résit}) = \frac{(1 + K_S)}{2} \iff K_S = 2 \Re(\text{résit}) - 1.
\]
A Convenient Notation.

Let \( \alpha \mapsto \beta \) be a twice differentiable (or holomorphic) map with \( \frac{d\beta}{d\alpha} \neq 0 \).

Define the **nonlinearity** of \( \alpha \mapsto \beta \) to be the ratio

\[
((\alpha, \beta)) = \frac{\frac{d^2 \beta}{d\alpha^2}}{\frac{d\beta}{d\alpha}}.
\]

Thus \( ((\alpha, \beta)) = 0 \iff \beta = c_1\alpha + c_2 \).

The Chain Rule for \( \alpha \mapsto \beta \mapsto \gamma \):

\[
((\alpha, \gamma)) = ((\alpha, \beta)) + ((\beta, \gamma)) \frac{d\beta}{d\alpha}.
\]

This follows from the identity

\[
\log \frac{d\gamma}{d\alpha} \equiv \log \frac{d\beta}{d\alpha} + \log \frac{d\gamma}{d\beta} \quad \text{(mod } 2\pi i)\),
\]

by differentiating with respect to \( \alpha \).
A Simple Example.

The chain rule for the composition $c\alpha \mapsto \alpha \mapsto \beta$ yields

$$(((c\alpha, \beta)) = (((c\alpha, \alpha)) + ((\alpha, \beta)) \frac{d\alpha}{dc\alpha}$$

$$= 0 + (((\alpha, \beta))/c.$$
Curvature.

For a curve $s \mapsto w(s)$ parametrized by arclength, we have $|w'| = |dw/ds| = 1$, and

$$(s, w) = w''/w' = iK, \quad \text{hence} \quad K = \Im((s, w)).$$

For an arbitrary smooth parametrization $t \mapsto s \mapsto w$, it follows that $((t, w)) = ((t, s)) + iK \, ds/dt$, hence

$$\Im((t, w)) = 0 + K \frac{ds}{dt} = K \left| \frac{dw}{dt} \right|.$$
Again let \( g_\lambda(z) = z^2 + \lambda z \). 

Thus \( g_\lambda \) has a fixed point at \( z = 0 \) with multiplier \( \lambda \). If \( \lambda \approx \lambda_0 = e^{2\pi i q/p} \), then \( g_\lambda \) has a period \( p \) orbit near zero.

Let \( \mu \) be its multiplier. Then \( \text{ind}(g^o_\lambda, 0) = \left( \frac{1}{1-\lambda^p} + \frac{p}{1-\mu} \right) \).

\[
\implies \text{ind}(g^o_\lambda, 0) = \lim_{\lambda \to \lambda_0} \left( \frac{1}{1-\lambda^p} + \frac{p}{1-\mu} \right) .
\]

Corollaries:

1. \( \mu = 1 \) if and only if \( \lambda^p = 1 \).

2. \( \mu \) is locally a holomorphic function of \( \lambda \), or of \( \lambda^p \).

3. The derivative at \( \lambda_0 \) is \( d\mu/d\lambda^p = -p \),

\[
\iff \quad d \log \mu / d \log \lambda = -p^2 .
\]

4. \( \text{ind}(f^o_{\lambda_0}) = ((1 - \lambda^p, 1 - \mu))/2 \) evaluated at \( \lambda_0 \),

\[
= -((\lambda^p, \mu))/2 .
\]
Computation of the résidu itératif.

**Theorem:** For any $k \geq 1$ we have

$$\text{résit}(f_{\lambda_0}^k, 0) = \frac{((\log \lambda, \log \mu))}{2k}.$$ 

**Proof outline:** Start with $-\text{ind}(f_{\lambda_0}^p, 0) = (\lambda^p, \mu)/2$.

First express $((\lambda^p, \mu))$ as a linear function of $((\log \lambda, \mu))$, using the chain rule for the composition $\log \lambda^p \mapsto \lambda^p \mapsto \mu$ (where $\log(\lambda^p) = p \log(\lambda)$).

Then express $((\log \lambda, \mu))$ as a function of $((\log \lambda, \log \mu))$, using the chain rule for the composition $\log \lambda \mapsto \log \mu \mapsto \mu$.

The result will be

$$-\text{ind}(f_{\lambda_0}^p, 0) = \frac{((\log \lambda, \log \mu))}{2p} - \frac{p + 1}{2}.$$ 

Adding $\frac{p + 1}{2}$ to both sides, we obtain

$$\text{résit}(f_{\lambda_0}^p, 0) = \frac{((\log \lambda, \log \mu))}{2p}. \quad \square$$
For a holomorphically parametrized family of maps

\[ F_\xi : \mathbb{C} \to \mathbb{C}. \]

Suppose that:

(1) each \( F_\xi \) has a specified fixed point \( z_0(\xi) \) which varies holomorphically with \( \xi \),

(2) the multiplier \( \lambda = \lambda(\xi) \) of this fixed point satisfies \( d\lambda/d\xi \neq 0 \), and

(3) \( \xi_0 \) is a parameter for which \( z_0(\xi_0) \) is a fixed point of \textit{parabolic multiplicity} \( m = 1 \).

parabolic multiplicity \( m = 1 \) \hspace{5cm} \text{parabolic multiplicity} \( m = 2 \)
Cubic Examples

\[ f(z) = z^3 + iz, \quad \text{parabolic multiplicity} \quad m = 1 \]

\[ z \mapsto z^3 + iz^2 - z, \quad \text{parabolic multiplicity} \quad m = 2 \]
Recall the conditions for a family of maps $F_\xi : \mathbb{C} \to \mathbb{C}$.

Suppose that:

1. Each $F_\xi$ has a specified fixed point $z_0(\xi)$ which varies holomorphically with $\xi$,

2. The multiplier $\lambda = \lambda(\xi)$ of this fixed point satisfies $d\lambda/d\xi \neq 0$, and

3. $\xi_0$ is a parameter for which $z_0(\xi_0)$ is a fixed point of parabolic multiplicity one.

Theorem. Then

$$\text{résit}(F_{\xi_0}, z_0) = \frac{((\log \lambda, \log \mu))}{2}$$

$$= \frac{((\log \lambda, \xi)) + p^2((\log \mu, \xi))}{2} .$$
Curvature Again.

Make the substitutions \( \lambda = e^{i\phi} \) and \( \mu = e^{i\theta} \).

Thus real values of \( \phi \) (or \( \theta \)) parametrize \( \partial H \) (or \( \partial S \)).

Then

\[
\text{résit}(F_{\xi_0}) = \frac{((\log \lambda, \xi)) + p^2((\log \mu, \xi))}{2} = \frac{((\phi, \xi)) + p^2((\theta, \xi))}{2i}
\]

Corollary.

\[
\Re(\text{résit}(F_{\xi_0})) = \frac{K(\partial H, \xi_0) + K(\partial S, \xi_0)}{2} \left| \frac{d\xi}{d\lambda} \right|.
\]
Limiting Shape?

What can one say about the “sizes” and “shapes” of the various satellites $S(q/p)$ of the rounded Mandelbrot set?

**Question:** Given a sequence of fractions $q_j/p_j$ tending to a limit, when do the $S(q_j/p_j)$ have a limiting shape? Each $S(q/p)$ has a preferred center point $c = c(q/p)$, defined by the equation $\mu = 0$.

Define the “radius” $r = r(q/p)$ to be the distance $|c - \lambda_0|$, where $\lambda_0 = e^{2\pi i q/p}$ is the root point.

Then the product $r K_S$ associated with a given satellite is scale invariant measure of distortion, equal to one for a round disk.
Approximating 1/3 by Farey Neighbors

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<th>$2 , \text{résit}/p^2$</th>
<th>$r_S K_S$</th>
<th>$q/p$</th>
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<th>$r_S K_S$</th>
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Approximating $(\sqrt{5} - 1)/2$.

(Illustrating an ongoing project by D. Dudko, M. Lyubich and N. Selinger.)

<table>
<thead>
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<th>$q/p$</th>
<th>$2 \text{résit}/p^2$</th>
<th>$r_SK_S$</th>
<th>$q/p$</th>
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