Is Entropy Effectively Computable?

Given an explicit dynamical system and given $\epsilon > 0$, is it possible in principle to compute the associated entropy, either topological or measure-theoretic, with a maximum error of ϵ? In practice, is there an effective procedure to carry out this computation in a reasonable length of time? In the most general case, the answer to both questions is certainly no: Cellular automaton mappings from a Cantor set (namely a full shift) to itself have an explicit finite description, yet Hurd, Kari and Culik have shown that the associated topological entropy is not algorithmically computable in general. For iterated smooth mappings in dimension ≥ 2, or for smooth diffeomorphisms in dimension ≥ 3, Misiurewicz has pointed out that topological entropy does not always depend continuously on parameters.[1] This suggests that computation may be very difficult. On the other hand, for piecewise monotone interval mappings, perhaps the simplest interesting dynamical systems, there is an effective computation which depends only on being able to order finitely many forward images of the critical points. A proof is sketched in [19, §5.10], based on [1]. (Compare [14]. For unimodal or bimodal maps, the most efficient procedure is based on comparison with constant slope maps. Compare [4], [5], as well as [18].)

One quite general computational method, based on the exponential growth of length or volume, has been studied by Newhouse and Pignataro [22] (see also [21], [25]). As an example, they tabulate some entropy estimates for the Hénon family, but without any precise error bounds.

Diffeomorphisms of dimension two provide a rich family of reasonably stable examples with a great deal of available theory. (Compare [3], [6]-[13], [15], [17], [24].) Thus it seems natural to ask whether topological entropy can be effectively computed in this case. For orientation preserving diffeomorphisms F of the 2-sphere, every finite invariant set $S = F(S)$ with n elements determines a class β_S of elements in the n-stranded braid group. There is a minimum possible topological entropy $h_{\text{top}}(\beta)$ associated with any such braid-class; and an effective computation for this associated entropy has been given by Bestvina and Handel. The topological entropy $h_{\text{top}}(F)$ can be described as the supremum, over all finite F-invariant sets, of these braid-entropies.[2] Thus one way of looking for good lower bounds for $h_{\text{top}}(F)$ would be to search for periodic orbits and then compute the associated $h_{\text{top}}(\beta_S)$. It seems likely that one could find upper bounds which are good enough to prove that $h_{\text{top}}(F)$ is Turing computable; although it is not at all certain that one could find an algorithm which is fast enough to be useful. For other related ideas towards computation, see [10].

There are two well known families of 2-dimensional diffeomorphisms, namely the Hénon family on $\mathbb{R}^2$, and the "standard family" of torus diffeomorphisms. Either of these would provide excellent test cases.

For diffeomorphisms which preserve some standard area form, one can ask the same question about measure-theoretic entropy. Again Hénon maps and standard family maps

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[1] Compare [20] (but see also [25], [21]). One simple example is the family of maps $f_t(z) = tz^2$ from the closed unit disk to itself, with $h_{\text{top}}(f_1) > 0$, but $h_{\text{top}}(f_t) = 0$ for $|t| < 1$.

[2] This is proved in [6, Theorem 9.3], using [17]. However, in the case of *homeomorphisms*, Mary Rees has given an example on T^2 with $h_{\text{top}} > 0$, but with no periodic orbits.

seem like ideal objects to study.[3] The area preserving Hénon case (compare [12]) is harder to deal with, since to define $h(F)$ it is necessary to restrict F to the union $K(F)$ of all bounded orbits, and to require that $K(F)$ have positive area. Again, the question is whether entropy can be computed (in theory, and if possible in practice) up to an error which can be made arbitrarily small. According to Pesin, the measure-entropy of F can be computed as the limit as $n \to \infty$ of $1/n$ times the average of $\log \|DF^{\circ n}\|$. (Compare [2].) For torus diffeomorphisms, and probably also for area preserving Hénon maps, this gives a sequence of effectively computable upper bounds. However, I am not aware of any effective lower bound.

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[3] Both area preserving Hénon maps and standard maps can be put in the form

$$F(x_{n-1}\,,\,x_n) \;=\; (x_n\,,\,x_{n+1})\;, \qquad \text{where} \qquad x_{n-1}+x_{n+1}=f(x_n)\;.$$

For the Hénon case, x_n ranges over $\mathbb{R}$ and $f(x)$ is a non-linear polynomial function such as x^2+c ; while for the standard family, using a normal form due to B. V. Chirikov, one takes $x \in \mathbb{R}/\mathbb{Z}$ with $f(x) = 2x + k\,\sin(2\pi x)$. (Other trigonometric functions, such as $f(x) = a + b\sin(2\pi x)$ would surely also be of interest.) There are a number of web sites describing standard maps. See for example www.dynamical-systems.org , which includes numerical measure-entropy computations and a conjectured lower bound $h(F) \geq \log(\pi|k|)$, and also www.expm.t.u-tokyo.ac.jp/∼kanamaru/Chaos/ .

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