Understanding Cubic Maps

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TOULOUSE, DECEMBER 2009
Parameter Space

PROBLEM:  To study cubic polynomial maps \( F \) with a critical point which is periodic under \( F \).

—work with Araceli Bonifant and Jan Kiwi—

Normal form:

Any cubic polynomial map is affinely conjugate to a monic centered map

\[
F(z) = F_{a,v}(z) = (z - a)^2(z + 2a) + v.
\]

Here \( a \) is the marked critical point, and \( F(a) = v \) is the marked critical value.

The parameter space for the family of all such maps is the set of all pairs \( (a, v) \in \mathbb{C}^2 \).
The Period $p$ Curve

**Definition.** The **period $p$ curve** $S_p$ consists of those parameter pairs $(a, v) \in \mathbb{C}^2$ such that that marked critical point $a$ for $F = F_{a,v}$ has period exactly $p$.

(Conjecture: $S_p$ is irreducible for all $p \geq 1$.)

Degree computation: The set of parameter pairs $(a, v)$ which satisfy the polynomial equation

$$F^{\circ p}(a) = a$$

forms a smooth affine variety

$$S_p^\oplus = \bigsqcup_{n \mid p} S_n \subset \mathbb{C}^2.$$

Equation (1) has degree $3^{p-1}$. Hence the degree $d_p$ of $S_p$ can be computed inductively from the equation

$$\sum_{n \mid p} d_n = 3^{p-1}.$$

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 8, \quad d_4 = 24, \quad d_5 = 80, \quad \ldots.$$
Canonical Coordinates for $S_p$

Define $H_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$H_p(a, v) = F^p(a) - a, \quad \text{with} \quad F = F_{a,v}.$$ 

This vanishes everywhere on $S_p$, with $dH_p \neq 0$ on $S_p$.

Think of $H_p$ as a complex Hamiltonian function, and consider the Hamiltonian differential equation

$$\frac{da}{dt} = \frac{\partial H_p}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H_p}{\partial a}.$$ 

The local solutions $t \mapsto (a, v) = (a(t), v(t))$ are holomorphic, and lie in curves $H_p = \text{constant}$.

Those solutions which lie in $S_p$ provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Equivalently, the holomorphic 1-form

$$dt = \frac{da}{\partial H_p/\partial v} \quad \text{and/or} \quad -\frac{dv}{\partial H_p/\partial a}$$

is well defined and non-zero everywhere on $S_p$. 

More generally, any smooth affine curve $S \subset \mathbb{C}^2$ has such a canonical 1-form $dt$.

Such a curve can be decomposed (non-uniquely) into a compact subset, together with finitely many end regions $\mathcal{E}_h$, each conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$.

We can compactify, to obtain a smooth compact complex 1-manifold $\overline{S}$, by adding a single ideal point $\infty_h$ to each end region $\mathcal{E}_h$. 
The holomorphic 1-form $dt$ on $S$ becomes a meromorphic 1-form on $\overline{S}$, with zeros or poles only at the ideal points.

The **Euler characteristic** of $\overline{S}$ can be computed as follows:

$$\chi(\overline{S}) = \#(\text{poles}) - \#(\text{zeros}),$$

counted with multiplicity.

If $S$ is connected, then

$$\text{genus}(S) = \text{genus}(\overline{S}) = 1 - \chi(\overline{S})/2.$$
Special properties of the period $p$ curve

There is a dynamically defined compact subset of $S_p$, namely the **connectedness locus** $C(S_p)$ consisting of all maps $F \in S_p$ such that the Julia set $J(F)$ is connected.

Each connected component $\mathcal{E}_h \subset S_p \setminus C(S_p)$, called an **escape region** in $S_p$, is conformally isomorphic to $\mathbb{C} \setminus \overline{D}$. 
The winding number

**Theorem.** The residue of $dt$ at each ideal point $\infty_h \in \overline{S}_p$ is zero:

$$\frac{1}{2\pi i} \oint_{\infty_h} dt = 0.$$ 

Thus $t$ can be defined as a meromorphic function throughout any simply connected subset of $\overline{S}_p$.

**Normal form near an ideal point $\infty_h$:** We can choose a local parameter $\zeta$ for $\overline{S}_p$, and a canonical parameter $t$, so that

$$t = \zeta^{w_h}, \quad \text{with} \quad w_h \in \mathbb{Z}, \quad w_h \neq 0.$$ 

Here $w_h$ is the **winding number** of the $t$-plane around $\infty_h$.

As $\zeta \to 0$, note that $t \to \begin{cases} 0 & \text{if } w > 0, \\ \infty & \text{if } w < 0. \end{cases}$
Winding number: examples in $S_4$
Euler characteristic formulas

Since \( t = \zeta^{wh} \),

\[
dt = d(\zeta^{wh}) = w_h \zeta^{wh-1} d\zeta,
\]

with a zero of order \( w_h - 1 \) at the ideal point.

Thus the formula \( \chi = \#(\text{poles}) - \#(\text{zeros}) \) takes the form

\[
\chi(\overline{S}_p) = \sum_h (1 - w_h),
\]

summed over all ideal points. With a lot of work, this yields

\[
\chi(\overline{S}_p) = (2 - p)d_p + \text{(number of ideal points)}.
\]

(Key tool for the proof:
Kiwi’s theory of dynamics, including Branner-Hubbard puzzles,
over the completion of the field of formal Puiseux series.)

Examples:

\[
\chi(S_1) = \chi(S_2) = 2, \quad \chi(S_3) = 0, \quad \chi(S_4) = -28.
\]
Further computations of $\chi(\overline{S}_p)$ by Laura DeMarco

- Period 5: -184
- Period 6: -784
- Period 7: -3236
- Period 8: -11848
- Period 9: -42744
- Period 10: -147948
- Period 11: -505876
- Period 12: -1694848
- Period 13: -5630092
- Period 14: -18491088
- Period 15: -60318292
- Period 16: -195372312
- Period 17: -629500300
- Period 18: -2018178780
- Period 19: -6443997852
- Period 20: -20498523320
Filled Julia set for a map in the “rabbit” escape region of $S_3$. 
Sketch of the dynamic plane for a map belonging to any escape region $\mathcal{E}_h \subset S_p$.

Critical points: $a, -a$.

Cocritical points: $2a, -2a$, with $F(\pm2a) = F(\mp a)$.

Definition: $\theta = \theta(F) \in \mathbb{R}/\mathbb{Z}$ is the **cocritical angle**.
Rays in parameter space: Examples in $S_2$.

The indicated rays all land at parabolic maps, and have angles of the form $m/3n$. 
Each external ray in an escape region $\mathcal{E}_h \subset S_p$ is labeled by its cocritical angle $\theta(F) \in \mathbb{R}/\mathbb{Z}$.

**Theorem.** Every parameter ray with rational cocritical angle $\theta$ lands at a well defined map $F_0$ in the topological boundary $\partial \mathcal{E}_h \subset S_p$.

This landing map $F_0$ has a parabolic orbit

$\iff$ one of the two angles $\theta \pm 1/3$ is periodic.

$\iff$ $\theta$ has the form $\frac{m}{3n}$ with $3 \nmid m$ and $3 \nmid n$.

Complication: For each $\theta$, there are $\mu_h$ distinct parameter rays in $\mathcal{E}_h$ with label $\theta$, where $\mu_h \geq 1$ is an invariant called the **multiplicity** of $\mathcal{E}_h$. 
More examples in $S_2$
Here $F_0$ is the landing map for the 10/24, 11/24, 14/24, and 17/24 rays at the upper left of the previous figure.
Critically finite maps

Theorem: If the landing map for a rational parameter ray is not parabolic, then it is critically finite. **An example in** \( S_4 \):

The same rays land at \( F \in S_p \) as at \( 2a_F \in J(F) \).
Asymptotic similarity (as in Tan Lei)

$F \in S_p$ critically finite map, $\eta = \text{multiplier of postcritical cycle}$.

Koenigs: There is a Hausdorff limit $\lim_{n \to \infty} \eta^n (K(F) - 2a)$.

Linear equivalence: $\cong \lim_{n \to \infty} \eta^n (C(S_p) - F)$

(interpreting last expression using a local parameter).

Julia set parameter space (in $S_3$)

*Cubic Polynomial Maps with Periodic Critical Orbit:*


*Part III: External rays* (with Bonifant), in preparation.