Critically Periodic Cubic Polynomials

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IN MEMORY OF ADRIEN DOUADY

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THE PROBLEM: To study cubic polynomial maps $F$ with a marked critical point which is periodic under $F$.

—work in progress with Araceli Bonifant—

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) = F_{a,v}(z) = z^3 - 3a^2z + (2a^3 + v).$$

Here $a$ is the marked critical point, $F(a) = v$ is the marked critical value, $-a$ is the free critical point.

The set of all such maps $F = F_{a,v}$ will be identified with the parameter space, consisting of all pairs $(a, v) \in \mathbb{C}^2$. 

Parameter Space
Definition: the period p curve $S_p \subset \mathbb{C}^2$, consists of all maps $F = F_{a,v}$ such that the marked critical point $a$ has period exactly $p$.

Assertion. $S_p$ is a smooth affine curve in $\mathbb{C}^2$.

Complication: The genus of $S_p$ increases rapidly with $p$.

- $S_1$ has genus zero with one puncture ($\cong \mathbb{C}$),
- $S_2$ has genus zero with two punctures,
- $S_3$ has genus one with 8 punctures,
- $S_4$ has genus 15 with 20 punctures, ... 

We can simplify a little by passing to the moduli space $S_p/I$ of holomorphic conjugacy classes. Here $I$ is the involution $F(z) \leftrightarrow -F(-z)$, so that $F_{a,v} \leftrightarrow F_{-a,-v}$.

The genus of $S_p/I$ is smaller, but still increases with $p$. 
Picture of Part of $S_3$
Part of $S_3$, labeled
Let $\overline{S}_p$ be the smooth compact surface obtained from $S_p$ by filling in each puncture point.

**Conjecture.** There is a canonical cell subdivision of each $\overline{S}_p$. For $p \geq 2$, the 1-skeleton can be identified with the union of all simple closed regulated curves.

Sketch of a regulated curve:
Let $\mathcal{C}(S_p)$ be the **connectedness locus** in $S_p$.

Each connected component $\mathcal{E}$ of the complement $S_p \setminus \mathcal{C}(S_p)$ will be called an **escape region** in $S_p$.

**Theorem.** For each $\mathcal{E}$, there is a canonical covering map

$$\mathcal{E} \to \mathcal{C} \setminus \mathbb{D}.$$  

The degree of this covering map will be called the **multiplicity** $\mu \geq 1$ of the escape region.

We can talk about **equipotentials** and **parameter rays** in each escape region.

**Notation:** A parameter ray in the escape region $\mathcal{E}$ will be denoted by $\mathcal{R}_\mathcal{E}(t)$. Here $t \in \mathbb{R}/\mu \mathbb{Z}$.

If $\mu > 1$, then $t$ will be called a **generalized angle**.
For $F$ in the escape region $\mathcal{E}$, the equipotential through $2a$ and $-a$ is a figure eight curve. Here $2a$ is the free \textbf{cocritical point}, with $F(2a) = F(-a)$.

The Böttcher coordinate $\beta(2a) \in \mathbb{C} \setminus \overline{D}$ of the escaping cocritical point is well defined, and the correspondence $F \mapsto \beta(2a)$ is the required covering map

$$\mathcal{E} \rightarrow \mathbb{C} \setminus \overline{D}.$$
Let $U_0$ and $U_1$ be the two bounded regions cut out by the figure eight curve, with $a \in U_0$. Any bounded orbit $z_1 \mapsto z_2 \mapsto \cdots$ determines a sequence $\sigma_1, \sigma_2, \ldots$ of zeros and ones with

$$z_j \in U_{\sigma_j}.$$ 

Now take $z_1$ equal to the marked critical value $v = F(a)$. The associated sequence $\{\sigma_j\}$ will be called the kneading sequence of the escape region $\mathcal{E}$. Thus

$$F^{\circ j}(a) \in U_{\sigma_j} \quad \text{for } j \geq 1.$$
The Associated Quadratic Map.

The kneading sequence of any escape region $E \subset S_p$ is clearly periodic: its period $p_1$ divides $p$.

**Theorem (Branner and Hubbard).** Suppose that $F$ belongs to the escape region $E \subset S_p$. Then the Julia set $J(F)$ consists of countably many copies of a quadratic Julia set $J(Q)$, together with uncountably many single point components. Here the quadratic polynomial $Q = Q_E$ is critically periodic of period $p_2$ where

$$p = p_1 p_2.$$ 

In other words:

Period of marked critical point

$= \text{(kneading period)} \times \text{(associated quadratic period)}$. 
Here the kneading sequence is $00$, and the associated quadratic map is $z^2 - 1$ (the “basilica”).

Kneading sequence $10$, with associated quadratic $z^2$. 
Canonical Coordinates for $S_p$.

Consider the function

\[ H_p : \mathbb{C}^2 \to \mathbb{C}, \quad H_p(a, v) = F_{a,v}^p(a) - a \]

which vanishes everywhere on $S_p$. Think of $H_p$ as a “complex Hamiltonian function”, and consider the Hamiltonian differential equation

\[
\frac{da}{dt} = \frac{\partial H_p}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H_p}{\partial v}.
\]

There are holomorphic local solutions

\[ t \mapsto (a, v) = \Phi(t). \]

These lie in curves $H_p = \text{constant}$, parallel to $S_p$. Those solutions which lie in $S_p$ provide a local holomorphic parametrization, unique up to translation of the $t$-coordinate.

**Equivalent description:** There is a canonical 1-form $dt$ which is well defined and non-zero throughout $S_p$. 
Part of $S_4$ in canonical coordinates
Kneading sequence 1010 ⋯, with period $p_1 = 2$. $Q(z) = z^2 - 1$ with critical period $p_2 = 2$. 
Example in the Double-Basilica Region.

Kneading sequence $0000 \cdots$, with period $p_1 = 1$. $Q(z) = z^2 - 1.3107 \ldots$ with critical period $p_2 = 4$. 
Quadratic Julia sets:

Double-Basilica

Worm
Two More Quadratic Julia Sets

Kokopelli

(1/4)-Rabbit
Comparing Rays in the Mandelbrot Set

$H(1/3)$

$1/5$

$1/7$

$4/15$

$2/7$
Parameter Rays

Let $\mathcal{E} \subset S_p$ be any escape region.

**Theorem.** If the generalized angle $t_0$ is rational, then the ray $\mathcal{R}_\mathcal{E}(t_0)$ lands at a well defined point $F_0$ in the boundary $\partial \mathcal{E}$. Furthermore, $F_0$ is either critically finite, or parabolic.

Define $t \in \mathbb{Q}/\mathbb{Z}$ to be **co-periodic** if:

$t \pm 1/3$ is periodic under angle tripling,

$\Leftrightarrow$ $3t$ is periodic but $t$ is not periodic,

$\Leftrightarrow$ $t$ has the form $\frac{m}{3n}$ where $m$ and $n$ are not divisible by 3.

**Theorem.** If $t_0 \pmod{\mathbb{Z}}$ is co-periodic, then the landing point of $\mathcal{R}_\mathcal{E}(t_0)$ is parabolic.

We believe that this should be an if and only if statement:

$t_0$ co-periodic $\Leftrightarrow$ the landing point is parabolic.
If \( t \pm 1/3 \) has period \( q \), we say that \( t \) has \textbf{co-period} \( q \).

Note that any angle of co-period \( q \) can be written as a fraction

\[
    t = \frac{m}{3(3^q - 1)}.
\]

For example,

\[
    q = 1 \quad \Rightarrow \quad t = m/6,
\]

\[
    q = 2 \quad \Rightarrow \quad t = m/24.
\]

\textbf{Period} \( q \) \textbf{decomposition:} The collection of all rays of co-period \( q \), together with their landing points, decomposes the parameter curve \( S_p \) into a finite number of connected open sets \( U_j \).
Example: The Period 1 Decomposition of $S_2$.  

20.
Period 2 Decomposition of $S_2$.
Stability of Periodic Orbits.

Let \( U_j \) be any connected component of

\[ S_p \setminus \bigcup \text{rays of coperiod } q, \]

and let \( t_0 \in \mathbb{Q}/\mathbb{Z} \) have period \( q \).

As \( F \) varies over \( U_j \), the dynamic ray \( R_F(t_0) \) varies smoothly:

**Theorem.** For each \( F \in U_j \), and each angle \( t_0 \in \mathbb{Q}/\mathbb{Z} \) of period \( q \), the ray \( R_F(t_0) \) lands at a repelling periodic point \( z_F \in J(F) \subset \mathbb{C} \).

Furthermore, the correspondence \( F \mapsto z_F \) defines a holomorphic function \( U_j \to \mathbb{C} \).

The pattern of which dynamic rays of period \( q \) have a common landing point is the same for all \( F \in U_j \).

**Corollary.** Every parabolic map \( F_0 \in S_p \) is the landing point of at least one co-periodic ray.
Orbit Portraits for $F \in S_2$ (Periods 1 and 2).
A Small Mandelbrot Set in $S_4$
Detail of $J(F_0)$ near $2a$
(Empirical Claims)

Every Mandelbrot component $M \subset S_p$ has a well defined root point $F_0$, and every parabolic point $F_0 \in S_p$ is the root point of a unique Mandelbrot component $M \subset S_p$.

For $F \in M$, let $r_0$ be the root point of the Fatou component $U(2a)$ containing the cocritical point $2a$. Then a neighborhood of $F_0$ in $S_p$ is closely related to a neighborhood of $r_0$ in the dynamic plane for $F$. More precisely:

- The two closest parameter rays at $F_0$ which enclose $M$ have the same angles (modulo $\mathbb{Z}$) as the two closest dynamic rays at $r_0$ which enclose $U(2a)$.

- Furthermore, any parameter ray landing at $F_0$ has the same angle (modulo $\mathbb{Z}$) as some dynamic ray landing at $r_0$. 

Comparing Parameter Space and Julia Set
A Small Mandelbrot Set in $S_5$
Detail of corresponding Julia Set