Cylinder Maps and the Schwarzian

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Cylinder Maps

—work with Araceli Bonifant—

Let $\mathcal{C}$ denote the cylinder $(\mathbb{R}/\mathbb{Z}) \times I$.

We will study maps

$$F(x, y) = (kx, f(x)(y))$$

from $\mathcal{C}$ to itself, where $k \geq 2$ is a fixed integer, where each $f_x : I \rightarrow I$ is a diffeomorphism with $f_x(0) = 0$ and $f_x(1) = 1$, and where the Schwarzian $Sf_x(y)$ has constant sign for almost all $(x, y) \in \mathcal{C}$. 

The Schwarzian derivative of a $C^3$ interval diffeomorphism $f$ is defined by the formula

$$Sf(y) = \frac{f'''(y)}{f'(y)} - \frac{3}{2} \left( \frac{f''(y)}{f'(y)} \right)^2.$$  \hspace{1cm} (1)

On the left: Graph of a function $q_a(y) = y + ay(1 - y)$ ($a = 0.82$), with $S_{q_a} < 0$ everywhere.

Middle: Graph of $y \mapsto 3y/(1 + 2y)$, with $S \equiv 0$.

Right: Graph of $q_{-a}^{-1}(y)$, with $S > 0$ everywhere.
Let $\mathcal{A}_0 = (\mathbb{R}/\mathbb{Z}) \times 0$ and $\mathcal{A}_1 = (\mathbb{R}/\mathbb{Z}) \times 1$ be the two boundaries of $\mathcal{C}$. The **transverse Lyapunov exponent** of the boundary circle $\mathcal{A}_\iota$ can be defined as the average

$$\text{Lyap}(\mathcal{A}_\iota) = \int_{\mathbb{R}/\mathbb{Z}} \log \left( \frac{df_x}{dy} (x, \iota) \right) \, dx.$$ 

Let $\mathcal{B}_\iota = \mathcal{B}(\mathcal{A}_\iota)$ be the **attracting basin**: the union of all orbits which converge towards $\mathcal{A}_\iota$.

**Standard Theorem.** If $\text{Lyap}(\mathcal{A}_\iota) < 0$ then $\mathcal{B}_\iota$ has strictly positive measure. In this case, the boundary circle $\mathcal{A}_\iota$ will be described as a **measure-theoretic attractor**. However, if $\text{Lyap}(\mathcal{A}_\iota) > 0$ then $\mathcal{B}_\iota$ has measure zero.
Lemma. Suppose that $Sf(y)$ has constant sign (positive, negative or, zero) for almost all $(x, y)$ in $C$.

If $Sf > 0$ almost everywhere, then $f'(0)f'(1) > 1$.

If $Sf \equiv 0$, then $f'(0)f'(1) = 1$.

If $Sf < 0$ almost everywhere, then $f'(0)f'(1) < 1$.

Corollary. If $Sf_x(y)$ has constant sign for almost all $(x, y)$, then $\text{Lyap}(A_0) + \text{Lyap}(A_1)$ has this same sign.

For example, if $\text{Lyap}(A_0)$ and $\text{Lyap}(A_1)$ have the same sign, and if $Sf_x(y) < 0$ almost everywhere, then it follows that both boundaries are measure-theoretic attractors.
Standing Hypothesis: Always assume that
\( \text{Lyap}(A_0) \) and \( \text{Lyap}(A_1) \) have the same sign.

Theorem 1. Suppose also that \( Sf_x(y) < 0 \) almost everywhere. Then there is an almost everywhere defined measurable function \( \sigma : \mathbb{R}/\mathbb{Z} \to I \) such that:

\[
(x, y) \in B_0 \quad \text{whenever} \quad y < \sigma(x),
\]

and \( (x, y) \in B_1 \quad \text{whenever} \quad y > \sigma(x). \)

It follows that the union \( B_0 \cup B_1 \) has full measure.

More generally, the same statement is true if the \( k \)-tupling map on the circle is replaced by any continuous ergodic transformation \( g \) on a compact space with \( g \)-invariant probability measure.
The proof will make use of the cross-ratio

\[ \rho(y_0, y_1, y_2, y_3) = \frac{(y_2 - y_0)(y_3 - y_1)}{(y_1 - y_0)(y_3 - y_2)}. \]

We will take \( y_0 < y_1 < y_2 < y_3 \), and hence \( \rho > 1 \).

According to Allwright (1978):

Maps \( f_x \) with \( S(f_x) < 0 \) almost everywhere have the basic property of increasing the cross-ratio \( \rho(y_0, y_1, y_2, y_3) \) for all \( y_0 < y_1 < y_2 < y_3 \) in the interval.

Similarly, maps with \( S(f_x) \equiv 0 \) will preserve all such cross-ratios;

and maps with \( S(f_x) > 0 \) will decrease these cross-ratios.
Proof of Theorem 1

Since each $f_x$ is an orientation preserving homeomorphism, there are unique numbers

$$0 \leq \sigma_0(x) \leq \sigma_1(x) \leq 1$$

such that the orbit of $(x, y)$:

- converges to $A_0$ if $y < \sigma_0(x)$,
- converges to $A_1$ if $y > \sigma_1(x)$,
- does not converge to either circle if $\sigma_0(x) < y < \sigma_1(x)$.

Thus, the area of $B_0$ can be defined as $\int \sigma_0(x) \, dx$. Since this is known to be positive, it follows that the set of all $x \in \mathbb{R}/\mathbb{Z}$ with $\sigma_0(x) > 0$ must have positive measure.

On the other hand, this set is fully invariant under the ergodic map $x \mapsto kx$, using the identity $\sigma_0(kx) = f_x(\sigma_0(x))$.

Hence it must actually have full measure.

Similarly, the set of $x$ with $\sigma_1(x) < 1$ must have full measure.
To finish the argument, we must show that $\sigma_0(x) = \sigma_1(x)$ for almost all $x \in \mathbb{R}/\mathbb{Z}$. Suppose otherwise that $\sigma_0(x) < \sigma_1(x)$ on a set of $x$ of positive measure. Then a similar ergodic argument would show that

$$0 < \sigma_0(x) < \sigma_1(x) < 1$$

for almost all $x$.

Hence the cross-ratio

$$r(x) = \rho(0, \sigma_0(x), \sigma_1(x), 1)$$

would be defined for almost all $x$, with $1 < r(x) < \infty$. Furthermore, since maps of negative Schwarzian increase cross-ratios, we would have $r(kx) > r(x)$ almost everywhere.

This is impossible!
The inequality $1 < r(x) < r(kx)$ would imply that

$$\int_{\mathbb{R}/\mathbb{Z}} \frac{dx}{r(kx)} < \int_{\mathbb{R}/\mathbb{Z}} \frac{dx}{r(x)}.$$ 

But Lebesgue measure is invariant under push-forward by the map $x \mapsto kx$. It follows that

$$\int \phi(kx) \, dx = \int \phi(x) \, dx$$

for any bounded measurable function $\phi$. This contradiction proves that we must have $\sigma_0(x) = \sigma_1(x)$ almost everywhere. Defining $\sigma(x)$ to be this common value, this proves Theorem 1. \qed
For any measurable set \( S \subset \mathcal{C} \), let \( \mu_\ell(S) \) be the Lebesgue measure of the intersection \( B_\ell \cap S \). When Theorem 1 applies, \( \mu_0 \) and \( \mu_1 \) are non-zero measures on the cylinder, and have sum equal to Lebesgue measure.

**Definition.** The two basins \( B_0 \) and \( B_1 \) are **intermingled** if

\[
\mu_0(U) > 0 \quad \text{and} \quad \mu_1(U) > 0
\]

for every non-empty open set \( U \).

Equivalently, they are intermingled if both measures have support equal to the entire cylinder.

(Here the **support**, \( \text{supp}(\mu_\ell) \), is defined to be the smallest closed set which has full measure under \( \mu_\ell \).)
Example (Ittai Kan 1994)

Let
\[
q_a(y) = y + ay(1 - y),
\]
and let
\[
a = p(x) = \epsilon \cos(2\pi x), \quad \text{with } 0 < \epsilon < 1.
\]

**Theorem 2.** If \( k \geq 2 \), then the basins \( B_0 \) and \( B_1 \) for the map
\[
F(x, y) = \left( kx, q_{p(x)}(y) \right)
\]
are intermingled.
Proof of Theorem 2

Lemma. Suppose that there exist:

• an angle $x^- \in \mathbb{R}/\mathbb{Z}$, fixed under multiplication by $k$, and a neighborhood $U(x^-)$ such that
  
  $f_x(y) < y$ for all $x \in U(x^-)$ and all $0 < y < 1$, and

• an angle $x^+ \in \mathbb{R}/\mathbb{Z}$, fixed under multiplication by $k$, and a neighborhood $U(x^+)$ such that
  
  $f_x(y) > y$ for all $x \in U(x^+)$ and all $0 < y < 1$.

If $Sf_x < 0$ almost everywhere, and if $\text{Lyap}(A_\ell) < 0$ for both $A_\ell$, then the basins $B_0$ and $B_1$ are intermingled.

Kan’s example $F(x, y) = \left( kx, q_\epsilon \cos(2\pi x)(y) \right)$ satisfies this hypothesis for $k > 2$, since the angle $k$-tupling map has fixed points with $\cos(2\pi x) > 0$, and also fixed points with $\cos(2\pi x) < 0$.

For the case $k = 2$, we can replace $F$ by $F \circ F$ in order to obtain a fixed point with $\cos(2\pi x) < 0$.

Thus this Lemma will imply Theorem 2.
Proof of Lemma

Note that the support \( \text{supp}(\mu_i) \)
- is a closed subset of \( C \),
- is fully \( F \)-invariant, and
- has positive area.

We must prove that this support is equal to the entire cylinder.

To begin, choose any point \((x_0, y_0) \in \text{supp}(\mu_0)\) with \(0 < y_0 < 1\). Construct a backward orbit

\[
\cdots \mapsto (x_{-2}, y_{-2}) \mapsto (x_{-1}, y_{-1}) \mapsto (x_0, y_0)
\]

under \( F \) by induction, letting each \( x_{-(k+1)} \) be that preimage of \( x_{-k} \) which is closest to \( x^- \). Then this backwards sequence converges to the point \((x^-, 1)\).
Since $\text{supp}(\mu_0)$ is closed and $F$-invariant, it follows that $(x^-, 1) \in \text{supp}(\mu_0)$. Since the iterated pre-images of $(x^-, 1)$ are everywhere dense in the upper boundary circle $A_1$, it follows that $A_1$ is contained in $\text{supp}(\mu_0)$.

But if $(x, y)$ belongs to $\text{supp}(\mu_0)$, then clearly the entire line segment $x \times [0, y]$ is contained in $\text{supp}(\mu_0)$.

Therefore $\text{supp}(\mu_0)$ is the entire cylinder.

The proof for $\mu_1$ is completely analogous.

This proves the Lemma, and proves Theorem 2.
Now suppose that $S_{f_x} > 0$ almost everywhere. We will see that almost all orbits for the map \[ F(x, y) = (kx, f_x(y)) \] have the same asymptotic distribution.

**Definition.** An *asymptotic measure* $\nu$ for $F$ is a probability measure on the cylinder $C$ such that, for Lebesgue almost every orbit $(x_0, y_0) \mapsto (x_1, y_1) \mapsto \cdots$, and for every continuous test function $\psi : C \to \mathbb{R}$, the time average
\[
\frac{1}{n} \left( \sum_{i=0}^{n-1} \psi(x_i, y_i) \right)
\] converges to the space average $\int_C \psi(x, y) \, d\nu(x, y)$ as $n \to \infty$. Briefly: Almost every orbit is *uniformly distributed* with respect to the measure $\nu$.
Theorem 3. If $Sf_x(y) > 0$ almost everywhere, and if $\text{Lyap}(A_\iota) > 0$ for both $A_\iota$, then $F$ has a (necessarily unique) asymptotic measure.

Proof Outline. Let $\mathcal{G}_k$ be the solenoid consisting of all full orbits

$$\cdots \mapsto x_{-2} \mapsto x_{-1} \mapsto x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$$

under the $k$-tupling map.

Then $F$ lifts to a homeomorphism $\widetilde{F}$ of $\mathcal{G}_k \times I$.

Here $\widetilde{F}$ maps fibers to fibers with $S > 0$. Therefore $\widetilde{F}^{-1}$ maps fibers to fibers with $S < 0$.

Hence we can apply the argument of Theorem 1 to $\widetilde{F}^{-1}$.
In particular, there is an almost everywhere defined measurable section

$$\sigma : \mathcal{G}_k \rightarrow \mathcal{G}_k \times I$$

which separates the basins of $\mathcal{G}_k \times 0$ and $\mathcal{G}_k \times 1$ under $\tilde{F}^{-1}$.

Let $\tilde{\nu}$ be the push-forward under $\sigma$ of the standard shift-invariant probability measure on $\mathcal{G}_k$. Thus $\tilde{\nu}$ is an $\tilde{F}$-invariant probability measure on $\mathcal{G}_k \times I$.

**Assertion:** $\tilde{\nu}$ is an asymptotic measure for $\tilde{F}$.

Since almost all points are pushed away from the graph of $\sigma$ by the inverse map $\tilde{F}^{-1}$, it follows that they are pushed towards this graph by the map $\tilde{F}$.

Now push $\tilde{\nu}$ forward under the projection from $\mathcal{G}_k \times I$ to $\mathcal{C} = (\mathbb{R}/\mathbb{Z}) \times I$,

This yields the required asymptotic measure for $F$. \qed
Let \( F(x, y) = \left( kx, q_{\epsilon \cos(2\pi x)}^{-1}(y) \right) \).

50000 points of a randomly chosen orbit for \( F \).
§3. The Hard Case: Zero Schwarzian

Suppose that each orientation preserving diffeomorphism $f_x : I \to I$ has Schwarzian $Sf_x$ identically zero. Such a map is necessarily fractional linear, and can be written as

$$ y \mapsto ay \frac{1}{1 + (a - 1)y} \quad \text{with} \quad a > 0. \quad (2) $$

Here $a = a(x) = f_x'(0)$ is the derivative with respect to $y$ at $y = 0$. Note that each $f_x$ preserves the Poincaré distance

$$ d(y_1, y_2) = \left| \log \rho(0, y_1, y_2, 1) \right|. $$

Hence, by a change of variable, we can transform this fractional linear transformation of the open interval into a translation of the real line: Replace $y$ by the Poincaré arclength coordinate

$$ t(y) = \log \rho(0, 1/2, y, 1) = \log \frac{y}{1 - y}. $$

The map (2) then corresponds to the translation

$$ t \mapsto t + \log a. \quad (3) $$
A Pseudo-Random Walk.

Using this change of coordinate, the skew product map

$$(x, y) \mapsto (kx, f(x))$$

on $(\mathbb{R}/\mathbb{Z}) \times I$ takes the form

$$(x, t) \mapsto (kx, t + \log a(x)),$$

mapping $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ to itself.

Think of the $k$-tupling map as generating a sequence of pseudo-random numbers

$$\log a(x), \log a(kx), \log a(k^2x), \ldots.$$

Then the resulting sequence of $t$ values can be described as a "pseudo-random walk" on the real line. The condition that

$$\text{Lyap}(A_0) = \int_{\mathbb{R}/\mathbb{Z}} \log (a(x)) \, dx = 0$$

means that this pseudo-random walk is unbiased.
Suppose that $Sf_y \equiv 0$, with $\text{Lyap}(A_0) = \text{Lyap}(A_1) = 0$, and with $f_x(y) \not\equiv y$, then we conjecture that almost every orbit comes within any neighborhood of $A_0$ infinitely often, but also within any neighborhood of $A_1$ infinitely often, on such an irregular schedule that there can be no asymptotic measure!

More precisely, for almost every orbit

$$(x_1, y_1) \mapsto (x_2, y_2) \mapsto (x_3, y_3) \mapsto \cdots,$$

we have

$$\liminf \frac{y_1 + \cdots + y_n}{n} = 0 \quad \text{and} \quad \limsup \frac{y_1 + \cdots + y_n}{n} = 1.$$
The corresponding statement is known to be true for an honest random walk on $\mathbb{R}$, where the successive steps sizes are independent random variables with mean zero.

Conjecturally, our pseudo-random walk must behave enough like an actual random walk so that this behavior will persist.

THE END