Two Moduli Spaces

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December 4, 2017
Outline: Two Examples.

The object of this talk will be to describe two examples of smooth group actions on smooth manifolds.

Easier Example (Divisors on $\mathbb{P}^1$):

*The group $G(\mathbb{P}^1) = \text{PGL}_2(\mathbb{C})$ of Möbius automorphisms of the Riemann sphere $\mathbb{P}^1$ acts on the space $\mathcal{D}_n$ of effective divisors of degree $n$ on $\mathbb{P}^1$, with quotient space $\mathcal{D}_n/G(\mathbb{P}^1)$.*

Much Harder Example (Curves in $\mathbb{P}^2$):

*The group $G(\mathbb{P}^2) = \text{PGL}_3(\mathbb{C})$ of projective automorphisms of the complex projective plane $\mathbb{P}^2$, acts on the projective compactification $\mathcal{C}_n$ of the space of algebraic curves of degree $n$ in $\mathbb{P}^2$, with quotient space $\mathcal{C}_n/G(\mathbb{P}^2)$.*

In both cases, some parts of the quotient space are beautiful objects to study, but other parts are rather nasty.

**Basic Problem: Which parts are which?**
The additive group $G$ of real numbers acts on $\mathbb{R}^2$ by
$$g_t(x, y) = (e^t x, e^{-t} y).$$

Most orbits are smooth curves; but the origin is a single point orbit.

If we remove the origin, then the quotient space
$$\left(\mathbb{R}^2 \setminus \{(0, 0)\}\right)/G$$
is locally a smooth manifold.

But it is only locally Hausdorff.
Part 1. The Space $\mathcal{D}_n$ of Degree $n$ Divisors on $\mathbb{P}^1$.

**Definition:** An **effective divisor** $\mathcal{D}$ of degree $n$ on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ is a formal sum

$$\mathcal{D} = m_1\langle p_1 \rangle + \cdots + m_k\langle p_k \rangle,$$

where the $m_j > 0$ are integers with $\sum_j m_j = n$, and the $p_j$ are distinct points of $\mathbb{P}^1$.

Each such $\mathcal{D}$ can be identified with the set of zeros, counted with multiplicity, for some non-zero homogeneous polynomial

$$\Phi(x, y) = c_0x^n + c_1x^{n-1}y + \cdots + c_ny^n.$$

It follows that the space $\mathcal{D}_n$ of all such divisors is isomorphic to the projective space $\mathbb{P}^n(\mathbb{C})$.

The group $G = G(\mathbb{P}^1)$ of Möbius automorphisms of $\mathbb{P}^1$ acts on $\mathcal{D}_n$.

Two integer invariants under the action of $G$:

- **The number of points** $k = \# |\mathcal{D}|$ **in the support**
  $$|\mathcal{D}| = \{p_1, \ldots, p_k\} \subset \mathbb{P}^1.$$  
- **The maximum** $m_{\text{max}} = \max\{m_1, \ldots, m_k\}$ **of the multiplicities of the various points** of $|\mathcal{D}|$. 

Finite Stabilizers

Definition. The stabilizer $G_D$ of a divisor $D$ is the subgroup of $G$ consisting of all $g \in G$ with $g(D) = D$.

Lemma. The stabilizer $G_D$ is finite if and only if the support $|D| \subset \mathbb{P}^1$ contains at least three elements.

Proof. For any $D$ there is a natural homomorphism $G_D \rightarrow \mathcal{S}_{|D|}$, where $\mathcal{S}_{|D|}$ is the symmetric group consisting of all permutations of the finite set $|D|$. If $\#|D| \geq 3$, since any Möbius transformation which fixes three distinct points must be the identity, it follows that $G_D$ maps isomorphically onto a subgroup of $\mathcal{S}_{|D|}$.

Now suppose that $\#|D| \leq 2$. After a Möbius transformation, we may assume that $|D| \subset \{0, \infty\}$. (Here I am identifying the Riemann sphere with $\mathbb{C} \cup \{\infty\}$.) The group $G_D$ then contains infinitely many transformations of the form $g_\kappa(z) = \kappa z$ with $\kappa \neq 0$. □
Let $\mathcal{D}_n^{f\text{stab}}$ be the open subset of $\mathcal{D}_n$ consisting of all divisors with finite stabilizer (⇐⇒ all divisors with $\#|\mathcal{D}| \geq 3$).

**Definition.** The quotient $\mathcal{M}_n = \mathcal{D}_n^{f\text{stab}} / G$ will be called the *moduli space* for divisors, under the action of $G$.

**Proposition 1.** This quotient space $\mathcal{M}_n$ is a $T_1$-space, that is:

Every point of $\mathcal{M}_n$ is a closed subset,

$\iff$ Every $G$-orbit $((\mathcal{D})) = \{g(\mathcal{D}) ; g \in G\}$ in $\mathcal{D}_n^{f\text{stab}}$ is closed as a subset of $\mathcal{D}_n^{f\text{stab}}$.

In other words, every $\mathcal{D}' \in \mathcal{D}_n$ which belongs to the topological boundary $((\mathcal{D})) \setminus ((\mathcal{D}))$ must have infinite stabilizer.

*To prove Proposition 1, we must study elements of $G$ which are “close to infinity” in $G$.*
Distortion Lemma for Möbius Transformations.

Using the spherical metric on $\mathbb{P}^1$, let $N_{\varepsilon}(p)$ be the open $\varepsilon$-neighborhood of $p$.

**Lemma.** For any $\varepsilon > 0$ there is a large compact set $K = K_\varepsilon \subset G$ with the following property: For any $g \notin K$, there are (not necessarily distinct) points $p$ and $q$ such that $g(N_{\varepsilon}(p)) \cup N_{\varepsilon}(q) = \mathbb{P}^1$.

Thus points outside of $N_{\varepsilon}(p)$ map inside $N_{\varepsilon}(q)$.

**(Proof Outline.** The proof for the group of diagonal transformations $d(x : y) = (\kappa x : y)$ is easy. But any $g \in G$ can be written as a product $g = r \circ d \circ r'$ where $r$ and $r'$ are rotations of the Riemann sphere and $d$ is diagonal. . . .)
Proof of Proposition 1: Points of $\mathcal{M}_n$ are closed.

To prove: Every $G$-orbit $(\mathcal{D}) \subset \mathcal{D}_n^{\text{fstab}}$ is closed as a subset of $\mathcal{D}_n^{\text{fstab}}$.

Choose $\varepsilon$ small enough so that any two points of $|\mathcal{D}|$ have distance $> 2\varepsilon$ from each other.

$\implies$ No $\varepsilon$-ball contains more than one point of $|\mathcal{D}|$.

Given any $g \notin K_\varepsilon$, choose $p$ and $q$ as in the Distortion Lemma. It follows that:

all but possibly one of the points of $g(|\mathcal{D}|)$ lie in $N_\varepsilon(q)$.

Now suppose that we are given a sequence of points $g_j(\mathcal{D}) \in (\mathcal{D})$ converging to $\mathcal{D}' \in \mathcal{D}_n$.

Case 1. If all $g_j \in K \subset G$, then $\mathcal{D}' \in (\mathcal{D})$.

Case 2. If $g_j \in K_{\varepsilon_j}$ with $\varepsilon_j \to 0$, then $|\mathcal{D}'|$ has at most two points, so $\mathcal{D}' \notin \mathcal{D}_n^{\text{fstab}}$. 

\qed
The Cases \( n \leq 4 \) are very special.

\[ \mathcal{M}_3 \] is a single point.

\[ \mathcal{M}_4 \cong \mathbb{P}^1 \] is a 2-sphere.

Proof Outline: Four distinct points in \( \mathbb{P}^1 \) determine a 2-fold branched covering which is an elliptic curve; characterized by the classical invariant \( j(C) \in \mathbb{C} \). Thus the open subset corresponding to divisors with four distinct points is canonically isomorphic to \( \mathbb{C} \).

But there is one other \( G \)-orbit

\[ (2\langle p \rangle + \langle q \rangle + \langle r \rangle) \subset \mathcal{D}_4^{\text{stab}} \]

consisting of divisors with only three distinct points.

It follows easily that \( \mathcal{M}_4 \) is homeomorphic to the one point compactification \( \mathbb{C} \cup \{ \infty \} \cong \mathbb{P}^1 \).
Theorem. For $n \geq 5$, $\mathcal{M}_n$ has a unique maximal open subset $\mathcal{M}_n^{\text{Haus}}$ which is Hausdorff.

Here $\mathcal{M}_n^{\text{Haus}}$ is the set of all images $\pi(\mathcal{D}) \in \mathcal{M}_n$ where $\mathcal{D}$ is a divisor with maximum multiplicity $m_{\text{max}} < n/2$

(where $\pi : \mathcal{D}_{n}^{\text{f stab}} \to \mathcal{M}_n$ denotes the projection map).

$\mathcal{M}_n^{\text{Haus}}$ is compact if $n$ is odd;
but non-compact if $n$ is even.

$\mathcal{M}_n^{\text{Haus}}$ is an orbifold of complex dimension $n - 3$.

Points of $\mathcal{M}_n$ outside of $\mathcal{M}_n^{\text{Haus}}$
are not even locally Hausdorff.
Definition. The action of a Lie group $G$ on a space $X$ is \textit{proper} if, for every $x, y \in X$, there are neighborhoods $U$ and $V$ so that the set of group elements with $g(U) \cap V \neq \emptyset$ has compact closure within $G$.

Standard Theorem. The quotient $X/G$ of a Hausdorff space under a proper action is a Hausdorff space.

Using the Distortion Theorem, one can show that the action of $G(\mathbb{P}^1)$ on the space of divisors with $m_{\text{max}} < n/2$ is proper.
Non Locally Hausdorff Points for \( n > 4 \)

To fix ideas, let \( n = 5 \). Consider two divisors of the form
\[ D = D_2 + 3\langle\infty\rangle \quad \text{and} \quad D' = D_3 + 2\langle\infty\rangle \]
in \( D_5 \), where
\[ D_2 = \langle p \rangle + \langle q \rangle \quad \text{and} \quad D_3 = \langle p' \rangle + \langle q' \rangle + \langle r' \rangle. \]

Let \( g_\kappa(z) = \kappa^2/z \), with \( \kappa \gg 1 \);
so that \( |z| < \kappa \iff |g_\kappa(z)| > \kappa \).

Then the two divisors \( D_2 + g_\kappa(D_3) \) and \( D_3 + g_\kappa(D_2) \)
belong to the same \( G \)-orbit.

As \( \kappa \to \infty \), the first converges to \( D \)
and the second converges to \( D' \).

Thus every neighborhood of \( \pi(D) \in M_5 \)
intersects every neighborhood of \( \pi(D') \).

Since \( D' \) can be arbitrarily close to \( D \), this proves that \( M_5 \) is not locally Hausdorff at the point \( \pi(D) \).

Definition. An effective 1-cycle of degree $n \geq 1$ on the complex projective plane $\mathbb{P}^2$ is a formal sum

$$C = m_1 \cdot C_1 + \cdots + m_k \cdot C_k,$$

where each $C_j$ is an irreducible complex curve, where the $m_j \geq 1$ are integers, and where $n = \sum_j m_j \deg(C_j)$.

The space $\mathcal{C}_n$ of all effective 1-cycles can be given the structure of a complex projective space of dimension $n(n + 3)/2$. (In fact each non-zero homogeneous polynomial $\Phi(x, y, z)$ of degree $n$ has a zero locus consisting of irreducible curves $C_j$, each counted with some multiplicity $m_j \geq 1$; yielding a 1-cycle.)

The group $G = G(\mathbb{P}^2) = \text{PGL}_3(\mathbb{C})$ of all automorphisms of $\mathbb{P}^2$ acts on $\mathbb{P}^2$ and hence on the space $\mathcal{C}_n$.

The stabilizer $G_C$ of $C \in \mathcal{C}_n$ is just the group consisting of all projective automorphisms which map $C$ to itself.

This stabilizer $G_C$ may be either finite or infinite.
W-curves (and cycles).

Curves with infinite stabilizer were first studied by Felix Klein and Sophus Lie, who called them *W-curves*.

Some examples:

Let $W_n \subset C_n$ be the algebraic set consisting of all cycles with infinite stabilizer. ($W_n$ is a union of finitely many maximal irreducible subvarieties of $C_n$, of varying dimension.)

**Note:** $C$ has finite stabilizer if and only if the $G$-orbit $((C)) \subset C_n$ has dimension 8.

In fact $\dim ((C)) + \dim (G_C) = \dim (G) = 8$, where $\dim (G_C) = 0$ $\iff$ $G_C$ is finite.
The Moduli Space $\mathcal{M}_n$.

The complement $C^{\text{f stab}}_n = C_n \setminus \mathcal{W}_n$ is the open set consisting of all cycles with \textit{finite stabilizer}.

**Definition.** The quotient space $\mathcal{M}_n = C^{\text{f stab}}_n / G$, will be called the \textit{moduli space} for plane cycles of degree $n$.

**Examples.** $\mathcal{M}_1 = \mathcal{M}_2 = \emptyset$.

The moduli space $\mathcal{M}_3$ for cubic curves in $\mathbb{P}^2$ is canonically isomorphic to the moduli space $\mathcal{M}_4$ for divisors in $\mathbb{P}^1$. Each has two “ramified points” corresponding to points with extra symmetry (= larger stabilizer). Each also has one “improper point” where the group action is not proper.

Thus $\mathcal{M}_3 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$. 

\[\begin{array}{c}
\infty \\
\mathcal{M}_3
\end{array} \overset{\sim}{\longrightarrow} \begin{array}{c}
\mathbb{C} \\
\mathbb{C} \cup \{\infty\} \\
\mathbb{P}^1
\end{array}\]
\( \mathbb{M}_n \) is a \( T_1 \)-space.

Cartoon of \( \mathcal{C}_n \), showing a typical \( G \)-orbit in red:

**Theorem.** The topological boundary of any \( G \)-orbit in \( \mathcal{C}_n \) is contained in the closed subset \( \mathcal{W}_n \).

[Ghizzetti 1936; Aluffi and Faber 2010.]

\( \implies \) Every \( G \)-orbit of cycles with finite stabilizer is closed as a subset of \( \mathcal{C}^{\text{f stab}}_n \).

\( \implies \) Every point in \( \mathbb{M}_n \) is a closed set.
The Distortion Lemma for $\mathbb{P}^2$.

**Lemma.** Given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset G(\mathbb{P}^2)$ with the following property.

For any $g \notin K_\varepsilon$ there exists either:

1. a point $p \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$ such that $g(N_\varepsilon(p)) \cup N_\varepsilon(L) = \mathbb{P}^2$

   (so that $g$ maps every point outside of $N_\varepsilon(p)$ into $N_\varepsilon(L)$),

   or 2. a line $L' \subset \mathbb{P}^2$ and a point $q \in \mathbb{P}^2$ such that $g(N_\varepsilon(L')) \cup N_\varepsilon(q) = \mathbb{P}^2$

   (so that $g$ maps every point outside of $N_\varepsilon(L')$ into $N_\varepsilon(q)$).
The Genus Invariant of a Singularity.

Let \( p \) be a singular point of a complex curve \( C \subset \mathbb{P}^2 \). Let \( N_\varepsilon \) be a small round ball centered at \( p \).

If \( C' \) is a smooth curve which closely approximates \( C \), then
\[
S_p = C' \cap N_\varepsilon
\]
is a compact connected Riemann-surface-with-boundary.

Its genus \( g(S_p) \) will be called **the genus of the singularity** \( p \in C \).

**Examples:** For a cusp singularity \( x^p = y^q \) the genus is \((p - 1)(q - 1)/2\).

If \( C \) is locally the union of \( k \) smooth branches \( B_j \), then the genus is
\[
-1 + \sum_{i<j} B_i \cdot B_j.
\]
Two Properties of the Genus.

**Monotonicity.** Suppose that

\[ S = S_1 \cup \cdots \cup S_k \subset S', \]

where the \( S_j \) are disjoint compact Riemann-surfaces-with-boundary, and \( S' \) is another compact Riemann surface, possibly with boundary. Then

\[ g(S) := \sum g(S_j) \leq g(S). \]

**Scissors and Paste.** Suppose that \( k \) disjoint embedded curves cut the closed Riemann surface \( S \) into \( \ell \) subspaces with boundary \( S_j \). Then

\[ g(S) = k + 1 - \ell + \sum g(S_j). \]

(This follows from the Euler characteristic identity

\[ \chi(S) = \sum \chi(S_j). \])
A Hypothesis which implies Proper Action.

For any line $L \subset \mathbb{P}^2$ and any specified curve $C$ we can form the intersection $S_L = C' \cap \overline{N}_\varepsilon(L)$, where $\varepsilon$ is small and $C'$ is a very close generic approximation to $C$.

**Lemma.** If

$$g(C' \setminus S_p) > g(S_L)$$

for every $p \in |C|$ and every $L \subset \mathbb{P}^2$,

then the action of $G$ is locally proper at $C$. 

Let $S_p^* = \overline{C'} \setminus S_p$. Then

$$S_p \cup S_p^* = C', \quad S_p \cap S_p^* = (\text{union of } k \text{ circles}).$$

Therefore

$$g(S_p) + g(S_p^*) + k - \ell + 1 = g(C') = \binom{n-1}{2}.$$

Here $\ell \geq 2$ is the number of components of $S_p$ plus the number of components of $S_p^*$.

Define the **augmented genus** of $S_p$ to be

$$g^+(S_p) = g(S_p) + k - 1.$$

Together the Lemma, this formula yields:

**Theorem.** If $g^+(S_p) + g(S_L) < g(C')$ for every $p \in C$ and every $L \subset \mathbb{P}^2$,

then the action of $G$ is locally proper at $C$. 

Scissors and Paste.
Let \( \mathcal{U}_n \subset \mathcal{C}_n \) be the open set consisting of curves with no singularities other than simple double points and cubic cusps.

**Corollary.** *If \( n \geq 4 \) then the action of \( G(\mathbb{P}^2) \) is locally proper throughout \( \mathcal{U}_n \).*

In fact the action is proper throughout \( \mathcal{U}_n \), so the quotient space \( \mathcal{U}_n / G(\mathbb{P}^2) \subset \mathcal{M}_n \) is a Hausdorff space.


A. Ghizzetti, *Determinazione delle curve limiti di un sistema continuo $\infty^1$ di curve piane omografiche*, Rendiconti Reale accademia dei Lincei. 23 (1936) 261–264.

