The Poincaré Conjecture 99 Years Later: A Progress Report

The topology of 2-dimensional manifolds or *surfaces* was well understood in the 19th century.[1] In fact, there is a simple list of all possible smooth compact orientable surfaces. Any such surface has a well-defined *genus* $g \geq 0$, which can be described intuitively as the number of holes; two such surfaces can be put into a smooth one-to-one correspondence with each other if and only if they have the same genus.

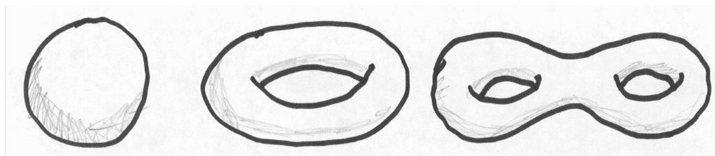

Figure 1. Sketches of smooth surfaces of genus 0, 1, and 2.

The corresponding question in higher dimensions is much more difficult. Henri Poincaré was perhaps the first to try to make a similar study of 3-dimensional manifolds. The most basic example of such a manifold is the 3-dimensional *unit sphere*, that is, the locus of all points (x, y, z, w) in 4-dimensional Euclidean space which have distance exactly 1 from the origin:

$$x^2 + y^2 + z^2 + w^2 = 1 .$$

He noted that a distinguishing feature of the 2-dimensional sphere is that every simple closed curve in the sphere can be deformed continuously to a point without leaving the sphere. In 1904, he asked a corresponding question in dimension three.[2] In more modern language, it can be phrased as follows:

> *If a smooth compact 3-dimensional manifold M^3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3 ?*

He commented, with considerable foresight, "*Mais cette question nous entraînerait trop loin*". Since then, the hypothesis that every simply connected closed 3-manifold is homeomorphic to the 3-sphere has been known as the *Poincaré Conjecture.* It has inspired topologists ever since, and attempts to prove it have led to many advances in our understanding of the topology of manifolds.[3]

Early Missteps.

From the first, the apparently simple nature of this statement has led mathematicians to overreach. Four years earlier, in 1900, Poincaré himself had been the first to err, stating a false theorem that can be phrased as follows.[4]

> *Every compact polyhedral manifold with the homology of an n-dimensional sphere is actually homeomorphic to the n-dimensional sphere.*

[1] For definitions and other background material, see for example Massey, or Munkres 1975, as well as Thurston 1997. (Names in small caps refer to the list of references at the end.)

[2, 4] See Poincaré, pages 486, 498; and also 370.

[3] For a representative collection of attacks on the Poincaré Conjecture, see Papakyriakopoulos, Birman, Jakobsche, Thickstun, Gillman and Rolfsen, Gabai 1995, Rourke, and Poénaru.

With further study, Poincaré found a beautiful counterexample to his own claim. This example can be described geometrically as follows. Consider all possible regular icosahedra inscribed in the 2-dimensional unit sphere. In order to specify one particular icosahedron in this family, we must provide three parameters. For example, two parameters are needed to specify a single vertex on the sphere, and then another parameter to specify the direction to a neighboring vertex. *Thus each such icosahedron can be considered as a single "point" in the 3-dimensional manifold M^3 consisting of all such icosahedra.*[5] This manifold meets Poincaré's preliminary criterion: By the methods of homology theory, it cannot be distinguished from the 3-dimensional sphere. However, he could prove that it is not a sphere by constructing a simple closed curve that cannot be deformed to a point within M^3. The construction is not difficult: Choose some representative icosahedron and consider its images under rotation about one vertex through angles $0 \le \theta \le 2\pi/5$. This defines a simple closed curve in M^3 that cannot be deformed to a point.

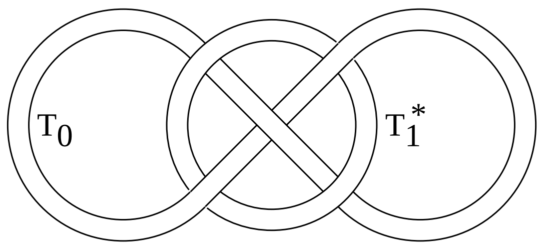

Figure 2. The Whitehead link.

The next important false theorem was by Henry Whitehead in 1934. As part of a purported proof of the Poincaré Conjecture, he claimed that every open 3-dimensional manifold that is *contractible* (that is, can be continuously deformed to a point) is homeomorphic to Euclidean space. Following in Poincaré's footsteps, he then substantially increased our understanding of the topology of manifolds by discovering a counterexample to his own theorem. (See WHITEHEAD, pp. 21–50.) His counterexample can be briefly described as follows. Start with two disjoint solid tori T_0 and T_1^* in the 3-sphere that are embedded as shown in Figure 2, so that each one individually is unknotted, but so that the two are linked together with linking number zero. Since T_1^* is unknotted, its complement $T_1 = S^3 \smallsetminus \mathrm{interior}(T_1^*)$ is another unknotted solid torus that contains T_0. Choose a homeomorphism h of the 3-sphere that maps T_0 onto this larger solid torus T_1. Then we can inductively construct an increasing sequence of unknotted solid tori

$$T_0 \subset T_1 \subset T_2 \subset \cdots$$

in S^3 by setting $T_{j+1} = h(T_j)$. The union $M^3 = \bigcup T_j$ of this increasing sequence is the required Whitehead counterexample, a contractible manifold that is not homeomorphic to Euclidean space. To see that $\pi_1(M^3) = 0$, note that every closed loop in T_0 can be shrunk to a point (after perhaps crossing through itself) within the larger solid torus T_1. But every closed loop in M^3 must be contained in some T_j, and hence can be shrunk to a point within

[5] In more technical language, this M^3 can be defined as the coset space $\mathrm{SO}(3)/\mathrm{I}_{60}$ where $\mathrm{SO}(3)$ is the group of all rotations of Euclidean 3-space and where I_{60} is the subgroup consisting of the 60 rotations which carry a standard icosahedron to itself. The fundamental group $\pi_1(M^3)$, consisting of all homotopy classes of loops from a point to itself within M^3, is a perfect group of order 120.

$T_{j+1} \subset M^3$. On the other hand, M^3 is not homeomorphic to Euclidean 3-space since, if $K \subset M^3$ is any compact subset which is large enough to contain T_0 , one can check that the difference set $M^3 \setminus K$ is not simply connected.

Since this time, many false proofs of the Poincaré Conjecture have been proposed, some of them relying on errors that are rather subtle and difficult to detect. For a delightful presentation of some of the pitfalls of 3-dimensional topology, see Bing.

Higher Dimensions.

The late 1950s and early 1960s saw an avalanche of progress with the discovery that higher dimensional-manifolds are actually easier to work with than 3-dimensional ones. One reason for this is the following: The fundamental group plays an important role in all dimensions even when it is trivial, and relations between generators of the fundamental group correspond to 2-dimensional disks, mapped into the manifold. In dimension 5 or greater, such disks can be put into general position so that they are disjoint from each other, with no self-intersections, but in dimension 3 or 4 it may not be possible to avoid intersections, leading to serious difficulties.

Stephen Smale announced a proof of the Poincaré Conjecture in high dimensions in 1960. He was quickly followed by John Stallings, who used a completely different method, and by Andrew Wallace, who had been working along lines quite similar to those of Smale.

Let me first describe the Stallings result, which has a weaker hypothesis and easier proof, but also a weaker conclusion. He assumed that the dimension is 7 or greater, but Christopher Zeeman later extended his argument to dimensions 5 and 6.

> **Stallings-Zeeman Theorem.** *If* M^n *is a finite simplicial complex of dimension* $n \geq 5$ *which has the homotopy type*[6] *of the sphere* S^n *and is locally piecewise linearly homeomorphic to the Euclidean space* $\mathbb{R}^n$ *, then* M^n *is homeomorphic to* S^n *under a homeomorphism which is piecewise linear except at a single point. In other words, the complement* $M^n \setminus (\text{point})$ *is piecewise linearly homeomorphic to* $\mathbb{R}^n$ *.*

(The method of proof consists of pushing all of the difficulties off toward a single point; hence there can be no control near that point.)

The Smale proof, and the closely related proof given shortly afterward by Wallace, depended rather on differentiable methods, building a manifold up inductively, starting with an n-dimensional ball, by successively adding handles. Here a k-*handle* can be added to a manifold M^n with boundary by first attaching a k-dimensional cell, using an attaching homeomorphism from the $(k-1)$-dimensional boundary sphere into the boundary of M^n , and then thickening and smoothing corners to obtain a larger manifold with boundary. The proof is carried out by rearranging and canceling such handles. (Compare the presentation in Milnor, Siebenmann and Sondow.)

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[6] In order to check that a manifold M^n has the same "homotopy type" as the sphere S^n , we must check not only that it is simply connected, $\pi_1(M^n) = 0$, but also that it has the same homology as the sphere. The example of the product $S^2 \times S^2$ shows that it is not enough to assume that $\pi_1(M^n) = 0$ when $n > 3$.

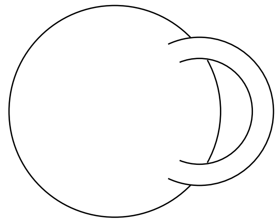

Figure 3. A 3-dimensional ball with a 1-handle attached.

Smale Theorem. *If M^n is a differentiable homotopy sphere of dimension $n \geq 5$, then M^n is homeomorphic to S^n. In fact M^n is diffeomorphic to a manifold obtained by gluing together the boundaries of two closed n-balls under a suitable diffeomorphism.*

This was also proved by Wallace, at least for $n \geq 6$. (It should be noted that the 5-dimensional case is particularly difficult.)

The much more difficult 4-dimensional case had to wait twenty years, for the work of Michael Freedman. Here the differentiable methods used by Smale and Wallace and the piecewise linear methods used by Stallings and Zeeman do not work at all. Freedman used wildly non-differentiable methods, not only to prove the 4-dimensional Poincaré Conjecture for topological manifolds, but also to give a complete classification of all closed simply connected topological 4-manifolds. The integral cohomology group H^2 of such a manifold is free abelian. Freedman needed just two invariants: The cup product $\beta : H^2 \otimes H^2 \to H^4 \cong \mathbb{Z}$ is a symmetric bilinear form with determinant ± 1, while the *Kirby-Siebenmann invariant* κ is an integer mod 2 that vanishes if and only if the product manifold $M^4 \times \mathbb{R}$ can be given a differentiable structure.

Freedman Theorem. *Two closed simply connected 4-manifolds are homeomorphic if and only if they have the same bilinear form β and the same Kirby-Siebenmann invariant κ. Any β can be realized by such a manifold. If $\beta(x \otimes x)$ is odd for some $x \in H^2$, then either value of κ can be realized also. However, if $\beta(x \otimes x)$ is always even, then κ is determined by β, being congruent to one eighth of the signature of β.*

In particular, if M^4 is a homotopy sphere, then $H^2 = 0$ and $\kappa = 0$, so M^4 is homeomorphic to S^4. It should be noted that the piecewise linear or differentiable theories in dimension 4 are much more difficult. It is not known whether every smooth homotopy 4-sphere is diffeomorphic to S^4; it is not known which 4-manifolds with $\kappa = 0$ actually possess differentiable structures; and it is not known when this structure is essentially unique. The major results on these questions are due to Simon Donaldson. As one indication of the complications, Freedman showed, using Donaldson's work, that $\mathbb{R}^4$ admits uncountably many inequivalent differentiable structures. (Compare Gompf.)

In dimension 3, the discrepancies between topological, piecewise linear, and differentiable theories disappear (see Hirsch, Munkres 1960, and Moise). However, difficulties with the fundamental group become severe.

The Thurston Geometrization Program.

In the 2-dimensional case, each smooth compact surface can be given a beautiful geometrical structure, as a round sphere in the genus 0 case, as a flat torus in the genus 1 case, and as a surface of constant negative curvature when the genus is 2 or more. A far-reaching conjecture by William THURSTON in 1983 claims that something similar is true in dimension 3. His conjecture asserts that every compact orientable 3-dimensional manifold can be cut up along 2-spheres and tori so as to decompose into essentially unique pieces, each of which has a simple geometrical structure. There are eight possible 3-dimensional geometries in Thurston's program. Six of these are now well understood,[7] and there has been a great deal of progress with the geometry of constant negative curvature.[8] However, the eighth geometry, corresponding to constant positive curvature, remains largely untouched. For this geometry, we have the following extension of the Poincaré Conjecture.

> **Thurston Elliptization Conjecture.** *Every closed 3-manifold with finite fundamental group has a metric of constant positive curvature, and hence is homeomorphic to a quotient* S^3/Γ *, where* $\Gamma \subset \mathrm{SO}(4)$ *is a finite group of rotations that acts freely on* S^3 *.*

The Poincaré Conjecture corresponds to the special case where the group $\Gamma \cong \pi_1(M^3)$ is trivial. The possible subgroups $\Gamma \subset \mathrm{SO}(4)$ were classified long ago by HOPF, but this conjecture remains wide open.

Approaches through Differential Geometry and Differential Equations.

In recent years there have been several attacks on the geometrization problem (and hence on the Poincaré Conjecture) based on a study of the geometry of the infinite-dimensional space consisting of all Riemannian metrics on a given smooth 3-dimensional manifold.

By definition, the length of a path γ on a Riemannian manifold is computed, in terms of the *metric tensor* g_{ij} , as the integral $\int_\gamma ds = \int_\gamma \sqrt{\sum g_{ij}dx^i dx^j}$. From the first and second derivatives of this metric tensor, one can compute the *Ricci curvature tensor* R_{ij} , and the *scalar curvature* R . (As an example, for the flat Euclidean space one gets $R_{ij} = R = 0$, while for a round 3-dimensional sphere of radius r , one gets Ricci curvature $R_{ij} = 2g_{ij}/r^2$ and scalar curvature $R = 6/r^2$.)

One approach, due to Michael ANDERSON and building on earlier work by Hidehiko YAMABE, studies the *total scalar curvature*

$$\mathcal{S} \;=\; \iiint_{M^3} R\, dV$$

as a functional on the space of all smooth unit volume Riemannian metrics. The critical points of this functional are the metrics of constant curvature.

Another approach, due to Richard HAMILTON, studies the *Ricci flow*, that is, the solutions to the differential equation

$$\frac{dg_{ij}}{dt} \;=\; -2R_{ij} \;.$$

In other words, the metric is required to change with time so that distances decrease in directions of positive curvature. This is essentially a parabolic differential equation and

[7] See, for example, GORDON AND HEIL, AUSLANDER AND JOHNSON, SCOTT, TUKIA, GABAI 1992, and CASSON AND JUNGREIS.

[8] See SULLIVAN, MORGAN, THURSTON 1986, MCMULLEN, and OTAL. The pioneering papers by HAKEN and WALDHAUSEN provided the basis for much of this work.

behaves much like the heat equation studied by physicists: If we heat one end of a cold rod, then the heat will gradually flow throughout the rod until it attains an even temperature. Similarly, the initial hope for a 3-manifold with finite fundamental group was that, under the Ricci flow, positive curvature would tend to spread out until, in the limit (after rescaling to constant size), the manifold would attain constant curvature.

If we start with a 3-manifold of positive Ricci curvature, Hamilton was able to carry out this program and construct a metric of constant curvature, thus solving a very special case of the Elliptization Conjecture. In the general case there are serious difficulties, since this flow may tend toward singularities. He conjectured that these singularities must have a very special form, however, so that the method could still be used to construct a constant curvature metric.

Three months ago, Grisha Perelman in St. Petersburg posted a preprint describing a way to resolve some of the major stumbling blocks in the Hamilton program and suggesting a path toward a solution of the full Elliptization Conjecture. The initial response of experts to this claim has been carefully guarded optimism, although, in view of the long history of false proofs in this area, no one will be convinced until all of the details have been carefully explained and verified. Perelman is planning to visit the United States in April, at which time his arguments will no doubt be subjected to detailed scrutiny.

I want to thank the many mathematicians who helped me with this report.

John Milnor, Stony Brook University, February 2003

References.

M. T. Anderson, *Scalar curvature and geometrization conjectures for 3-manifolds*, M.S.R.I Publ. 30, 1997.

——, *On long-time evolution in general relativity and geometrization of 3-manifolds*, Comm. Math. Phys. 222 (2001), no. 3, 533–567.

L. Auslander and F.E.A. Johnson, *On a conjecture of C.T.C. Wall*, J. Lond. Math. Soc. **14** (1976) 331–332.

R. H. Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, in "Lectures on Modern Mathematics II", edit. T. L. Saaty, Wiley, 1964.

J. Birman, *Poincaré's conjecture and the homeotopy group of a closed, orientable* 2*-manifold*, Collection of articles dedicated to the memory of Hanna Neumann, VI. J. Austral. Math. Soc. **17** (1974), 214–221.

A. Casson and D. Jungreis, *Convergence groups and Seifert fibered* 3*-manifolds*, Invent. Math. 118 (1994) 441–456.

S. K. Donaldson, *Self-dual connections and the topology of smooth* 4*-manifolds*, Bull. Amer. Math. Soc. **8** (1983) 81–83.

M. H. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.

D. Gabai, *Convergence groups are Fuchsian groups*, Annals of Math. 136 (1992) 447–510.

———, *Valentin Poenaru's program for the Poincaré conjecture*, Geometry, topology, & physics, 139–166, Conf. Proc. Lecture Notes Geom. Topology, VI, Internat. Press, Cambridge, MA, 1995.

D. Gillman and D. Rolfsen, *The Zeeman conjecture for standard spines is equivalent to the Poincaré conjecture*, Topology 22 (1983), no. 3, 315–323.

R. Gompf, *An exotic menagerie*, J. Differential Geom. **37** (1993) 199–223.

C. Gordon and W. Heil, *Cyclic normal subgroups of fundamental groups of 3-manifolds*, Topology 14 (1975) 305–309.

W. Haken, *Über das Homöomorphieproblem der 3-Mannigfaltigkeiten I*, Math. Z. 80 (1962) 89–120.

R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.

———, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.

———, *Non-singular solutions of the Ricci flow on three-manifolds* Comm. Anal. Geom. 7 (1999), no. 4, 695–729.

M. Hirsch, *Obstruction theories for smoothing manifolds and maps*, Bull. Amer. Math. Soc. **69** (1963) 352–356.

H. Hopf, *Zum Clifford-Kleinschen Raumproblem*, Math. Annalen **95** (1925–26) 313–319.

W. Jakobsche, *The Bing-Borsuk conjecture is stronger than the Poincaré conjecture*, Fund. Math. **106** (1980) 127–134.

W. S. Massey, “Algebraic Topology: An Introduction”, Harcourt Brace, 1967; Springer, 1977; or “A Basic Course in Algebraic Topology”, Springer, 1991.

C. McMullen, *Riemann surfaces and geometrization of 3-manifolds*, Bull. Amer. Math. Soc. **27** (1992) 207–216.

J. Milnor with L. Siebenmann and J. Sondow, “Lectures on the h-Cobordism Theorem”, Princeton Math. Notes, Princeton U. Press, 1965.

E. E. Moise, “Geometric Topology in Dimensions 2 and 3”, Springer, 1977.

J. Morgan, *On Thurston’s uniformization theorem for three-dimensional manifolds*, pp. 37–125 of “The Smith Conjecture”, edit. Bass and Morgan, Pure and Appl. Math. 112, Academic Press, 1984.

J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms,* Annals Math. **72** (1960) 521–554.

———, “Topology: A First Course,” Prentice Hall, 1975.

J.-P. Otal, *The hyperbolization theorem for fibered 3-manifolds*, Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, 7. American Mathematical Society, Providence, RI; Socit Mathmatique de France, Paris, 2001.

C. Papakyriakopoulos, *A reduction of the Poincaré conjecture to group theoretic conjectures*, Annals Math. **77** (1963) 250–305.

G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, (preprint available form the Los Alamos National Laboratory archive: math.DG/0211159v1, 11 Nov 2002).

V. Poénaru, *A program for the Poincaré conjecture and some of its ramifications*, Topics in low-dimensional topology (University Park, PA, 1996), 65–88, World Sci. Publishing, River Edge, NJ, 1999.

H. Poincaré, Œuvres, Tome VI, Paris, 1953.

C. Rourke, *Algorithms to disprove the Poincaré conjecture*, Turkish J. Math. **21** (1997) 99–110.

P. Scott, *A new proof of the annulus and torus theorems*, Amer. J. Math. 102 (1980) 241–277.

———, *There are no fake Seifert fibre spaces with infinite* π_1, Annals of Math. 117 (1983) 35–70

——, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. 15 (1983) 401–487.

S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Annals Math. **74** (1961) 391-406. (See also: Bull. Amer. Math. Soc. **66** (1960) 373–375.)

———, *The story of the higher dimensional Poincaré conjecture* (*What actually happened on the beaches of Rio*), Math. Intelligencer **12** (1990) 44–51.

J. Stallings, *Polyhedral homotopy spheres,* Bull. Amer. Math. Soc. **66** (1960) 485–488.

D. Sullivan, *Travaux de Thurston sur les groupes quasi-fuchsiens et sur les variétés hyperboliques de dimension 3 fibrées sur le cercle,* Sém. Bourbaki 554, Lecture Notes Math. 842, Springer, 1981.

T. L. Thickstun, *Open acyclic* 3*-manifolds, a loop theorem and the Poincaré conjecture*, Bull. Amer. Math. Soc. (N.S.) 4 (1981) 192–194.

W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, in "The Mathematical heritage of Henri Poincaré", Proc. Symp. Pure Math. **39** (1983), Part 1. (Also in Bull. Amer. Math. Soc. **6** (1982) 357–381.)

———, *Hyperbolic structures on 3-manifolds, I, deformation of acyclic manifolds* , Annals of Math. **124** (1986) 203–246

———, "Three-Dimensional Geometry and Topology", Vol. 1. edit. Silvio Levy. Princeton Mathematical Series **35**. Princeton University Press, 1997.

P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988) 1–54.

F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Annals of Math. 87 (1968) 56–88.

A. Wallace, *Modifications and cobounding manifolds, II*, J. Math. Mech **10** (1961) 773–809.

J. H. C. Whitehead, "Mathematical Works, Volume II", Pergamon, 1962.

H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960) 21–37.

E. C. Zeeman, *The Poincaré conjecture for* $n \geq 5$, in "Topology of 3-Manifolds and Related Topics" Prentice Hall, 1962. (See also Bull. Amer. Math. Soc. **67** (1961) 270.)