Pasting Together Julia Sets — a worked out example of mating

John Milnor, December 18, 1997

Abstract. *The operation of "mating" two suitable complex polynomial maps f_1 and f_2 constructs a new dynamical system by carefully pasting together the boundaries of their filled Julia sets so as to obtain a copy of the Riemann sphere, together with a rational map $f_1 \perp\!\!\!\perp f_2$ from this sphere to itself. This construction is particularly hard to visualize when the filled Julia sets $K(f_i)$ are dendrites, with no interior. This note will work out an explicit example of this type, with effectively computable maps from $K(f_1)$ and $K(f_2)$ onto the Riemann sphere.*

Contents.

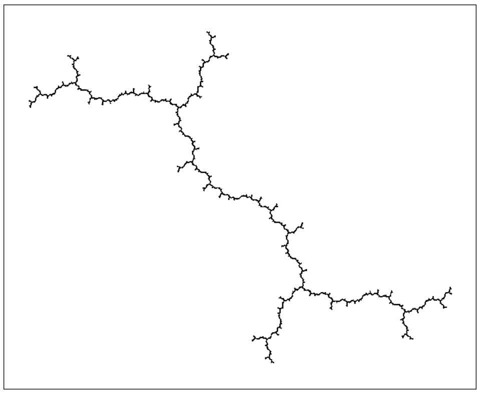


Figure 1. The Julia set $K = \partial K$ for $f(z) = z^2 + c$, where $c \cong -0.228 + 1.115i$.

§1. Introduction.

The operation of *mating*, first described by [Douady 1982], has been shown to exist for suitable pairs of quadratic polynomial maps by [Tan Lei 1992], [Rees], and [Shishikura]. (See §2.) In an attempt to understand this construction, this paper concentrates on one very special example. We consider the (filled) Julia set $K = K(f)$ which is illustrated in Figure 1 and described more precisely in §2. The mating $f \perp\!\!\!\perp f$ exists according to Shishikura. This means that we can form a full Riemann sphere by pasting two copies of $K = \partial K$

together, in such a way that each copy of K covers the full Riemann sphere, while the map f on each copy corresponds to a smooth quadratic rational map from this sphere to itself. We will give a computationally effective description for this particular example, showing just how such a dendrite can map onto a sphere. The construction is closely related to a well known measure-preserving area filling curve, with associated fractal self-similar tiling,[1] which is known as the "Heighway Dragon". The resulting rational map $F \cong f \perp\!\!\!\perp f$, where $F(z) = (i/2)\,(z+z^{-1})$, can also be described as a *Lattès mapping*, that is as the quotient of a rigid expanding map on a torus. (This is not a new remark: it has been known to experts for many years.) It is this juxtaposition of these two quite different constructions which makes the explicit description possible. Section 3 will describe this example, and Section 4 will characterize and compute the associated semiconjugacy from the angle doubling map on the circle to this rational map F. Section 5 shows that this semiconjugacy carries 1-dimensional measure on the circle to 2-dimensional measure on the sphere, section 6 discusses associated fractal tilings, and the last section asks further questions. There are four appendices. The second describes further examples, supplied by Shishikura, showing that every quadratic Lattès mapping can be given the structure of a mating in one or more ways, and the last describes some exotic topological conjugacies between filled Julia set, suggested by Douady.

Acknowledgement. I want to thank Adrien Douady, Adam Epstein, Mitsuhiro Shishikura, Tan Lei, and Saeed Zakeri for their substantial help in writing this paper, and also the Gabriella and Paul Rosenbaum Foundation and the National Science Foundation (Grant DMS-9505833) for their support of mathematical activity at Stony Brook.

§2. The Mating Construction.

Some standard definitions. (See for example [Milnor 1990], as well as [Goldberg & Milnor].) Let $f : \mathbf{C} \to \mathbf{C}$ be a polynomial map of degree $d \geq 2$. The *filled Julia set* $K = K(f) \subset \mathbf{C}$ can be defined as the union of all bounded orbits. Its topological boundary ∂K is equal to the *Julia set* of f. If K is connected, then its complement is conformally isomorphic to the complement of the closed unit disk, and this conformal isomorphism

$$\varphi : \mathbf{C} \smallsetminus \overline{\mathbf{D}} \;\to\; \mathbf{C} \smallsetminus K$$

can be chosen so as to conjugate the d-th power map on $\mathbf{C} \smallsetminus \overline{\mathbf{D}}$ to the map f on $\mathbf{C} \smallsetminus K$. That is:

$$\varphi(w^d) \;=\; f(\varphi(w))\;. \tag{1}$$

If K is also locally connected, then according to Carathéodory φ extends continuously over the boundary, to yield a map from the unit circle $\partial\mathbf{D}$ onto the Julia set ∂K, still satisfying (1). It will be convenient to parametrize the unit circle by the real numbers modulo one. The resulting map

$$\gamma \;:\; \mathbf{R}/\mathbf{Z} \;\to\; \partial K(f)\;,$$

defined by

$$\gamma(t) \;=\; \gamma_f(t) \;=\; \lim_{r \to 1} \varphi(re^{2\pi i t})\;,$$

will be called the *Carathéodory semiconjugacy* from the circle of reals modulo one to the Julia

[1] See 4.7. and Figures 15, 16. This construction was discovered by John Heighway, a physicist at NASA, circa 1966. Compare [Davis & Knuth], [Edwards], and even [Crichton].

set. With this notation, the semiconjugacy identity (1) in the degree d case takes the form

$$\gamma(d \cdot t) \;=\; f(\gamma(t)) \;. \tag{1'}$$

Equivalently, if we define the *external ray* $\mathcal{R}_t = \mathcal{R}_t(K)$ of angle t to be the curve consisting of all points $\varphi(re^{2\pi i t}) \in \mathbf{C} \smallsetminus K$ with $1 < r < \infty$, then we can describe $\gamma(t)$ as the *landing point* of this ray, that is its limiting value as $r \to 1$. As an example, Figure 2 shows several examples of external rays for the Julia set of Figure 1.

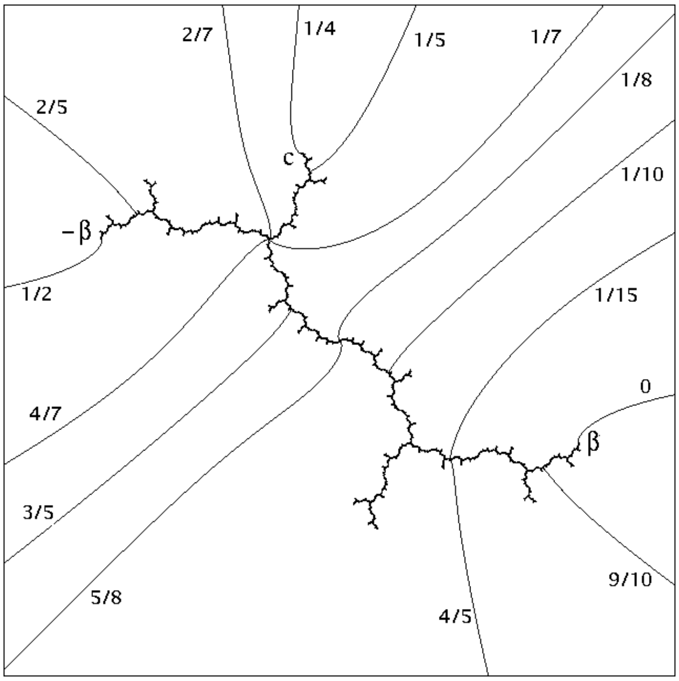


Figure 2. Julia set $K = \partial K$ for the map $f(z) = z^2+c$ of Figure 1, showing selected external rays. Here c is chosen in the upper half-plane so that $f(c) + f^{\circ 2}(c) = 0$, or equivalently so that $c^3 + 2c^2 + 2c + 2 = 0$. The landing point of the zero ray is called the β fixed point. In this example, the $1/4$-ray lands at the critical value c and its image, the $1/2$-ray, lands at $-\beta$, while the other fixed point α is the landing point of the $1/7$, $2/7$ and $4/7$-rays.

2.1. Topological-mating. Now suppose that f_1 and f_2 are quadratic polynomials, for example of the form $f_j(z) = z^2 + c_j$. If both filled Julia sets $K_j = K(f_j)$ are locally connected, then the *topological-mating* $f_1 \perp\!\!\!\perp f_2$ is a continuous map from an associated compact space $K_1 \perp\!\!\!\perp K_2$ onto itself, constructed as follows. Let $\gamma_j : \mathbf{R}/\mathbf{Z} \to \partial K_j$ be the Carathéodory semiconjugacy from the circle of reals modulo one onto the Julia set of f_j. Form the disjoint union of K_1 and K_2, and let $K_1 \perp\!\!\!\perp K_2$ be the quotient space in which the image $\gamma_1(t) \in \partial K_1$ is identified with $\gamma_2(-t) \in \partial K_2$ for every $t \in \mathbf{R}/\mathbf{Z}$. (More precisely, let $\sim$ be the smallest equivalence relation on the disjoint union of K_1 and K_2 such that

$$\gamma_1(t) \;\sim\; \gamma_2(-t) \quad \text{for every} \quad t \in \mathbf{R}/\mathbf{Z}\;,$$

and let $K_1 \perp\!\!\!\perp K_2$ be the quotient topological space in which each equivalence class is identified to a point.) Using the semiconjugacy identity $\gamma_j(2t) = f_j(\gamma_j(t))$, we see that the map f_1 on K_1 and the map f_2 on K_2 fit together to yield the required continuous map $f_1 \perp\!\!\!\perp f_2$ from this quotient space onto itself. In particular, there are canonical semiconjugacies $K_1 \to K_1 \perp\!\!\!\perp K_2$ and $K_2 \to K_1 \perp\!\!\!\perp K_2$ from f_1 and f_2 to $f_1 \perp\!\!\!\perp f_2$.

In this generality, there is no reason to expect this space $K_1 \perp\!\!\!\perp K_2$ to be particularly well behaved. None-the-less, in many cases it turns out that $K_1 \perp\!\!\!\perp K_2$ is a topological sphere, and furthermore that we can give this sphere a conformal structure so that $f_1 \perp\!\!\!\perp f_2$ becomes a holomorphic map, rational of degree two.

Here is an alternative description of $K_1 \perp\!\!\!\perp K_2$ which provides some additional information. Let S^2 be the unit sphere in $\mathbf{C} \times \mathbf{R}$. Let us identify the dynamic plane for f_1 with the northern hemisphere H^+ of S^2 and the dynamic plane for f_2 with the southern hemisphere H^-, under the gnomonic[2] projections

$$\nu_1 : \mathbf{C} \to H_+ \,, \qquad \nu_2 : \mathbf{C} \to H_- \,,$$

where

$$\nu_1(z) = (z, 1)/\sqrt{|z|^2+1} \,, \qquad \nu_2(z) = (\overline{z}, -1)/\sqrt{|z|^2+1} \,.$$

Note that ν_2 can be described as the composition of ν_1 with the $180°$ rotation about the x-axis

$$(x+iy\,,\,h) \mapsto (x-iy\,,\,-h)\,. \tag{2}$$

If we assume that the polynomial f_1 has leading coefficient $+1$, then it is not hard to check that the image of the external ray $\mathcal{R}_t(K_1)$ in H_+ limits on the point $(e^{2\pi it}, 0)$ on the equator. Similarly, if f_2 is also monic then $\nu_2(\mathcal{R}_{-t}(K_2))$ limits at this same point $(e^{2\pi it}, 0)$ on the equator. It follows that the map $\nu_1 \circ f_1 \circ \nu_1^{-1}$ on the northern hemisphere and the map $\nu_2 \circ f_2 \circ \nu_2^{-1}$ on the southern hemisphere tend to the same limiting values $(z, 0) \mapsto (z^2, 0)$ as we approach the equator. Thus these two maps fit together so as to yield a smooth map from the entire 2-sphere to itself. (Figure 3.) Let us use the notation $f_1 \uplus f_2$ for this induced map on S^2.

Define the *ray equivalence relation* to be the smallest equivalence relation $\overset{\text{ray}}{\sim}$ on S^2 such that the closure of the image $\nu_1(\mathcal{R}_t(K_1))$, as well as the closure of $\nu_2(\mathcal{R}_{-t}(K_2))$, lies in a single equivalence class. Then it is easy to see that the quotient space $S^2/\overset{\text{ray}}{\sim}$ is canonically homeomorphic to the quotient space $K_1 \perp\!\!\!\perp K_2$ described above, and that the map $f_1 \uplus f_2$ on S^2 corresponds to the map $f_1 \perp\!\!\!\perp f_2$ on this quotient space. However, this new description has several advantages. In particular, it allows us to make use of the following classical result. (Compare [Daverman].)

Let $\simeq$ be any equivalence relation on the sphere S^2 which is topologically closed. (That is, we assume that the set of all pairs (x, y) with $x \simeq y$ forms a closed subset of $S^2 \times S^2$.) Assume also that each equivalence class is connected, but is not the entire sphere.

2.2. Theorem of R. L. Moore. *The quotient space $S^2/\simeq$ is itself homeomorphic to S^2 if and only if no equivalence class separates the sphere into two or more connected components.*

---

[2] The *gnomonic projection* from a plane to the unit sphere has the characteristic property, useful in navigation, of carrying straight lines in the plane to great circle arcs in the sphere. In the case of a plane not passing through the origin in $\mathbf{R}^3$ it can be defined by the simple formula $\nu(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$.

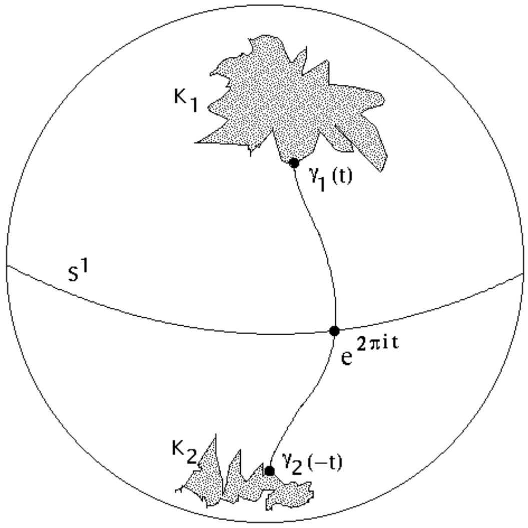


Figure 3. The images of the external rays $\mathcal{R}_t(K_1)$ and $\mathcal{R}_{-t}(K_2)$ come together at a common point $(e^{2\pi it}, 0)$ on the equator S^1. By definition, these two rays collapse to a single point $\hat{\gamma}(t)$ in the quotient space $S^2/\overset{\text{ray}}{\sim}$, which is homeomorphic to $K_1 \perp\!\!\!\perp K_2$.

Further, under the conditions of Moore's theorem, when $S^2/\simeq$ is a topological sphere, it can be shown that the quotient map $S^2 \to S^2/\simeq$ induces isomorphisms of homology, and hence imposes a preferred orientation on this quotient sphere. We can now formulate the following, with $K_1 \perp\!\!\!\perp K_2$ defined as above.

2.3. Definition. A quadratic rational map $F : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is called a *geometric mating* of the quadratic polynomials f_1 and f_2, or briefly a *mating*, if there exists a topological conjugacy h from the map $f_1 \perp\!\!\!\perp f_2$ on the space $K_1 \perp\!\!\!\perp K_2$ to the rational map F on the Riemann sphere $\hat{\mathbf{C}}$, where $h : K_1 \perp\!\!\!\perp K_2 \to \hat{\mathbf{C}}$ is an orientation preserving homeomorphism, holomorphic on the interior (if any) of K_1 and K_2. Thus $h \circ (f_1 \perp\!\!\!\perp f_2) = F \circ h$. We will often write briefly $F \cong f_1 \perp\!\!\!\perp f_2$.

In all known cases, this homeomorphism h is unique up to composition with a Möbius automorphism, so that we can speak of *the* geometric mating of f_1 and f_2. The uniqueness question for matings is part of a larger rigidity question: If two quadratic rational maps are topologically conjugate under an orientation preserving homeomorphism which is holomorphic on the Fatou set, does it follow that they are holomorphically conjugate? (Compare Lyubich [1995, §5].) The only known counterexamples to the corresponding statement in higher degrees are Lattès mappings of degree $d = n^2 \geq 4$. (Compare Appendix B.5(a).)

Here is a trivial example. Suppose that $f_2(z) = z^2$, so that K_2 is the closed unit disk. Pasting the boundaries of K_1 and K_2 together, as described above, we simply obtain the Riemann sphere with the original polynomial map f_1. In other words, any f_1 with locally connected Julia set can be mated with the standard map $z \mapsto z^2$ in such a way that the resulting rational map

$$f_1 \perp\!\!\!\perp (z \mapsto z^2)$$

is holomorphically conjugate to f_1. For non-trivial examples, see 2.6 as well as [Wittner], [Luo]. Note that there exist *shared matings*, where a given rational map can be described as

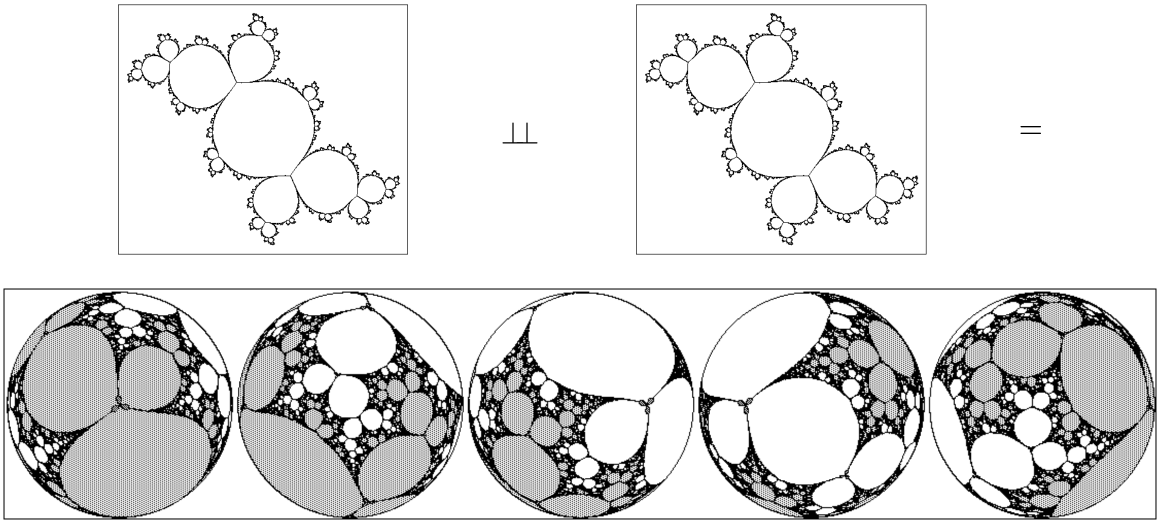


Figure 4. A conjectured example of mating, with $f_1 = f_2$ *. The Julia set for* $f(z) = z^2 + \lambda z$ *, with* $\lambda = e^{2\pi i/3}$ *, is shown above, and the Julia set for a rational map* $F(w) = w(w+\lambda)/(\lambda w + 1)$ *is shown below. It is believed that* $F \cong f \perp\!\!\!\perp f$ *. (Compare 2.9.) This figure shows five projections as the sphere rotates to the left. The presumed image of one of the two filled Julia sets has been shaded.*

a geometric mating in essentially distinct ways. (See [Wittner] as well as Appendix B.9.)

In order to describe some elementary properties of this construction, we will need the following.

2.4. Definition. To every 2-fold branched covering $g : M \to M'$ there is associated the *canonical involution* $\tau = \tau_g : M \to M$ which interchanges the two preimages of any point of M', so that

$$g^{-1}(g(x)) \;=\; \{x\,,\,\tau(x)\}$$

for every $x \in M$. Here M and M' could be arbitrary manifolds, but for us g will always be a self-covering, with $M = M'$ of real or complex dimension one. Note that the fixed points (if any) of τ are precisely the critical points of g.

Examples. To any quadratic polynomial in the form $z \mapsto z^2 + c$ we associate the involution $\tau(z) = -z$, fixing the critical point. More generally, if F is any quadratic rational map, then $\tau = \tau_F$ is the unique Möbius involution which fixes the two critical points. For example if $F(z) = a(z + z^{-1}) + b$, then $\tau(z) = 1/z$. As a final example, if g is the angle doubling map on $\mathbf{R}/\mathbf{Z}$ then $\tau_g(t) = t + 1/2$.

Given two such branched coverings $g : M \to M$ and $f : N \to N$, a semiconjugacy $h : M \to N$, $h \circ g = f \circ h$, is called τ*-equivariant* if $h \circ \tau_g = \tau_f \circ h$. As an example, for $f(z) = z^2 + c$, it is easy to check that: *The Carathéodory semiconjugacy* $\gamma : \mathbf{R}/\mathbf{Z} \to \partial K$ *is always* τ*-equivariant.* In fact, the involution τ for f maps the entire ray $\mathcal{R}_t(K)$ onto $\mathcal{R}_{\tau(t)}(K) = \mathcal{R}_{t+1/2}(K)$. As another example, a linear map $h(t) = kt$ from $\mathbf{R}/\mathbf{Z}$ to itself is τ-equivariant if and only if its degree k is odd.

Recall that the β *fixed point* of a map $f(z) = z^2 + c$ with connected Julia set is defined to be the landing point $\gamma(0)$ of the ray $\mathcal{R}_0(K)$.

2.5. Lemma. *If $F \cong f_1 \perp\!\!\!\perp f_2$ is a geometric mating, then:*

• *The β fixed points of f_1 and f_2 are glued together in $K_1 \perp\!\!\!\perp K_2$, but no other point in K_1 or K_2 is identified with these β fixed points.*

• *Similarly the points $\tau(\beta) = -\beta$ in K_1 and K_2 are glued together, but are not identified with any other point.*

• *The critical points of f_1 and f_2 correspond to the two critical points of F. In particular, these two points always remain distinct under the mating.*

• *Furthermore the two associated semiconjugacies $K_1 \to K_1 \perp\!\!\!\perp K_2$ and $K_2 \to K_1 \perp\!\!\!\perp K_2$ are τ-equivariant.*

Proof. The first statement follows from the general theory of external rays landing at a repelling periodic point. Such rays are always periodic with a common period. But the zero-ray for a quadratic map is the only ray of period one, so no other ray can land at the same point β. Applying the involution τ which carries $\mathcal{R}_t(K)$ to $\mathcal{R}_{t+1/2}(K)$, we obtain a corresponding statement for $\tau(\beta)$.

If we assume that f_1 and f_2 are polynomials of the form $f_j(z) = z^2 + c_j$ with critical point at the origin, then the corresponding critical points of $f_1 \uplus f_2$ will be at the north and south poles of S^2. In this case, the canonical involution associated with $f_1 \uplus f_2$ is the $180°$ rotation $\tau(z,h) = (-z,h)$. Clearly this rotation is compatible with the ray equivalence relation on S^2, and hence gives rise to a well defined continuous involution τ' of the quotient space $K_1 \perp\!\!\!\perp K_2 = S^2/\overset{\text{ray}}{\sim}$, with $(f_1 \perp\!\!\!\perp f_2) \circ \tau' = f_1 \perp\!\!\!\perp f_2$.

First let us show that the north and south poles belong to different ray equivalence classes, and hence correspond to distinct points of $K_1 \perp\!\!\!\perp K_2$. By construction, two points of $\nu_1(K_1) \cup \nu_2(K_2)$ map to a common point of $K_1 \perp\!\!\!\perp K_2$ if and only if there is a path made up out of finitely many external rays which leads from one to the other. If there were such a path leading from the north pole to the south pole, then its image under rotation would be another such path, and together these paths would disconnect the sphere. Since we have assumed that the mating exists, it follows by Moore's criterion that this is impossible.

To complete the proof, showing that τ' is indeed the canonical involution associated with $f_1 \perp\!\!\!\perp f_2$, we must show that it has no fixed points other than the two critical points. In other words, we must show that the points (z,h) and $\tau(z,h) = (-z,h)$ map to different points of $K_1 \perp\!\!\!\perp K_2$ unless they both belong to the same ray equivalence class as one of the poles. But, if there were a ray path from (z,h) to $(-z,h)$ which misses both poles, then together with its image under rotation this path would disconnect the sphere. Again this is impossible. It follows that the projection from S^2 to $S^2/\overset{\text{ray}}{\sim}$ is τ-equivariant, and that the poles of S^2 map to the unique fixed points of the canonical involution τ_F, and hence to the critical points of F. □

2.6. Existence and Uniqueness Results: Tan Lei, Rees, and Shishikura. Tan Lei studied matings under the hypothesis that both $f_1(z) = z^2 + c_1$ and $f_2(z) = z^2 + c_2$ have periodic critical orbit. She showed that the map $f_1 \uplus f_2$, as described above, is "Thurston equivalent" (see [Douady and Hubbard]) to some rational map F if and only if:

> Condition NC: *The points c_1 and c_2 do* not *belong to complex conjugate limbs of the Mandelbrot set.*

When Condition NC is satisfied, F is described as a "formal mating" of f_1 and f_2. To

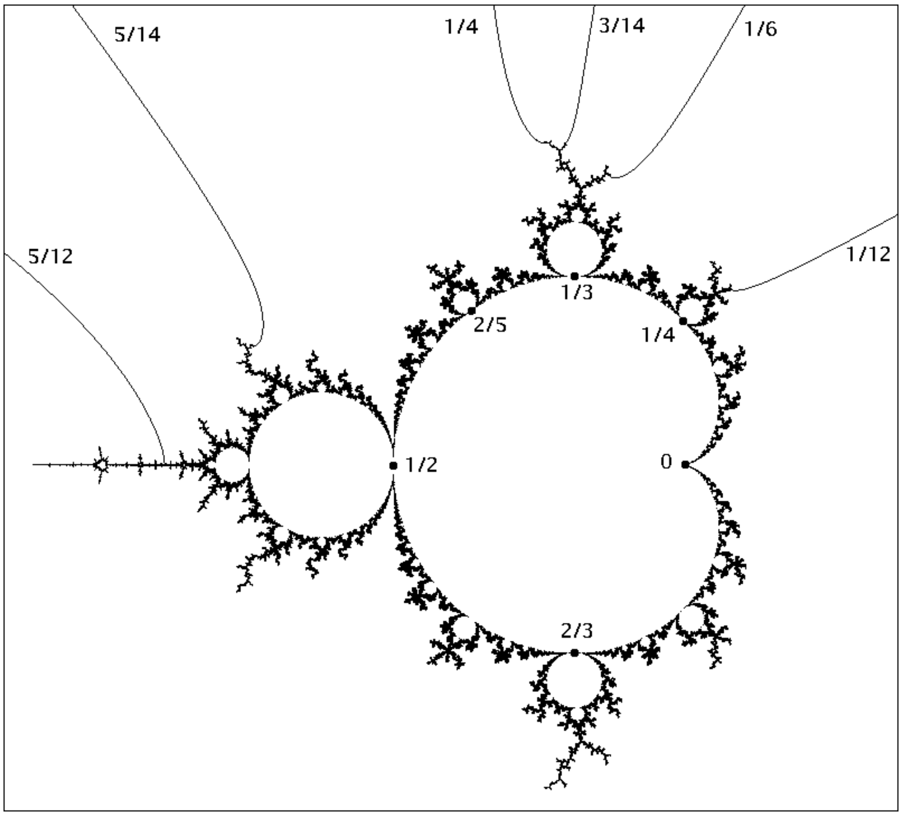


Figure 5. Picture of the Mandelbrot set boundary, together with selected external rays. Labels of the form p/q along the cardioid indicate points $c(\lambda)$ such that $f(z) = z^2 + c(\lambda)$ has a fixed point of multiplier $\lambda = e^{2\pi ip/q}$.

explain this condition, note that for each complex number λ there is one and only one polynomial $f(z) = z^2 + c$ having a fixed point $z = f(z)$ with multiplier $f'(z) = \lambda$. In fact since $z^2 + c = z$ and $\lambda = 2z$, we can solve for $c = c(\lambda) = \lambda/2 - (\lambda/2)^2$. As λ varies over the unit circle, the corresponding parameter $c(\lambda)$ varies over the *cardioid* which is prominently visible in any picture of the Mandelbrot set. Now for each root of unity $\lambda = e^{2\pi ip/q} \neq 1$ it turns out that there is a connected component $M(p/q)$ of $M \smallsetminus (\text{cardioid})$ which lies outside of the cardioid but is attached to it at the point $c(\lambda)$. The closure $\overline{M}(p/q)$ is called the (p/q)-*limb* of M . The characteristic property for polynomials $f(z) = z^2 + c$ with c in this (p/q)-limb is that there are q external rays $\mathcal{R}_{t_j}(K)$ landing at a common fixed point of f , with angles $0 < t_1 < \cdots < t_q < 1$, such that $f(\mathcal{R}_{t_j}) = \mathcal{R}_{t_k}$ where $k \equiv j + p \pmod{q}$. These angles t_j are uniquely determined by p/q . This common landing point is called the α-fixed point of f . As an example, if c belongs to the $(1/3)$-limb $\overline{M}(1/3)$, then the three external rays $\mathcal{R}_t(K)$ with angles $t = 1/7$, $2/7$, and $4/7$ all land at the α-fixed point of K .

Now suppose that c_1 belongs to the p/q-limb $M(p/q)$ while c_2 belongs to the complex conjugate limb $M(-p/q) = M((q-p)/q)$. Then there are at least two distinct rays $\mathcal{R}_t(K_1)$ and $\mathcal{R}_s(K_1)$ landing at the α-fixed point of $K_1 = K(z^2 + c_1)$. Similarly the rays $\mathcal{R}_{-t}(K_2)$ and $\mathcal{R}_{-s}(K_2)$ land at the α-fixed point of $K_2 = K(z^2 + c_2)$. In the sphere S^2 of Figure 3, these four rays fit together to form a closed loop which separates the sphere. Hence Moore's criterion is not satisfied. The quotient space $K_1 \perp\!\!\!\perp K_2$ is not a topological sphere, and the geometric mating certainly cannot exist.

Mary Rees sharpened Tan Lei's result. Still assuming that f_1 and f_2 have periodic critical orbit and do not belong to complex conjugate limbs, she showed that the rational function constructed by Tan Lei really is a geometric mating of f_1 and f_2. Using quasi-conformal surgery, it is not difficult to extend this result to arbitrary hyperbolic polynomials $z^2 + c$ with connected Julia set.

Shishikura extended this work to the *postcritically finite* case. Still assuming that c_1 and c_2 do not belong to complex conjugate limbs, he showed that the geometric mating exists and is unique up to holomorphic conjugacy whenever both maps have eventually periodic critical orbits. (In this generality, we can no longer say that $f_1 \uplus f_2$ is Thurston-equivalent to a rational map. In fact Thurston's algorithm, applied to $f_1 \uplus f_2$, may not converge in the conventional sense, using the Teichmüller topology on the space of *embeddings* of the postcritical set into $\hat{\mathbf{C}}$ which are defined up to Möbius automorphisms of $\hat{\mathbf{C}}$. However, it still converges in a weaker sense, where we allow suitably restricted mappings of the postcritical set into $\hat{\mathbf{C}}$.) Thus:

For postcritically finite quadratic polynomials, the geometric mating exists
$\iff$ *Condition* NC *is satisfied*
$\iff$ $K_1 \perp\!\!\!\perp K_2$ *is a topological sphere.*

(However, [Shishikura and Tan Lei] have described a cubic example where the geometric mating does not exist, even though $K_1 \perp\!\!\!\perp K_2$ is a topological sphere.)

2.7. Further discussion. In order to describe a possible extension of this Tan Lei-Rees-Shishikura work to more general polynomials, it is convenient to define the *t-limb* for an irrational number $0 < t < 1$ as the single point $c(\lambda)$ with $\lambda = e^{2\pi i t}$. The corresponding polynomial $z^2 + c(\lambda)$ has either a Siegel disk or a Cremer point with multiplier $\lambda \in \partial\mathbf{D}$. Note that the geometric mating of $z^2 + c(\lambda)$ and $z^2 + c(\overline{\lambda})$ cannot exist, since no quadratic rational map can have distinct fixed points of multiplier λ and λ^{-1}.

2.8. Geometric Mating Problem. If f_1 and f_2 are quadratic polynomial maps, not belonging to complex conjugate limbs of the Mandelbrot set, and if their Julia sets are locally connected, does such a geometric mating $f_1 \perp\!\!\!\perp f_2$ always exist? Is it unique up to a Möbius automorphism of $\hat{\mathbf{C}}$? (It had earlier been conjectured that such a mating operation not only exists, but depends continuously on the maps f_1 and f_2. See for example [Milnor, 1993]. However, [Epstein] has shown that this is false: In many cases, the mating operation between two hyperbolic components of the Mandelbrot set does not extend continuously to the boundary.)

The only progress on this question which is known to me is due to [Luo], who has constructed geometric matings in many cases which are not postcritically finite or hyperbolic.

2.9. Some Problematic Examples. Here is a family of examples which is not covered by the known results. Suppose that f_1 and f_2 are polynomials of the form $f_j(z) = z^2 + a_j z$ with $|a_j| = 1$. In this case, the condition that f_1 and f_2 do not belong to conjugate limbs of the Mandelbrot set reduces to the inequality $a_1 a_2 \neq 1$. A candidate rational map is given by the formula

$$F(w) \; = \; w \, \frac{w - a_1}{a_2 \, w - 1} \; ,$$

with fixed points of multiplier a_1 and a_2 at zero and infinity respectively. If $J(f_1)$ and $J(f_2)$ are locally connected,[3] then it seems quite likely that F is indeed a geometric mating of f_1 and f_2. An example with $a_1 = a_2 = e^{2\pi i/3}$ is shown in Figure 4.

2.10. The Brolin and Lyubich Measures. According to [Lyubich 1983], for any rational map $F : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of degree $d \geq 2$ there exists a unique F-invariant probability measure on $\hat{\mathbf{C}}$ which has maximal entropy, equal to $\log d$. This measure $\mathbf{m}$ can be characterized as the unique invariant probability measure, supported on the Julia set $J(F)$, with the following property:

If $X \subset \hat{\mathbf{C}}$ is a measurable set, such that $F|_X$ is one-to-one, then

$$\mathbf{m}(F(X)) \;=\; d\,\mathbf{m}(X)\;. \tag{3}$$

As an example, if $U \subset \hat{\mathbf{C}}$ is a simply-connected open set containing no critical values, then each of the d connected components of $f^{-1}(U)$ qualifies, and hence has measure equal to $\mathbf{m}(U)/d$. In the special case of a polynomial map, this measure $\mathbf{m}$ was first studied by [Brolin], and is known as the *Brolin measure*. In the general case, we will call it the *Lyubich measure*.

We will need the following. Let J_1 and J_2 be Julia sets for quadratic rational maps, and let $\mathbf{m}_1$ and $\mathbf{m}_2$ be their Lyubich measures. As in 2.4, let τ_k be the canonical involution which interchanges the two preimages of any point of J_k.

2.11 Lemma. *If $\varphi : J_1 \to J_2$ is a continuous semiconjugacy from J_1 onto J_2 with $\varphi\circ\tau_1 = \tau_2\circ\varphi$, then the push-forward $\varphi_*(\mathbf{m}_1) = \mathbf{m}_1\circ\varphi^{-1}$ is equal to $\mathbf{m}_2$.*

Proof. We must show that the set function $X \mapsto \mathbf{m}_1(\varphi^{-1}X)$ satisfies the characterizing conditions for the measure $\mathbf{m}_2$, and hence is equal to $\mathbf{m}_2$. It follows easily from (3), together with the finiteness of our measures, that every point $z \in J_2$ has measure $\mathbf{m}_2(\{z\}) = 0$. Similarly, the preimage $\varphi^{-1}(z)$ must have measure $\mathbf{m}_1\varphi^{-1}(z) = 0$. If z is non-periodic under F_2, this follows since the sets $F_1^{-k}\varphi^{-1}(z)$ are pairwise disjoint with equal measure. On the other hand, if z is periodic then $\tau_2(z)$ is non-periodic, hence $\varphi^{-1}(z) = \tau_1\varphi^{-1}\tau_2(z)$ has measure zero. Let $U \subset \hat{\mathbf{C}}$ be a simply-connected open set which contains no critical values of F_2. Then both of the connected components U' and U'' of $F_2^{-1}(U)$ map bijectively onto U, and $\tau_2(U') = U''$. It follows that τ_1 maps $\varphi^{-1}(U')$ onto the disjoint set $\varphi^{-1}(U'')$, and that the union of these two sets is $F_1^{-1}(\varphi^{-1}U)$. Therefore

$$\mathbf{m}_1(\varphi^{-1}F_2(U')) \;=\; 2\,\mathbf{m}_1(\varphi^{-1}(U'))\;,$$

and the same is true for any measurable subset of U'. It follows easily that the measure $\mathbf{m}_1\circ\varphi^{-1}$ is equal to $\mathbf{m}_2$, as required. □

A Question. It seems possible that the following is true: If $\varphi : J_1 \to J_2$ is any topological semiconjugacy from the Julia set of one non-linear rational function onto another, then the degrees of these rational functions must be equal, and $\varphi_*(\mathbf{m}_1)$ must be equal to $\mathbf{m}_2$. However, I cannot prove this more general statement.

Here is an application of 2.11: To any geometric mating of two quadratic polynomials

[3] To make sense of "mating" for Julia sets which are not locally connected one would need some different definition. (Two possible proposals for an alternative definition are given in [Milnor, 1993 and 1994].)

there is associated the following commutative diagram of topological semiconjugacies

$$\begin{tikzcd}
\mathbf{R}/\mathbf{Z} \arrow[rr, "(-1)\cdot"] \arrow[d, "\gamma_1"] & & \mathbf{R}/\mathbf{Z} \arrow[d, "\gamma_2"] \\
J(f_1) \arrow[dr] & & J(f_2) \arrow[dl] \\
& J(f_1 \perp\!\!\!\perp f_2) &
\end{tikzcd} \tag{4}$$

using the angle doubling map on $\mathbf{R}/\mathbf{Z}$ (which is topologically conjugate to $z \mapsto z^2$ on its Julia set S^1). *Since each map satisfies the conditions of* 2.11 *, it follows that the Lebesgue measure on* $\mathbf{R}/\mathbf{Z}$ *pushes forward to the Brolin measure on either* $J(f_1)$ *or* $J(f_2)$ *, which in turn pushes forward to the Lyubich measure on* $J(f_1 \perp\!\!\!\perp f_2)$. This diagram of semiconjugacies will play a central role in §4.

§3. The Example, a Lattès Mapping.

We now concentrate on one very explicit example. Let $c = c_{1/4}$ be the landing point of the $1/4$-ray $\mathcal{R}_{1/4}(M)$ for the Mandelbrot set (Figure 5), and let $f(z) = f_{1/4}(z) = z^2 + c$. According to the Douady-Hubbard correspondence between parameter plane and dynamic plane, it follows that the critical value $c = f(0) \in K(f)$ is equal to the landing point $\gamma(1/4)$ of the ray $\mathcal{R}_{1/4}(K)$ in the dynamic plane. Therefore, by $(1')$, the critical orbit for f has the form

$$0 \;\mapsto\; \gamma(1/4) \;\mapsto\; \gamma(1/2) \;\mapsto\; \gamma(0) \circlearrowleft \;,$$

or in other words

$$0 \quad\mapsto\quad c \quad\mapsto\quad -\beta \quad\mapsto\quad \beta \circlearrowleft \;,$$

where the fixed point β is the landing point of the zero-ray for $K(f)$, and $\tau(\beta) = -\beta$ is the landing point of the $(1/2)$-ray. (Compare 2.4 and Figures 1, 2.) From the resulting polynomial equation $f^{\circ 2}(c) + f(c) = 0$, one sees easily that $c = -0.22815549 + 1.115142508\,i$ is the unique root of the equation $c^3 + 2c^2 + 2c + 2 = 0$ in the upper half plane.

According to Shishikura's Theorem, as described in 2.6, the geometric self-mating

$$F \;\cong\; f_{1/4} \perp\!\!\!\perp f_{1/4}$$

exists, and is unique up to holomorphic conjugacy. To fix our ideas, suppose that we choose a representative of this holomorphic conjugacy class so that the critical points are at ± 1 and the image $\hat{\beta}$ of the β-fixed points is at infinity. Then we will prove the following.

3.1. Lemma. *The resulting rational function* F *is given by*

$$F(z) \;=\; \pm\frac{i}{2}\,(z + z^{-1})$$

for some choice of sign. The critical orbits of this mapping F *have the form*

$$\pm 1 \;\mapsto\; \pm i \;\mapsto\; 0 \;\mapsto\; \infty \;\circlearrowleft\; . \tag{5}$$

Proof. For any quadratic rational map F , normalized so that the two critical points are at ± 1 , with one of the three fixed points at infinity, it is not hard to check that F has the form

$$F(z) \;=\; a(z + z^{-1}) + b \tag{6}$$

for some $a \neq 0$ and b. (Compare [Milnor 1993]. In fact the canonical involution for F must be a Möbius involution which fixes the two critical points, and hence must be given by $\tau(z) = 1/z$. Since infinity is a fixed point, the point $\tau(\infty) = 0$ maps to infinity, and it follows that F must have the form $F(z) = (az^2 + bz + c)/z = az + b + c/z$. Setting $F'(\pm 1) = 0$, it then follows that $a = c$, as required.)

Now suppose that F commutes with some non-identity Möbius automorphism $\sigma : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ which fixes the point at infinity. Since σ cannot fix three distinct points, it must interchange the two critical points, and hence must be the involution $\sigma(z) = -z$. Hence $F(z)$ must be equal to $-F(-z) = a(z + z^{-1}) - b$. *Thus the coefficient b is equal to zero if and only if F commutes with some Möbius automorphism which keeps ∞ fixed, and which necessarily interchanges the two critical points,*

In particular, an arbitrary geometric mating $F \cong f_1 \perp\!\!\!\perp f_2$ can be put in the normal form (6). In the special case of a self-mating, with $f_1 = f_2$, there evidently exists an automorphism fixing the fixed point $\hat{\beta} = \infty$ and interchanging the two critical points, so it follows that $F(z) = a(z + z^{-1})$ with $b = 0$. In our particular case, the equation

$$F(F(\pm 1)) \;=\; F(\pm 2a) \;=\; 0$$

must also be satisfied. The equation $F(z) = a(z + z^{-1}) = 0$ has solutions $z = \pm i$, so it follows that $2a = \pm i$, as required. $\square$

The distinction between the maps $F(z) = (i/2)(z + z^{-1})$ and $F(z) = (-i/2)(z + z^{-1})$ is much more subtle. For the moment, let us simply state the following without proof. Recall that c_t is the landing point of the ray $\mathcal{R}_t(M)$ in the Mandelbrot set.

3.2. Assertion. *The geometric mating of $z \mapsto z^2 + c_{1/4}$ with itself is given by $F(z) = (i/2)(z + z^{-1})$, while the geometric mating of the complex conjugate map $z \mapsto z^2 + c_{3/4}$ with itself is given by $F(z) = (-i/2)(z + z^{-1})$.*

See Appendix B.11. (An intuitive proof that this is the right choice of sign can be derived by noting that Figures 12 and 13 have compatible orientations.)

This map $z \mapsto (i/2)\,(z + z^{-1})$ is one of a collection of examples which can be thoroughly understood using the following constructions.

3.3. From Torus to Sphere. Following [Lattès], let us start with a lattice $\Lambda \subset \mathbf{C}$, that is a free additive subgroup generated by two elements which are linearly independent over $\mathbf{R}$, and form the quotient torus $\mathbf{T} = \mathbf{C}/\Lambda$. Now form a further identification space by identifying each $w \in \mathbf{T}$ with $-w$, or equivalently form the space of orbits for the group of transformations

$$w \;\mapsto\; \pm w + \lambda$$

of $\mathbf{C}$, where λ ranges over the lattice Λ. *The resulting quotient space, which we denote by $\mathbf{T}/\pm$, is a Riemann surface of genus zero, and hence is conformally isomorphic to the Riemann sphere.* To see this, note that the involution $w \mapsto -w$ of $\mathbf{T}$ has just four fixed points, namely the four points w_j modulo Λ such that

$$w_j \;\equiv\; -w_j \pmod{\Lambda} \quad\Longleftrightarrow\quad 2w_j \;\in\; \Lambda \quad\Longleftrightarrow\quad w_j \;\in\; \frac{1}{2}\Lambda\,.$$

These four fixed points in $\mathbf{T}$ are the critical points of the projection map $\mathbf{T} \to \mathbf{T}/\pm$. (As local uniformizing parameter for the quotient $\mathbf{T}/\pm$ in the neighborhood of the image of

w_j we can use the expression $(w - w_j)^2$, where w ranges over a neighborhood of w_j in $\mathbf{C}$.) Since $\mathbf{T} \to \mathbf{T}/\pm$ is a map of degree two with four critical points, the *Riemann-Hurwitz formula* asserts that the Euler characteristic $\chi(\mathbf{T}) = 0$ can be computed from the Euler characteristic of the quotient by the formula

$$\chi(\mathbf{T}) \;=\; 2\,\chi(\mathbf{T}/\pm) \;-\; 4\;.$$

(This can be proved by triangulating the quotient with the four critical values as vertices, and noting that each simplex other than the four critical values is covered by exactly two simplexes in $\mathbf{T}$.) Thus $\chi(\mathbf{T}/\pm) = 2$, as required.

In fact one specific isomorphism $\mathbf{T}/\pm \to \hat{\mathbf{C}}$ is induced by the *Weierstrass $\wp$-function* which is associated with the lattice Λ. This is a holomorphic mapping $\wp : \mathbf{C} \to \hat{\mathbf{C}}$ which satisfies

$$\wp(w) \;=\; \wp(w') \qquad \Longleftrightarrow \qquad w' \;\equiv\; \pm w \pmod{\Lambda}\;.$$

Furthermore, $\wp$ is an even function, $\wp(w) = \wp(-w)$, with $\wp(0) = \infty$. See Appendix A for details.

3.4. Lattès Mappings. Let η be any non-zero complex number with the property that $\eta\,\Lambda \subset \Lambda$ and let κ be a complex constant. Then we can define a linear map $L : \mathbf{T} \to \mathbf{T}$ from the torus $\mathbf{T} = \mathbf{C}/\Lambda$ to itself by the formula

$$L(w) \;\equiv\; \eta\, w \,+\, \kappa \pmod{\Lambda}\;.$$

This map has degree $d = |\eta|^2$, since it multiplies areas by the factor $|\eta|^2$. Note that every period p point $w \equiv L^{\circ p}(w)$ has multiplier $(L^{\circ p})' = \eta^p$. If $|\eta| > 1$, then these periodic points are repelling and it is easy to check that they are everywhere dense, hence the Julia set $J(L)$ is equal to the entire torus.

Now suppose that $2\kappa \in \Lambda$, so that $L(-w) \equiv -L(w) \pmod{\Lambda}$. Then L induces a holomorphic map from $\mathbf{T}/\pm \cong \hat{\mathbf{C}}$ to itself, also of degree $d = |\eta|^2$. More explicitly we can define $F = F_L$ to be the rational map $F = \wp \circ L \circ \wp^{-1} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, so that

$$F :\; \wp(w) \;\mapsto\; \wp(L(w))\;.$$

In other words, the diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{L} & \mathbf{T} \\ \wp\downarrow & & \wp\downarrow \\ \hat{\mathbf{C}} & \xrightarrow{F} & \hat{\mathbf{C}} \end{array} \qquad (7)$$

is commutative. If the degree $|\eta|^2$ is two or more, then the resulting F, or any map holomorphically conjugate to it, is called a *Lattès map*.

3.5. Remark. It is easy to check that the Lyubich measure for F (compare 2.10) is just the push-forward under $\wp$ of the normalized Lebesgue measure on the torus $\mathbf{T}$. In particular, it follows that this Lyubich measure has a density function ρ, so that $\mathbf{m}(S) = \int\int_S \rho(x+iy)\,dx\,dy$, where ρ is smooth except at the four critical values of $\wp$. (Compare Appendix A.5.)

We are primarily interested in one particular degree two example. However, it is almost as easy to describe the general case. As the simplest example of this Lattès construction we can take $L(w) = 2w$, with $\eta = 2$, so that the induced mapping on the Riemann sphere has degree $2^2 = 4$. In this case there is no restriction at all on the lattice Λ . Thus, as Λ varies, we obtain a one complex parameter family of essentially distinct maps of degree four, all with the entire Riemann sphere as Julia set.

We will first prove the following. By a *simple* critical point of a rational map F we mean a point where the local degree of F is equal to two. (Thus a point $\omega \neq \infty$ is a simple critical point if and only if $F'(\omega) = 0$ but $F''(\omega) \neq 0$.) By a *postcritical point* for F we mean any $F^{\circ k}(\omega)$ where ω is a critical point and $k \geq 1$.

3.6. Lemma. *Let $V = \wp(\frac{1}{2}\Lambda)$ be the set of critical values for the holomorphic map $\wp : \mathbf{T} \to \hat{\mathbf{C}}$, and let $F = \wp \circ L \circ \wp^{-1}$ with L as above. Then $F(V) \subset V$. Furthermore, V is equal to the set of all postcritical points of F , while $F^{-1}(V) \smallsetminus V = \wp(\frac{1}{2\eta}\Lambda) \smallsetminus \wp(\frac{1}{2}\Lambda)$ is the set of all critical points of F . These critical points are all simple.*

Proof. This follows easily by inspecting the diagram (7), and noting that the local degree function satisfies

$$\deg(L\,,\,w)\cdot\deg(\wp\,,\,L(w)) \;=\; \deg(\wp\circ L\,,\,w) \;=\; \deg(F\circ\wp\,,\,w) \;=\; \deg(\wp\,,\,w)\cdot\deg(F\,,\,\wp(w))\,,$$

where the local degree of L is always one, and the local degree of $\wp$ is two at its critical points. □

3.7. Remark. In fact we can sharpen this statement into a complete characterization of Lattès maps as follows:

A rational function is a Lattès map if and only every critical point is simple, and there are exactly four postcritical points, none of which is also critical.

The proof will be given in Appendix B.1.

For most rational maps, it is difficult to see any structure in the collection of multipliers of the various periodic orbits. However, in the case of a Lattès map there is a very simple description.

3.8. Lemma. *If F is a Lattès map of the form $\wp\circ L\circ\wp^{-1}$, with $L(w) = \eta w + \kappa$, then the multiplier for a periodic orbit $z = F^{\circ p}(z)$ is equal to η^{2p} whenever this orbit is contained in the postcritical set V of F , and is equal to $\pm\eta^p$ otherwise.*

(We cannot distinguish between $+\eta$ and $-\eta$, since the linear maps L and $-L$ give rise to the same Lattès map.)

Proof of 3.8. Recall that $\wp(w_1) = \wp(w_2)$ if and only if $w_2 \equiv \pm w_1 \pmod{\Lambda}$. Therefore, for any $\lambda_0 \in \Lambda$, differentiating the identity $\wp(w) = \wp(\pm w + \lambda_0)$ we see that $\wp'(w_1) = \pm\wp'(w_2)$ whenever $\wp(w_1) = \wp(w_2)$. Now if $z = \wp(w_1)$ has period p and is not in V , so that $\wp'(w_1) \neq 0$, then applying the chain rule to the identity $\wp \circ L^{\circ p} = F^{\circ p} \circ \wp$, we see that $(L^{\circ p})'(w_1) = \eta^p$ is equal to $\pm(F^{\circ p})'(z)$, as required.

In the case of a period p point with $\wp(w_1) \in V$, so that $w_1 \in \frac{1}{2}\Lambda$, we proceed as follows. The Taylor series for $\wp(w_1 + h) = \wp(w_1 - h)$ contains only even powers of h , so we can use h^2 as a local uniformizing parameter for $\hat{\mathbf{C}}$ in a neighborhood of $\wp(w_1)$. Since $L^{\circ p}(w_1 + h) = w_2 + \eta^p h \equiv \pm w_1 + \eta^p h \pmod{\Lambda}$, the local uniformizing parameter h^2 maps to $\eta^{2p}h^2$, and the conclusion follows. □

3.9. The Example. Let $\Lambda = \mathbf{Z}[i] = \mathbf{Z} + i\mathbf{Z}$ be the lattice of *Gaussian integers*, and let $L(w) = (1-i)w$. Then the condition that $(1-i)\Lambda \subset \Lambda$ is clearly satisfied. Since $|1-i|^2 = 2$, the associated Lattès map $z \mapsto \wp \circ L_{1-i} \circ \wp^{-1}(z)$ is a well defined quadratic rational map. Let c_1 and c_2 be the critical points of this map. (Using 3.6, it is not hard to check that these two critical points are equal to $\wp((1 \pm i)/4)$.) It will be convenient to work with a modified Weierstrass function of the form $\hat{\wp}(w) = p\wp(w) + q$, where the complex coefficients $p \neq 0$ and q are chosen so that

$$p\,c_1 + q \;=\; +1\;, \qquad p\,c_2 + q \;=\; -1\;.$$

Then evidently the map

$$F \;=\; \hat{\wp} \circ L \circ \hat{\wp}^{-1}$$

is holomorphically conjugate to $\wp \circ L \circ \wp^{-1}$, and has critical points ± 1 . Therefore, proceeding as in 3.1, we can set

$$F(z) \;=\; a(z + z^{-1}) + b$$

for suitable coefficients $a \neq 0$ and b . To compute the coefficient a , note that $\hat{\wp}(0) = \infty$ is a postcritical fixed point, with multiplier equal to $(1-i)^2 = -2i$ by 3.8. Since the multiplier at infinity using this normal form is $1/a$, this yields $a = 1/(-2i) = i/2$. To compute b , note that the linear map L commutes with the linear automorphism $w \mapsto iw$ of the torus $\mathbf{T}$. It follows that F commutes with the corresponding automorphism $\sigma(z) = \hat{\wp}(i\hat{\wp}^{-1}(z)$, which must fix the point at infinity and interchange the two critical points, and hence be given by $\sigma(z) = -z$. It follows as in 3.1 that $b = 0$. Thus $\hat{\wp} \circ L \circ \wp^{-1}$ coincides with the map $F(z) = (i/2)(z + z^{-1})$ of 3.2, with $F \cong f_{1/4} \perp\!\!\!\perp f_{1/4}$.

§4. Semiconjugacies from the Angle Doubling Map.

Suppose that $F \cong f_1 \perp\!\!\!\perp f_2$ is a geometric mating between quadratic polynomials. It follows that there is a commutative diagram of semiconjugacies

$$\begin{tikzcd} & \mathbf{R}/\mathbf{Z} \arrow[loop right, "2\cdot"] \arrow[dl, "\gamma_1"] \arrow[dr, "\gamma_2 \circ -"] & \\ K_1 \arrow[loop right, "f_1"] \arrow[dr, "\mu_1"] & & K_2 \arrow[loop right, "f_2"] \arrow[dl, "\mu_2"] \\ & \hat{\mathbf{C}} \arrow[loop right, "F"] & \end{tikzcd} \tag{8}$$

(Compare the diagram (4).) Here $\gamma_2 \circ -$ stands for the semiconjugacy $t \mapsto \gamma_2(-t)$ from $\mathbf{R}/\mathbf{Z}$ onto ∂K_2 , and $\mu_j : K_j \to \hat{\mathbf{C}}$ is the composition of the natural map of K_j into $K_1 \perp\!\!\!\perp K_2$ composed with the homeomorphism $K_1 \perp\!\!\!\perp K_2 \xrightarrow{\cong} \hat{\mathbf{C}}$ which conjugates $f_1 \perp\!\!\!\perp f_2$ to F . Going either way around this diagram, we obtain a semiconjugacy $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$, onto the Julia set of F , where

$$\hat{\gamma}(t) \;=\; \mu_1 \circ \gamma_1(t) \;=\; \mu_2 \circ \gamma_2(-t)\;. \tag{8$'$}$$

4.1. Definitions. We will call $\hat{\gamma}$ the *mating semiconjugacy* which is associated with the mating $F \cong f_1 \perp\!\!\!\perp f_2$. It will be convenient to say that a semiconjugacy $\gamma : \mathbf{R}/\mathbf{Z} \to J$ is *primitive* if $\gamma^{-1}(\gamma(0))$ is equal to the single point zero (mod $\mathbf{Z}$).

A quadratic rational map F is called *symmetric* if there exists a Möbius involution σ of $\hat{\mathbf{C}}$ which commutes with F and interchanges the two critical points of F. (Compare [Milnor 1993].) The semiconjugacy γ will be called *symmetric* if F is symmetric, and if $\gamma(-t) = \sigma(\gamma(t))$. Recall that γ is τ*-equivariant* if $\gamma(t+1/2) = \tau(\gamma(t))$, where τ is the *canonical involution* which interchanges the points of $F^{-1}(z)$. (See 2.4.)

4.2. Lemma. *The mating semiconjugacy $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ is a primitive and τ-equivariant semiconjugacy from the doubling map on $\mathbf{R}/\mathbf{Z}$ onto the Julia set of F. In the special case of a self-mating with $f_1 = f_2$, this semiconjugacy $\hat{\gamma}$ is also symmetric.*

Proof. It follows immediately from 2.5 that $\hat{\gamma}$ is primitive and τ-equivariant. As in 3.1, we may assume that F is in the normal form

$$F(z) \;=\; a(z+z^{-1})+b\,,$$

with the image of the critical points of f_1 and f_2 at -1 and $+1$ respectively, and with the image $\hat{\beta}$ of the two β fixed points at infinity. If we interchange the roles of f_1 and f_2, then $F(z)$ will be replaced by $-F(-z) = a(z+z^{-1}) - b$, with the two critical points interchanged. Thus, in the special case of a self-mating, with $f_1 = f_2$ hence $\gamma_1 = \gamma_2$, we must have $b = 0$, so that F commutes with the symmetry $\sigma(z) = -z$, which fixes the points $\hat{\beta}$ and $\tau(\hat{\beta})$ and interchanges the two critical points. This symmetry must correspond to the $180°$ rotation (2) of S^2. Hence $\mu_1 = \sigma \circ \mu_2 = -\mu_2$. The equation

$$\hat{\gamma}(t) \;=\; \mu_1(\gamma_1(t)) \;=\; \mu_2(\gamma_2(-t)) \;=\; -\mu_1(\gamma_1(-t))$$

then says that $\hat{\gamma}(t) = -\hat{\gamma}(-t)$, which proves that $\hat{\gamma}$ is symmetric. $\square$

Now let us specialize to the map $F(z) = (i/2)(z+z^{-1})$ of 3.2 and 3.9. We will prove the following.

4.3. Theorem. *For this F, there is one and up to sign only one semiconjugacy $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ which is symmetric, τ-equivariant, and primitive.*

The proof, which gives an explicit construction of this map $\hat{\gamma}$, will make no mention of mating. To begin the argument, we will prove the following. Let $\hat{\gamma}$ be any symmetric τ-invariant semiconjugacy from the doubling map on $\mathbf{R}/\mathbf{Z}$ to this F.

4.4. Lemma. *The value $\hat{\gamma}(1/8)$ is necessarily equal to ± 1. If we normalize so that $\hat{\gamma}(1/8) = +1$, then we have the following table, with t a multiple of $1/8$ and with $\hat{\gamma}(t)$ in the set of critical or postcritical points.*

$t \equiv$	0	$1/8$	$1/4$	$3/8$	$1/2$	$5/8$	$3/4$	$7/8$
$\hat{\gamma}(t) =$	∞	1	i	-1	0	1	$-i$	-1

If we assume also that $\hat{\gamma}$ is primitive, then these are the only angles in $\mathbf{R}/\mathbf{Z}$ which map to critical or postcritical points of F.

Proof. Since $\hat{\gamma}$ is symmetric, and since $z \mapsto -z$ is the only automorphism commuting with F, we have $\hat{\gamma}(-t) = -\hat{\gamma}(t)$, hence the fixed point $\hat{\gamma}(0)$ must be the point ∞. Hence $\hat{\gamma}(1/2)$ must be equal to $\tau(\infty) = 0$, and $\hat{\gamma}(1/4)$ must belong to $F^{-1}(0) = \{\pm i\}$. Suppose to fix our ideas, that $\hat{\gamma}(1/4) = +i$. Then $\hat{\gamma}(1/8) \in F^{-1}(i) = \{1\}$. Further details of the argument are straightforward. $\square$

Let $\hat{\wp} : \mathbf{T} \to \hat{\mathbf{C}}$ be the Weierstrass map associated with the lattice $\Lambda = \mathbf{Z}[i]$ of Gaussian integers, normalized as in 3.9 so that

$$F(z) \;=\; \hat{\wp} \circ L \circ \hat{\wp}^{-1}(z) \;=\; (i/2)(z + z^{-1})$$

where $L(w) = (1-i)w$. The four critical values of $\hat{\wp}$ are equal to the four postcritical points of F by 3.6. It is not hard to see that the critical orbits correspond to:

$$\begin{array}{llllllll} w \equiv \pm & (1+i)/4 \mapsto 1/2 \mapsto (1+i)/2 \mapsto 0\,, & & (1-i)/4 \mapsto i/2 \mapsto (1+i)/2 \mapsto 0\,, \\ \hat{\wp}(w) = & 1 \mapsto i \mapsto 0 \mapsto \infty\,, & & -1 \mapsto -i \mapsto 0 \mapsto \infty\,. \end{array}$$

Note the identities

$$\begin{aligned} \hat{\wp}(iw) \;&=\; -\hat{\wp}(w)\,, \\ \hat{\wp}\big(w + (1+i)/2\big) &= 1/\hat{\wp}(w)\,. \end{aligned} \tag{9}$$

In fact the first equation follows as in 3.9 since multiplication by i maps $\mathbf{Z}[i]$ isomorphically onto itself, and the second follows since the canonical involution 2.4 for the degree two map $w \mapsto (1-i)w$ is given by $\tau_L(w) = w + 1/(1-i) = w + (1+i)/2$, while the canonical involution for F is $\tau_F(z) = 1/z$.

We want to lift $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ to a map $g : \mathbf{R}/\mathbf{Z} \to \mathbf{T}$ with $\hat{\wp} \circ g = \hat{\gamma}$. Suppose that we subdivide $\mathbf{R}/\mathbf{Z}$ as a cell complex with vertices at the four points $0,\ 1/4,\ 1/2,\ 3/4$. Then each vertex maps to a critical value of $\hat{\wp}$, which lifts uniquely, and we must have:

$$\begin{array}{rccccl} t \;\equiv & 0 & 1/4 & 1/2 & 3/4 & (\mathrm{mod}\ \mathbf{Z}) \\ g(t) \equiv & 0 & 1/2 & (1+i)/2 & i/2 & (\mathrm{mod}\ \mathbf{Z}[i]\,)\,. \end{array} \tag{10}$$

However, each edge of this cell complex can be lifted in two different ways, hence there are sixteen possible liftings in all. Whatever choice we make, the hypotheses of 4.3 translate to the following four conditions, making use of (9):

- **semiconjugacy:** $g(2t) \;\equiv\; \pm(1-i)g(t) \quad (\mathrm{mod}\ \mathbf{Z}[i]\,)$,
- **symmetry:** $g(-t) \equiv \pm i\, g(t)$.
- **τ-equivariance:** $g(t + 1/2) \;\equiv\; \pm g(t) + 1/(1-i) \;\equiv\; \pm g(t) + g(1/2)$,
- **primitivity:** $g(t) \equiv 0 \ \ (\mathrm{mod}\ \mathbf{Z}[i]\,) \quad \Longleftrightarrow \quad t \equiv 0 \ \ (\mathrm{mod}\ \mathbf{Z})$,

In fact let us restrict to two out of the sixteen possible liftings, as follows. Start with either of the two liftings g restricted to the interval $[0\,,\,1/4]$. Then $g(2t) \equiv \eta\, g(t)$ for $0 \le t \le 1/8$, where the coefficient $\eta = \pm(1-i)$ remains constant by continuity. Choose the lifting g on $[1/4\,,\,1/2]$ by requiring that $g(2t) = \eta\, g(t)$ for $1/8 \le t \le 1/4$, with the same constant η. Finally, extend to the interval $[1/2\,,\,1]\ (\mathrm{mod}\ \mathbf{Z})$ by setting

$$g(t + 1/2) \;\equiv\; g(t) + g(1/2) \qquad \text{for all} \quad t\,.$$

This makes sense, since $g(1/2) + g(1/2) \equiv 0 \ \ (\mathrm{mod}\ \mathbf{Z}[i])$.

As a final step, note that any map $g : \mathbf{R}/\mathbf{Z} \to \mathbf{C}/\mathbf{Z}[i]$ with $g(0) \equiv 0$ lifts uniquely to a map $\tilde{g} : \mathbf{R} \to \mathbf{C}$ between the universal covering spaces with

$$\tilde{g}(0) \;=\; 0\,, \qquad \text{and} \qquad \tilde{g}(t) \equiv g(t) \ \ (\mathrm{mod}\ \mathbf{Z}[i]\,) \quad \text{for all} \quad t\,.$$

Here the identity

$$\tilde{g}(t+1) \;=\; \tilde{g}(t) + \tilde{g}(1)$$

is always satisfied. Evidently the maps g and $\tilde{g}$ determine each other uniquely. For any g with the above four properties we will prove the following.

4.5. Lemma. *This lifted map* $\tilde{g}:\mathbf{R}\to\mathbf{C}$ *satisfies the following four conditions:*

(a) $\tilde{g}(2t) = (1-i)\tilde{g}(t)$ *for* $0\le t\le 1/4$,

(b) $\tilde{g}(-t) = i\,\tilde{g}(t)$ *for* $0\le t\le 1/4$,

(c) $\tilde{g}(t+1/2) = \tilde{g}(t)+\tilde{g}(1/2)$ *for all* t , *and*

(d) $\tilde{g}(t)\equiv 0 \pmod{\mathbf{Z}[i]} \iff t\equiv 0 \pmod{\mathbf{Z}}$.

Proof. It is straightforward to check that conditions (c) and (d) are satisfied. Similarly, it is straightforward to check that (a) and (b) are satisfied up to sign. That is:

$$\begin{aligned}\tilde{g}(2t) &= \epsilon(1-i)\,\tilde{g}(t) \qquad \text{for} \qquad t\in[0,1/4] \quad \text{and}\\ \tilde{g}(-t) &= \epsilon' i\,\tilde{g}(t) \qquad \text{for} \qquad t\in[0,1/4]\ ,\end{aligned}$$

where $\epsilon=\pm1$ and $\epsilon'=\pm1$ are fixed signs. Combining these facts with (c), we see that

$$\tilde{g}(1/2)=\epsilon(1-i)\,\tilde{g}(1/4)\ ,\quad \tilde{g}(-1/4)=\epsilon' i\,\tilde{g}(1/4)\ ,\quad \text{and}\quad \tilde{g}(1/2)+\tilde{g}(-1/4)-\tilde{g}(1/4)=0\ .$$

Substituting the two equations on the left into the right hand one, we have

$$(\epsilon(1-i)\ +\ \epsilon' i\ -\ 1)\,\tilde{g}(1/4)\ =\ 0\ ,$$

and hence, since $\tilde{g}(1/4)\ne 0$ by (d),

$$\epsilon(1-i)\ +\ \epsilon' i\ -\ 1\ =\ 0\ .$$

But this last equation implies that $\epsilon=\epsilon'=+1$. Hence conditions (a) and (b) are satisfied also. □

Now let us temporarily forget condition (d), and study functions satisfying the remaining three conditions.

4.6. Lemma. *Given any constant* $\tilde{g}(1)\in\mathbf{C}$ *, there exists one and only one continuous function* $\tilde{g}:\mathbf{R}\to\mathbf{C}$ *satisfying conditions* (a), (b), (c) *of* 4.5 .

Proof of uniqueness. Suppose that there were two functions $\tilde{g}$ and h satisfying these same conditions, with $\tilde{g}(1)=h(1)$ and hence $\tilde{g}(1/2)=h(1/2)$ by (c). Let K be the maximum of $|\tilde{g}-h|$ on the interval $[0\,,\,1/4]$. Then using (b) and (c) we see that $|\tilde{g}(t)-h(t)|\le K$ for all t , and using (a) we see that $|\tilde{g}(t)-h(t)|\le K/|1-i|=K/\sqrt{2}$ for $0\le t\le 1/4$. This proves that $K=0$, as required.

Proof of existence. Construct a sequence of continuous functions $\tilde{g}_n:\mathbf{R}\to\mathbf{C}$ as follows. Start with the linear function $\tilde{g}_1(t)=k\,t$, where $k=\tilde{g}(1)$. Suppose inductively that we have a continuous function $\tilde{g}_n$ which satisfies $g_n(t+1/2)=\tilde{g}_n(t)+k/2$ for all t , with $\tilde{g}_n(0)=0$. Construct $\tilde{g}_{n+1}$ in three steps, as follows. Let

$$\tilde{g}_{n+1}(t)\ =\ \tilde{g}_n(2t)/(1-i) \qquad \text{for} \qquad 0\le t\le 1/4\ ,$$

and note that $(1-i)\tilde{g}_{n+1}(1/4)=\tilde{g}_n(1/2)=k/2$. Then extend over the interval $[-1/4\,,\,1/4]$ by setting

$$\tilde{g}_{n+1}(-t)\ =\ i\,g_{n+1}(t) \qquad \text{for} \qquad 0\le t\le 1/4\ ,$$

and note that the difference $\tilde{g}_{n+1}(1/4)-\tilde{g}_{n+1}(-1/4)=(1-i)\tilde{g}_{n+1}(1/4)$ is also equal to $k/2$. Hence there is a unique extension $\tilde{g}_n:\mathbf{R}\to\mathbf{C}$ which satisfies the required equation $\tilde{g}_{n+1}(t+1/2)\ =\ \tilde{g}_{n+1}(t)\ +\ k/2$ for all real numbers t . It is not difficult to check that

$$|\tilde{g}_2(t)\ -\ \tilde{g}_1(t)|\ \le\ |\tilde{g}_2(1/4)-\tilde{g}_1(1/4)|\ <\ |k|/\sqrt{2}$$

for all t, and it follows inductively that

$$|\tilde{g}_{n+1}(t) - \tilde{g}_n(t)| \; < \; \left|\frac{k}{(1-i)^n}\right| \; = \; \frac{|k|}{\sqrt{2^n}} \; .$$

Thus the sequence of functions $\tilde{g}_n$ converges uniformly to a function $\tilde{g}$, which clearly satisfies all of the required conditions. $\square$

4.7. A Geometric Description: The Heighway Dragon. The proof of 4.6 has been completely constructive, and is easily implemented on a computer. (See 4.10.) However, the formal construction has obscured some fascinating fractal geometry. We can describe the proof more geometrically as follows. (Compare the discussion of the "Heighway Dragon" in [Edgar].) The function $\tilde{g}$ on the interval $0 \le t \le 1/2$ is the limit of a sequence of piecewise linear functions $\tilde{g}_n : [0\,,\,1/2] \to \mathbf{C}$ which are defined inductively, with the following properties: If we subdivide $[0\,,\,1/2]$ into 2^{n-1} subintervals of length $1/2^n$, then $\tilde{g}_n$ will be linear on each subinterval, with constant speed $|d\tilde{g}_n(t)/dt| = \sqrt{2^n}$. Hence the image $G_n = \tilde{g}_n[0\,,\,1/2]$ will be a union of 2^{n-1} line segments, each of length $1/\sqrt{2^n}$.

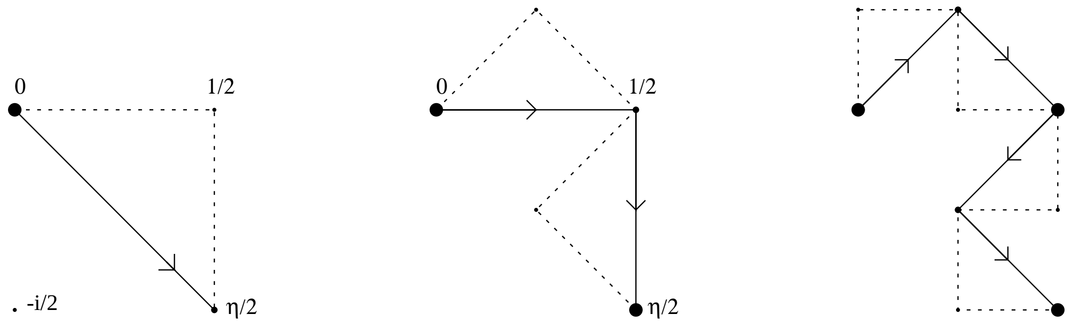


Figure 6. The images $G_n = \tilde{g}_n[0\,,\,1/2]$ for $n = 1, 2, 3$, with G_{n+1} shown as a dotted path. The "old" vertices, which already existed in G_{n-1} when $n > 0$, are indicated by heavy dots.

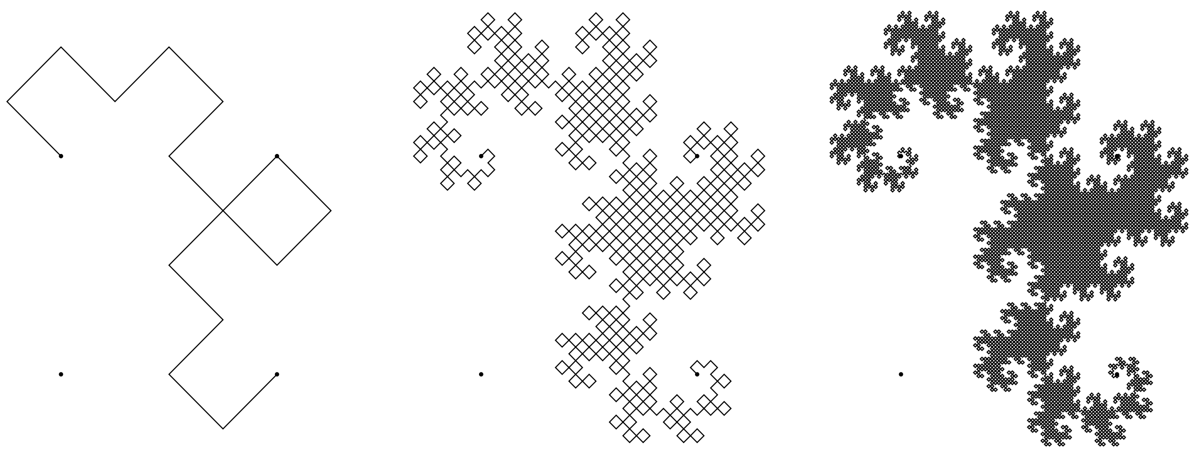


Figure 7. Successive approximations G_5, G_{11}, and G_{15} to the image $G = \tilde{g}\,[0, 1/2] \subset \mathbf{C}$. (Compare Figure 16.)

To begin the inductive definition, let $\tilde{g}_1(t) = \eta t$ for $t \in [0\,,\,1/2]$, with $\eta = 1 - i$, so that $G_1 = \tilde{g}_1[0\,,\,1/2]$ is a straight line segment leading from from 0 to $\eta/2$, as indicated by the solid line in Figure 6(left). By definition, 0 will be called an "old" vertex and $\eta/2$ a "new" vertex. For the inductive step, suppose that G_n is given as a piecewise linear path, where the vertices are alternately "old" and "new". To construct G_{n+1} we replace any line segment of G_n leading from v to v' by a broken path, leading from v to v'' to v' , where

$$v'' \;=\; \begin{cases} \frac{1}{2}(v'+v) \,+\, \frac{i}{2}(v'-v) & \text{if } v' \text{ is an "new" vertex} \\ \frac{1}{2}(v'+v) \,-\, \frac{i}{2}(v'-v) & \text{if } v' \text{ is an "old" vertex.} \end{cases}$$

For the next stage of the construction, both v and v' will be considered as old vertices, while v'' is a new vertex. This completes the inductive construction.

With this geometric definition, it again follows that the sequence of functions $\tilde{g}_n : [0\,,\,1/2] \to \mathbf{C}$ converges uniformly and geometrically. In fact the sharp estimate

$$|\tilde{g}_{n+1}(t) - \tilde{g}_n(t)| \;\le\; \frac{1}{2\sqrt{2^n}} \qquad \text{for all} \quad t$$

can be verified by induction on n . Now extending the limit function $\tilde{g}$ over $\mathbf{R}$ so that $\tilde{g}(t+1/2) = \tilde{g}(t) + \tilde{g}(1/2)$, it is not difficult to check that the resulting function satisfies all of the conditions of 4.6. With this geometric description, we can also prove primitivity:

4.8. Lemma. *This function* $\tilde{g} : \mathbf{R} \to \mathbf{C}$, *with* $\tilde{g}(1) = 1 - i$, *satisfies the condition that*

$$\tilde{g}(t) \;\equiv\; 0 \pmod{\mathbf{Z}[i]} \quad\Longleftrightarrow\quad t \equiv 0 \pmod{\mathbf{Z}} \ .$$

Proof. Let us start with any line segment of length $s = 1/\sqrt{2^n}$ in the graph G_n . Examining Figure 6, we see that, passing to G_{n+3} , this line segment will be replaced by eight line segments of length $s/\sqrt{8}$, all lying within a neighborhood of radius $s/2$ of the original. Similarly, passing to G_{n+3k} , the original segment will be replaced by 8^k segments of length $s/\sqrt{8^k}$, all lying within a neighborhood of radius

$$\frac{s}{2}\left(1 + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8^2}} + \cdots \frac{1}{\sqrt{8^{k-1}}}\right) \ .$$

Passing to the limit as $k \to \infty$, the corresponding segment of G will lie within a neighborhood of radius

$$\frac{s}{2(1-1/\sqrt{8})} \;\approx\; \frac{s}{1.29} \;<\; s$$

of the original line segment.

Now let us apply this argument to the vertical line segment $\tilde{g}_2[1/4, 1/2]$ of length 1/2, joining $1/2$ to $(1-i)/2$. This argument proves that the image $\tilde{g}[1/4, 1/2]$ lies within a neighborhood of radius strictly less than $1/2$ of the original segment. Hence $\tilde{g}[1/4, 1/2]$ cannot contain any lattice point. Similarly, the image $\tilde{g}(0, 1/4]$ cannot contain any lattice point. For if $\tilde{g}(t) \equiv 0 \pmod{\mathbf{Z}[i]}$ with $0 < t \le 1/4$, then choosing k so that $1/4 \le 2^k t \le 1/2$ it would follow by 4.5(a) that $\tilde{g}(2^k t) \equiv 0 \pmod{\mathbf{Z}[i]}$, which we have seen is impossible. The case $1/2 \le t < 1$ is handled with a similar argument. □

Proof of Theorem 4.3. Let $\tilde{g} = \tilde{g}_0 : \mathbf{R} \to \mathbf{C}$ be the function of 4.6 and 4.8, with $\tilde{g}_0(1) = 1 - i$, and let g_0 be the induced function from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{C}/\mathbf{Z}[i]$. Then we will first prove that the composition

$$\hat{\gamma}_0 \;=\; \hat{\wp}\circ g_0 \;:\; \mathbf{R}/\mathbf{Z} \;\to\; \hat{\mathbf{C}}$$

is a primitive, symmetric, τ-equivariant semiconjugacy from the doubling map on $\mathbf{R}/\mathbf{Z}$ to the map $F(z)=(i/2)(z+z^{-1})$ on $\hat{\mathbf{C}}$. In fact follows immediately from the construction that:

- $\hat{\gamma}_0(2t)=F(\hat{\gamma}_0(t))$ for $0\le t\le 1/4$,
- $\hat{\gamma}_0(-t)=-\hat{\gamma}_0(t)$ for $0\le t\le 1/4$, and
- $\hat{\gamma}_0(t+1/2)=1/\hat{\gamma}_0(t)$ for all t .

Using the last condition, it is not hard to check that $\hat{\gamma}_0(-t)=-\hat{\gamma}_0(t)$ for all t . It then follows easily that the semiconjugacy equation $\hat{\gamma}_0(2t)=F(\hat{\gamma}_0(t))$ also holds for all t . Since $\hat{\gamma}_0$ is primitive by 4.8, it has all of the specified properties.

Note also that the image $\Gamma=\hat{\gamma}_0(\mathbf{R}/\mathbf{Z})$ must be equal to the entire Riemann sphere. In fact using the semiconjugacy condition together with τ-equivariance, we see that Γ is compact and fully F-invariant, $\Gamma=F^{-1}(\Gamma)$. Since iterated preimages of any point of the Julia set are dense in the Julia set, and since the Julia set $J(F)$ is the entire Riemann sphere, this proves that $\Gamma=\hat{\mathbf{C}}$.

Conversely, let $\hat{\gamma}:\mathbf{R}/\mathbf{Z}\to\hat{\mathbf{C}}$ be any primitive, symmetric, τ-equivariant semiconjugacy from the doubling map to F . By 4.4, 4.5, and 4.6, the corresponding lifted map $\tilde{g}:\mathbf{R}\to\mathbf{C}$ must be a multiple of $\tilde{g}_0$, say

$$\tilde{g}(t) \;=\; k\,\tilde{g}_0(t) \qquad \text{for all} \quad t\ .$$

Now $\tilde{g}(1/2)\equiv(1-i)/2 \pmod{\mathbf{Z}[i]}$ by (10), and $\tilde{g}_0(1/2)$ is equal to $(1-i)/2$. Therefore $\tilde{g}(1/2)=\tilde{g}_0(1/2)+\lambda$ for some $\lambda\in\mathbf{Z}[i]$, and we have

$$k \;=\; \frac{\tilde{g}(1/2)}{\tilde{g}_0(1/2)} \;=\; 1+\frac{\lambda}{(1-i)/2} \;=\; 1+(1+i)\,\lambda\ .$$

In particular, this constant k must be a non-zero element of the lattice $\mathbf{Z}[i]$.

Case 1. If $|k|=1$ then either $k=\pm1$ and $\hat{\gamma}=\hat{\wp}\circ\tilde{g}=\hat{\gamma}_0$ or $k=\pm i$ and $\hat{\gamma}=\hat{\wp}\circ\tilde{g}=-\hat{\gamma}_0$.

Case 2. If $|k|>1$, then $1/k\notin\mathbf{Z}[i]$. Since $\hat{\gamma}_0=\hat{\wp}\circ\tilde{g}_0$ maps $\mathbf{R}/\mathbf{Z}$ onto the Riemann sphere, we can choose $0<t<1$ so that $\tilde{g}_0(t)=\pm1/k$. It follows that $\tilde{g}(t)=\pm1\in\mathbf{Z}[i]$, hence $\hat{\gamma}(t)=\hat{\wp}\circ\tilde{g}(t)=\infty$, contradicting the hypothesis that $\hat{\gamma}$ is primitive. Hence Case 2 cannot occur, which completes the proof of 4.3. $\square$

4.9. Some Related Semiconjugacies. For any constant $k\ne0$ in $\mathbf{Z}[i]$, the map

$$t \;\mapsto\; \gamma(t) \;=\; \hat{\wp}(k\,\tilde{g}_0(t))$$

is clearly another symmetric semiconjugacy from the dynamical system $(\mathbf{R}/\mathbf{Z}\,,\,2\cdot)$ onto $(\hat{\mathbf{C}}\,,\,F)$. This semiconjugacy can also be written as

$$\gamma \;=\; (\hat{\wp}\circ L_k\circ\hat{\wp}^{-1})\circ(\hat{\wp}\circ\tilde{g}_0) \;=\; (\hat{\wp}\circ L_k\circ\hat{\wp}^{-1})\circ\gamma_0\ ,$$

where $L_k(w)=k\,w$ so that $F_k=\hat{\wp}\circ L_k\circ\hat{\wp}^{-1}$ is a Lattès map of degree $|k|^2$ which commutes with F . More generally, for any constant $k'\ne0$ in $\mathbf{Z}$ we can also consider the semiconjugacy

$$t \;\mapsto\; \gamma(k'\,t) \;=\; F_k\circ\gamma_0(k't)$$

from $\mathbf{R}/\mathbf{Z}$ onto $\hat{\mathbf{C}}$. Note however that this related semiconjugacy is not primitive when either $|k|>1$ or $|k'|>1$.

4.10. A note on computation. We conclude this section by noting that it is quite easy to compute the function $\tilde{g}(t)$. For any real number t_0, let us set

$$2t_0 \;=\; \epsilon\, t_1 + k \qquad \text{with} \qquad \epsilon = \pm 1\;, \quad t_1 \in [0\,,\,1/2]\;, \quad \text{and} \quad k \in \mathbf{Z}\;.$$

Then it follows easily from 4.5 that

$$\tilde{g}(t_0) \;=\; \sqrt{\epsilon}\,\tilde{g}(t_1)/\eta \,+\, k\,\eta/2\;, \tag{10}$$

where $\eta = 1-i$, taking $\sqrt{+1}$ to be $+1$ and $\sqrt{-1}$ to be $+i$. For computational purposes, we may assume that t_0 is a dyadic rational, say $t_0 = m/2^n$. Since t_1 has smaller denominator than t_0, and since $\tilde{g}(0)$ is zero by definition, this yields a recursive definition which is easily implemented.

It is interesting to note that the correspondence $t_0 \mapsto t_1$, restricted to the interval $[0\,,\,1/2]$, is just the familiar tent map on this interval. For an arbitrary rational number t_0, note that the orbit $t_0 \mapsto t_1 \mapsto t_2 \mapsto \cdots$ under this tent map is eventually periodic. Hence (10) yields a finite set of linear equations which we can solve for $\tilde{g}(t_0)$. It follows that $\tilde{g}(t_0)$ necessarily belongs to the field $\mathbf{Q}[i]$ of Gaussian rational numbers.

Here is an example. Since the external rays $\mathcal{R}_{1/7}$, $\mathcal{R}_{2/7}$ and $\mathcal{R}_{4/7}$ for the polynomial $z \mapsto z^2 + c_{1/4}$ of §3 all land at a common point, namely the α-fixed point, we know that the values $\tilde{g}(1/7)$, $\tilde{g}(2/7)$ and $\tilde{g}(4/7)$ must all represent the same point in the quotient space $\hat{\wp}(\mathbf{C}) = \hat{\mathbf{C}}$. In other words, we must have $\tilde{g}(1/7) \equiv \pm\tilde{g}(2/7) \equiv \pm\tilde{g}(4/7) \pmod{\mathbf{Z}[i]}$. In fact, computing by the algorithm described above, it turns out that

$$\tilde{g}(1/7) \;=\; \frac{2+i}{5}\,, \qquad \tilde{g}(2/7) \;=\; \tilde{g}(4/7) \;=\; \frac{3-i}{5}\,,$$

with $\tilde{g}(1/7) \equiv \pm\tilde{g}(2/7)$ as expected.

§5. Measure Properties.

We know from 2.11 and 3.5 that:

- *One-dimensional Lebesgue measure on* $\mathbf{R}/\mathbf{Z}$ *pushes forward, under the mating semiconjugacy* $\hat{\gamma}$ *from* $(\mathbf{R}/\mathbf{Z}\,,\,2\cdot)$ *to* $(\hat{\mathbf{C}}\,,\,F)$, *to Lyubich measure on* $\hat{\mathbf{C}}$.
- *Two-dimensional Lebesgue measure on* $\mathbf{C}/\mathbf{Z}[i]$ *pushes forward under the normalized Weierstrass map* $\hat{\wp}$ *to this same Lyubich measure on* $\hat{\mathbf{C}}$.

Since $\hat{\gamma}$ is equal to the composition

$$\mathbf{R}/\mathbf{Z} \;\xrightarrow{g}\; \mathbf{C}/\mathbf{Z}[i] \;\xrightarrow{\hat{\wp}}\; \hat{\mathbf{C}}\,,$$

this might suggest that the 1-dimensional Lebesgue measure on $\mathbf{R}/\mathbf{Z}$ pushes forward under $g : \mathbf{R}/\mathbf{Z} \to \mathbf{C}/\mathbf{Z}[i]$ to the 2-dimensional Lebesgue measure on the torus. However this is not quite right. In fact the map $\hat{\wp}$ is two-to-one, and the image $g(\mathbf{R}/\mathbf{Z})$ covers only about half of the torus. The correct statement is as follows.

5.1. Lemma. *Let $\{\pm 1\} \times \mathbf{R}/\mathbf{Z}$ be the union of two disjoint circles, mapped to $\mathbf{C}/\mathbf{Z}[i]$ by the correspondence $(\pm 1\,,\,t) \mapsto \pm g(t)$. Then the push forward $\mathbf{m}$ of 1-dimensional Lebesgue measure on $\{\pm 1\} \times \mathbf{R}/\mathbf{Z}$ is equal to twice the 2-dimensional Lebesgue measure on the torus.*

Proof Outline. If we subdivide $\mathbf{R}/\mathbf{Z}$ into 2^n intervals of length $1/2^n$, then the approximation g_n of 4.6 or 4.7 maps each of these to an interval of length $s = 1/\sqrt{2^n}$ in the torus. Combining these with the corresponding 2^n intervals for $-g_{n1}$, we obtain a rectilinear configuration consisting of 2^{n+1} edges of length s which subdivide the torus into 2^n squares of area $s^2 = 1/2^n$, as illustrated in Figure 8. (It is important to note that there are twice as many edges as squares.) Now consider a region U of area A with piecewise smooth boundary in this torus. For large values of n the number of squares in U is asymptotic to $A/s^2 = 2^n A$. Since there are twice as many edges as squares,

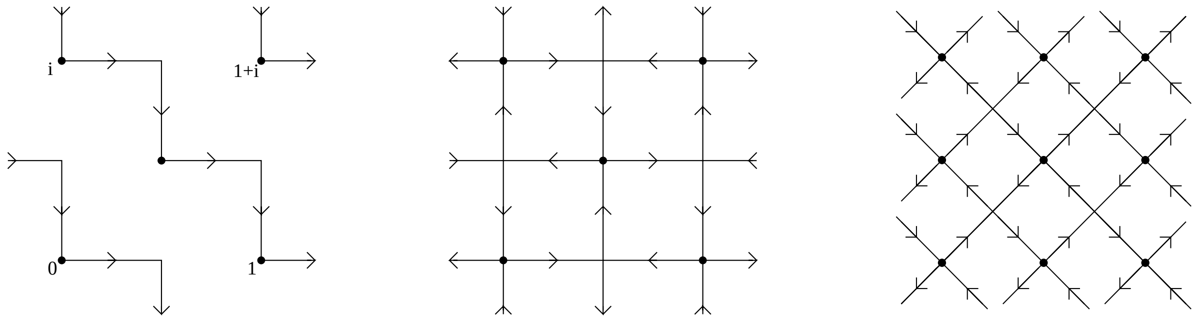


Figure 8. The left hand figure shows the image $g_2(\mathbf{R}/\mathbf{Z}) \subset \mathbf{C}/\mathbf{Z}[i]$, lifted to the universal covering space $\mathbf{C}$. Here, as in Figure 6, the "old" vertices are indicated by heavy dots. In the middle figure, the rotated image $-g_2(\mathbf{R}/\mathbf{Z})$ has been added. Evidently the union $g_2(\mathbf{R}/\mathbf{Z}) \cup -g_2(\mathbf{R}/\mathbf{Z})$, lifted to $\mathbf{C}$, forms a full rectilinear grid. In the right hand picture, the inductive construction of Figure 6 has been applied to each edge, to obtain $g_3(\mathbf{R}/\mathbf{Z}) \cup -g_3(\mathbf{R}/\mathbf{Z})$. The resulting picture in the universal covering is isomorphic to that for $g_2(\mathbf{R}/\mathbf{Z}) \cup -g_2(\mathbf{R}/\mathbf{Z})$, except for a 45° rotation and a scale change of $1/\sqrt{2}$. Continuing inductively, we get an analogous picture for any $n \geq 2$.

the number of edges is asymptotic to $2^{n+1}A$, and the push forward of the Lebesgue measure on $\{\pm 1\} \times \mathbf{R}/\mathbf{Z}$, evaluated on U, is asymptotic to this number of edges multiplied by the 1-dimensional Lebesgue measure $1/2^n$ of each edge in the preimage. Hence the total is asymptotic to $\mathbf{m}(U) = 2A$, as required. $\square$

Remark. This last figure illustrates the image $g_n(\mathbf{R}/\mathbf{Z}) \cup -g_n(\mathbf{R}/\mathbf{Z})$ without giving any clue as to whether we must turn left or right upon reaching a vertex, as we traverse one of the two copies of $\mathbf{R}/\mathbf{Z}$. In fact the required pattern is extremely complicated. It can best be visualized by choosing a suitable approximation to g_n.

5.2. Lemma. *The mating semiconjugacy $\hat{\gamma}$ from $\mathbf{R}/\mathbf{Z}$ onto $\hat{\mathbf{C}}$ can be uniformly approximated by a topological embedding.*[4] *Similarly each map $(\pm 1\,,\,t) \mapsto \pm\tilde{g}_n(t)$ from $\{\pm 1\} \times \mathbf{R}/\mathbf{Z}$ onto $\mathbf{C}/\mathbf{Z}[i]$ can be uniformly approximated by a topological embedding.*

[4] This statement is conjectured to be true for any mating, but the following proof applies only to our special example.

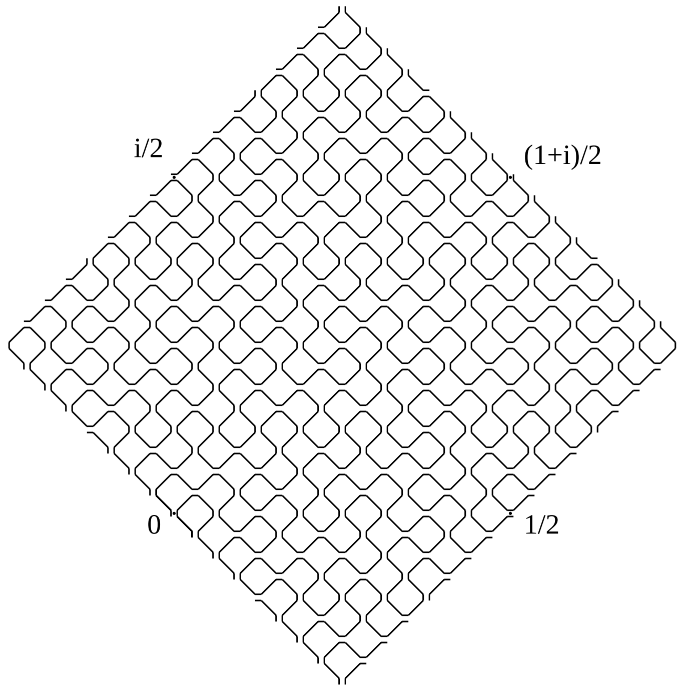


Figure 9. The image of an approximation to $g_9 : \mathbf{R}/\mathbf{Z} \to \mathbf{T}/\pm$ *, where* $\mathbf{T} = \mathbf{C}/\mathbf{Z}[\mathbf{i}]$ *. The region shown is a fundamental domain for the action of* $\{\pm 1\}$ *on the torus, with ramification points at the midpoints of the four edges. The quotient* $\mathbf{T}/\pm$ *can be obtained from this region by identifying each edge with itself under a* $180°$ *rotation about its midpoint. The grid* $\pm g_9(\mathbf{R}/\mathbf{Z})$ *cuts this region into* 2^8 *small squares, each of area* $1/2^9$ *.*

Proof. Examining the construction above we see that the image $g_n(\mathbf{R}/\mathbf{Z}) \cup -g_n(\mathbf{R}/\mathbf{Z})$ never crosses itself. It has many double points, but these are always places where two segments of this image come together and then bounce off at right angles, without crossing. Hence by a slight deformation, we can get rid of all of these double points. For example, if we replace $g_n(t)$ by the average $(g_n(t) + g_n(t+\epsilon))/2$, then for ϵ sufficiently small we obtain the required embedding which approximates g . (Compare Figure 9.) The corresponding statement for $\hat{\gamma} = \hat{\wp} \circ g$ follows easily. (Figure 10.) $\square$

Although the area filling curve $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ (or $g : \mathbf{R}/\mathbf{Z} \to \mathbf{T}/\pm$) has many self-intersections, we will show that it is one-to-one almost everywhere. We must first consider the corresponding question for the Carathéodory semiconjugacy $\gamma : \mathbf{R}/\mathbf{Z} \to J(f)$.

Let $f : \mathbf{C} \to \mathbf{C}$ be any polynomial which is postcritically finite. To simplify the exposition, we assume also that f has no attracting cycles, so that the Julia set $J(f)$ is equal to the filled Julia set $K(f)$.

Definition. The minimal *Hubbard tree* H_0 of f is the smallest connected subset of $J(f)$ which contains the orbits of the critical points. We will also need the enlarged trees $H_n = f^{-n}(H_0) \subset J(f)$. Here are some basic properties:

5.3. Lemma. *Each of these sets* $H_0 \subset H_1 \subset H_2 \subset \cdots$ *is a finite topological tree, with* $f(H_n) = H_{n-1}$ *for* $n > 0$ *and* $f(H_0) = H_0$ *. Furthermore, the union* $\bigcup H_n$ *is dense in* $J(f)$ *. If two distinct external rays land at a point*

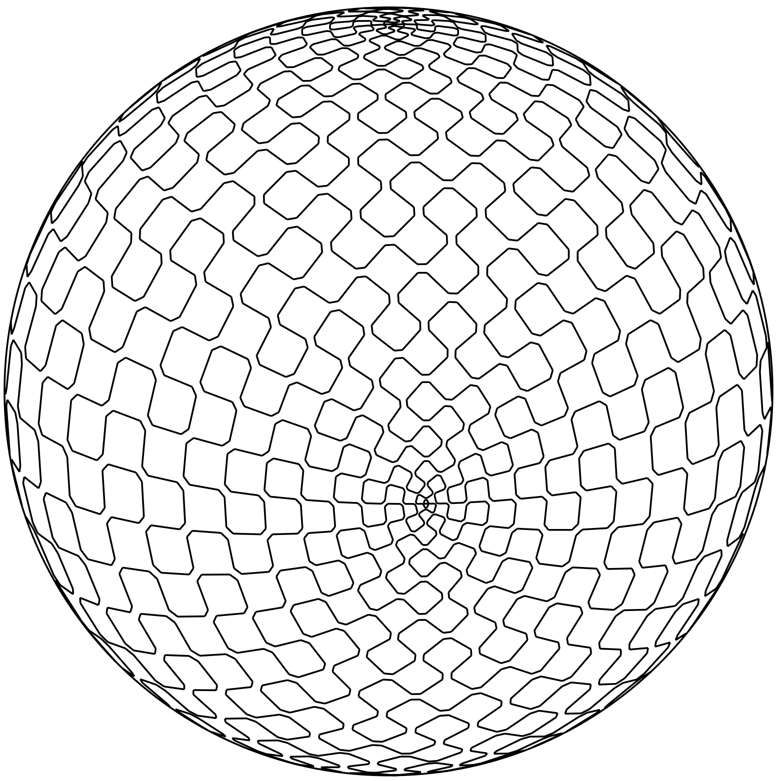


Figure 10. An analogous approximation to $\hat{\wp}\circ g_{11}(\mathbf{R}/\mathbf{Z})\approx\hat{\gamma}(\mathbf{R}/\mathbf{Z})$ on the Riemann sphere (orthonormal projection). The image is a Jordan curve which cuts the sphere into two simply connected regions, each made up out of 2^9 approximate squares of Lyubich area $1/2^{10}$. Note that this image becomes highly compressed and distorted around the critical values of $\hat{\wp}$, where Lyubich measure tends to be concentrated. (The sphere has been rotated so that two of these four critical values are visible.)

$z\in J(f)$, then z must belong to some H_n. If we exclude Chebyshev[5] maps such as $f(z)=z^2-2$, for which $H_0(f)=J(f)$, then each H_n is a set of Brolin measure zero in $J(f)$. It follows that each $\gamma^{-1}(H_n)$ has Lebesgue measure zero in $\mathbf{R}/\mathbf{Z}$, and that the Carathéodory semiconjugacy $\gamma:\mathbf{R}/\mathbf{Z}\to J(f)$ is one-to-one almost everywhere in the sense that $\gamma^{-1}(\gamma(t))=\{t\}$ for Lebesgue almost every t.

To the naive eye, the Hubbard tree seems to occupy a large part of the Julia set. (Compare Figures 1, 2, 11, 12.) However, in terms of Brolin measure, it occupies a negligibly small part.

Remark 5.4. [Zakeri] has proved the more general statement that $\gamma:\mathbf{R}/\mathbf{Z}\to J(f)$ is one-to-one almost everywhere provided only that f is quadratic and not a Chebyshev map, with $J(f)$ locally connected.

Proof of 5.3. The first statement is straightforward. The union is dense since the preimages of any point are dense in the Julia set $J(f)$. If two rays land at z, then it follows from the Theorem of F. and M. Riesz that these two rays, together with their landing point,

[5] See Appendix B.2.

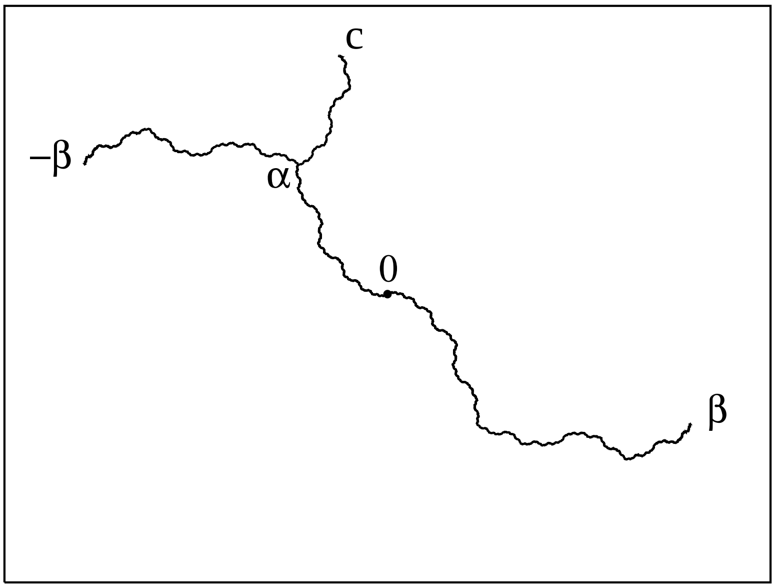


Figure 11. The Hubbard tree $H_0 = H_0(f_{1/4}) \subset J(f_{1/4})$.

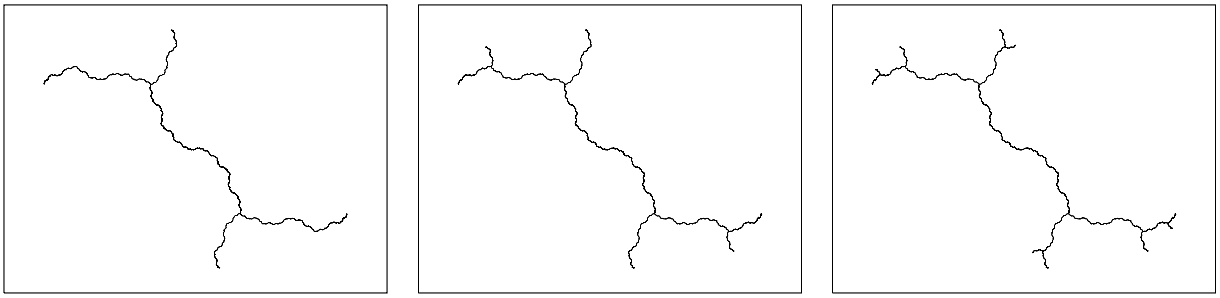


Figure 12. Enlarged Hubbard trees $H_n = f^{-n}(H_0)$ *for* $n = 1\,,\ 2\,,\ 3$.

cut $J(f)$ into two non-degenerate subsets. Since the union of the H_n is dense, some H_n must intersect both of these subsets, hence $z \in H_n$. Finally, since the Brolin measure $\mathbf{m}$ is f-invariant, we have $\mathbf{m}(H_0) = \mathbf{m}(H_1)$, hence the difference $H_1 \smallsetminus H_0$ has measure zero. If $H_0 \neq J$, then the forward images of $H_1 \smallsetminus H_0$ cover H_0 . For otherwise, if H_0 contained a point z which is not in any $f^{\circ n}(H_1 \smallsetminus H_0)$, then all of the iterated preimages of z would belong to H_0 , hence the closure $\overline{H}_0 = H_0$ would be the entire Julia set.

Since Brolin measure has the property that the image of any set of measure zero also has measure zero (compare 2.10, formula (3)), it follows that H_0 has measure zero. Therefore every H_n has measure zero. Since the push-forward of Lebesgue measure on $\mathbf{R}/\mathbf{Z}$ is the Brolin measure on $J(f)$, the last statement follows. □

5.5. Corollary to 5.4. *If* $F \cong f_1 \perp\!\!\!\perp f_2$ *is a geometric mating between non-Chebyshev quadratic polynomials, then the mating semiconjugacy* $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ *is one-to-one almost everywhere, using Lebesgue measure on* $\mathbf{R}/\mathbf{Z}$ *and Lyubich measure on* $J(F) \subset \hat{\mathbf{C}}$ *. If* $I \subset \mathbf{R}/\mathbf{Z}$ *is any closed line segment of length* ℓ *, it follows that the image* $\hat{\gamma}(I)$ *is a compact set of Lyubich measure* ℓ *, with boundary of measure zero.*

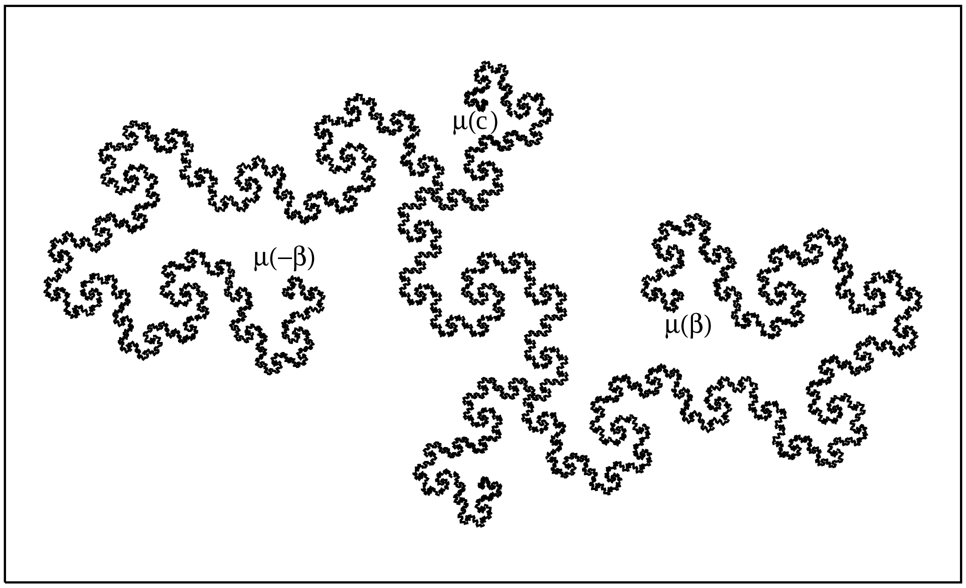


Figure 13. Image $\mu_1(H_1)$ of the symmetric Hubbard tree in the complex plane, using a normalization with the two critical points at zero and infinity. This image is topologically embedded (see Appendix C) as a subset of measure zero.

Proof. Let γ_1 and γ_2 be the Carathéodory semiconjugacies for f_1 and f_2. By Zakeri's statement 5.4, there are subsets X_1 and X_2 of measure zero in $\mathbf{R}/\mathbf{Z}$ so that $\gamma_j^{-1}(\gamma_j(t))$ is the single point t for $t \notin X_j$. It follows easily that $\hat{\gamma}^{-1}(\hat{\gamma}(t)) = \{t\}$ for $t \notin X_1 \cup (-X_2)$. Therefore the image $X = \hat{\gamma}(X_1 \cup (-X_2))$ is a set of Lyubich measure zero in $\hat{\mathbf{C}}$, such that $\hat{\gamma}^{-1}(z)$ is a single point for every $z \notin X$.

Now given an interval $I \subset \mathbf{R}/\mathbf{Z}$, since $\hat{\gamma}^{-1} \circ \hat{\gamma}(I)$ is equal to I together with a set of measure zero, it follows that $\mathbf{m}(\hat{\gamma}(I))$, which is equal to the Lebesgue measure of $\hat{\gamma}^{-1}\hat{\gamma}(I)$, must be equal to $\ell(I)$. If $I' = \overline{\mathbf{R}/\mathbf{Z} \smallsetminus I}$, then since $\mathbf{m}(\hat{\gamma}(I)) + \mathbf{m}(\hat{\gamma}(I')) = 1$, it follows that the common boundary must have measure zero. $\square$

5.6. Corollary. *The map $\tilde{g} : \mathbf{R} \to \mathbf{C}$ of 4.8 carries any closed line segment of length ℓ to a compact set of Lebesgue area equal to $\ell/2$, with boundary of Lebesgue measure zero.*

Proof. This follows by applying 5.3 or 5.5 to the mating $F \cong f_{1/4} \perp\!\!\perp f_{1/4}$ of §3. $\square$

§6. Fractal Tiling with Hubbard Tree Boundary.

Let $F \cong f_1 \perp\!\!\perp f_2$ be any geometric mating, where the Julia sets of f_1 and f_2 are full, so that $J = K$, and so that the Julia set $J(F)$ is the entire sphere $\hat{\mathbf{C}}$. Then corresponding to any partition of $\mathbf{R}/\mathbf{Z}$ into non-overlapping intervals I_j we get a partition of $\hat{\mathbf{C}}$ into compact subsets $T_j = \hat{\gamma}(I_j)$. Note that any overlap $T_j \cap T_k$ has Lyubich measure zero. In particular, such intersections have no interior.

The simplest partition divides $\mathbf{R}/\mathbf{Z}$ into the two intervals $[0, 1/2]$ and $[1/2, 1]$. The corresponding sets T_1 and T_2 both map onto the whole sphere under F, and map to each other under the canonical involution τ which fixes both critical points. In the symmetric case, when $f_1 = f_2$, they also map to each other under the symmetry involution, which interchanges the two critical points. For a Lattès mating, this tiling of $\hat{\mathbf{C}}$ lifts to a tiling of the branched covering space $\mathbf{C}$.

The common boundary $\partial T_1 = \partial T_2 = T_1 \cap T_2$ can be described as follows. By the *spine* S_j of the Julia set $J_j = J(f_j)$ we will mean the unique arc $S_j \subset J_j$ which joins the β fixed point to its preimage $-\beta$. (This definition makes sense for any quadratic Julia set which is locally connected and full.) Recall the notations from (8) in §4. We have maps $\mu_j : K_j \to \hat{\mathbf{C}}$ with

$$\hat{\gamma}(t) \;=\; \mu_1 \circ \gamma_1(t) \;=\; \mu_2 \circ \gamma_2(-t) \;,$$

and with $F \circ \mu_j = \mu_j \circ f_j$.

6.1 Lemma. *The intersection $T_1 \cap T_2$ is equal to the union $\hat{S} = \mu_1(S_1) \cup \mu_2(S_2)$ of the images of the spines of the two Julia sets.*

Note that these two spine images certainly meet at their endpoints $\hat{\beta}$ and $\tau(\hat{\beta})$. If $\mu_1(S_1)$ and $\mu_2(S_2)$ met only at these endpoints then T_1 and T_2 would be Jordan domains, but in practice the situation is more complicated. For example for the mating $f_{1/4} \perp\!\!\!\perp f_{1/4}$ of §3, since the rays of angle $2/5$ and $-2/5 \equiv 3/5$ both land on the spine in Figure 2, it follows that $\mu_1(S_1)$ and $\mu_2(S_2)$ meet also at the points $\hat{\gamma}(2/5) = \mu_1(\gamma(2/5)) = \mu_2(\gamma(3/5))$. Similarly they meet at the sequence of points $\hat{\gamma}(t_n)$ and at the sequence of points $\hat{\gamma}(1/2 - t_n)$, where $t_n = 1/(2^n \cdot 10)$ for $n \ge 0$. These intersections between the two boundary curves explain the pinchings which are visible in Figures 7, 14, 15, 16. See Appendix C for details.

Proof of 6.1. If $J = J(f)$ is full and locally connected, then the union

$$S(J) \;\cup\; \mathcal{R}_0(J) \;\cup\; \mathcal{R}_{1/2}(J) \quad \subset \quad \mathbf{C}$$

cuts the complex plane into two open sets, one containing all rays $\mathcal{R}_t(J)$ with $0 < t < 1/2$ and the other containing all rays $\mathcal{R}_t(J)$ with $1/2 < t < 1$. *It follows that the intersection $\gamma[0, 1/2] \cap \gamma[1/2, 1]$, for the associated semiconjugacy $\gamma = \gamma_f$, is precisely equal to the spine $S(J)$.* Now consider a mating $F \cong f_1 \perp\!\!\!\perp f_2$ with f_1 and f_2 as above. Evidently the associated $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ satisfies the following:

The image $\hat{\gamma}(t)$ is equal to $\hat{\gamma}(t')$ if and only if there exists a finite chain

$$t = t_0\,,\ t_1\,,\ \ldots\,,\ t_n = t'$$

so that, for every i between 1 and n , either $\gamma_1(t_{i-1}) = \gamma_1(t_i)$ or $\gamma_2(-t_{i-1}) = \gamma_2(-t_i)$.

In particular, if this condition is satisfied with $0 \le t \le 1/2 \le t' \le 1$ then there must exists some i with $0 \le t_{i-1} \le 1/2 \le t_i \le 1$, and it follows that $\hat{\gamma}(t)$ belongs either to $\mu_1(S_1)$ or to $\mu_2(S_2)$, as required. □

Similarly, if we partition $\mathbf{R}/\mathbf{Z}$ into four equal intervals $[j/4\,,\,(j+1)/4]$, then we obtain a corresponding partition of the Riemann sphere into four tiles. The analogue of 6.1 is the statement that the union of the boundaries of these four tiles is equal to $\hat{S} \cup F^{-1}(\hat{S})$, or in

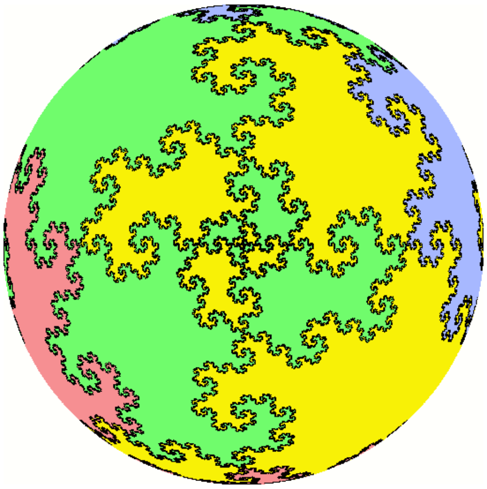


Figure 14. Tiling of the Riemann sphere by the sets $\hat\gamma[0,1/4]\,,\,\hat\gamma[1/4\,,\,1/2]$, $\hat\gamma[1/2\,,\,3/4]$ *and* $\hat\gamma[3/4,\,,\,1]$. *Here* $\hat\gamma$ *is the semiconjugacy of* 4.3*, associated with the mating* $f_{1/4} \perp\!\!\!\perp f_{1/4}$ *of* 3.1.

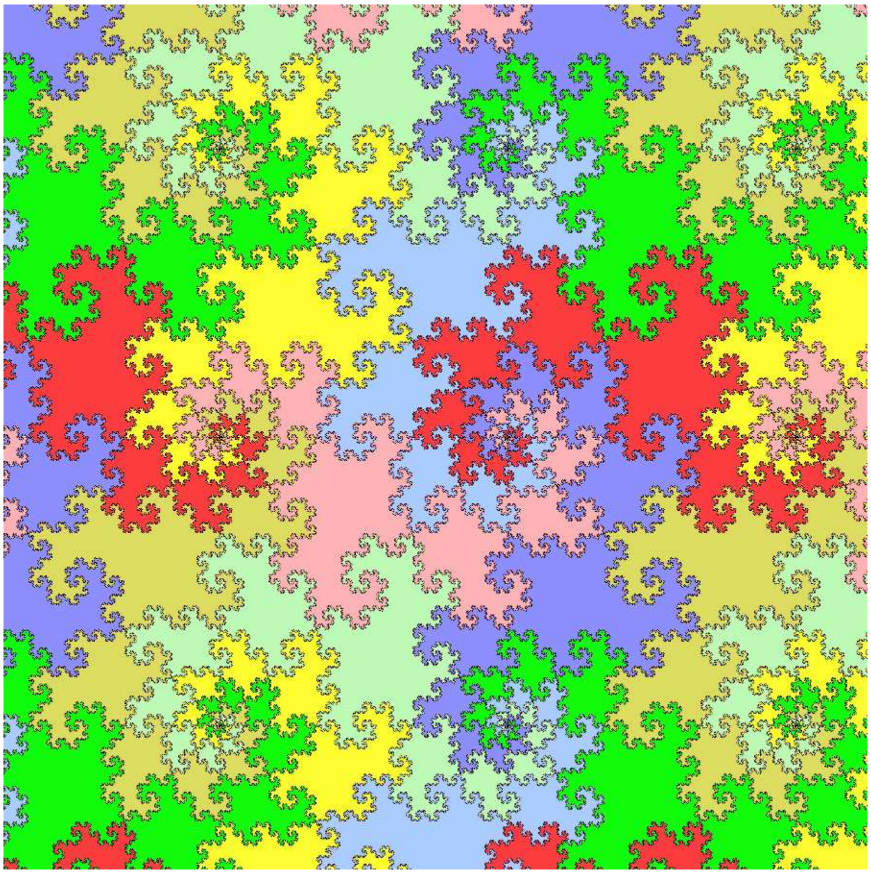


Figure 15. Corresponding tiling lifted to $\mathbf{C}$ *via the* $\wp$*-function. (The number of colors has been doubled, since each tile lifts to the torus in two different ways.) The illustrated square is somewhat larger than a fundamental domain for the lattice* $\mathbf{Z}[i]$. *The pattern is invariant under* 180° *rotations about the critical points of* $\hat\wp$. *Furthermore, a suitably chosen translation or rotation will carry regions of one color into regions of any other color.*

other words is equal to the union of the set $\mu_1(S_1 \cup f_1^{-1}(S_1))$ with the corresponding set for f_2. In fact the set $S_1 \cup f_1^{-1}(S_1)$, together with the rays of angle 0, $1/4$, $1/2$, $3/4$, cuts the complex plane up into four regions, and the discussion proceeds as above. For the special case $F \cong f_{1/4} \perp\!\!\perp f_{1/4}$ of §3, the situation is particularly simple, since the set $S(f) \cup f^{-1}(S(f))$ is just the symmetric Hubbard tree H_1 of Figure 12. Hence the union of the boundaries of the regions in Figure 14 is made up out of two copies of the set $\mu(H_1)$ of Figure 13. If these two copies intersected only at the points $\pm\hat{\beta}$ and $\hat{\gamma}(\pm 1/4)$, then they would cut the sphere up neatly into four Jordan domains. However, as noted above and in Appendix C, there are many other intersections and the situation is much more complicated. These extra intersections occur precisely at the angles

$$t \;=\; \cdots 1/40\,,\ 1/20\,,\ 1/10\,,\ 1/5\,,\ 2/5\,,\ 9/20\,,\ 19/40\,,\ \cdots\,. \tag{11}$$

Under the associated Weierstrass function $\hat{\wp} : \mathbf{T} \to \hat{\mathbf{C}}$, each of these four tiles lifts in two ways. Thus we obtain a tiling of the torus $\mathbf{T}$ by eight Heighway dragons, as illustrated in Figure 15. Similarly, if we subdivide each $[j/4\,,\,(j+1)/4]$ into 2^n equal intervals, then each dragon of area $1/8$ will be subdivided into 2^n dragons of area $1/(8\cdot 2^n)$. See Figure 16 to get some idea of how these dragons map under multiplication by $1-i$. (For the theory of such self-similar tilings, compare [Kenyon].)

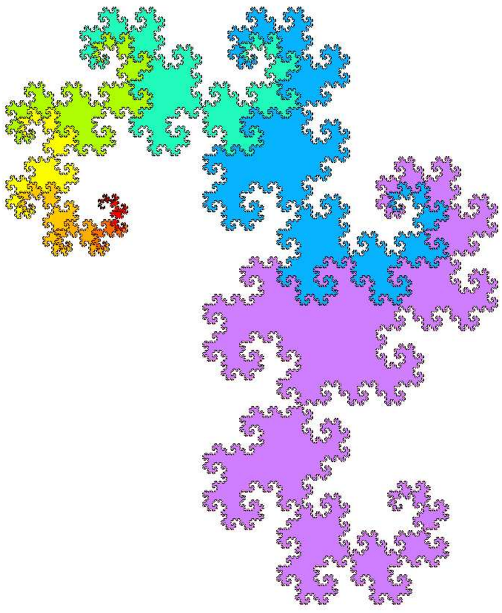


Figure 16. A picture of the dragon $\hat{g}[0,1/2]$ of Figure 7, subdivided into images $\hat{g}[1/2^n\,,\,1/2^{n-1}]$ in order to illustrate the dynamics. Each of these regions (except the last) maps isomorphically onto the next under multiplication by $1-i$ The last maps isomorphically onto a set $i\hat{g}[0,1/2]-i$ which can be rotated isomorphically onto the whole set $\hat{g}[0,1/2]$.

§7. Further Questions.

The basic question raised by this paper is the following: What is the relationship between semiconjugacy and mating? First one can ask: *Does the semiconjugacy* $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to \hat{\mathbf{C}}$ *associated with a mating* $F \cong f_1 \perp\!\!\!\perp f_2$ *uniquely determine the polynomials* f_1, f_2 *and the mating homeomorphism* $K_1 \perp\!\!\!\perp K_2 \xrightarrow{\cong} \hat{\mathbf{C}}$ *?* This seems very likely, but I don't have a proof.

What conditions on a semiconjugacy are needed in order to conclude that it comes from some mating? We know that $\hat{\gamma}$ must be primitive and τ-equivariant, with $\hat{\gamma}(2t) = F(\hat{\gamma}(t))$. However, A. Douady has pointed out to me that these conditions are not sufficient. His example is based on the following observations.

There exist quadratic polynomials f and g which are topologically conjugate on their filled Julia sets,

$$h \,:\, K(g) \overset{\cong}{\to} K(f) \qquad \text{with} \qquad h \circ g = f \circ h \,,$$

even though g is not topologically conjugate to f on the whole complex plane. Some examples are described in Appendix D. Given such an h, we can construct an exotic semiconjugacy $\eta : \mathbf{R}/\mathbf{Z} \to \partial K(f)$ by setting $\eta = h \circ \gamma_g$. Now if the mating $F \cong f_1 \perp\!\!\!\perp f_2$ is defined, with $f = f_1$, then in place of the mating semiconjugacy $\hat{\gamma}(t) = \mu_1 \circ \gamma_1(t) = \mu_2 \circ \gamma_2(-t)$ we can consider the exotic semiconjugacy $\hat{\eta} = \mu_1 \circ \eta$ from $\mathbf{R}/\mathbf{Z}$ to $J(F)$. There is no reason to expect that this $\hat{\eta}$ is the semiconjugacy associated with any mating. Taking account of such examples, Douady suggests the following further requirement:

> **Almost Embedding Condition.** It must be possible to uniformly approximate the semiconjugacy $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to J(F)$ by topological embeddings of the circle into the sphere. (Compare 5.2 and Figure 10.)

We can then ask: *Are the primitive* τ*-equivariant semiconjugacies satisfying this Douady almost embedding condition exactly the ones which arise from matings?*

Even in the special case of the mating $f_{1/4} \perp\!\!\!\perp f_{1/4}$ which has been studied above, there remain many questions: *Is it necessary to assume symmetry in order to prove* 4.3*? What semiconjugacies from* $(\mathbf{R}/\mathbf{Z}\,,\, 2\cdot)$ *to* $(\hat{\mathbf{C}}\,,\, F)$ *actually exist? Furthermore, can one carry out a similar program for the other Lattès matings described in Appendix B? How much of this program can be carried out for more general matings?*

Appendix A: The Weierstrass $\wp$-function.

Let Λ be some fixed lattice in $\mathbf{C}$, and let $\mathbf{T}$ be the quotient torus $\mathbf{C}/\Lambda$. It will be convenient to use the term *Weierstrass* $\wp$*-function* loosely to mean any holomorphic function $\wp : \mathbf{T} \to \hat{\mathbf{C}}$ of degree two which satisfies

$$\wp(-w) \;=\; \wp(w)\;, \qquad \text{and} \qquad \wp(0) \;=\; \infty\;.$$

Evidently any such $\wp$ induces a conformal isomorphism between the quotient $\mathbf{T}/\pm$ and the Riemann sphere. It will be convenient to write the Laurent series for $\wp$ as

$$\wp(w) \;=\; (a/w)^2 \;+\; b \;+\; \sum_1^\infty c_k w^{2k}\;,$$

where $a \neq 0$. We can then set $\wp(w) = a^2\,\wp_0(w) + b$, where $\wp_0$ is the *standard Weierstrass function*, satisfying $\wp_0(w) = 1/w^2 + O(w^2)$. (Compare [Ahlfors].)

A.1. A rapidly converging series. After a linear change of the variable w, we may assume that the lattice Λ is generated by 1 and τ, with $\tau \in \mathbf{C} \smallsetminus \mathbf{R}$. One choice of $\wp$-function for the lattice $\Lambda = \mathbf{Z} + \tau\,\mathbf{Z}$ is then given by the series

$$\wp_1(w) \;=\; \sum_{n\in\mathbf{Z}} \eta(w+n\tau)\;, \qquad \text{where} \qquad \eta(w) \;=\; 1+\cot^2(\pi w)\;. \tag{12}$$

This series converges very rapidly. In fact we will prove the asymptotic estimate

$$|\eta(u+iv)| \;\sim\; 4/e^{2\pi|v|}\;, \tag{13}$$

which holds uniformly in u as $|v|\to\infty$. (Note that the constant $e^{2\pi}\approx 535.49$ is quite large.) Since $\eta(w)=\eta(-w)=\eta(w+1)$, it follows that the sum of this series satisfies

$$\wp_1(w) \;=\; \wp_1(-w) \;=\; \wp_1(w+1) \;=\; \wp_1(w+\tau)\;.$$

It is often convenient to make the substitution

$$\mathcal{E} \;=\; e^{2\pi i w}\;, \qquad \cot(\pi w) \;=\; i\,\frac{e^{\pi i w}+e^{-\pi i w}}{e^{\pi i w}-e^{-\pi i w}} \;=\; i\,\frac{\mathcal{E}+1}{\mathcal{E}-1}\;, \tag{14}$$

hence

$$\eta(w) \;=\; 1-\left(\frac{\mathcal{E}+1}{\mathcal{E}-1}\right)^2 \;=\; \frac{-4\mathcal{E}}{(\mathcal{E}-1)^2} \;=\; \frac{4}{2-\mathcal{E}-\mathcal{E}^{-1}}\;.$$

In particular, note that

$$\eta(w) \;=\; \infty \quad\Longleftrightarrow\quad \mathcal{E} \;=\; 1 \quad\Longleftrightarrow\quad w \;\in\; \mathbf{Z}\;.$$

It follows that $\wp_1$ has poles only at the points of $\mathbf{Z}+\tau\mathbf{Z}$. If $w=u+iv$, then $|\mathcal{E}|=e^{-2\pi v}$. As v tends to $+\infty$, $\mathcal{E}$ tends rapidly to zero, hence

$$\eta(w) \;\sim\; -4\mathcal{E}\;, \qquad |\eta(w)| \;\sim\; |4\mathcal{E}| \;=\; 4/e^{2\pi v}\;.$$

This proves (13) as $v\to+\infty$, and the proof when $v\to-\infty$ is similar. Thus the series (12) converges. Its sum clearly gives rise to an even, degree two map from $\mathbf{T}$ to $\hat{\mathbf{C}}$, with $\wp_1(0)=\infty$. Therefore $\wp_1(w)=a^2\,\wp_0(w)+b$ for suitable $a\neq 0$ and b. In fact since

$$\eta(w) \;=\; 1/(\pi w)^2 \,+\, O(1) \qquad \text{as} \qquad w\to 0\;,$$

it follows that $a=1/\pi$. For the computation of b, see A.4 below.

A.2. Remark on computation. To actually compute, one uses the series

$$\wp_1(w) \;=\; -4\sum_{n\in\mathbf{Z}} \frac{c^n\mathcal{E}}{(c^n\mathcal{E}-1)^2} \;=\; \sum \frac{4}{2-c^n\mathcal{E}-(c^n\mathcal{E})^{-1}}\;, \quad \text{where} \quad c \;=\; e^{2\pi i\tau}\;, \tag{12$'$}$$

with $\mathcal{E}$ as in (14). For the application in §4, it was convenient to choose a specific Weierstrass function $\hat\wp$ by specifying the values $\hat\wp(w_1)$ and $\hat\wp(w_2)$ at two designated points w_1 and w_2 of $\mathbf{C}/\Lambda$. Any such function can be evaluated as $\hat\wp(w)=\alpha\,\wp_1(w)+\beta$, where the coefficients α and β can be computed by solving the linear equations $\hat\wp(w_j)=\alpha\,\wp_1(w_j)+\beta$.

A.3. The inverse function. Given distinct points v_1, v_2, v_3 in the complex plane, and given a constant $a \neq 0$, consider the elliptic integral

$$w(z) \;=\; a\int_\infty^z \frac{d\zeta}{\sqrt{4(\zeta - v_1)(\zeta - v_2)(\zeta - v_3)}} \;. \tag{15}$$

We can make sense of this many-valued function as follows. Let T be the smooth projective variety consisting of all pairs $(z, r) \in \mathbf{C}^2$ with

$$r^2 \;=\; f(z)\;, \qquad \text{where} \qquad f(z) = 4(z - v_1)(z - v_2)(z - v_3)\;,$$

together with one point at infinity. Then T is a 2-fold branched covering of $\hat{\mathbf{C}}$ under the projection $(z, r) \mapsto z$. There are four branch points, hence T is a surface of genus one by the Riemann-Hurwitz formula. Given any smooth path P in T which leads from the point at infinity to (z, r), the integral

$$w \;=\; a\int_P dz/r \;=\; a\int_P 2\,dr/f'(z) \tag{16}$$

is well defined. (If we exclude the point at infinity, note that we can use z as local parameter except at the three points where $r = 0$, and that we can use r as a local parameter except at the two points where $f'(z) = 0$. The ramification point at infinity is more complicated, and will be discussed in the proof of A.4.) If we pass to the universal covering space $\tilde{T}$, then this integral does not depend on the choice of path, so we obtain a well defined holomorphic mapping from $\tilde{T}$ to $\mathbf{C}$,

$$(\tilde{z}\,,\,\tilde{r}) \;\mapsto\; w(\tilde{z}\,,\,\tilde{r}) \qquad \text{for} \qquad (\tilde{z}\,,\,\tilde{r}) \;\in \tilde{T}\;.$$

Now map the fundamental group $\pi_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ into $\mathbf{C}$ by mapping any closed loop in T to the integral (16) around this loop. Then the image $\Lambda \subset \mathbf{C}$ of $\pi_1(T)$ is an additive group Λ with two generators. In fact Λ must be a lattice, that is its generators must be linearly independent over $\mathbf{R}$. For otherwise, if both generators were contained in a real 1-dimensional sub vector space $\mathbf{R}\Lambda \subset \mathbf{C}$, then the correspondence $(z, r) \mapsto w \pmod{\mathbf{R}\Lambda}$ would be an open mapping from the compact set T to the quotient vector space $\mathbf{C}/\mathbf{R}\Lambda$, which is impossible. Thus the quotient $\mathbf{C}/\Lambda$ is also a torus, and we have a holomorphic mapping from T to $\mathbf{C}/\Lambda$ which induces an isomorphism of fundamental groups. But any holomorphic map from one torus to another is necessarily linear, since the first derivative is a well defined holomorphic map from torus to $\mathbf{C}$, and hence is constant. This proves that our correspondence

$$(z\,,\,r) \;\mapsto\; w(\tilde{z}\,,\,\tilde{r}) \;=\; a\int_\infty^{(z,r)} dz/r \pmod{\Lambda}$$

maps the Riemann surface T biholomorphically onto the torus $\mathbf{C}/\Lambda$.

We will prove the following.

A.4. Lemma. *The inverse mapping*

$$\wp \,:\, w(\tilde{z}\,,\,\tilde{r}) \;\mapsto\; z \,\in\, \hat{\mathbf{C}}$$

is a Weierstrass function of the form $\wp(w) = a^2\wp_0(w) + b$, *with* a *as in* (15), *and with* b *equal to the average* $(v_1 + v_2 + v_3)/3$ *of the finite critical values.*

Proof. To understand behavior of the integral (15) near infinity, we introduce a local uniformizing parameter t, where

$$z = 1/t^2, \qquad r = \sqrt{f(1/t^2)} = 2\big(1 - 3b\,t^2/2 + \cdots\big)/t^3,$$

with $b = (v_1 + v_2 + v_3)/3$. A brief computation shows that

$$w = a\int dz/r = -a\int dt/(1 - 3b\,t^2/2 + \cdots) = -a\,t\big(1 + b\,t^2/2 + \cdots\big),$$

hence

$$(a/w)^2 = \frac{1}{t^2}\big(1 - bt^2 + O(t^4)\big) = z - b + O(t^2),$$

or $z = a^2/w^2 + b + O(w^2)$, as required. $\square$

A.5. The preferred area form on $\wp(\mathbf{T})$. If we push forward the Lebesgue area form $du\,dv$ on the torus under the map $\wp : \mathbf{T} \to \hat{\mathbf{C}}$, then we obtain an area form $\rho(x+iy)\,dx\,dy$ on the Riemann sphere. Since $dw = a\,dz/\sqrt{f(z)}$, the density function is easily computed as

$$\rho(z) = 2|a^2/f(z)|.$$

(The factor of two arises since every point of the sphere has two preimages on the torus, counting multiplicity.) Thus ρ is smooth except at the three finite critical values of $\wp$. Using 3.5, we obtain the following.

A.6. Corollary. *The Lyubich measure for any Lattès mapping*

$$F(z) = \wp \circ L \circ \wp^{-1}(z)$$

with finite postcritical points v_1, v_2, v_3 is given by the area form $\rho(x+iy)\,dx\,dy$ where

$$\rho(z) = \frac{k}{|(z-v_1)(z-v_2)(z-v_3)|}.$$

Here the normalizing constant k is equal to $|a|^2/2$ divided by the area of $\mathbf{T}$.

Note that there are infinitely many such Lattès maps for any such torus; however we obtain this same Lyubich measure for all of them. As one example, using the specific Weierstrass map $\hat{\wp} : \mathbf{C}/\mathbf{Z}[i] \to \hat{\mathbf{C}}$, with critical values $\pm i$ and 0, it follows that the Lyubich measure for the rational map $F(z) = (i/2)(z + z^{-1})$ is given by $\rho(x+iy)\,dx\,dy$ with density

$$\rho(z) = \left|\frac{a^2/2}{z^3 + z}\right|.$$

A.7. The centered spherical metric. We conclude this appendix with a digression. The *standard spherical metric*

$$2|dz|/(1 + |z|^2)$$

on the Riemann sphere can be obtained by pulling back the usual Riemannian metric on the unit sphere in 3-space under *stereographic projection*

$$\vec{s} : \hat{\mathbf{C}} \xrightarrow{\cong} S^2 \subset \mathbf{R}^3, \qquad \vec{s}(x+iy) = \frac{(2x,\, 2y,\, x^2 + y^2 - 1)}{x^2 + y^2 + 1}.$$

A Möbius transformation $z \mapsto (az+b)/(cz+d)$ will be called a Möbius *rotation* if it preserves this metric. Its matrix of coefficients then belongs to the projective unitary group, with

$$d = \overline{a}\,, \quad b = -\overline{c}\,,$$

up to a constant factor. The *antipodal map* for this metric is given by $z \mapsto -1/\overline{z}$.

Definition. We will say that a rational map F is *centered* if the centroid, with respect to the Lyubich measure $\mathbf{m}$, of the image of its Julia set under stereographic projection to S^2 is equal to the origin:

$$\int\int_{\hat{\mathbf{C}}} \vec{s}(z)\, d\,\mathbf{m}(z) = \vec{0}\ .$$

Intuitively, this means that the interesting features of its Julia set are distributed in a balanced way around the sphere S^2. Using the methods of [Douady and Earle], we see that: *Every rational map is conjugate to one which is centered, and this centered map is unique up to conjugation by a rotation.* Pulling back the standard spherical metric by this conjugacy, we conclude that: *There is a preferred spherical metric for any rational Julia set.*

In the case of a quadratic rational map, note that the critical points are always antipodal with respect to this preferred metric. To see this, suppose that the critical points are at 0 and ∞, so that the canonical involution τ_F corresponds to the $180°$ rotation of S^2 about its poles. Since Lyubich measure is invariant under τ_F, it follows that the centroid $\int\int \vec{s}(z) d\,\mathbf{m}(z)$ lies on the axis through the poles. A suitable scale change, replacing $F(z)$ by $F(cz)/c$, will then move this centroid to the origin. (Note however that the *critical values* of F can be arbitrarily close to each other in the preferred metric.)

In the special case of a *symmetric* quadratic map, it is not hard to check that the normal form $z \mapsto a(z+z^{-1})$ is always centered. However, in general, the operation of "centering" a rational map seems computationally awkward.

Similarly, we can say that a Weierstrass $\wp$-function is *centered* if

$$\int\int_{\mathbf{T}} \vec{s}(\wp(u+iv))\, du\, dv = \vec{0}\ .$$

so that the push forward of Lebesgue measure is distributed in a balanced way around the unit 2-sphere. Evidently, a Weierstrass function is centered if and only if all of its associated Lattès maps are centered. Any $\wp$-function can be centered by composing it with some Möbius transformation, which is unique up to rotation. In fact, this centering operation for Weierstrass functions is computationally straightforward: To every lattice $\mathbf{T} = \mathbf{C}/\Lambda$ there is associated a commutative group consisting of the three translations $L : w \mapsto w + \lambda/2$ of order two, together with the identity map. These give rise to three commuting involutions

$$z \mapsto \wp \circ L \circ \wp^{-1}(z)$$

of the Riemann sphere, each of which has two easily computed fixed points. *The map $\wp$ is centered if and only if the fixed points of each of these involutions are antipodal, so that the involution is a $180°$ rotation.* To achieve this condition, we can for example compose $\wp$ with a Möbius transformation which carries these three pairs to $\{\pm 1\}$, $\{\pm i\}$, and $\{0, \infty\}$ respectively. In order to satisfy the usual requirement that $\wp(0) = \infty$, we can then compose with a further Möbius rotation.

Appendix B. Lattès Maps and Matings.

We first discuss Lattès maps of arbitrary degree, and then specialize to the quadratic case. According to Lemma 3.6, every Lattès map F has the following two properties:

- F is a rational map of degree $d \geq 2$ with only simple critical points, so that there are exactly $2d-2$ critical points.
- F has exactly four postcritical points, and none of these four points is also critical.

Conversely we have the following, as promised in 3.7.

B.1. Lemma. *Any F with these two properties is a Lattès map.*

Proof. First note that *every* immediate preimage of one of the four postcritical points is either critical or postcritical. In other words, if V is the set of postcritical points, then $F^{-1}(V) \smallsetminus V$ is the set of all critical points. In fact there are $4d$ elements of $F^{-1}(V)$ counted with multiplicity, where d is the degree and where each critical point must be counted with multiplicity two. Since there are $2d-2$ critical points by the Riemann-Hurwitz formula, and 4 postcritical points, we can account for all $2(2d-2)+4 = 4d$ of the elements of $F^{-1}(V)$. Hence every point in this set must be either critical or postcritical.

Proceeding as in A.3, we form the 2-fold branched covering T of $\hat{\mathbf{C}}$, branched over the four postcritical points. Now however, it will be more convenient to assume that the critical and postcritical points are all finite, defining T to be the set of all $(z, r) \in \mathbf{C}^2$ with

$$r^2 \;=\; p(z) \qquad \text{where} \qquad p(z) = \prod_{v_j \in V} (z - v_j) \;,$$

together with two points at infinity corresponding to the two branches of the function $\sqrt{p(z)}$ as $|z| \to \infty$, with $r \sim +z^2$ or $r \sim -z^2$ respectively. By the Riemann-Hurwitz formula, $\chi(T) = 0$, hence T is conformally isomorphic to $\mathbf{C}/\Lambda$ for some lattice Λ. Choosing this conformal isomorphism so that the zero point in $\mathbf{C}/\Lambda$ corresponds to the postcritical point $v_1 \in V$, it follows that the involution $(z, r) \mapsto (z, -r)$ of T must correspond to some involution of $\mathbf{C}/\Lambda$, which can only be $w \mapsto -w \pmod{\Lambda}$.

We must show that F lifts to a holomorphic map $L : T \to T$, which is unique up to composition with the involution $(z, r) \mapsto (z, -r)$. First consider the local problem, near some point $(z_0\,,\, r_0) \in T$. There are four cases, as follows:

(a) If z_0 is neither a critical point nor a postcritical point nor a pole of F, then $p(F(z_0)) \neq 0$. Hence we can simply set

$$L(z, r) \;=\; \Big(F(z)\,,\, \pm\sqrt{p(F(z))} \Big) \;, \tag{17}$$

making some consistent choice of sign throughout a neighborhood of $(z_0\,,\, r_0)$.

(b) If $F(z_0) = \infty$, the argument is similar.

(c) Now suppose that z_0 is a critical point, with $F(z_0) = v_j$. Then the Taylor expansion for F around z_0 has the form $F(z_0 + h) = v_j + c_2 h^2 + c_3 h^3 + \cdots$ with $c_2 \neq 0$. Since $p'(v_j) \neq 0$, the Taylor expansion for $p \circ F$ has the form

$$p \circ F(z_0 + h) \;=\; c_2' h^2 + c_3' h^3 + \cdots$$

with $c'_2 \neq 0$. Again we can make a consistent choice of sign in (17) for $z = z_0 + h$ in some neighborhood of $(z_0\,,\, r_0)$.

(d) Finally, if $z_0 \in V$ is a postcritical point, then since the derivative $p'(z_0)$ is non-zero. we can solve locally for z as a smooth function $z = p^{-1}(p(z)) = p^{-1}(r^2)$. Furthermore, $(p \circ F)'(z_0) \neq 0$, so the composition $r \mapsto p \circ F \circ p^{-1}(r^2)$ has Taylor series of the form $r \mapsto c_2 r^2 + c_4 r^4 + \cdots$ with $c_2 \neq 0$. Therefore we can set

$$L(z\,,\,r) \;=\; \Big(F(z)\,,\;\pm\sqrt{p\circ F\circ p^{-1}(r^2)}\,\Big)\;,$$

again making a consistent choice of sign throughout some neighborhood.

Thus, near any point $z_0 \in \hat{\mathbf{C}}$ there are exactly two possible liftings, and these local liftings form an unbranched two-sheeted covering of the Riemann sphere $\hat{\mathbf{C}}$. Since $\hat{\mathbf{C}}$ is simply-connected, this means that there exists a global lifting $L : T \to T$. Since every holomorphic map from a torus to itself is linear, it follows that F is indeed a Lattès map. □

B.2. Chebyshev maps. The theory of Chebyshev maps is quite similar to the theory of Lattès maps. Let $M = \mathbf{C}/\mathbf{Z}$ be the infinite cylinder, and let $M/\pm$ be the quotient space in which each w (modulo $\mathbf{Z}$) is identified with $-w$. The analogue of the Weierstrass $\wp$-function for this quotient is the function $w \mapsto 2\cos(2\pi w)$ which maps $M/\pm$ biholomorphically onto the complex plane $\mathbf{C}$. Equivalently, the function $w \mapsto e^{2\pi i w}$ maps M biholomorphically onto $\mathbf{C} \smallsetminus \{0\}$, and if we identify $z = e^{2\pi i w}$ with $1/z = e^{-2\pi i w}$ then the correspondence $z \mapsto z + z^{-1} = 2\cos(2\pi w)$ maps the quotient space biholomorphically onto $\mathbf{C}$. For any integer $d \geq 2$ the linear map $w \mapsto d\,w$ (mod $\mathbf{Z}$) from M to itself (or $z \mapsto z^d$ from $\mathbf{C} \smallsetminus \{0\}$ to itself) induces a monic polynomial map

$$2\cos(\theta) \;\mapsto\; \mathrm{Q}_d(2\cos\theta) \;=\; 2\cos(d\,\theta)\;,$$

or equivalently

$$z + z^{-1} \;\mapsto\; \mathrm{Q}_d(z + z^{-1}) \;=\; z^d + z^{-d}\;,$$

which is called the *degree d Chebyshev map*. As examples,

$$\mathrm{Q}_2(z) = z^2 - 2\;, \qquad \mathrm{Q}_3(z) = z^3 - 3z\;, \qquad \mathrm{Q}_4(z) = z^4 - 4z^2 + 2\;.$$

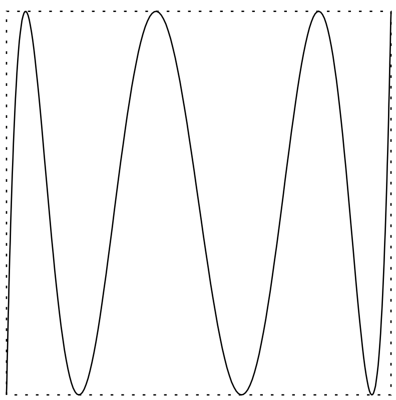


Figure 17, Graph of $\mathrm{Q}_7(x) = x^7 - 7x^5 + 14x^3 - 7x$ *in the square* $[-2, 2] \times [-2, 2]$.

The analogue of B.1 is the following statement.

B.3. Lemma. *A polynomial map f of degree d is linearly conjugate to $\pm\mathrm{Q}_d$ if and only if it has $d-1$ distinct critical points and exactly two postcritical points in the finite plane $\mathbf{C}$, neither of these postcritical points being also critical.*

Proof Outline. After a linear conjugation, we may assume that the postcritical points are ± 2. Form the 2-fold branched covering $z \mapsto s = z + z^{-1}$, branched over $s = \pm 2$. Proceeding as in the proof of B.1, the polynomial $s \mapsto f(s)$ lifts to a rational function $z \mapsto F(z)$ which has critical points and critical values only at zero and infinity. Such a rational function, with $F(2) = \pm 2$, must be given by $F(z) = \pm z^{\pm d}$, and it follows that $f(s) = \pm \mathrm{Q}_d(s)$. □

It is not difficult to check that the Julia set $J(\pm\mathrm{Q}_d)$ is equal to the interval $[-2\,,\,2]$. Conversely:

B.4. Lemma. *Any degree d polynomial f whose Julia set is homeomorphic to an interval (or more generally to a finite topological tree) is linearly conjugate to $\pm\mathrm{Q}_d$.*

Proof. Since J is connected, it contains all finite critical points. Define the *valence* $v(z)$ at a point $z \in J$ to be the number of connected components of $J \smallsetminus \{z\}$, and note that $v(z)$ is equal to $v(f(z))$ multiplied by the local degree of f at z. It follows that the tree J must actually be a simple arc. For otherwise there would be at least one point z with $v(z) \geq 3$. Hence, taking iterated preimages, there would be infinitely many such points, which is impossible. It now follows easily that the local degree is two if z is a critical point, which must belong to the open arc and map to an endpoint, and is one otherwise. The $d-1$ critical points cut J into d closed intervals, each of which maps onto J. It follows that both endpoints must be postcritical, and the conclusion then follows from B.3. □

B.5. Algebraic description of Lattès maps. Let us identify T with the quotient torus $\mathbf{T} = \mathbf{C}/\Lambda$ for some lattice $\Lambda \subset \mathbf{C}$ and write this linear map as

$$L(w) \;=\; \eta w + \kappa \qquad (\mathrm{mod}\ \Lambda)\,.$$

As in 3.4, we must have $\eta\Lambda \subset \Lambda$ and $2\kappa \in \Lambda$. Without loss of generality, we may assume that $1 \in \Lambda$, and hence that $\eta\,,\,\eta^2\,,\,\eta^3\,,\,\ldots \in \Lambda$. In other words, Λ must contain the additive group $\mathbf{Z}[\eta]$ generated by all the powers η^k, $k \geq 0$. Since this additive group is finitely generated, it follows that η must be an algebraic integer, satisfying a polynomial equation with integer coefficients and with leading coefficient 1. There are now two possibilities:

(a) If η is a rational integer, $\eta \in \mathbf{Z}$, then there is no restriction at all on the lattice Λ. The maps $L : \mathbf{T} \to \mathbf{T}$ and $F : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ have degree $d = |\eta|^2 \geq 4$.

(b) Otherwise the additive group $\mathbf{Z}[\eta] \subset \mathbf{C}$ has rank two, and it follows that η is a quadratic algebraic integer, satisfying an equation of the form

$$\eta^2 \;-\; c\eta \;+\; d \;=\; 0 \tag{18}$$

with integer coefficients. In particular, it follows that $\mathbf{Z}[\eta] = \mathbf{Z} + \eta\mathbf{Z}$. Here the constant $d = |\eta|^2 \geq 2$ is the degree, and c is the real part of 2η. Note that η is an invariant of the Lattès map only up to sign, since the two linear maps $L(w)$ and $-L(w)$ give rise to the same $F = \wp \circ L \circ \wp^{-1}$. To eliminate this ambiguity, we will often list η^2 rather than η. Changing the sign of η if necessary, we may assume that $c \geq 0$. Since η is assumed to be non-real, the discriminant $c^2 - 4d$ must be negative, hence

$$0 \leq c < 2\sqrt{d} . \tag{19}$$

B.6. The degree two case. We will show that there are exactly seven distinct Lattès maps of degree two, up to holomorphic conjugation. (More precisely, there are three pairs of complex conjugate Lattès maps, plus one real Lattès map.) First note, using (18) and (19) , that either:

$$\begin{aligned} c &= 0 , & \eta^2 + 2 &= 0 , & \text{with} \quad \eta^2 &= -2 , & \text{or} \\ c &= 1 , & \eta^2 - \eta + 2 &= 0 , & \text{with} \quad \eta^2 &= (-3 \pm i\sqrt{7})/2 , & \text{or} \\ c &= 2 , & \eta^2 - 2\eta + 2 &= 0 , & \text{with} \quad \eta^2 &= \pm 2i . \end{aligned} \tag{20}$$

Thus there are five distinct possibilities for η^2 . (Correspondingly η can take the values $\pm i\sqrt{2}$, $\pm(1 \pm i\sqrt{7})/2$, and $\pm(1 \pm i)$.)

Next we must ask which lattices are possible, for a given η . By a scale change, we can always assume that the minimum distance between distinct lattice elements is equal to 1 , so that $|\lambda| \geq 1$ for all non-zero $\lambda \in \Lambda$. Furthermore, by rotating the coordinates we can then assume that $1 \in \Lambda$ and hence that $\mathbf{Z}[\eta] \subset \Lambda$. *In the degree two case, these two conditions suffice to guarantee that* $\Lambda = \mathbf{Z}[\eta]$. In fact, in each of the cases listed in (20) it is not hard to choose a compact fundamental domain for $\mathbf{Z}[\eta]$ which is strictly contained in the open unit disk. Hence it is not possible to add more points to the lattice $\mathbf{Z}[\eta]$ without violating the condition that $|\lambda| \geq 1$ for $\lambda \neq 0$.

Finally, fixing the multiplier η and the lattice $\Lambda = \mathbf{Z}[\eta]$, we must consider the additive constant $\kappa \in \frac{1}{2}\Lambda$. This is not an invariant of the Lattès map. In fact if we replace $L(w) = \eta w + \kappa$ by

$$L(w + \lambda/2) - \lambda/2 = \eta\, w + \kappa' \qquad \text{with} \qquad \kappa' = \kappa + (\eta - 1)\,\lambda/2 ,$$

then we obtain a holomorphically conjugate Lattès map, provided that $\lambda/2$ is a fixed point of the involution $w \mapsto -w \pmod{\Lambda}$, or in other words provided that $\lambda \in \Lambda$. In the cases $c = 0$ and $c = 2$, it is not hard to check that every element of $\frac{1}{2}\Lambda$ can be written as $(\eta - 1)\lambda/2 \pmod{\Lambda}$ for some $\lambda \in \Lambda$. Hence in these cases we can always choose the origin in $\mathbf{C}/\Lambda$ so that $\kappa = 0$. However, when $\eta = \pm(1 \pm \sqrt{-7})/2$ the equation $(\eta - 1)\lambda/2 \equiv 1/2 \pmod{\Lambda}$ has no solution $\lambda \in \Lambda$. Hence the two linear maps

$$L(w) = \eta w \qquad \text{and} \qquad L(w) = \eta w + 1/2$$

give rise to distinct Lattès maps.

The census of quadratic Lattès maps can now be tabulated as follows. Here the bottom two lines of the table give the number of postcritical fixed points for F , and the information as to whether or not F commutes with some Möbius automorphism.

B.7. Lemma. *Up to holomorphic conjugacy there are exactly seven distinct Lattès maps of degree two, corresponding to seven linear maps* $L(w) = \eta\, w + \kappa$ *with the following descriptions:*

$\eta^2 =$	-2	$(-3 \pm i\sqrt{7})/2$	$(-3 \pm i\sqrt{7})/2$	$\pm 2i$
$\kappa =$	0	0	$1/2$	0
postcrit. f.p.	1	2	0	1
symmetric?	*no*	*yes*	*yes*	*yes.*

The number of postcritical fixed points is the most elementary and easily computable invariant for a Lattès map. If there is just one postcritical fixed point, then (in the degree two case) it is easy to check that we must have the following schematic diagram for the critical orbits.

$$\begin{array}{ccccccccc} & & & & \odot & & & & \\ & & & & \uparrow & & & & \\ \mathbf{c}_1 & \mapsto & \mathbf{v}_1 & \mapsto & \cdot & \leftarrow\!\shortmid & \mathbf{v}_2 & \leftarrow\!\shortmid & \mathbf{c}_2 \end{array}$$

Here the $\mathbf{c}_j$ are the critical points and $\mathbf{v}_j$ the corresponding critical values, while the fixed point is indicated by a circled dot. If there are two postcritical fixed points, then the diagram splits into two parts as follows.

$$\mathbf{c}_1 \;\mapsto\; \mathbf{v}_1 \;\mapsto\; \odot \qquad\qquad \odot \;\leftarrow\!\shortmid\; \mathbf{v}_2 \;\leftarrow\!\shortmid\; \mathbf{c}_2$$

Finally, if there are no postcritical fixed points, then there must be a postcritical cycle of period two, and the diagram takes the following form.

$$\mathbf{c}_1 \;\mapsto\; \mathbf{v}_1 \;\mapsto\; \cdot \;\leftrightarrow\; \cdot \;\leftarrow\!\shortmid\; \mathbf{v}_2 \;\leftarrow\!\shortmid\; \mathbf{c}_2$$

To distinguish between these three possibilities for the seven examples of B.7, it is simply necessary to compute the mapping $w \mapsto \eta\, w + \kappa$ on the four element set $\frac{1}{2}\Lambda/\Lambda \cong V$. This computation will be left to the reader.

In each case, we can give explicit examples of rational maps in the Lattès conjugacy class as follows. This will also enable us to decide which of these maps are symmetric.

First suppose that $\eta^2 = -2$. Putting the critical points at ± 1 and the postcritical fixed point of multiplier η^2 at infinity, the map must have the form

$$F(z) \;=\; (z + z^{-1})/\eta^2 \,+\, b \;=\; -(z + z^{-1})/2 \,+\, b$$

for some constant b . Thus the critical points ± 1 map to $b \mp 1$, which must map to the preimage 0 of ∞ . This yields the two equations $2b \;=\; (b \mp 1) \,+\, (b \mp 1)^{-1}$, with solution $b = \pm\sqrt{2}$. Thus F is a rational map with real coefficients. For example, taking $b = +\sqrt{2}$, we have

$$F(z) \;=\; -(z + z^{-1})/2 + \sqrt{2} \tag{21}$$

with critical orbits $\pm 1 \;\mapsto\; \sqrt{2} \mp 1 \;\mapsto\; 0 \;\mapsto\; \infty$. Since $b \neq 0$, this map is not symmetric.

Next suppose that $\eta^2 = (-3 \pm i\sqrt{7})/2$, with two postcritical fixed points. Any quadratic rational map with fixed points of multiplier α and β can be put in the normal form $z \mapsto z(z + \alpha)/(\beta z + 1)$, with these designated fixed points at zero and infinity. (Compare [Milnor 1993].) In the special case $\alpha = \beta$, this map commutes with the involution $\sigma(z) = 1/z$, and hence is symmetric. In our case, the map has two postcritical fixed points of multiplier η^2 . Hence it has the form

$$F(z) \;=\; z\frac{z + \eta^2}{\eta^2 z + 1} \qquad \text{with} \qquad \eta^2 \;=\; (-3 \pm i\sqrt{7})/2\;, \tag{22}$$

and commutes with $\sigma(z) = 1/z$. Now consider the composition

$$F \circ \sigma(z) \;=\; \sigma \circ F(z) \;=\; \frac{\eta^2 z + 1}{z(z+\eta^2)}\,. \tag{23}$$

Evidently this map has exactly four postcritical points, with a postcritical cycle $0 \leftrightarrow \infty$ of multiplier η^4. Hence it is the required Lattès map, corresponding to $L(w) = \eta\, w + 1/2$. Evidently this map is also symmetric.

Finally suppose that $\eta^2 = \pm 2i$. Then as described in §3, we can put the critical points at ± 1 and the postcritical fixed point of multiplier η^2 at infinity, to obtain the symmetric normal form

$$F(z) \;=\; (z + z^{-1})/\eta^2\,, \qquad \text{with} \qquad \eta^2 = 2i \tag{24}$$

and with critical orbits $\pm 1 \;\mapsto\; \mp i \;\mapsto\; 0 \;\mapsto\; \infty$. This completes the proof of B.7. $\square$

B.8. Lattès matings. This section will give examples of matings which satisfy the conditions of B.1, and hence can also be described as Lattès mappings. I am indebted to Shishikura for providing the following table, which gives more examples and more precise information than I was able to obtain. Recall the notation

$$f_{p/q}(z) \;=\; z^2 \,+\, c_{p/q}$$

where $c_{p/q}$ is the landing point of the p/q-ray in the Mandelbrot set. (Compare Figure 5.) The first two columns of this table list pairs p/q and r/s for a mating $f_{p/q} \perp\!\!\!\perp f_{r/s}$, while the remaining two columns list the constants η^2 for the associated linear map $L(w) = \eta\, w + \kappa$. Here only the examples with η^2 in the upper half-plane have been listed. In each case, a complex conjugate Lattès mating can be obtained by changing the signs of the angles. For example the mating $f_{1/4} \perp\!\!\!\perp f_{1/4}$ of §3, with $\eta = 1 - i$, $\eta^2 = -2i$, corresponds to the complex conjugate of the first entry. With these conventions, here is Shishikura's list.

p/q	r/s	η^2	κ
3/4	3/4	$2i$	0
1/12	5/12	-2	0
5/6	5/6	$(-3+i\sqrt{7})/2$	1/2
1/6	5/14	$(-3+i\sqrt{7})/2$	0
3/14	3/14	$(-3+i\sqrt{7})/2$	0
3/14	1/2	$(-3+i\sqrt{7})/2$	0
5/6	1/2	$(-3+i\sqrt{7})/2$	0

B.9. Remark. The last four rows of this table describe a noteworthy example: *They show that the Lattès map* (22) *with two postcritical fixed points can be presented as a mating in four essentially different ways.* Furthermore, most of these mating structures are non-symmetric, even though the map itself is symmetric. Thus there are seven different mating structures for this Lattès map if we mark the critical points and hence distinguish between $f_{p/q} \perp\!\!\!\perp f_{r/s}$ and $f_{r/s} \perp\!\!\!\perp f_{p/q}$.

Here is a case by case discussion, showing that each of these seven matings yields a

Lattès map. The more difficult question as to precisely which mating corresponds to which Lattès map will be discussed in B.11 below.

Examples with one postcritical fixed point. The mating $f_{3/4} \perp\!\!\!\perp f_{3/4}$ is clearly symmetric, with postcritical orbits

$$\hat{\gamma}(3/4) \mapsto \hat{\gamma}(1/2) \mapsto \hat{\gamma}(0) \hookleftarrow \qquad \text{and} \qquad \hat{\gamma}(-3/4) \mapsto \hat{\gamma}(1/2) \mapsto \hat{\gamma}(0) \hookleftarrow \ .$$

(Compare §3.) The mating $f_{1/12} \perp\!\!\!\perp f_{5/12}$ has postcritical orbits

$$\hat{\gamma}(1/12) \mapsto \hat{\gamma}(1/6) \mapsto \hat{\gamma}(1/3) \hookleftarrow \ ,$$

where $\hat{\gamma}(1/3) = \hat{\gamma}(2/3)$ is a fixed point since $c_{5/12}$ belongs to the $1/2$-limb of the Mandelbrot set, and

$$\hat{\gamma}(-5/12) \mapsto \hat{\gamma}(1/6) \mapsto \hat{\gamma}(1/3) \hookleftarrow \ .$$

(The proof that this example is not symmetric is non-trivial. Note that $f_{1/12} \perp\!\!\!\perp f_{5/12} = f_{1/12} \perp\!\!\!\perp f_{7/12}$ since $c_{5/12} = c_{7/12} \in \mathbf{R}$.)

Example with a postcritical orbit of period two. For $f_{5/6} \perp\!\!\!\perp f_{5/6}$ we have

$$\hat{\gamma}(5/6) \mapsto \hat{\gamma}(2/3) \leftrightarrow \hat{\gamma}(1/3) \mapsfrom \hat{\gamma}(-5/6) \ .$$

Examples with two postcritical fixed points. For the mating $f_{5/6} \perp\!\!\!\perp f_{1/2}$ we have

$$\hat{\gamma}(5/6) \mapsto \hat{\gamma}(2/3) \hookleftarrow \qquad \text{and} \qquad \hat{\gamma}(-1/2) \mapsto \hat{\gamma}(0) \hookleftarrow \ .$$

(Again $\hat{\gamma}(2/3)$ is a fixed point because $c_{1/2}$ belongs to the $1/2$-limb. Note that

$$f_{1/2}(z) \ = \ z^2 - 2 \ , \quad f_{1/6}(z) \ = \ z^2 + i \ , \quad f_{5/6}(z) = z^2 - i \ .$$

Compare B.10.) For $f_{1/6} \perp\!\!\!\perp f_{5/14}$:

$$\hat{\gamma}(1/6) \mapsto \hat{\gamma}(1/3) \hookleftarrow \qquad \text{and} \qquad \hat{\gamma}(-5/14) \mapsto \hat{\gamma}(2/7) \hookleftarrow \ ,$$

where $\hat{\gamma}(1/3)$ is fixed because $c_{-5/14}$ belongs to the $1/2$-limb, and $\hat{\gamma}(2/7)$ is fixed because $c_{1/6}$ belongs to the $1/3$-limb. For $f_{3/14} \perp\!\!\!\perp f_{3/14}$:

$$\hat{\gamma}(3/14) \mapsto \hat{\gamma}(3/7) \hookleftarrow \qquad \text{and} \qquad \hat{\gamma}(-3/14) \mapsto \hat{\gamma}(-3/7) \hookleftarrow \ ,$$

using the fact that $c_{3/14}$ belongs to the $1/3$-limb while $c_{-3/14}$ belongs to the $2/3$-limb. Similarly, for $f_{3/14} \perp\!\!\!\perp f_{1/2}$ we have

$$\hat{\gamma}(3/14) \mapsto \hat{\gamma}(3/7) \hookleftarrow \qquad \text{and} \qquad \hat{\gamma}(1/2) \mapsto \hat{\gamma}(0) \hookleftarrow \ .$$

B.10. Remark. Matings with the Chebyshev map $f_{1/2}(z) = z^2 - 2$ deserve special mention. First, these are the only matings such that the associated $\hat{\gamma} : \mathbf{R}/\mathbf{Z} \to J(F)$ is not one-to-one almost everywhere, but rather satisfies $\hat{\gamma}(t) = \hat{\gamma}(-t)$. (Compare 5.5.) Also, they are closely related to self-matings. C. Petersen has pointed out that for any quadratic rational map which is symmetric, $\sigma \circ F = F \circ \sigma$, we can collapse the Riemann sphere under the involution σ to obtain a new rational map, which we may denote by F/σ , on the quotient Riemann surface $\hat{\mathbf{C}}/\sigma \cong \hat{\mathbf{C}}$. If we use the normal form $F(z) = a(z + z^{-1})$ with $\sigma(z) = -z$, then we can introduce the coordinate $Z = z^2$ on $\hat{\mathbf{C}}/\sigma$, with associated map

$$Z \ \mapsto \ F\left(\sqrt{Z}\right)^2 \ = \ a^2\,(Z + Z^{-1} + 2) \ .$$

Here the two critical points $z = \pm 1$ for F correspond to the single critical point $Z = +1$, while a new preperiodic critical point appears, namely $Z = -1 \mapsto 0 \mapsto \infty \circlearrowleft$. Conversely, any quadratic rational map having a critical point for which the second forward image is a fixed point arises in this way from a symmetric map.

If $F \cong f \perp\!\!\!\perp f$ is a self-mating, then it is easy to check that the associated F/σ can be identified with $f \perp\!\!\!\perp f_{1/2}$. As noted above, the matings $f_{3/14} \perp\!\!\!\perp f_{1/2} \cong f_{5/6} \perp\!\!\!\perp f_{1/2}$ can be given a Lattès structure. On the other hand, the mating $f_{1/4} \perp\!\!\!\perp f_{1/2}$ does not have a Lattès structure, since one critical point maps to the other. However, it behaves very much like a Lattès mapping in that it has a flat orbifold structure. (For a discussion of orbifold structure, see for example [Milnor 1990, §14].)

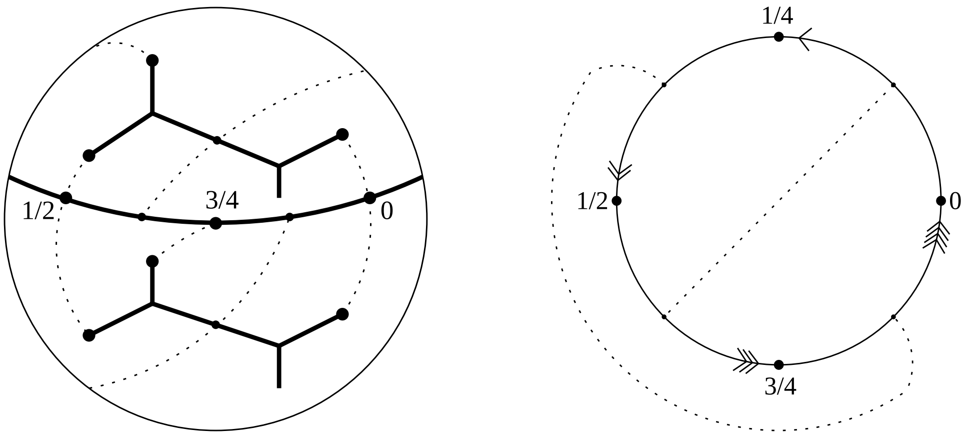


Figure 18. Schematic diagrams for the mating $f_{1/4} \perp\!\!\!\perp f_{1/4}$.

B.11. The algorithm. Here is an outline of a slightly modified form of Shishikura's procedure for determining which mating corresponds to which Lattès map. It is best carried out with a set of colored markers and a quantity of blank paper. Start with a schematic diagram, similar to Figure 3, for the two Julia sets $J_{p/q}$ and $J_{r/s}$ embedded in the sphere S^2 (Figure 18-left). Then make a simplified version which includes only the following key features: the equator (represented by a circle in Figure 18-right), with points in the critical and postcritical ray classes marked, and with any ray pairs A_j joining one of these marked points to another within the northern or southern hemisphere drawn in. (These are the dotted curves in Figure 18-right. Note that each such A_j corresponds to a single point in the quotient sphere $K_{p/q} \perp\!\!\!\perp K_{r/s} = S^2/\overset{\text{ray}}{\sim}$.) Choose four base points among these marked points, one in each postcritical ray equivalence class. These cut the equator into four arcs, which we number consecutively as E_1 through E_4. (In the figure, each E_j is indicated by an arrow with j heads.) Next form the 2-fold covering torus T, branched over these four points. Then each E_i will be covered by a simple closed curve $\hat{E}_i$ in T, where $\hat{E}_1$ is parallel to $\hat{E}_3$ but perpendicular to $\hat{E}_2$ and $\hat{E}_4$. Thus the homology group $H_1(\mathbf{T};\mathbf{Q})$ is spanned by the classes $(\hat{E}_1) = (\hat{E}_3)$ and $(\hat{E}_2) = (\hat{E}_4)$ of these curves. In the universal covering of T, we obtain a rectilinear grid with lines covering $\hat{E}_1$ and $\hat{E}_3$ in one direction and lines covering $\hat{E}_2$ and $\hat{E}_4$ at right angles. The rectangles in this grid represent alternately the northern and southern hemispheres of S^2. Choose a fundamental domain for T consisting of four rectangles from this grid, and carefully lift each of the arcs A_j to each of the two rectangles corresponding to its hemisphere. (Figure 19-left.) Next determine the preimage of E_1 under the angle doubling map from the equator to itself. Lift both components of this

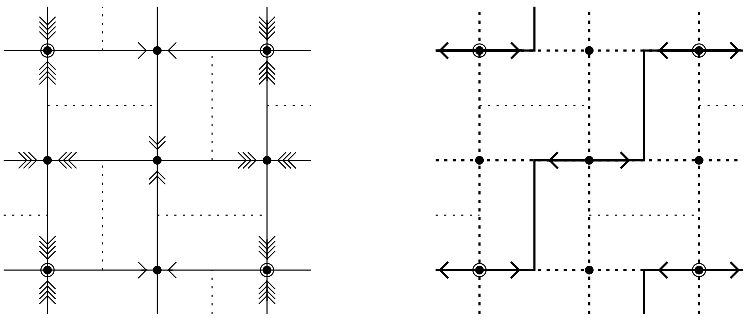


Figure 19. This shows Figure 18-right lifted to the torus, and the preimage of E_1 lifted to the torus.

preimage to $\mathbf{T}$ in two ways, and join them up by following the lifted arcs A_j as necessary, to form a simple closed curve or pair of simple closed curves $\hat{E}'_1$ in T. (Figure 19-right.) Now do the same for E_2. The required linear map L on the torus induces an automorphism $L_* : H_1(\mathbf{T};\mathbf{Q}) \to H_1(\mathbf{T};\mathbf{Q})$ of rational homology. Evidently L_*^{-1} carries each homology class $(\hat{E}_j)$ to some easily determined multiple of $(\hat{E}'_j)$. For example, in the figure, $L_*^{-1}(\hat{E}_1) = (\hat{E}'_1)/2 = (\hat{E}_1)/2 + (\hat{E}_2)/2$ up to sign, and a similar computation shows that $L_*^{-1}(\hat{E}_2) = -(\hat{E}_1)/2 + (\hat{E}_2)/2$ up to sign, Since the determinant of L_*^{-1} must be positive, it follows that the matrix of this linear transformation is $\pm \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$, so that the coefficient η^{-1} must be $\pm(1+i)/2$. Further details will be omitted.

Appendix C. External angles and the Hubbard tree.

This will be a brief outline of how one computes the external angles of points on the Hubbard tree $H_0 = H_0(f_{1/4})$. (For much more detail on how one computes such things, see [Douady, 1986].) Let us start with the following schematic diagram of H_0.

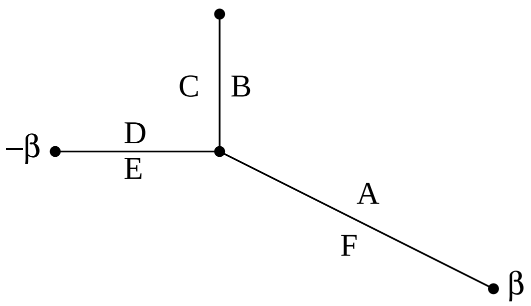


Figure 20. Diagram of the Hubbard tree for $f_{1/4}$, with labels for the two sides of the three edges.

Here the two sides of each edge in H_0 have been labeled separately since an external ray must land on one side or the other (if we exclude the four vertices). For any external ray which lands at a point $z \in H_0$ we obtain an infinite sequence of symbols in $\{A, B, C, D, E, F\}$ by following its orbit under $f_{1/4}$. The possible transitions between these six symbols under the map $f_{1/4}$ can be described briefly by the diagram

$$\begin{array}{ccccccc} C & \to & E & \to & A & \to & A \cup B \cup C \\ B & \to & D & \to & F & \to & B \cup C \cup F\ , \end{array}$$

or more conveniently by the following diagram

$$\begin{tikzcd} D \arrow[dr] & B \arrow[l] & A \arrow[l] \arrow[d] \arrow[loop right] & \\ & F \arrow[u] \arrow[r] \arrow[loop left] & C \arrow[r] & E\ . \arrow[ul] \end{tikzcd} \tag{27}$$

(Compare Figure 21. As an example, A maps onto the union $A \cup B \cup C$, but B maps only to D.) For each ray landing at an interior point z of some edge in H_0, there is a different ray which lands at the same point z from the opposite side of this edge. Its symbol sequence is obtained from the original sequence by permuting the six symbols according to the scheme

$$A \leftrightarrow F\ , \qquad B \leftrightarrow C\ , \qquad D \leftrightarrow E\ . \tag{28}$$

This corresponds to a rotation of the transition diagram (27) by $180°$.

Now replace each of these six symbols by a zero or one according as it lies above or below the path from $-\beta$ to β, so that

$$A\,,\,B\,,\,C\,,\,D \mapsto 0\ , \qquad E\,,\,F \mapsto 1\ .$$

Then we obtain the following transition diagram.

$$\begin{tikzcd} 0 \arrow[dr] & 0 \arrow[l] & 0 \arrow[l] \arrow[d] \arrow[loop right] & \\ & 1 \arrow[u] \arrow[r] \arrow[loop left] & 0 \arrow[r] & 1 \arrow[ul] \end{tikzcd} \tag{29}$$

Starting anywhere in this diagram and following the arrows, making an arbitrary choice whenever there is more than one outward arrow, we obtain an infinite sequence of zeros and ones which represents the binary expansion of a corresponding external angle. The angles obtained in this way are precisely those whose rays land on H_0. As an example, the sequence $FFBD$ repeated periodically, denoted briefly by $\overline{FFBD}$, leads to the periodic binary expansion $.\overline{1100} = 4/5$, hence the $4/5$-ray lands on H_0. If we apply the involution (28), we obtain the periodic sequence $\overline{AACE}$, corresponding to $.\overline{0001} = 1/15$. *Thus the* $1/15$ *and* $4/5$*-rays land at exactly the same point of* H_0. (Figure 2.)

We would like to know when the rays $\mathcal{R}_t$ and $\mathcal{R}_{-t}$ both land on H_0. As one example, note that the sequence $\overline{BDFF}$ corresponds to $.\overline{0011} = 1/5$. We have just seen that the ray $\mathcal{R}_{-1/5} = \mathcal{R}_{4/5}$ lands on H_0. *Thus the* $1/5$ *and the* $-1/5$*-rays both land on the tree*

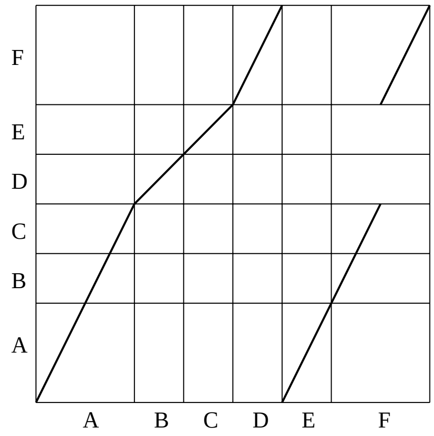


Figure 21. Graph of a piecewise linear circle map which represents the allowed transitions between edges in Figure 20. This graph has a jump discontinuity, since nothing in F maps to D or E .

H_0 . (They land at different points since the first lands on edge B while the second lands on F . Compare Figure 2.) Similarly, the $2/5$ and $-2/5$-rays land on H_0 , as do the $1/10$ and $-1/10$ rays, the $1/20$ and $-1/20$-rays, and so on. These coincidences lead to infinitely many intersections between the images $\mu_1(H_0)$ and $\mu_2(H_0)$ in $\hat{\mathbf{C}}$, and hence to the complications seen in Figure 7 and 14-16. Here is a precise statement. (Compare §6(11).)

C.1. Lemma. *The external rays $\mathcal{R}_t$ and $\mathcal{R}_{-t}$ for $J(f_{1/4})$ both land on the Hubbard tree $H_0 \subset J(f_{1/4})$ if and only if t is either 0 , $1/2$, or an angle of the form $\pm 1/(2^n\,5)$ or $\pm(1 - 1/(2^n\,5))/2$ with $n \ge 0$.*

Proof. Note that the bit sequence for $-t$ is obtained from the sequence for t by reversing all bits, so that $0 \leftrightarrow 1$. In the case of a ray $\mathcal{R}_t$ landing on H_0 , we see by inspecting (29) that the sequence 1011 can never occur in the binary expansion of t . Now suppose that $\mathcal{R}_{-t}$ lands on H_0 . Then the dual sequence 0100 cannot occur in the expansion of t . Comparing (27) and (29) , this implies that the symbol C cannot occur in the symbol sequence for t . It follows that the sequence 101 cannot occur in its binary expansion, and that 10^n1 with $n > 2$ cannot occur except as an initial segment. Now suppose that both $\mathcal{R}_t$ and $\mathcal{R}_{-t}$ land on H_0 . Then the sequence 010 cannot occur, and 01^n0 with $n > 2$ can occur only as an initial segment. It is now straightforward to check that the only possible sequences are

$$.1^n\overline{1100} \leftrightarrow .0^n\overline{0011} \qquad \text{and} \qquad .10^n\overline{0011} \leftrightarrow .01^n\overline{1100}$$

with $n \ge 0$, corresponding to the angles listed. □

Remark C.2. Generically there are two external rays landing on each point of H_0 . The only exceptions are the fixed point α and its iterated preimages where three rays land, and the three points $c \mapsto -\beta \mapsto \beta$ where only one ray lands. The angle of an external ray determines its symbol sequence except in the case of the iterated preimages of α and β . As examples, the two symbol sequences $\overline{BDF}$ and $\overline{ACE}$ both determine the angle $.\overline{001} = 1/7$ with $\gamma(1/7) = \alpha$, while the two symbol sequences $ACE\overline{A} \mapsto .001\overline{0}$ and $ABD\overline{F} \mapsto .000\overline{1}$ both correspond to the ray $\mathcal{R}_{1/8}$ which lands at $\gamma(1/8) = 0$.

If two different rays land on a common point z, recall from 5.3 that z must belong to the union $\bigcup H_n$ of the iterated preimages of H_0. We can supplement C.1 as follows.

C.3. Lemma. *If both $\mathcal{R}_t$ and $\mathcal{R}_{-t}$ land on $\bigcup H_n$, then either t is a dyadic rational $p/2^m$, or else the binary expansion of t is eventually periodic with period $\overline{0011}$. In the latter case, there is a unique angle $s \not\equiv t$ with $\gamma(s) = \gamma(t)$ and its binary expansion has eventual period $\overline{0100}$. Similarly, there is a unique $u \not\equiv t$ with $\gamma(-u) = \gamma(-t)$ and its binary expansion has eventual period $\overline{1011}$.*

More explicitly, since $.\overline{0011} = 1/5$ and $.\overline{0101} = 4/15$, it follows that the orbits of t and s under angle doubling are eventually periodic, of the form

$$\begin{array}{ccccccccccccccccc} t & \mapsto & 2t & \mapsto & \cdots & \mapsto & 1/5 & \mapsto & 2/5 & \mapsto & 4/5 & \mapsto & 3/5 & \mapsto & 1/5 \\ s & \mapsto & 2s & \mapsto & \cdots & \mapsto & 4/15 & \mapsto & 8/15 & \mapsto & 1/15 & \mapsto & 2/15 & \mapsto & 4/15\,. \end{array}$$

The same is true for $-t$ and $-u$.

Proof of C.3. After doubling the angles sufficiently often, we may assume that both $\gamma(t)$ and $\gamma(-t)$ belong to H_0, so that C.1 applies. The proof is then straightforward. $\square$

Now consider the mating $F \cong f_{1/4} \perp\!\!\!\perp f_{1/4}$ and the associated semiconjugacy

$$\hat{\gamma} : (\mathbf{R}/\mathbf{Z}, 2\cdot) \to (\hat{\mathbf{C}}, F)\,.$$

It follows from 2.1 that $\hat{\gamma}(t) = \hat{\gamma}(t')$ if and only if there exists a chain $t = t_1, t_2, \ldots, t_n = t'$ such that:

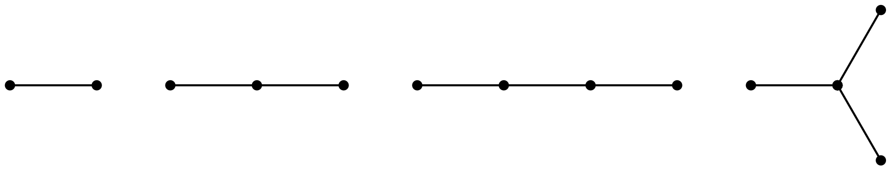
$$\text{either} \qquad \gamma(t_i) = \gamma(t_{i+1}) \qquad \text{or} \qquad \gamma(-t_i) = \gamma(-t_{i+1}) \tag{30}$$

for each i between 1 and $n-1$. Here we may assume that the t_i are distinct and that the $+$ and $-$ conditions alternate, since otherwise t and t' could be joined by a shorter such chain.

C.4. Corollary. *For this particular mating, it suffices to consider chains of length $n \le 3$ in order to test whether $\hat{\gamma}(t) = \hat{\gamma}(t')$.*

Proof. Given a chain t_1, t_2, t_3, t_4 of length four satisfying (30), first suppose that t_2 is not a dyadic rational. Then it follows from C.3, applied to t_1, t_2, t_3, that the binary expansion of t_2 has eventual period $\overline{0011}$ while the binary expansion of t_3 has eventual period $\overline{0001}$ or $\overline{0111}$. But C.3 applied to t_2, t_3, t_4 shows similarly that t_3 must have eventual period $\overline{0011}$, yielding a contradiction. Now suppose that t_2 is a dyadic rational, with say $\gamma(t_1) = \gamma(t_2) = z$. Then the orbit of z must pass through the critical point, since otherwise we would obtain two distinct rays landing at the β-fixed point. It follows that the orbit of $\gamma(-t_2)$ does not pass through the critical point, hence the condition $\gamma(-t_2) = \gamma(-t_3)$ cannot be satisfied with $t_2 \not\equiv t_3$. This contradiction completes the proof. $\square$

Remark C.5. We can use this same argument to verify the condition of Moore's Theorem 2.2 as applied to this particular mating. Recall that the ray equivalence relation on the 2-sphere was defined as the smallest equivalence relation $\overset{\text{ray}}{\sim}$ such that the closure of each $\nu_1(\mathcal{R}_t(J)) \cup \nu_2(\mathcal{R}_{-t}(J))$ lies in a single equivalence class. (Compare Figure 3.) The arguments above show that each ray equivalence class has one of the following four forms, where each edge represents such a ray pair closure, joining a point of $\nu_1(J)$ to a point of $\nu_2(J)$.



Each end vertex in one of these four graphs represents the image under ν_1 or ν_2 of a point in J where only one external ray lands, while each interior vertex represents the image of a point where two different rays land. Thus the first three graphs represent chains of length one, two, and three. In the last two graphs the vertices are all eventually periodic. The eventual periods involve only angles with denominator 5 and 15 in one case, and 7 in the other (corresponding to the α fixed point).

Since none of these four graphs can separate the plane, we see that the conditions of Moore's Theorem are satisfied, so that $(S^2/\overset{\text{ray}}{\sim}) = J \perp\!\!\!\perp J$ is indeed a topological sphere. (Of course this is only one step in Shishikura's proof that this topological-mating can be given a holomorphic structure.)

Here is another consequence. (Compare Figure 13.)

C.6. Corollary. *The symmetric Hubbard tree* $H_1 = H_0 \cup (-H_0)$ *is embedded injectively into the Riemann sphere by the map* $\mu_1 : J(f_{1/4}) \to J(F) = \hat{\mathbf{C}}$.

Proof. If $\gamma(t)$ and $\gamma(t')$ are two distinct points of H_1 which map to the same point under μ_1 , then $\hat{\gamma}(t) = \hat{\gamma}(t')$, hence t and t' are joined by a chain satisfying (30) . If there is more than one ray landing at $\gamma(t)$ and at $\gamma(t')$, then combining these with a chain from t to t' we get a chain of length ≥ 4 , contradicting C.4. On the other hand, if only one ray lands on $\gamma(t)$, then $t \in \{0\,,\,1/4\,,\,1/2\,,\,3/4\}$. Hence only one ray lands on $\gamma(-t)$ also, and the equivalence class is a singleton. □

Note however that the preimage $H_2 = f^{-1}(H_1)$ is not mapped injectively, since $\gamma(3/8)$ and $\gamma(7/8)$ are distinct points of H_2 but $\gamma(-3/8) = \gamma(-7/8) = 0$, hence

$$\mu_1(\gamma(3/8)) \;=\; \mu_1(\gamma(7/8))\ .$$

C.7. A note on computation. In order to actually plot an image of the Hubbard tree, it is convenient to work with an associated circle map, as graphed in Figure 18. For each point of the associated circle which parametrizes H_0 , it is not difficult to iterate this circle map, and hence compute the binary expansion of the corresponding angle t . It is then easy to plot the image $\mu_1(\gamma(t)) = \wp(\hat{g}(t))$, using A.2 and Appendix C. (See Figure 13.) It is much harder to compute the point $\gamma(t)$ itself, since the operation of following an external ray to locate its landing point is rather slow. One quite general method for plotting Hubbard trees of full Julia sets has been suggested to me by Zakeri: First make a raster file for the Julia set, and then use an algorithm to search for a minimal path between specified pixels within this set. A quite different procedure was actually used for Figures 11, 12, as follows. Given any dyadic rational t_0 with binary expansion $t_0 = .b_1 b_2 \cdots b_n$, let $t_k = .b_{k+1} \cdots b_n \equiv 2^k t_0$, and let $z_k = \gamma(t_k)$. Since $z_{k+1} = f(z_k)$ and $z_n = \beta$, we can try to solve for

$$z_k \;=\; \pm\sqrt{z_{k+1} - c} \;\in\; f^{-1}(z_{k+1})$$

by backwards induction. The problem is to make the correct choice of sign at each step. Inspecting Figure 2, we see that to a first approximation the point z_k lies in the left half-plane if and only if its angle t_k lies in the interval $[1/8\,,\,5/8]$. This observation gives a simple rule for choosing the sign, and yields a good first approximation to the required picture. However this choice of sign may well be wrong when z_k is very close to the origin. To make a correct choice in this case, one needs the observation that J is asymptotically self-similar near the preperiodic point 0 with expansion factor of $\sqrt{f'(\beta)} = \sqrt{2\beta}$. (Compare [Tan Lei, 1990].) Further details will be omitted.

Appendix D. Some non-standard topological conjugacies.

I am indebted to A. Douady for pointing out that there exists topological conjugacies between filled Julia sets which cannot be extended over any neighborhood in $\mathbf{C}$. This appendix will describe the simplest such examples.

For any rational number $0 < p/q < 1$, let $c(p/q)$ be the "center point" for the p/q-limb of the Mandelbrot set (or more precisely the center point for that hyperbolic component in the p/q-limb which is an immediate satellite attached to the central cardioid). Let $K(p/q)$ be the filled Julia set for the corresponding polynomial $z \mapsto z^2 + c(p/q)$. This polynomial has a periodic critical orbit of period q. Furthermore, the periodic Fatou components are arranged around their common boundary point α in a cyclic order as if they corresponded under a rotation through the angle of $p/q \in \mathbf{R}/\mathbf{Z}$. (Compare Figure 22 for the case $p/q = 2/5$.)

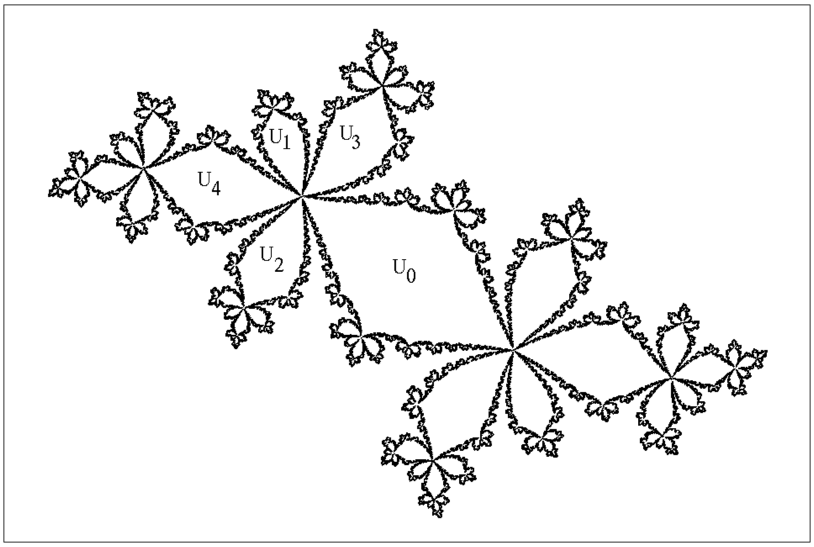


Figure 22. Filled Julia set $K(2/5)$ for the center point of the $2/5$-limb of the Mandelbrot set. The periodic Fatou components have been labeled so that $f(U_i) = U_{i+1}$, with $0 \in U_0 = U_5$.

Theorem D.1. *If $0 < p/q < p'/q < 1$ are distinct fractions in lowest terms with the same denominator, then there exists a unique topological conjugacy $K(p/q) \to K(p'/q)$ which is holomorphic on the interior. However, this conjugacy cannot be extended as a homeomorphism over any neighborhood of $K(p/q)$.*

(Compare [Branner and Fagella] for a quite different homeomorphism between $K(p/q)$ and $K(p'/q)$ which is compatible with the embedding into $\mathbf{C}$ but not with the dynamics.)

To prove D.1, we will show how to construct $K = K(p/q)$ as a topological space, together with its dynamics, given only the denominator q, without any reference to the numerator p. In fact, let K_0 be the union of the closures of the bounded periodic Fatou components of f, and let $K_n = f^{-n}(K_0)$. We will first give a description of K_0 which depends only on q, and then extend inductively to give a corresponding description of K_n. Finally, we will show that the entire filled Julia set can be described as the inverse limit of an appropriate sequence of maps $K_0 \leftarrow K_1 \leftarrow K_2 \leftarrow \cdots$.

First note that the closure $\overline{U}$ of an arbitrary bounded Fatou component U is canonically homeomorphic to the closed unit disk $\overline{\mathbf{D}}$. In the case of a periodic Fatou component, this canonical homeomorphism can be identified with the Böttcher coordinate for the first return map $f^{\circ q}$. In the case of an arbitrary component U, simply choose the smallest n so that $f^{\circ n}(U)$ is periodic, and then use the homeomorphism $f^{\circ n} : \overline{U} \to f^{\circ n}(\overline{U})$ to pull back the Böttcher coordinate. In all cases but one, note that the map $f : \overline{U} \to f(\overline{U})$ preserves this canonical coordinate. However, in the exceptional case when U is the central component, a point in $\overline{U}$ with coordinate w maps to a point in $f(\overline{U})$ with coordinate w^2.

Let us number the closures of the periodic Fatou components as

$$K_{0,1} \xrightarrow{\cong} K_{0,2} \xrightarrow{\cong} \cdots \xrightarrow{\cong} K_{0,q} \to K_{0,1} \ ,$$

where $K_{0,q}$ is the central component containing the critical point, and $K_{0,1}$ contains the critical value. Then K_0 is equal to the union $K_{0,1} \cup \cdots \cup K_{0,q}$. Since each $K_{0,j}$ is canonically homeomorphic to $\overline{\mathbf{D}}$, and since the $K_{0,j}$ intersect only at their common ***root point*** α, which has coordinate $+1$ in each $K_{0,j}$, this yields the required description of K_0, together with the map $f|_{K_0}$, without any reference to the numerator p.

Next we will construct $K_1 = f^{-1}(K_0) = K_0 \cup \tau(K_0)$. Here τ is the involution $z \mapsto -z$, so that $f \circ \tau = f$. Then K_1 can be obtained from K_0 by adjoining new copies $K_{1,j} = \tau(K_{0,j})$ of $\overline{\mathbf{D}}$ to K_0 for $1 \le j < q$. Each of these new disks is to be attached by identifying its root point with coordinate $+1$ to the point $\tau(\alpha) \in K_{0,q}$, which has coordinate -1 in $K_{0,q}$. The map f extends over K_1 by setting $f(\tau(z)) = f(z)$ for all $\tau(z) \in K_{1,j} = \tau(K_{0,j})$.

Now suppose inductively that we have constructed $K_0 \subset K_1 \subset \cdots \subset K_n$ together with the map $f|_{K_n} : K_n \to K_{n-1}$, which is exactly two-to-one except at the critical point. Suppose further that K_n is obtained from K_{n-1} by adjoining closed topological disks $K_{n,j}$ for $1 \le j \le 2^{n-1}(q-1)$, where these disks are attached by their root points to corresponding points $z(n,j) \in \partial K_{n-1}$, Then each $f^{-1}(z(n,j))$ consists of two points in ∂K_n, which we will call $z(n+1\,,\,2j-1)$ and $z(n+1\,,\,2j)$ respectively. Form K_{n+1} from K_n by attaching a copy $K_{n+1\,,\,k}$ of $\overline{\mathbf{D}}$ at each of these $2^n(q-1)$ points $z(n+1\,,\,k)$, and extend $f|_{K_n}$ to a map from K_{n+1} to K_n which carries both $K_{n+1\,,\,2j-1}$ and $K_{n+1\,,\,2j}$ onto $K_{n-1\,,\,j}$, preserving the canonical homeomorphism with $\overline{\mathbf{D}}$. This completes the inductive construction.

Let $r_n : K_n \to K_{n-1}$ be the retraction which collapses each attached disk $K_{n,j}$ to its point of attachment $z(n,j)$, while fixing every point of K_{n-1}, and let $\hat{K}$ be the inverse limit of the sequence

$$K_0 \xleftarrow{r_1} K_1 \xleftarrow{r_2} K_2 \xleftarrow{r_3} \cdots \ .$$

Then $\hat{K}$ is a compact topological space. Using the commutative diagram

$$\begin{array}{ccc} K_{n+1} & \xrightarrow{f} & K_n \\ \downarrow r_{n+1} & & \downarrow r_n \\ K_n & \xrightarrow{f} & K_{n-1} \end{array}$$

we see that the maps $f|K_n : K_n \to K_{n-1}$ give rise to a map $\hat{f} : \hat{K} \to \hat{K}$ in the limit.

We must show that this limit $\hat{K}$ can be identified with the original filled Julia set $K = K(p/q)$. (Compare [Douady, 1993].) Let $\mathcal{O} \subset K_0$ be the critical orbit of period $q \geq 2$. Then the Riemann surface $\mathbf{C} \smallsetminus \mathcal{O}$ has a Poincaré metric which is strictly expanding on the Julia set, and also on every disk $K_{n,j}$ with $n > 1$. (More precisely, there exists a constant $k > 1$ so that $\|f'(z)\| \geq k$ for every z in the compact set $\bigcup_{n>1} \bigcup_j K_{n,j}$, using the Poincaré metric at z and $f(z)$ to define the norm of such a derivative.) Let d_n be the maximum of the diameters of the disks $K_{n,j}$ in this metric. Then the sequence $d_1 > d_2 > d_3 > \cdots$ tends geometrically to zero, and it follows easily that every sequence of points

$$z_0 \overset{r_1}{\hookleftarrow} z_1 \overset{r_2}{\hookleftarrow} z_2 \overset{r_3}{\hookleftarrow} z_3 \overset{r_4}{\hookleftarrow} \cdots$$

converges to a unique point of K. This yields the required homeomorphism $\hat{K} \overset{\cong}{\to} K$.

Since this description of $K = K(p/q)$ makes no mention of p, it yields a homeomorphism $K(p/q) \overset{\cong}{\to} K(p'/q)$ which is holomorphic on the interior and compatible with the dynamics. For $p \neq p'$ it cannot be extended as a homeomorphism over any neighborhood of the α fixed point, since the various $K_{0,j}$ are arranged in a different cyclic order around α in these two filled Julia sets. $\square$

References

L. Ahlfors, "Complex Analysis", McGraw-Hill 1966.

B. Branner and N. Fagella, to appear.

H. Brolin, Invariant sets under iteration of rational functions, Arkiv för Math. **6** (1965) 103-144.

M. Crichton, "Jurassic Park", Ballantine 1990.

R. Daverman, "Decompositions of Manifolds", Academic Press 1986.

A. Douady Systèmes dynamiques holomorphes, Séminar Bourbaki, 35^e année (1982-3) n^o 599; Astèrisque **105-6** (1983) 39-63.

A. Douady, Algorithms for computing angles in the Mandelbrot set, "Chaotic Dynamics and Fractals", ed. Barnsley & Demko, Acad. Press (1986) 155-168.

A. Douady, Descriptions of compact sets in $\mathbf{C}$, pp. 429-465 of "Topological Methods in Modern Mathematics", edit. Goldberg and Phillips, Publish or Perish 1993.

A. Douady and C. Earle, Conformally natural extension of homeomorphisms of the circle. Acta Math. 157 (1986) 23-48.

A. Douady and J. H. Hubbard, A proof of Thurston's topological characterization of rational functions, Acta Math. **171** (1993) 263-297.

G. A. Edgar, "Measure, Topology, and Fractal Geometry", Springer 1990.

A. Epstein, Counterexamples to the quadratic mating conjecture, (in preparation).

R. Kenyon, The construction of self-similar tilings. Geom. Funct. Anal. 6 (1996), 471–488.

S. Lattès, Sur l'iteration des substitutions rationelles et les fonctions de Poincaré, C. R. Acad. Sci. Paris **16** (1918) 26-28.

Jiaqi Luo, Combinatorics and holomorphic dynamics: captures, matings, Newton's method; thesis, Cornell University 1995.

M. Lyubich, Entropy properties of rational endomorphisms of the Riemann sphere, Erg. Th. & Dy. Sy. **3** (1983) 351-385.

M. Lyubich, Renormalization ideas in conformal dynamics, pp. 109-133 of "Current Developments in Mathematics, 1995", edit. Bott, Jaffe, Yau, International Press 1995.

J. Milnor, Dynamics in One Complex Variable: Introductory Lectures, IMS preprints 1990/5.

J. Milnor, Geometry and dynamics of quadratic rational maps, Experim. Math. **2** (1993) 37-83. (See p. 54.)

J. Milnor, Thurston's algorithm without critical finiteness?, in "Linear and Complex Analysis Problem Book 3, vol. II" (eds. V. P. Havin and N. K. Nikolskii), Springer 1994; or in "Problems in Holomorphic Dynamics", (eds. B. Bielefeld and M. Lyubich) IMS preprint 1992/7.

R. L. Moore, Concerning upper semi-continuous collections of continua, Trans. Amer. Math. Soc. **27** (1925) 416-428.

M. Rees, A partial description of the parameter space of rational maps of degree two: Part 1, Acta Math. **168** (1992) 11-87. (See also: Realization of matings of polynomials as rational maps of degree two, preprint 1986.)

M. Shishikura, On a theorem of M. Rees for matings of polynomials, Preprint IHES 1990.

M. Shishikura and Tan Lei, A family of cubic rational maps and matings of cubic polynomials, Preprint, Max-Planck-Institut Bonn, 1988.

Tan Lei, Similarity between the Mandelbrot set and Julia sets. Comm. Math. Phys. 134 (1990) 587–617.

Tan Lei, Matings of quadratic polynomials, Ergodic Th. & Dy. Sy. **12** (1992) 589-620.

B. Wittner, On the bifurcation loci of rational maps of degree two, Thesis, Cornell University 1988.

S. Zakeri, Biaccessibility in quadratic Julia sets I: The locally-connected case, Stony Brook IMS Preprints, to appear.

John Milnor
Institute for Mathematical Sciences
State University of New York
Stony Brook NY 11794-3660
email: jack@math.sunysb.edu
web: http://www.math.sunysb.edu/~jack