Cylinder Maps and the Schwarzian

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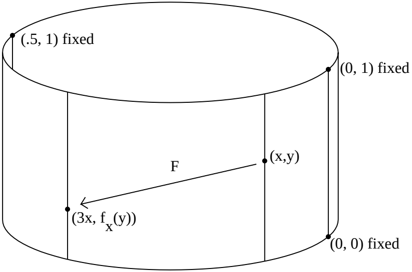
—work with Araceli Bonifant—

Let $\mathcal{C}$ denote the cylinder $(\mathbb{R}/\mathbb{Z}) \times I$.



We will study maps

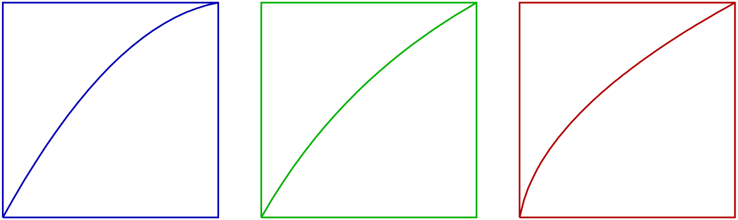
$$F(x, y) = \big(kx,\ f_x(y)\big)$$

from $\mathcal{C}$ to itself, where $k \geq 2$ is a fixed integer, where each $f_x : I \to I$ is a diffeomorphism with $f_x(0) = 0$ and $f_x(1) = 1$, and where the Schwarzian $\mathcal{S}f_x(y)$ has constant sign for almost all $(x, y) \in \mathcal{C}$.

Schwarzian derivative

The **Schwarzian derivative** of a C^3 interval diffeomorphism f is defined by the formula

$$\mathcal{S}f(y) \;=\; \frac{f'''(y)}{f'(y)} - \frac{3}{2}\left(\frac{f''(y)}{f'(y)}\right)^2 . \qquad (1)$$



On the left: Graph of a function $q_a(y) = y + ay(1-y)$ $(a = 0.82)$, with $\mathcal{S}q_a < 0$ everywhere.

Middle: Graph of $y \mapsto 3y/(1+2y)$, with $\mathcal{S} \equiv 0$.

Right: Graph of $q_{-a}^{-1}(y)$, with $\mathcal{S} > 0$ everywhere.

The Transverse Lyapunov Exponent

Let $\mathcal{A}_0 = (\mathbb{R}/\mathbb{Z}) \times 0$ and $\mathcal{A}_1 = (\mathbb{R}/\mathbb{Z}) \times 1$ be the two boundaries of $\mathcal{C}$. The **transverse Lyapunov exponent** of the boundary circle $\mathcal{A}_\iota$ can be defined as the average

$$\mathrm{Lyap}(\mathcal{A}_\iota) \;=\; \int_{\mathbb{R}/\mathbb{Z}} \log\left(\frac{df_x}{dy}(x,\,\iota)\right)\,dx\,.$$

Let $\mathcal{B}_\iota = \mathcal{B}(\mathcal{A}_\iota)$ be the **attracting basin**: the union of all orbits which converge towards $\mathcal{A}_\iota$.

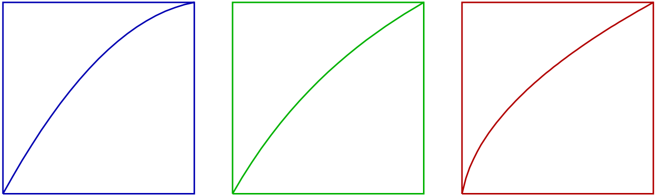
Standard Theorem. If $\mathrm{Lyap}(\mathcal{A}_\iota) < 0$ then $\mathcal{B}_\iota$ has strictly positive measure. In this case, the boundary circle $\mathcal{A}_\iota$ will be described as a "**measure-theoretic attractor**".
However, if $\mathrm{Lyap}(\mathcal{A}_\iota) > 0$ then $\mathcal{B}_\iota$ has measure zero.

Lemma. Suppose that $\mathcal{S}f(y)$ has constant sign (positive, negative or, zero) for almost all (x, y) in $\mathcal{C}$.

If $\mathcal{S}f > 0$ almost everywhere, then $f'(0)f'(1) > 1$.

If $\mathcal{S}f \equiv 0$, then $f'(0)f'(1) = 1$.

If $\mathcal{S}f < 0$ almost everywhere, then $f'(0)f'(1) < 1$.



Corollary. If $\mathcal{S}f_x(y)$ has constant sign for almost all (x, y), then $\mathrm{Lyap}(\mathcal{A}_0) + \mathrm{Lyap}(\mathcal{A}_1)$ has this same sign.

For example, if $\mathrm{Lyap}(\mathcal{A}_0)$ and $\mathrm{Lyap}(\mathcal{A}_1)$ have the same sign, and if $\mathcal{S}f_x(y) < 0$ almost everywhere, then it follows that **both** boundaries are measure-theoretic attractors.

Standing Hypothesis: *Always assume that* $\mathrm{Lyap}(\mathcal{A}_0)$ *and* $\mathrm{Lyap}(\mathcal{A}_1)$ *have the same sign.*

Theorem 1. Suppose also that $\mathcal{S}f_x(y) < 0$ almost everywhere. Then there is an almost everywhere defined measurable function $\sigma : \mathbb{R}/\mathbb{Z} \to I$ such that:

$$\begin{array}{llll} & (x, y) \in \mathcal{B}_0 & \textit{whenever} & y < \sigma(x)\,, \\ \text{and} & (x, y) \in \mathcal{B}_1 & \textit{whenever} & y > \sigma(x)\,. \end{array}$$

It follows that the union $\mathcal{B}_0 \cup \mathcal{B}_1$ has full measure.

More generally, the same statement is true if the k-tupling map on the circle is replaced by any continuous ergodic transformation g on a compact space with g-invariant probability measure.

The proof will make use of the cross-ratio

$$\rho(y_0, y_1, y_2, y_3) = \frac{(y_2 - y_0)(y_3 - y_1)}{(y_1 - y_0)(y_3 - y_2)}.$$

We will take $y_0 < y_1 < y_2 < y_3$, and hence $\rho > 1$.

According to Allwright (1978):

Maps f_x with $\mathcal{S}(f_x) < 0$ almost everywhere have the basic property of **increasing** the cross-ratio $\rho(y_0, y_1, y_2, y_3)$ for all $y_0 < y_1 < y_2 < y_3$ in the interval.

Similarly, maps with $\mathcal{S}(f_x) \equiv 0$ will **preserve** all such cross-ratios;

and maps with $\mathcal{S}(f_x) > 0$ will **decrease** these cross-ratios.

Proof of Theorem 1

Since each f_x is an orientation preserving homeomorphism, there are unique numbers

$$0 \le \sigma_0(x) \le \sigma_1(x) \le 1$$

such that the orbit of (x, y):

$$\begin{array}{rll} \text{converges to } \mathcal{A}_0 & \text{if} & y < \sigma_0(x), \\ \text{converges to } \mathcal{A}_1 & \text{if} & y > \sigma_1(x), \\ \text{does not converge to either circle} & \text{if} & \sigma_0(x) < y < \sigma_1(x)\,. \end{array}$$

Thus, the area of $\mathcal{B}_0$ can be defined as $\int \sigma_0(x)\,dx$. Since this is known to be positive, it follows that the set of all $x \in \mathbb{R}/\mathbb{Z}$ with $\sigma_0(x) > 0$ must have positive measure.
On the other hand, this set is fully invariant under the ergodic map $x \mapsto kx$, using the identity $\sigma_0(kx) = f_x\big(\sigma_0(x)\big)$.
Hence it must actually have full measure.
Similarly, the set of x with $\sigma_1(x) < 1$ must have full measure.

Proof of Theorem 1(continued)

To finish the argument, we must show that $\sigma_0(x) = \sigma_1(x)$ for almost all $x \in \mathbb{R}/\mathbb{Z}$. Suppose otherwise that $\sigma_0(x) < \sigma_1(x)$ on a set of x of positive measure. Then a similar ergodic argument would show that

$$0 < \sigma_0(x) < \sigma_1(x) < 1 \qquad \text{for almost all} \quad x\,.$$

Hence the cross-ratio

$$r(x) = \rho(0,\, \sigma_0(x),\, \sigma_1(x),\, 1)$$

would be defined for almost all x, with $1 < r(x) < \infty$. Furthermore, since maps of negative Schwarzian increase cross-ratios, we would have $r(kx) > r(x)$ almost everywhere.

This is impossible!

Proof of Theorem 1 (conclusion)

The inequality $1 < r(x) < r(kx)$ would imply that

$$\int_{\mathbb{R}/\mathbb{Z}} \frac{dx}{r(kx)} < \int_{\mathbb{R}/\mathbb{Z}} \frac{dx}{r(x)} .$$

But Lebesgue measure is invariant under push-forward by the map $x \mapsto kx$. It follows that

$$\int \phi(kx)\, dx = \int \phi(x)\, dx$$

for any bounded measurable function ϕ. This contradiction proves that we must have $\sigma_0(x) = \sigma_1(x)$ almost everywhere. Defining $\sigma(x)$ to be this common value, this proves Theorem 1. $\square$

For any measurable set $S \subset \mathcal{C}$, let $\mu_\iota(S)$ be the Lebesgue measure of the intersection $\mathcal{B}_\iota \cap S$. When Theorem 1 applies, μ_0 and μ_1 are non-zero measures on the cylinder, and have sum equal to Lebesgue measure.

Definition. The two basins $\mathcal{B}_0$ and $\mathcal{B}_1$ are **intermingled** if

$$\mu_0(U) > 0 \qquad \text{and} \qquad \mu_1(U) > 0$$

for every non-empty open set U.

Equivalently, they are intermingled if both measures have support equal to the entire cylinder.

(Here the *support*, **supp**(μ_ι), is defined to be the smallest closed set which has full measure under μ_ι.)

Example (Ittai Kan 1994)

Let

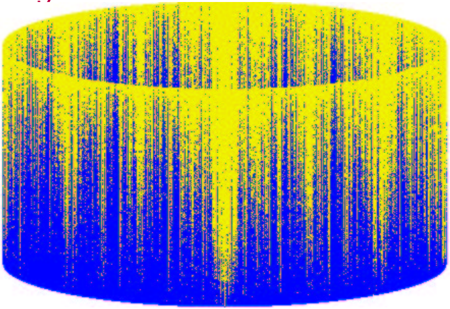
$$q_a(y) = y + ay(1 - y)\,,$$

and let $a = p(x) = \epsilon \cos(2\pi x)$, with $0 < \epsilon < 1$.

Theorem 2. *If $k \geq 2$, then the basins $\mathcal{B}_0$ and $\mathcal{B}_1$ for the map*

$$F(x, y) = \Big(kx,\; q_{p(x)}(y)\Big)$$

are intermingled.



Proof of Theorem 2

Lemma. Suppose that there exist:

• an angle $x^- \in \mathbb{R}/\mathbb{Z}$, fixed under multiplication by k, and a neighborhood $U(x^-)$ such that

$$f_x(y) < y \text{ for all } x \in U(x^-) \text{ and all } 0 < y < 1,$$ and

• an angle $x^+ \in \mathbb{R}/\mathbb{Z}$, fixed under multiplication by k, and a neighborhood $U(x^+)$ such that

$$f_x(y) > y \text{ for all } x \in U(x^+) \text{ and all } 0 < y < 1.$$

If $\mathcal{S}f_x < 0$ almost everywhere, and if $\mathrm{Lyap}(\mathcal{A}_\iota) < 0$ for both $\mathcal{A}_\iota$, then the basins $\mathcal{B}_0$ and $\mathcal{B}_1$ are intermingled.

Kan's example $F(x, y) = \Big(kx,\ q_{\epsilon\cos(2\pi x)}(y)\Big)$ *satisfies this hypothesis for* $k > 2$, *since the angle* k*-tupling map has fixed points with* $\cos(2\pi x) > 0$, *and also fixed points with* $\cos(2\pi x) < 0$.

For the case $k = 2$, *we can replace* F *by* $F \circ F$ *in order to obtain a fixed point with* $\cos(2\pi x) < 0$.

Thus this Lemma will imply Theorem 2.

Proof of Lemma

Note that the support $\mathbf{supp}(\mu_\iota)$

- is a closed subset of $\mathcal{C}$,
- is fully F-invariant, and
- has positive area.

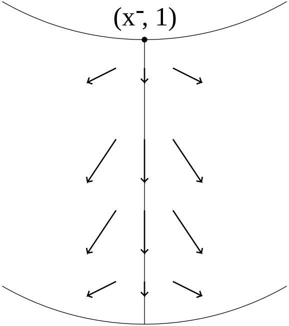
We must prove that this support is equal to the entire cylinder.

To begin, choose any point $(x_0, y_0) \in \mathbf{supp}(\mu_0)$ with $0 < y_0 < 1$. Construct a backward orbit

$$\cdots \mapsto (x_{-2}, y_{-2}) \mapsto (x_{-1}, y_{-1}) \mapsto (x_0, y_0)$$

under F by induction, letting each $x_{-(k+1)}$ be that preimage of x_{-k} which is closest to x^-. Then this backwards sequence converges to the point $(x^-, 1)$.

Proof (conclusion)



Since **supp**(μ_0) is closed and F-invariant, it follows that $(x^-, 1) \in$ **supp**(μ_0). Since the iterated pre-images of $(x^-, 1)$ are everywhere dense in the upper boundary circle $\mathcal{A}_1$, it follows that $\mathcal{A}_1$ is contained in **supp**(μ_0).

But if (x, y) belongs to **supp**(μ_0), then clearly the entire line segment $x \times [0, y]$ is contained in **supp**(μ_0).

Therefore **supp**(μ_0) is the entire cylinder.

The proof for μ_1 is completely analogous.

This proves the Lemma, and proves Theorem 2. $\square$

§2. Postive Schwarzian: Asymptotic Measure

Now suppose that $\mathcal{S}f_x > 0$ almost everywhere.
We will see that almost all orbits for the map

$$F(x,y) = (kx,\ f_x(y))$$

have the same asymptotic distribution.

Definition. An **asymptotic measure** ν for F is a probability measure on the cylinder $\mathcal{C}$ such that, for Lebesgue almost every orbit $(x_0, y_0) \mapsto (x_1, y_1) \mapsto \cdots,$ and for every continuous test function $\psi : \mathcal{C} \to \mathbb{R},$ the time average

$$\frac{1}{n}\Big(\sum_{i=0}^{n-1} \psi(x_i,\ y_i)\Big)$$

converges to the space average $\int_{\mathcal{C}} \psi(x,y)\, d\nu(x,y)$ as $n \to \infty$.

Briefly: Almost every orbit is **uniformly distributed** with respect to the measure ν.

Theorem 3. If $\mathcal{S}f_x(y) > 0$ almost everywhere,
and if $\mathrm{Lyap}(\mathcal{A}_\iota) > 0$ for both $\mathcal{A}_\iota$,
then F has a (necessarily unique) asymptotic measure.

Proof Outline. Let $\mathfrak{S}_k$ be the **solenoid** consisting of all **full orbits**

$$\cdots \mapsto x_{-2} \mapsto x_{-1} \mapsto x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$$

under the k-tupling map.

Then F lifts to a homeomorphism $\widetilde{F}$ of $\mathfrak{S}_k \times I$.

Here $\widetilde{F}$ maps fibers to fibers with $\mathcal{S} > 0$.
Therefore $\widetilde{F}^{-1}$ maps fibers to fibers with $\mathcal{S} < 0$.

Hence we can apply the argument of Theorem 1 to $\widetilde{F}^{-1}$.

Outline Proof (conclusion)

In particular, there is an almost everywhere defined measurable section

$$\sigma : \mathfrak{S}_k \to \mathfrak{S}_k \times I$$

which separates the basins of $\mathfrak{S}_k \times 0$ and $\mathfrak{S}_k \times 1$ under $\widetilde{F}^{-1}$.

Let $\widetilde{\nu}$ be the push-forward under σ of the standard shift-invariant probability measure on $\mathfrak{S}_k$. Thus $\widetilde{\nu}$ is an $\widetilde{F}$-invariant probability measure on $\mathfrak{S}_k \times I$.

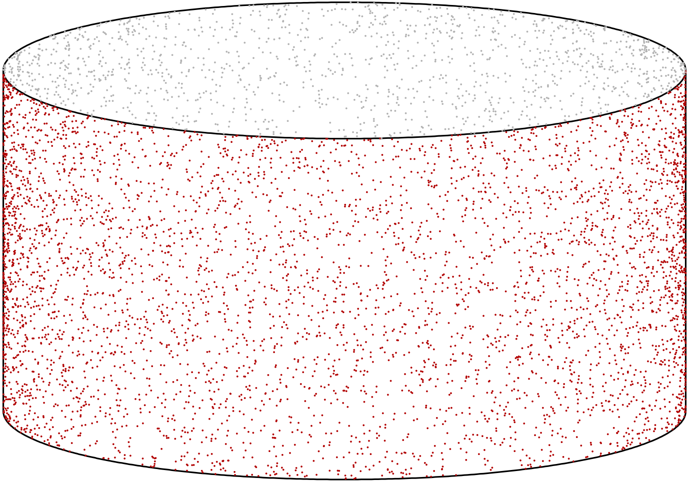
Assertion: $\widetilde{\nu}$ is an asymptotic measure for $\widetilde{F}$.

Since almost all points are pushed away from the graph of σ by the inverse map $\widetilde{F}^{-1}$, it follows that they are pushed towards this graph by the map $\widetilde{F}$.

Now push $\widetilde{\nu}$ forward under the projection from $\mathfrak{S}_k \times I$ to $\mathcal{C} = (\mathbb{R}/\mathbb{Z}) \times I$,
This yields the required asymptotic measure for F. □

Example

Let $F(x,y) = \left(kx,\ q^{-1}_{\epsilon\cos(2\pi x)}(y)\right).$



50000 points of a randomly chosen orbit for F.

§3. The Hard Case: Zero Schwarzian

Suppose that each orientation preserving diffeomorphism $f_x : I \to I$ has Schwarzian $\mathcal{S} f_x$ identically zero.
Such a map is necessarily fractional linear, and can be written as

$$y \;\mapsto\; \frac{ay}{1+(a-1)y} \qquad \text{with} \qquad a \;>\; 0\,. \tag{2}$$

Here $a = a(x) = f_x'(0)$ is the derivative with respect to y at $y = 0$. Note that each f_x preserves the **Poincaré distance**

$$\mathbf{d}(y_1,\, y_2) \;=\; \left|\log\ \rho(0,\, y_1,\, y_2,\, 1)\right|.$$

Hence, by a change of variable, we can transform this fractional linear transformation of the open interval into a translation of the real line: Replace y by the **Poincaré arclength coordinate**

$$t(y) \;=\; \log\ \rho(0,\, 1/2,\, y,\, 1) \;=\; \log \frac{y}{1-y}\,.$$

The map (2) then corresponds to the translation

$$t \;\mapsto\; t \;+\; \log a\,. \tag{3}$$

A Pseudo-Random Walk.

Using this change of coordinate, the skew product map $(x, y) \mapsto \big(kx,\ f_x(y)\big)$ on $(\mathbb{R}/\mathbb{Z}) \times I$ takes the form

$$(x, t) \ \mapsto \ \big(kx,\ t + \log a(x)\big)\,,$$

mapping $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ to itself.

Think of the k-tupling map as generating a sequence of pseudo-random numbers

$$\log a(x),\ \ \log a(kx),\ \ \log a(k^2 x),\ \ \ldots\,.$$

Then the resulting sequence of t values can be described as a **"pseudo-random walk"** on the real line. The condition that

$$\mathrm{Lyap}(\mathcal{A}_0) \ = \ \int_{\mathbb{R}/\mathbb{Z}} \log\big(a(x)\big)\, dx \ = \ 0$$

means that this pseudo-random walk is **unbiased.**

Conjectured Behavior

Suppose that $\mathcal{S} f_y \equiv 0$*,*
with $\mathrm{Lyap}(\mathcal{A}_0) = \mathrm{Lyap}(\mathcal{A}_1) = 0$*,*
and with $f_x(y) \not\equiv y$*,*
then we conjecture that almost every orbit comes within any neighborhood of $\mathcal{A}_0$ *infinitely often, but also within any neighborhood of* $\mathcal{A}_1$ *infinitely often, on such an irregular schedule that there can be no asymptotic measure!*

More precisely, for almost every orbit

$$(x_1, y_1) \mapsto (x_2, y_2) \mapsto (x_3, y_3) \mapsto \cdots ,$$

we have

$$\liminf \frac{y_1 + \cdots + y_n}{n} = 0 \quad \text{and} \quad \limsup \frac{y_1 + \cdots + y_n}{n} = 1 .$$

Intuitive Argument

The corresponding statement is known to be true for an honest random walk on $\mathbb{R}$, where the successive steps sizes are independent random variables with mean zero.

Conjecturally, our pseudo-random walk must behave enough like an actual random walk so that this behavior will persist.

THE END