On Lattès Maps

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Dedicated to Bodil Branner.

Abstract. An exposition of the 1918 paper of Lattès, together with its historical antecedents, and its modern formulations and applications.

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§1. Introduction: Quotients of Affine Maps. In 1918, some months before his death of typhoid fever, Samuel Lattès published a brief paper describing an extremely interesting class of rational maps. Similar examples had been described by Schröder almost fifty years earlier (see §7), but Lattès’ name has become firmly attached to these maps, which play a basic role in the holomorphic dynamics literature.

His starting point was the “Poincaré function” \( \theta : \mathbb{C} \to \hat{\mathbb{C}} \) associated with a repelling fixed point \( z_0 = f(z_0) \) of a rational function \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). This can be described as the inverse of the classical Königs linearization around \( z_0 \), extended to a globally defined meromorphic function. (Compare [La], [P], [K], [M3].) Assuming for convenience that \( z_0 \neq \infty \), it is characterized by the identity

\[
f(\theta(t)) = \theta(\mu t)
\]

for all complex numbers \( t \), with \( \theta(0) = z_0 \), and normalized by the condition that \( \theta'(0) = 1 \). Here \( \mu = f'(z_0) \) is the multiplier at \( z_0 \), with \( |\mu| > 1 \). This Poincaré function can be computed explicitly by the formula

\[
\theta(t) = \lim_{n \to \infty} f^{\circ n}(z_0 + t/\mu^n).
\]

Its image \( \theta(\mathbb{C}) \subset \hat{\mathbb{C}} \) is equal to the Riemann sphere \( \hat{\mathbb{C}} \) with at most two points removed. In fact if \( f \) has degree at least two, then the complement \( \hat{\mathbb{C}} \setminus \theta(\mathbb{C}) \) is precisely equal to the exceptional set \( E_f \), consisting of all points with finite grand orbit under \( f \).

In general this Poincaré function \( \theta \) has very complicated behavior. In particular, the Poincaré functions associated with different fixed points or periodic points are usually quite incompatible. However, Lattès pointed out that in special cases \( \theta \) will be periodic or doubly periodic, and will provide a simultaneous linearization for all but finitely many of the periodic points.

It will be convenient to make a very mild generalization of this Lattès construction, replacing the linear map \( t \mapsto \mu t \) by an affine map \( t \mapsto at + b \). Let \( A \subset \mathbb{C} \) be a discrete
additive subgroup. In the cases of interest, this subgroup will have rank either one or two, so that the quotient surface \( \mathbb{C}/\Lambda \) is either a cylinder or a torus.

**Definition 1.1.** A rational map \( f \) will be called a *quotient of an affine map* if there is flat surface \( \mathbb{C}/\Lambda \) and a finite-to-one holomorphic map \( \Theta : \mathbb{C}/\Lambda \to \mathbb{C} \setminus E_f \) which satisfies the semiconjugacy relation \( f \circ \Theta = \Theta \circ L \), where \( L(t) = at + b \) is some affine map from \( \mathbb{C}/\Lambda \) to itself. Thus the following diagram must commute:

\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{L} & \mathbb{C}/\Lambda \\
\Theta \downarrow & & \Theta \downarrow \\
\hat{\mathbb{C}} \setminus E_f & \xrightarrow{f} & \hat{\mathbb{C}} \setminus E_f.
\end{array}
\] (1)

It follows for example that any periodic orbit of \( L \) must map to a periodic orbit of \( f \), and conversely that every periodic orbit of \( f \) within \( \hat{\mathbb{C}} \setminus E_f \) is the image of a periodic orbit of \( L \). However, the periods are not necessarily the same.

We will always assume that the map \( f \) has degree two or more. (In fact, most maps of degree one are rather trivially “quotients of affine maps”.) Here is a simple example.

**Example 1.1.** The function \( \Theta(t) = 2 \cos t = e^{it} + e^{-it} \) can be considered as a holomorphic map of degree two from the cylinder \( \mathbb{C}/2\pi \mathbb{Z} \) onto the complex numbers \( \mathbb{C} \). Setting \( L(t) = 2t \), it follows from the identity

\[
\cos(2t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1
\]

that the composition \( \Theta \circ L \) can be written as \( f \circ \Theta \), where the map

\[
f(z) = z^2 - 2
\]

is described as the degree two *Chebyshev polynomial*. (Compare §6.)

More generally, the possible quotients of affine maps with degree \( d_f \geq 2 \) will be classified as either Lattès maps, Chebyshev maps, or power maps, according as the number of exceptional points for \( f \) is zero, one, or two. In Sections 2 through 5 we will concentrate on the Lattès case where \( \mathbb{T} = \mathbb{C}/\Lambda \) is a torus, so that \( \Theta \) necessarily maps onto the entire Riemann sphere, and there are no exceptional points. Section 2 will provide a more concrete characterization of Lattès maps. Section 3 will describe the associated flat orbifold structure, while §4 will provide a more detailed classification following Douady and Hubbard [DH, §9], and §5 will describe a number of concrete examples. Section 6 will describe the closely related classes of Chebyshev maps and power maps; §7 will describe the history of these ideas before Lattès; and §8 will describe some of the developments since his time.

### §2. Cyclic Group Actions on the Torus

The following result provides a more explicit description of all of the possible Lattès maps.

**Theorem 2.1.** A rational map is Lattès\(^1\) if and only if it is conformally conjugate to a map of the form \( L/G_n : \mathbb{T}/G_n \to \mathbb{T}/G_n \) where:

- \( \mathbb{T} \cong \mathbb{C}/\Lambda \) is a flat torus,
- \( G_n \) is the group of \( n \)-th roots of unity acting on \( \mathbb{T} \) by rotation around a base point, with \( n \) equal to either 2, 3, 4, or 6.

\(^1\)Caution: In earlier publications the author has used the term “Lattès map” with a more restricted meaning, allowing only the case \( n = 2 \).
• $\mathbb{T}/G_n$ is the quotient space provided with its natural structure as a smooth Riemann surface of genus one,

• $L$ is an affine map from $\mathbb{T}$ to itself which commutes with a generator of $G_n$, and

• $L/G_n$ is the induced holomorphic map from the quotient surface to itself.

In other words, in Diagram (1) we can always suppose that the projection map $\Theta$ has the form $\mathbb{T} \to \mathbb{T}/G_n \cong \hat{\mathbb{C}}$, with degree $n$ equal to 2, 3, 4, or 6.

To begin the discussion, let $\theta : \mathbb{C} \to \hat{\mathbb{C}}$ be a doubly periodic meromorphic function, and let $\Lambda \subset \mathbb{C}$ be its lattice of periods so that $\lambda \in \Lambda$ if and only if $\theta(t + \lambda) = \theta(t)$ for all $t \in \mathbb{C}$. Then the canonical flat metric $|dt|^2$ on $\mathbb{C}$ pushes forward to a corresponding flat metric on the torus $\mathbb{T} = \mathbb{C}/\Lambda$. If $\ell(t) = at + b$ is an affine map of $\mathbb{C}$ satisfying the identity $f \circ \theta = \theta \circ \ell$, then for $\lambda \in \Lambda$ and $t \in \mathbb{C}$ we have

$$\theta(at + b) = f(\theta(t)) = f(\theta(t + \lambda)) = \theta(a(t + \lambda) + b).$$

This shows that $a\Lambda \subset \Lambda$. Thus the maps $\ell$ and $\theta$ on $\mathbb{C}$ induce corresponding maps $L$ and $\Theta$ on $\mathbb{T}$, so that we have a commutative diagram of holomorphic maps

$$
\begin{array}{ccc}
\mathbb{T} & \xrightarrow{L} & \mathbb{T} \\
\downarrow \Theta & & \downarrow \Theta \\
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\
\end{array}
$$

We will think of $\mathbb{T}$ as a branched covering of the Riemann sphere with projection map $\Theta$. Since $L$ carries a small region of area $A$ to a region of area $|a|^2 A$, it follows that the map $L$ has degree $|a|^2$. Using Diagram (2), we see that the degree $d_f \geq 2$ of the map $f$ must also be equal to $|a|^2$.

Since periodic points of $L$ are dense in the torus, it follows that periodic points of $f$ are dense in the Riemann sphere. In particular, the Julia set of $f$ must be the entire sphere. Another easily derived property is the following. Let $C_f$ be the set of critical points of $f$ and let $V_f = f(C_f)$ be the set of critical values. Similarly, let $V_\Theta = \Theta(C_\Theta)$ be the set of critical values for the projection map $\Theta$.

**Lemma 2.2.** Every Lattès map $f$ is postcritically finite. In fact the postcritical set

$$P_f = V_f \cup f(V_f) \cup f^2(V_f) \cup \cdots$$

is precisely equal to the finite set $V_\Theta$ consisting of all critical values for the projection $\Theta : \mathbb{T} \to \hat{\mathbb{C}}$.

**Proof.** Let $d_f(z)$ be the local degree of the map $f$ at a point $z$. Thus

$$1 \leq d_f(z) \leq d_f,$$

where $d_f(z) > 1$ if and only if $z$ is a critical point of $f$. Given points $\tau_j \in \mathbb{T}$ and $z_j \in \hat{\mathbb{C}}$ with

$$
\begin{array}{ccc}
\tau_1 & \xrightarrow{L} & \tau_0 \\
\Theta \downarrow & & \Theta \downarrow \\
z_1 & \xrightarrow{f} & z_0,
\end{array}
$$
where $a$ is then a Königs linearizing map for a finite set must map to a fixed point of $f$ or both. Since the maps $L$ and $\Theta$ are surjective, it follows that

$$V_\Theta = V_f \cup f(V_\Theta),$$

and hence that $P_f \subset V_\Theta$.

On the other hand, if some critical point $\tau_0$ of $\Theta$ had image $\Theta(\tau_0)$ outside of the postcritical set $P_f$, then all of the infinitely many iterated preimages $\cdots \mapsto \tau_2 \mapsto \tau_1 \mapsto \tau_0$ would have the same property. This is impossible, since $\Theta$ can have only finitely many critical points. □

We will prove the following statement, with notations as in Diagram (2).

**Lemma 2.3.** If $f$ is a Lattès map, then there is a finite cyclic group $G$ of rigid rotations of the torus $\mathbb{T}$ about some base point, so that $\Theta(\tau') = \Theta(\tau)$ if and only if $\tau' = g \tau$ for some $g \in G$. Thus $\Theta$ induces a canonical homeomorphism from the quotient space $\mathbb{T}/G$ onto the Riemann sphere.

**Remark 2.4.** Such a quotient $\mathbb{T}/G$ can be given two different structures which are distinct, but closely related. Suppose that a point $\tau_0 \in \mathbb{T}$ is mapped to itself by a subgroup of $G$, necessarily cyclic, of order $r > 1$. Then we can take the power $(\tau - \tau_0)^r$ as a local uniformizing parameter for $\mathbb{T}/G$ near $\tau_0$ (thinking of the difference $\tau - \tau_0$ as a real number). In this way, the quotient becomes a smooth Riemann surface. On the other hand, if we want to carry the flat Euclidean structure of $\mathbb{T}$ over to $\mathbb{T}/G$, then the image of $\tau_0$ must be considered as a singular “cone point”, as described in the next section. The integer $r$, equal to the local degree $d_\Theta(\tau_0)$, is called the **ramification index** of the image point $\Theta(\tau_0)$.

**Proof of 2.3.** Choose a periodic point $z_0$ in $\hat{\mathbb{C}} \setminus P_f$. Replacing $f$ by some iterate and replacing $L$ by the corresponding iterate, we may assume without changing the map $\Theta$ that:

- $z_0$ is actually a fixed point of $f$, and that
- every periodic point in the finite set $\Theta^{-1}(z_0)$ is actually a fixed point of $L$.

Let $U$ be a simply connected neighborhood of $z_0$ in $\hat{\mathbb{C}} \setminus P_f$. Then the preimage $\Theta^{-1}(U)$ is the union $U_1 \cup \cdots \cup U_n$ of $n$ disjoint regions, where $n = d_\Theta$ is the degree of the projection map $\Theta$. If $\Theta_j : U_j \to U$ is the restriction of $\Theta$ to $U_j$, then each $\Theta_j$ is a diffeomorphism from $U_j$ onto $U$. We will first prove that each composition

$$\Theta_k^{-1} \circ \Theta_j : U_j \to U_k,$$

is an isometry from $U_j$ onto $U_k$, using the standard flat metric on the torus.

Since $L$ maps the $n$ element set $\Theta^{-1}(z_0)$ to itself, every point $\tau_j = \Theta^{-1}(z_0)$ in this finite set must map to a fixed point of $L$ under the iterate $L^{\circ n}$. The affine map

$$\phi_j(\tau) = L^{\circ n}(\tau) - L^{\circ n}(\tau_j)$$

is then a Königs linearizing map for $L$ near $\tau_j$. That is,

$$\phi_j(L(\tau)) = a \phi_j(\tau) \quad \text{with} \quad \phi_j(\tau_j) = 0,$$

where $a$ is the derivative $L'$. It follows that $\phi_j \circ \Theta_j^{-1}$ is a Königs linearizing map for $f$
near \( z_0 \). Since such a linearizing map is unique up to multiplication by a constant, we have

\[
\phi_k \circ \Theta_k^{-1} = c_{jk} \phi_j \circ \Theta_j^{-1}
\]

for some constant \( c_{jk} \neq 0 \). Composing with \( \Theta_j \), it follows that

\[
\phi_k \circ \Theta_k^{-1} \circ \Theta_j(\tau) = c_{jk} \phi_j(\tau)
\]

for \( \tau \) near \( \tau_j \). Since \( \phi_j \) and \( \phi_k \) are affine maps with the same constant derivative \( a^n \), this proves that \( \Theta_k^{-1} \circ \Theta_j \) is an affine map with constant derivative \( c_{jk} \) near \( \tau_j \). Choosing a local lifting of \( \Theta_k^{-1} \circ \Theta_j \) to the universal covering space \( \mathbb{T} \cong \mathbb{C} \) and continuing analytically, we obtain an affine map \( \ell_{jk} \) from \( \mathbb{C} \) to itself with derivative \( \ell'_{jk} = c_{jk} \), satisfying the identity \( \theta \circ \ell_{jk} = \theta \).

We must prove that \( |c_{jk}| = 1 \), so that this affine transformation is an isometry. Let \( \tilde{G} \) be the group consisting of all affine transformations \( \tilde{g} \) of \( \mathbb{C} \) which satisfy the identity \( \theta \circ \tilde{g} = \theta \). Then the translations \( t \mapsto t + \lambda \) with \( \lambda \in \Lambda \) constitute a normal subgroup of \( \tilde{G} \), and the quotient \( G = \tilde{G}/\Lambda \) acts as a finite group of complex affine automorphisms \( g : \mathbb{T} \to \mathbb{T} \). If the derivative \( g' \in \mathbb{C} \) had absolute value different from one, then \( g \) would increase or decrease area, which is clearly impossible. Thus \( G \) is a group of isometries. The derivative map \( g \mapsto g' \) is a homomorphic embedding of \( G \) into the unit circle, so the finite group \( G \) must be cyclic. Furthermore, a generator of \( G \) must have a fixed point in the torus, so \( G \) can be considered as a group of rotations about this fixed point. This completes the proof of 2.3. \( \square \)

Evidently this group \( G \) has \( n \) elements. If we translate coordinates so that the fixed base point is the origin of \( \mathbb{T} = \mathbb{C}/\Lambda \), then we can identify \( G \) with the group \( G_n \) of \( n \)-th roots of unity, acting by multiplication on \( \mathbb{T} \).

**Lemma 2.5.** The order of such a cyclic group of rotations of the torus is necessarily either 2, 3, 4, or 6.

**Proof.** Thinking of a rotation through angle \( \alpha \) as a real linear map, it has eigenvalues \( e^{\pm i\alpha} \) and trace \( e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha) \). On the other hand, if such a rotation carries the lattice \( \Lambda \) into itself, then its trace must be an integer. The function \( \alpha \mapsto 2\cos(\alpha) \) is monotone decreasing for \( 0 < \alpha \leq \pi \) and takes only the following integer values:

\[
\begin{align*}
r & = 6 & 4 & 3 & 2 \\
2\cos(2\pi/r) & = 1 & 0 & -1 & -2
\end{align*}
\]

This proves 2.5. \( \square \)

Now to complete the proof of Theorem 2.1, we must only note that an affine map \( L(\tau) = a\tau + b \) induces a well defined map of the quotient \( \mathbb{T}/G_n \) if and only if \( L \) commutes with a generator of \( G_n \). In the more general case of an arbitrary discrete group acting on \( \mathbb{T} \), the appropriate condition would be that for every \( g_1 \in G \) there exists \( g_2 \in G \) so that \( L \circ g_1 = g_2 \circ L \). However, in our case the group \( G_n \) acts simply by multiplication, so we require that \( L(\omega_1 \tau) = \omega_2 L(\tau) \). Differentiating this identity, we see that \( \omega_1 \) must equal \( \omega_2 \). This completes the proof of 2.1. \( \square \)

The following helps to demonstrate the extremely rigid geometry associated with Lattès maps. Nothing like it is true for more general rational maps.

**Corollary 2.6.** If \( z_0 = f(z_0) \) is a fixed point with ramification index \( r \geq 1 \),
then the multiplier of \( f \) at \( z_0 \) can be written as \((\omega a)^r\) where \( \omega \) is some \( n \)-th root of unity. More generally, the ramification index \( r \) is constant along any periodic orbit, and the multiplier \( \mu \) of such an orbit has the form \((\omega a^n)^r\), where \( p \) is the period. In particular, \(|\mu|^2\) is always a power of the degree \(|a|^2\).

**Proof.** Let \( \Theta(\tau_0) = z_0 \). As in 2.4, we can take \( \zeta = (\tau - \tau_0)^r \) as local uniformizing parameter near \( z_0 \). On the other hand, since \( f(z_0) = z_0 \) we have \( \tau_0 = \omega L(\tau_0) = \omega (a \tau_0 + b) \) for some \( \omega \in G_n \). Thus \( f \) corresponds to the linear map

\[
\tau - \tau_0 \mapsto \omega a (\tau - \tau_0) \quad \text{or to} \quad \zeta \mapsto (\omega a)^r \zeta.
\]

Applying the same argument of the \( p \)-th iterate of \( f \), we get a corresponding identity for a period \( p \) orbit. \( \square \)

For example for a periodic orbit of maximal ramification \( r = n \) the multiplier is simply \( a^{pn} \). In the generic case of a periodic orbit with \( r = 1 \), the multiplier has the form \( \omega a^p \).

### §3. Flat Orbifold Metrics

We can give another characterization of Lattès maps as follows.

**Definition.** By a **flat orbifold metric** on the Riemann sphere will be meant a metric which is smooth, conformal and locally isometric to the standard flat metric on \( \mathbb{C} \), except at finitely many “cone points”, where it has cone angle of the form \( 2\pi/r \). Here a cone point with cone angle \( 0 < \alpha < 2\pi \), is an isolated singular point of the metric which can be visualized by cutting an angle of \( \alpha \) out of a sheet of paper and then gluing the two edges together. (A more formal definition will be left to the reader.) In the special case where \( \alpha \) is an angle of the form \( 2\pi/r \), we can identify such a cone with the quotient space \( \mathbb{C}/G_r \) where \( G_r \) is the group of \( r \)-th roots of unity acting by multiplication on the complex numbers, and where the flat metric on \( \mathbb{C} \) corresponds to a flat metric on the quotient, except at the cone point.

![Figure 1. Model for a cone point with cone angle \( \alpha \).](image)

Evidently the quotient \( \mathbb{T}/G_n \simeq \hat{\mathbb{C}} \) of Theorem 2.1 can be given a canonical flat orbifold metric. Thus near any non cone point we can choose a local coordinate \( t \) so that the metric takes the form \( |dt|^2 \). I will say that such a metric linearizes the map \( f \) since, in terms of such preferred local coordinates, \( f \) is an affine map with constant derivative. An equivalent property is that \( f \) maps any curve of length \( \delta \) to a curve of length \( k \delta \) where \( k > 1 \) is constant.\(^2\)

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\(^2\) In fact lengths are multiplied by \( |a| = \sqrt{\det f} \). The more usual terminology for a map \( f \) which is linearized by such a flat orbifold metric would be that \( f \) has “parabolic” or “Euclidean” orbifold. Thurston has shown that the set \( \hat{\mathbb{C}} \setminus E_f \) for a postcritically finite map can be given a preferred orbifold metric with
If $\tau_0$ is a critical point of $\Theta : \mathbb{T} \to \mathbb{C}$ with local degree $r$, then the subgroup consisting of elements of $G$ which fix $\tau_0$ must be generated by a rotation through $2\pi/r$ about $\tau_0$. Hence the flat metric on $\mathbb{T}$ pushes forward to a flat metric on $\mathbb{T}/G$ with $\tau_0$ corresponding to a cone point $z_0 = \theta(\tau_0)$ of angle $2\pi/r$. As in §2, this integer $r = r(z_0) > 1$ is called the ramification index of the cone point, where we set $r(z) = 1$ if $z$ is not a cone point. Thus $r(z_0)$ is equal to the local degree $d_{\Theta}(\tau_0)$. There may be several different points in $\Theta^{-1}(z_0)$, but this argument shows that they must all have the same local degree. With this notation, Equation (4) takes the form

$$r(f(z)) = d_f(z) \cdot r(z).$$

(6)

By Lemma 2.2, the set of cone points (or equivalently the set of $z$ with $r(z) > 1$) is precisely the postcritical set $P_f$.

We will need a piecewise linear form of the Gauss-Bonnet Theorem. For this lemma only, we allow cone angles which are greater than $2\pi$.

**Lemma 3.1.** If a flat metric with finitely many cone points on a compact Riemann surface $S$ has cone angles $\alpha_1, \ldots, \alpha_k$, then

$$(2\pi - \alpha_1) + \cdots + (2\pi - \alpha_k) = 2\pi \chi(S),$$

where $\chi(S)$ is the Euler characteristic. In particular, if $\alpha_j = 2\pi/r_j$ and if $S$ is the Riemann sphere, then it follows that $\sum (1 - 1/r_j) = \chi(S) = 2$.

**Proof.** Choose a rectilinear triangulation, where the cone points will necessarily be among the vertices. Let $V$ be the number of vertices, $E$ the number of edges, and $F$ the number of faces (i.e., triangles). Then $2E = 3F$ since each edge bounds two triangles and each triangle has three edges. Thus

$$\chi(S) = V - E + F = V - F/2.$$

(8)

The sum of the internal angles of all of the triangles is clearly equal to $\pi F$. On the other hand, the $j$-th cone point contributes $\alpha_j$ to the total, while each non-cone vertex contributes $2\pi$. Thus

$$\pi F = \alpha_1 + \cdots + \alpha_k + 2\pi(V - k).$$

(9)

Multiplying equation (8) by $2\pi$ and using (9), we obtain the required equation (7). □

**Corollary 3.2.** The collection of ramification indices for a flat orbifold metric on the Riemann sphere must be either $\{2,2,2,2\}$ or $\{3,3,3\}$ or $\{2,4,4\}$ or $\{2,3,6\}$. In particular, the number of cone points must be either four or three.

**Proof.** Using the inequality $1/2 \leq (1 - 1/r_j) < 1$, it is easy to check that the required equation

$$\sum_j (1 - 1/r_j) = \chi(\mathbb{C}) = 2,$$

has only these solutions in integers $r_j > 1$. □

This corollary provides an alternative proof for Lemma 2.5. Note that the integer $n$ of 2.1 and 2.5 can be described either as the least common multiple of the ramification indices $r_j$, or as the largest of these indices.

constant curvature $\leq 0$. (Compare [DH] or [M3].) This metric has curvature zero if and only if $f$ can be expressed as a quotient of an affine map.
**Lemma 3.4.** The flat orbifold metric on the Riemann sphere is uniquely determined, up to multiplication by a constant, by the Lattès map $f$. Furthermore, the branched covering space $\mathbb{T}$ and the action of $G_n$ on $\mathbb{T}$ are also uniquely determined, up to a conformal equivalence.

**Proof.** The construction of the flat metric is based on the use of Koenigs linearization in Lemma 2.3, and hence is uniquely determined by the rational map $f$, up to a change of scale. Given this flat metric, we can define the holonomy homeomorphism

$$\eta : \pi_1(\hat{\mathbb{C}} \setminus P_f) \to G_n$$

as follows. If we start with a unit tangent vector at a base point in $\hat{\mathbb{C}} \setminus P_f$ and parallel transport it around any loop $\alpha$, then it will be rotated by the required root of unity $\eta(\alpha)$. The torus $\mathbb{T}$ can now be described as the metric completion of the associated $G_n$-cyclic covering of $\hat{\mathbb{C}} \setminus P_f$. \qed

It is then not difficult to lift the map $f$ to an affine map $L$ of this torus. However, there are three sources of ambiguity:

- The affine map $L(\tau) = a\tau + b$ is determined only up to multiplication by $n$-th roots of unity. In particular, its derivative $L' = a$ can be multiplied by any $n$-th root of unity.

- The construction required a choice of base point among the points of $\mathbb{T}$ which are fixed by the action of $G_n$, or equivalently among the points with maximal ramification index in $P_f$. In the four possible cases, $\{2,3,6\}$ or $\{2,4,4\}$ or $\{3,3,3\}$ or $\{2,2,2,2\}$, there are respectively one, two, three, or four such points (or $4\sin^2(\pi/n)$ points in all cases).

- If $\hat{\mathbb{C}}$ with its orbifold structure admits a symmetry $\sigma$ then the map $\sigma \circ f \circ \sigma^{-1}$ will be conformally conjugate to $f$, but may have a different associated map $L$.

We must deal with all three of these in the next section.

### §4. Classification of Lattès Maps.

By taking a closer look at the construction for Theorem 2.1, we can give a complete classification. We first concentrate on the cases $n \geq 3$ where there are exactly three postcritical points or equivalently exactly three cone points. Thus the collection of ramification indices must be either $\{2,3,6\}$ or $\{2,4,4\}$ or $\{3,3,3\}$. Each of these three possibilities is associated with a rigidly defined flat orbifold geometry which can be described as follows. Join each pair of cone points by a minimal geodesic. Evidently these geodesics cannot cross each other; and no geodesic can pass through a cone point since our cone angles are strictly less than $2\pi$. In this way, we obtain three edges which cut our locally flat manifold into two Euclidean triangles. Since these two triangles have the same edges, they must be precise mirror images of each other. In particular, the two edges which meet at a cone point of angle $2\pi/r_j$ must cut it into two Euclidean angles of $\pi/r_j$. Passing to the branched covering space $\mathbb{T}$ or its universal covering $\tilde{\mathbb{T}}$, we obtain a tiling of the torus or the Euclidean plane\(^3\) by triangles with angles $\pi/r_1$, $\pi/r_2$ and $\pi/r_3$. These tilings are illustrated in Figures 2, 3, 4.

In each case, each pair of adjacent triangles are mirror images of each other, and together form a fundamental domain for the action of $G_n$ on the plane, or for the action of $G_n$ on

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\(^3\) More generally, for any triple of integers $r_j \geq 2$ there is an associated tiling, either of the Euclidean or hyperbolic plane or of the 2-sphere depending on the sign of $1/r_1 + 1/r_2 + 1/r_3 - 1$. See for example [M1].
Figure 2. The \{2, 3, 6\}-tiling of the plane. In each of these diagrams, the points of ramification have been marked, with circles around the lattice points.

Figure 3. The \{2, 4, 4\}-tiling.

Figure 4. The \{3, 3, 3\}-tiling, with one tile and its images under $\tilde{G}_3$ labeled.
the torus. For each vertex of this diagram, corresponding to a cone point of angle $2\pi/r_j$, there are $r_j$ lines through the vertex, and hence $2r_j$ triangles which meet at the vertex. The subgroup of $\tilde{G}_n$ (or $G_n$) which fixes such a point has order $r_j$ and is generated by a rotation through the angle $\alpha_i = 2\pi/r_j$.

The subgroup $\Lambda \subset \tilde{G}_n$ consists of all translations of the plane which belong to $\tilde{G}_n$. Note that the integer $n$ can be described as the maximum of the $r_j$. The $2n$ triangles which meet at any maximally complicated vertex form a fundamental domain for the action of this subgroup $\Lambda$. In the $\{2,3,6\}$ and $\{3,3,3\}$ cases, this fundamental domain is a regular hexagon, while in the $\{2,4,4\}$ case it is a square. In all three cases, the torus $\mathbb{T}$ can be obtained by identifying opposite faces of this fundamental domain under the appropriate translations. Thus when $n \geq 3$, the torus $\mathbb{T}$ is uniquely determined by $n$, up to conformal diffeomorphism.

In the $\{2,3,6\}$ case, the integers $r_j$ are all distinct, so it is easy to distinguish the three kinds of vertices. However, in the $\{2,4,4\}$ case there are two different kinds of vertices of index 4. In order to distinguish them, one kind has been marked with dots and the other with circles. Similarly in the $\{3,3,3\}$ case, the three kinds of vertices have been marked in three different ways. In this last case, half of the triangles have also been labeled. In all three cases, the points of the lattice $\Lambda$, corresponding to the base point in $\mathbb{T}$, have been circled. For all three diagrams, the group $G_n$ can be described as the group of all rigid Euclidean motions which carry the marked diagram to itself, and the lattice $\Lambda$ can be identified with the subgroup consisting of translations which carry this marked diagram to itself.

Thus we have a precise description of the geometry in each of these cases. To complete the discussion, we must also give a precise description of all affine torus maps which commute with the action of $G_n$. It will be convenient to use the notation

$$\mathbb{Z}[G_n] = \mathbb{Z}[\exp(2\pi i/n)]$$

for the ring generated by the $n$-th roots of unity. We will only be interested in the cases $\mathbb{Z}[G_4] = \mathbb{Z}[i]$ and $\mathbb{Z}[G_3] = \mathbb{Z}[G_6]$ where $\mathbb{Z}[G_n]$ is a quadratic extension of $\mathbb{Z}$. In all three cases with $n \geq 3$, the torus $\mathbb{T}$ is uniquely determined up to conformal diffeomorphism, and can be identified with the quotient $\mathbb{C}/\mathbb{Z}[G_n]$.

**Definition.** Let $\text{Fix}(G_n, \mathbb{T}) \subset \mathbb{T}$ be the finite subgroup consisting of torus elements which are fixed under the action of $G_n$. This group can be described more explicitly as follows: It is trivial in the $\{2,3,6\}$ case, is cyclic of order two generated by $(1+i)/2$ in the $\{2,4,4\}$ case, and is cyclic of order three generated by $i/\sqrt{3}$ in the $\{3,3,3\}$ case. The corresponding points in the universal covering space are precisely the marked vertices of Figures 2, 3, 4.

**Lemma 4.1.** A Lattès map $f$ with three postcritical points is uniquely determined up to conformal conjugacy by the following:

- The integer $n$, equal to 3, 4, or 6.
- The invariant $a^n$, where $a$ must belong to the ring $\mathbb{Z}[G_n]$ with $|a| > 1$.
- The information as to whether $f$ does or does not have a postcritical fixed point for which the ramification index $r$ is equal to $n$.

(However, in the special case where every element of the group $\text{Fix}(G_n, \mathbb{T})$ belongs to the subgroup $(a-1)\text{Fix}(G_n, \mathbb{T})$ this last data is redundant, since $f$
Outline Proof. By the construction in §2, the zero point in \( \mathbb{T} \) corresponds to a cone point \( \Theta(0) = z_0 \) of maximal ramification index \( r = n \). The image \( f(z_0) \) must also be a cone point of maximal index, and hence must also lift uniquely to a point in the set \( \text{Fix}(G_n, \mathbb{T}) \). Choosing some lifting of \( f \) to a neighborhood of \( z_0 \), there is no obstruction to extending this lifting to a holomorphic map \( L \) from \( \mathbb{T} \) to itself. Such a holomorphic torus map is necessarily affine, since its derivative is a bounded holomorphic function from \( \mathbb{T} \) to \( \mathbb{C} \).

If \( f \) has a fixed point with \( r = n \), then we can choose this point to be \( \Theta(0) \), and the affine map \( L \) will actually be linear, \( L(\tau) = a\tau \). On the other hand, if there is no such fixed point then we must have \( L(\tau) = a\tau + b \) with \( b \neq 0 \). However, \( b \) must belong to the subgroup \( \text{Fix}(G_n, \mathbb{T}) \) since \( L \) commutes with the action of \( G_n \). Furthermore, \( b \) cannot be an element of the subgroup \( (a - 1)\text{Fix}(G_n, \mathbb{T}) \). For if \( b \) were equal to \( 1 - a \) times \( \tau_0 \in \text{Fix}(G_n, \mathbb{T}) \), then the computation \( L(\tau_0) = a\tau_0 + b = \tau_0 \) would show that \( \tau_0 \) must be a fixed point, contradicting the hypothesis.

In the \( \{3,3,3\} \) case the group \( \text{Fix}(G_3, \mathbb{T}) \) has two distinct non-zero elements. However, it doesn’t matter which of the two we use. In fact, conjugating \( L(\tau) = a\tau + b \) by the torus automorphism \( \tau \mapsto -\tau \) we obtain \( -L(-\tau) = a\tau - b \). \( \square \)

Now consider the case \( n = 2 \), with four postcritical points. There is a very easy characterization of such Lattès maps, as follows.

**Lemma 4.2.** A rational map with exactly four postcritical points is Lattès if and only its critical points are all simple (i.e., with local degree two), and none is postcritical.

Outline Proof. These conditions are necessary, since otherwise there would be a point of ramification index greater than two. But they are also sufficient. Given four distinct points of the Riemann sphere, there is a uniquely determined 2-fold covering surface which is branched at these four points. This surface has genus zero, and hence has a flat conformal metric which is unique up to scale, with a canonical \( G_2 \)-action. The conditions of 4.2 are exactly the ones needed to guarantee that \( f \) lifts to an affine torus map. (See [M5, Appendix B] for details.) \( \square \)

Remarks. There is a similar characterization of Lattès maps with \( n = 3 \). Another closely related characterization of Lattès maps with \( n = 2 \) is the following. It follows from equation (6) that for every Lattès map \( f \) we have \( f^{-1}P_f = C_f \cup P_f \). Conversely, a rational map with \( \nu \geq 4 \) postcritical points satisfies this condition if and only if it is Lattès. In fact it is not hard to show that the number of points in \( f^{-1}P_f \setminus P_f \) is always equal to \( (\nu - 2)(d_f - 1) \), and that the number of critical points counted with multiplicity is \( 2(d_f - 1) \). Hence the number of points in \( f^{-1}P_f \setminus (C_f \cup P_f) \) is greater than or equal to \( (\nu - 4)(d_f - 1) \), with equality only if every critical point is simple, and none is postcritical. \( \square \)

To specify the possibilities further, let us describe the possible constants \( L' = a \) and the possible associated tori. By definition, a complex number \( a \) is called an **imaginary quadratic integer** if it satisfies an equation \( a^2 + qa + d = 0 \) with integer coefficients so that

\[
a = \left(-q \pm \sqrt{q^2 - 4d}\right)/2,
\]
where \( q^2 - 4d < 0 \) so that \( a \not\in \mathbb{R} \) and so that \( |a|^2 = d \). Evidently the imaginary quadratic integers form a discrete subset of the complex plane. In fact for each choice of \( |a|^2 = d \) there are roughly \( 4\sqrt{d} \) possible choices for \( q \), and twice that number for \( a \).

**Lemma 4.3.** A given number \( a \in \mathbb{C} \) can occur as the derivative \( L' \) associated with an affine torus map if and only if it is either a rational integer \( a \in \mathbb{Z} \), or an imaginary quadratic integer. If \( a \in \mathbb{Z} \) then any torus can occur, but if \( a \not\in \mathbb{Z} \) then there are only finitely many possible tori up to conformal diffeomorphism. For any imaginary quadratic integer \( a \), there is a one-to-one correspondence between conformal diffeomorphism classes of such tori and ideal classes in the ring \( \mathbb{Z}[a] \).

**Proof.** Let \( T = \mathbb{C}/\Lambda \). The condition that \( a\Lambda \subset \Lambda \) means that \( \Lambda \) must be a module over the ring \( \mathbb{Z}[a] \) generated by \( a \). If \( a \in \mathbb{Z} \) then this condition is satisfied for every lattice. On the other hand, if \( a \not\in \mathbb{Z} \) we first show that \( a \) is an algebraic integer. Without loss of generality, we may assume that \( 1 \in \Lambda \) and hence that all powers of \( a \) belong to \( \Lambda \). If \( \Lambda_k \) is the sublattice spanned by \( 1, a, a^2, \ldots, a^k \), then the lattices \( \Lambda_1 \subset \Lambda_2 \subset \cdots \Lambda \) cannot all be distinct. But if \( a^k \in \Lambda_{k-1} \) then \( a \) satisfies a monic equation with integer coefficients, and hence is an algebraic integer. On the other hand, \( a \) belongs to a quadratic number field since the three numbers \( 1, a, a^2 \in \Lambda \) must satisfy a linear relation with integer coefficients. Using the fact that the integer polynomials form a unique factorization domain, it follows that \( a \) satisfies a monic degree two polynomial.

Now given \( a \not\in \mathbb{Z} \) we must ask which lattices \( \Lambda \) are possible. Without loss of generality, we may assume that \( \Lambda \) is the lattice \( \mathbb{Z} \oplus \gamma \mathbb{Z} \) spanned by 1 and \( \gamma \), where \( \gamma \) belongs to the Siegel region

\[
|\gamma| \geq 1, \quad |\Re(\gamma)| \leq 1/2, \quad \Im(\gamma) > 0.
\]  

It follows from these inequalities that \( \Im(\gamma) \geq \sqrt{3}/2 \). Since \( a\Lambda \subset \Lambda \) it certainly follows that \( a \in \Lambda \). Thus we can write \( a = r + s\gamma \) with \( r, s \in \mathbb{Z} \), and we may assume that \( s > 0 \). It then follows that

\[
r = \Re(a) - s\Re(\gamma) \quad \text{and that} \quad s = \Im(a)/\Im(\gamma).
\]

If \( a \) has been specified, then the inequality \( \Im(\gamma) \geq \sqrt{3}/2 \) yields an upper bound of \( 2\Im(a)/\sqrt{3} \) for \( s \), and the inequality \( |\Re(\gamma)| \leq 1/2 \) then yields an upper bound for \( |r| \). Thus there are only finitely many possibilities for \( \gamma = (a-r)/s \).

Next note that the lattice \( I = s\Lambda \) is contained in the ring \( \mathbb{Z}[a] \), and is an ideal in this ring since \( aI \subset I \). Clearly the torus \( \mathbb{C}/\Lambda \) is isomorphic to \( \mathbb{C}/I \). If \( I' \) is another ideal in \( \mathbb{Z}[a] \), note that \( \mathbb{C}/I \cong \mathbb{C}/I' \) if and only if \( I' = cI \) for some constant \( c \neq 0 \). Such a constant must belong to the quotient field \( \mathbb{Q}[a] \), so by definition this means that \( I \) and \( I' \) represent the same ideal class. \( \Box \)

The analogue of 4.1 in this case is the following.

**Lemma 4.4.** Every Lattès map with four postcritical points is conformally conjugate to a quotient \( L/G_2 : \mathbb{T}/G_2 \rightarrow \mathbb{T}/G_2 \) where \( \mathbb{T} \cong \mathbb{C}/(\mathbb{Z} \oplus \gamma \mathbb{Z}) \) with \( \gamma \) in the Siegel region (10), and with \( L(\tau) = a\tau + b \). Here \( \gamma \) is uniquely determined up to the identification \( \gamma \leftrightarrow -\gamma \) on the boundary of the Siegel region, and the coefficient \( a \) satisfying the condition \( a(\mathbb{Z} \oplus \gamma \mathbb{Z}) \subset \mathbb{Z} \oplus \gamma \mathbb{Z} \) is uniquely determined up to sign. In general, the constant \( b \in \text{Fix}(G_2, \mathbb{T}) \) is also uniquely
determined. However, in the special case where $T \cong \mathbb{C}/\mathbb{Z}[G_3]$ or $\mathbb{C}/\mathbb{Z}[G_4]$ the torus $T$ (or the orbifold $T/G_2$) has an extra automorphism, so that we are also free to multiply $b$ by an element of $G_3$ or $G_4$ respectively.

The proof is similar to the proof of Lemma 4.1. 

Figure 5. A typical \{2, 2, 2\} tiling of the plane.

The analogue of Figures 2, 3, 4 for the tiling of the plane associated with a typical lattice $\Lambda = \mathbb{Z} \oplus \gamma \mathbb{Z}$ is illustrated in Figure 5. All of the vertices in this figure represent critical points for the projection $\theta : \mathbb{C} \to \hat{\mathbb{C}}$. Again lattice points have been circled. Any two adjacent small parallelograms form a fundamental region for the action of the group $\hat{G}$, which consists of 180° rotations around the vertices, together with lattice translations. The four small parallelograms adjacent to any vertex forms a fundamental domain under lattice translations. The corresponding flat orbifold is isometric either to some tetrahedron in Euclidean space or (when the fundamental parallelogram is a rectangle) to the double of a similar rectangular region. Compare [De].

§5. Examples. Lattès gave just one explicit example in the doubly periodic case. After correcting an obvious misprint, it can be written as follows.

Example 5.1. Let

$$f(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}, \quad (11)$$

This map has critical points $\pm i$ and $\pm 1 \pm \sqrt{2}$ mapping to the critical values 0 and $\pm 1$, which in turn map to the fixed point at infinity. Thus it follows from 4.2 that $f$ is a Lattès map. Since $f$ is an odd function, $f(-z) = -f(z)$, the involution $z \mapsto -z$ must lift to an automorphism $\tau \mapsto i\tau$ of the associated torus $\mathbb{T}$. Therefore $\mathbb{T}$ must be isomorphic to the square torus $\mathbb{C}/\mathbb{Z}[i]$. Note that infinity is a postcritical fixed point with multiplier 4. Setting this equal to $a^2$, we see that the associated affine map of $\mathbb{T}$ must be given by $L(\tau) = 2\tau$. In fact if $\wp : \mathbb{C} \to \hat{\mathbb{C}}$ is the Weierstrass function associated with the lattice $\mathbb{Z}[i]$, then using the identity $\wp(it) = -\wp(t)$ and normalizing by setting $\theta(t) = \wp(t)/\wp(1/2)$, we see that $\theta$ has critical values 0, $\pm 1$, and $\infty$. It follows that the map $f$ defined by the identity $f(\theta(t)) = \theta(2t)$ coincides with the map defined by (11). Note that the map $-f(z)$ satisfying $-f(\theta(t)) = \theta(2it)$ is dynamically distinct, with multiplier $(2i)^2 = -4$ at infinity.
Example 5.2. The maps $L/G_n$ for different values of $n$ are closely related to each other. For example, given a linear map $L(\tau) = a \tau$ of the torus $T = \mathbb{C}/\mathbb{Z}[G_3] = \mathbb{C}/\mathbb{Z}[G_6]$, since both $G_2$ and $G_3$ are subgroups of $G_6$, we obtain maps of type $\{2, 2, 2, 2\}$ and $\{3, 3, 3\}$ and $\{2, 3, 6\}$ which are related by a commutative diagram

$$
\begin{array}{ccc}
L & \to & L/G_2 \\
\downarrow & & \downarrow \\
L/G_3 & \to & L/G_6
\end{array}
$$

The quotient $T/G_2$, with its flat orbifold metric, is isometric to a regular tetrahedron with the four cone points as vertices. In the simplest case, with $a = i\sqrt{3}$, it is not hard to check that these cone points are precisely the fixed points of the Lattès map $L/G_2$. In fact this degree three map $L/G_2$ is beautifully symmetric, with critical points at the mid points of the faces, each mapping to a fixed point at the opposite vertex. (Compare [DMc] for a discussion of symmetric rational maps.) If we place these critical points on the Riemann sphere at the cube roots of $-1$ and at infinity, then this map takes the form

$$f(z) = \frac{6z}{z^3 - 2}.$$ 

The tetrahedral symmetry implies in particular that $f(\omega z) = \omega f(z)$ for all $\omega \in G_3$. Here $f(\infty) = f(0) = 0$, and $f(-1) = f(2) = 2$ hence

$$f(-\omega) = f(2\omega) = 2\omega$$

for all $\omega \in G_3$. The multiplier at each critical value fixed point is equal to $m^2 = -3$. It will be convenient to describe the critical and postcritical points of this map by the following diagram

$$
\begin{array}{cccc}
* & \mapsto & \bullet & 2 \\
* & \mapsto & \bullet & 2 \\
* & \mapsto & \bullet & 2 \\
* & \mapsto & \bullet & 2
\end{array}
$$

where each * stands for a simple critical point and each • for a postcritical fixed point. The 2 under each postcritical point represents its ramification index.

The corresponding Lattès map $L/G_6$ of type $\{2, 3, 6\}$ can be obtained from $L/G_2$ by identifying each $z$ with $\omega z$ for $\omega \in G_3$. If we introduce the new variable $\zeta = z^3$, then the corresponding map $L/G_6 = f/G_3$ is given by

$$\zeta = z^3 \mapsto g(\zeta) = \left(\frac{6z}{z^3 - 2}\right)^3 = \frac{6^3\zeta}{(\zeta - 2)^3}.$$ 

(12)

The three critical points at the cube roots of $-1$ now coalesce into a single critical point $-1$, with $g(-1) = g(8) = 8$. There is still a critical point at infinity with $g(\infty) = g(0) = 0$. But now infinity is also a critical value. In fact there if a double critical point at $\zeta = 2$, with $g(2) = \infty$. The corresponding diagram for the critical and postcritical points $2 \mapsto \infty \mapsto 0$ and $-1 \mapsto 8$ takes the form

$$
\begin{array}{cccc}
** & \mapsto & * & \mapsto \bullet & 3 \\
* & \mapsto & \bullet & 6 \\
* & \mapsto & \bullet & 2
\end{array}
$$

where the symbol ** stands for a critical point of multiplicity two. The multipliers at the two postcritical fixed points are $a^6 = -27$ and $a^2 = -3$ respectively.

Similarly we can study the Lattès map $L/G_3$. In this case the three points of $\text{Fix}(G_3, T)$ all map to zero. Thus the three cone points of the orbifold $T/G_3$ all map
to one of the three. The corresponding diagram has the following form.

\[
\begin{array}{cccc}
** & \mapsto & \bullet & \mapsto \bullet & \mapsto \bullet & \mapsto ** \\
3 & & 3 & & 3 &
\end{array}
\]

If we put the critical points at zero and infinity, and the postcritical fixed point at +1 (compare [M4]), then this map takes the form

\[
f(z) = \frac{z^3 + \omega}{\omega z^3 + 1},
\]

where \( \omega = (-1 + i\sqrt{3})/2 \) is a primitive cube root of unity, with critical orbits \( 0 \mapsto \omega \mapsto 1 \), and \( \infty \mapsto 1/\omega \mapsto 1 \), where 1 is a fixed point with multiplier \( f'(1) = -3i\sqrt{3} = a^3 \).

In contrast to \( L/G_2 \) and \( L/G_6 \), this is not a map with real coefficients. If we replace \( a = i\sqrt{3} \) by its complex conjugate, then \( f \) will be replaced by its complex conjugate or mirror image map.

The cyclic group of order 2 acts on this rational map by the involution \( z \mapsto 1/z \). The quotient under this involution is another model for \( L/G_6 \), and one can check that is is conformally conjugate to (12).

**Remark 5.3.** We might try to construct more general examples by replacing the cyclic group \( G_2 \) by some more complicated group. As an example, let \( G \) be the twelve element group consisting of rotations of a regular tetrahedron. Then \( G \) acts on the orbifold \( \mathbb{T}/G_2 \).

The quotient \( (\mathbb{T}/G_2)/G \) can be identified with \( \mathbb{T}/\hat{G} \) where \( \hat{G} \) is the centrally extended tetrahedral group with 24 elements. (See for example [M1].) However, Theorem 2.1 implies that we can get nothing new by such a construction. In fact the rational map \( L/\hat{G} \) on the orbifold \( \mathbb{T}/\hat{G} \) is conformally conjugate to the map \( L/G_6 \) on \( \mathbb{T}/G_6 \).

**Example 5.4.** Similarly, since \( G_2 \subset G_4 \), any Lattès map \( L/G_4 \) of type \( \{2, 4, 4\} \) is a quotient of a Lattès map \( L/G_2 \) of type \( \{2, 2, 2\} \). The simplest example is obtained by taking the linear map \( L(\tau) = (1 + i)\tau \) of the torus \( \mathbb{T} = \mathbb{C}/\mathbb{Z}[i] \). Note that \( \text{Fix}(G_2, \mathbb{T}) = \{0, 1/2, i/2, (1 + i)/2\} \) modulo \( \mathbb{Z}[i] \). For the associated Lattès map \( L/G_2 \) we have

\[
L(1/2) \equiv L(i/2) \equiv (1 + i)/2 \quad \text{and} \quad L((1 + i)/2) \equiv L(0) \equiv 0 \quad (\text{mod } \mathbb{Z}[i]).
\]

Thus the associated critical-postcritical diagram must take the form

\[
\begin{array}{cccc}
* & \mapsto & \bullet & \mapsto \bullet & \mapsto & \bullet & \mapsto * \\
\downarrow & & & & & & \\
\bullet & & & & & & 
\end{array}
\]

where each of the four postcritical points has ramification index 2. If we put the two critical points at \( \pm 1 \) and the postcritical fixed point at infinity, then this map will have the form

\[
f(z) = \alpha(z + z^{-1}) + \beta,
\]

with multiplier \( 1/\alpha \) at infinity. (Compare [M2].) Setting this equal to \( a^2 = 2i \), we see that \( \alpha = -i/2 \), and it is easy to check that \( \beta = 0 \). Thus \( f(z) = -i(z + z^{-1})/2 \) with

\[
-1 \mapsto i \mapsto 0 \mapsto -i \mapsto 1 \mapsto \infty
\]

The associated map \( L/G_4 \) can be obtained by identifying pairs of points of \( \mathbb{T}/G_2 \) under an
involution. In fact the map \( L/G_2 \) admits a unique involution \( z \mapsto -z \) which interchanges its two critical points. We can identify \( z \) with \(-z\) by introducing the variable \( \zeta = z^2 \), with
\[
\zeta \mapsto f(z)^2 = -(z^2 + 2 + z^{-2})/4.
\]
In other words
\[
\zeta \mapsto -(\zeta + 2 + \zeta^{-1})/4,
\]
with critical points \( \zeta = \pm 1 \). The critical orbits are given by \( 1 \mapsto -1 \mapsto 0 \mapsto \infty \), with schematic diagram
\[
* \mapsto * \mapsto \bullet \mapsto \bullet
\]
of type \( \{2, 4, 4\} \).

Both of these quadratic maps arise as examples of matings. In fact, there are exactly eight distinct quadratic Lattès maps. In the \( \{2, 2, 2, 2\} \) case, with a postcritical fixed point, the invariant \( a^2 \) can be either \( \pm 2i \) (as above) or \(-2\) or \((-3 \pm i\sqrt{7})/2\), while without a postcritical fixed point it can only be \((-3 \pm i\sqrt{7})/2\). In the \( \{2, 4, 4\} \) case, the unique example is described above, with \( a^4 = (2i)^2 = -4 \). Shishikura has shown that every one of these eight Lattès maps can be obtained in one or more ways as a mating. Compare [M5].

**Example 5.5.** The following example comes up in the study of maps of the complex projective plane with invariant elliptic curve. (See [BDM, §6.1].) If we take the quotient surface \( \mathbb{T}/G_3 \) as in 5.1, but take the linear map \( L(\tau) = -2\tau \), then \( L/G_3 \) is conjugate to the real fourth degree map
\[
f(z) = \frac{-z(z^3 + 2)}{2z^3 + 1},
\]
with double critical points at the cube roots of unity. Each critical point \( \omega \in G_3 \) maps to the fixed point \(-\omega\), so the appropriate diagram is:
\[
** \mapsto \bullet \quad ** \mapsto \bullet \quad ** \mapsto \bullet
\]
with diagram
\[
** \mapsto \bullet \quad ** \mapsto ** \quad ** \mapsto \bullet.
\]
This map commutes not only with the involution \( z \mapsto 1/z \) but also with the rotation \( z \mapsto \omega z \). However, if we replace the map \( L(\tau) = -2\tau \) by \( L(\tau) = 2\tau \), then we get a quite different map \( z \mapsto f(1/z) = 1/f(z) \), with diagram
\[
** \mapsto \bullet \quad ** \mapsto ** \quad ** \mapsto \bullet.
\]
In either case, we can obtain a model for \( L/G_6 \) by setting \( \zeta = z + 1/z \), so that the corresponding map on \( \mathbb{T}/G_6 \) is represented by
\[
\zeta \mapsto f(z) + 1/f(z) = \frac{-(\zeta^4 + 4\zeta - 6)}{2\zeta^3 - 6\zeta + 5}.
\]
In this case, two critical points \(-1 \pm \sqrt{3}\) map to the double critical point \(2\), which maps to the fixed point \(-2\). There is also a double critical point \(-1\) which maps to the fixed point \(+1\). The diagram follows:
\[
* \quad \quad \quad
** \mapsto \bullet \quad ** \mapsto \bullet
\]
\[
* \quad \quad \quad
2 \quad 6 \quad 3
\]
§6. Chebyshev Maps and Power Maps. In the spirit of the Lattès paper, we can consider a rational function \( f \) with exceptional set \( E_f \neq \emptyset \), together with a commutative diagram of the form

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ell} & \mathbb{C} \\
\theta \downarrow & & \theta \downarrow \\
\hat{\mathbb{C}} \setminus E_f & \xrightarrow{f} & \hat{\mathbb{C}} \setminus E_f,
\end{array}
\]

where \( \ell \) is to be an affine map \( \ell(t) = at + b \). We will assume that \( \theta \) is periodic, with lattice of periods precisely equal to the subgroup \( \mathbb{Z} \subset \mathbb{C} \), and that the induced map \( \Theta : \mathbb{C}/\mathbb{Z} \to \hat{\mathbb{C}} \setminus E_f \) is finite-to-one. Proceeding as in §2, we can then construct a finite cyclic group \( G_n \) of rotations of the cylinder \( \mathbb{C}/\mathbb{Z} \) so that \( \Theta(\tau') = \Theta(\tau) \) if and only if \( \tau' = \omega \tau \) for some \( \omega \in G_n \). Clearly the order of such a rotation group must be either one or two. Thus we are reduced to two cases:

**Case 1. Power Maps.** If \( G_n \) is the trivial group, so that \( \Theta \) maps \( \mathbb{C}/\mathbb{Z} \) diffeomorphically onto \( \hat{\mathbb{C}} \setminus E_f \), then the set \( E_f \) of exceptional points has two elements. After conjugating \( f \) by a Möbius automorphism, we may assume that \( E_f = \{0, \infty\} \), and take \( \theta(t) = \exp(2\pi it) \), using the linear map \( \ell(t) = at \) where \( a \) is a positive or negative integer with \( |a| \geq 2 \). The relation \( f_a(\theta(t)) = \theta(at) \) then implies that

\[
f_a(z) = z^a.
\]

Thus the degree \( d_{f_a} \) is equal to \( |a| \geq 2 \). The Julia set \( J(f_a) \) is the unit circle; and each period \( p \) orbit in the Julia set has multiplier \( a^p \).

**Case 2. Chebyshev Maps.** If \( n = 2 \) with \( G_2 = \{\pm 1\} \), then there is just one exceptional point which we may take to be the point at infinity so that \( f \) is a polynomial. Let

\[
\theta(t) = 2 \cos(2\pi t) = e^{2\pi it} + e^{-2\pi it}
\]

so that \( \theta(t') = \theta(t) \) if and only if \( t' \equiv t \pmod{\mathbb{Z}} \). The set of critical points of the induced map \( \Theta : \mathbb{C}/\mathbb{Z} \to \mathbb{C} \) coincides with the subgroup \( \text{Fix}(G_2, \mathbb{C}/\mathbb{Z}) = \{0, 1/2\} \) consisting of elements of order two in \( \mathbb{C}/\mathbb{Z} \). First consider the linear map \( L(\tau) = d\tau \) from \( \mathbb{C}/\mathbb{Z} \) to itself, where \( d \in \mathbb{Z}, \ d \geq 2 \). The associated map \( \Psi_d : \mathbb{C} \to \mathbb{C} \), defined by the identity

\[
\Psi_d(\theta(t)) = \theta(dt),
\]

is called the degree \( d \) **Chebyshev polynomial**.\(^5\) Alternatively, writing \( \theta(t) = 2 \cos(2\pi t) \) as \( w + 1/w \) where \( w = \exp(2\pi it) \), we can define \( \Psi_d \) by the identity

\[
\Psi_d(w + 1/w) = w^d + 1/w^d.
\]

For example

\[
\Psi_2(z) = z^2 - 2, \quad \Psi_3(z) = z^3 - 3z, \quad \Psi_4(z) = z^4 - 4z^2 + 2, \quad \ldots.
\]

The Julia set \( J(\Psi_d) \) is the closed interval \( \theta(\mathbb{R}) = [-2, 2] \), and the postcritical set \( \theta(\{0, 1/2\}) = \{-2\} \) is the boundary of this interval. The multiplier of any period \( p \) orbit in the interior of the interval has the form \( \pm d^p \).

---

\(^5\) The Russian letter \( \Psi \) is called “chi”, pronounced as in “chicken”. 
We can vary this construction by using the affine map \( L(\tau) = d\tau + 1/2 \). The corresponding polynomial map is \(-\Psi_d\), with
\[
-\Psi_d(\theta(t)) = \theta(at + 1/2).
\]
If the degree \( d \) is even the map \( \Psi_d \) is linearly conjugate to \(-\Psi_d\). However, for \( d \) odd these maps are dynamically distinct: The two postcritical points \( \pm 2 \) are fixed by \( \Psi_d \), but interchanged by \(-\Psi_d\). By definition, any map conjugate to either \( \Psi_d \) or \(-\Psi_d\) will be called a Chebyshev map.

A different approach would be to look for maps on a once or twice punctured sphere which are linearized by a complete flat orbifold metric. It is not hard to see that the maps listed above, namely
\[
z \mapsto z^{\pm d} \quad \text{and} \quad z \mapsto \pm \Psi_d(z),
\]
are the only possibilities up to conformal conjugacy. Compare [DH, §9].

§7. Some History: Lattès Maps before Lattès. Although the name of Lattès has become firmly attached to the construction studied in this paper, it actually occurs much earlier in the mathematical literature. Ernst Schröder, in a well known 1871 paper gave an explicit one-parameter family of examples as follows. Let \( x = \text{sn}(u) \) be the Jacobi sine function with elliptic modulus \( k \), defined by the equation
\[
\text{sn}(u) = \int_0^u \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}.
\]
This is a well defined doubly periodic meromorphic function. There are closely related doubly periodic functions \( \text{cn}(u) \) and \( \text{dn}(u) \) which satisfy
\[
\text{cn}^2(u) = 1 - \text{sn}^2(u) \quad \text{and} \quad \text{dn}^2(u) = 1 - k^2\text{sn}^2(u).
\]
Furthermore
\[
\text{sn}(2u) = \frac{2\text{sn}(u)\text{cn}(u)\text{dn}(u)}{1-k^2\text{sn}^4(u)}.
\]
(Compare [WW, §22.2].) Setting \( z = x^2 = \text{sn}^2(u) \) (and correcting a misprint in Schröder’s paper), it follows easily that there is a well defined rational function
\[
F(z) = \frac{4z(1-z)(1-k^2z)}{(1-k^2z^2)^2}
\]
of degree four which satisfies the semiconjugacy relation
\[
\text{sn}^2(2u) = F\left(\text{sn}^2(u)\right).
\]
In our terminology, \( F \) is a “Lattès map”, described 47 years before Lattès.

It is not hard to see that \( \text{sn}(u) \) has critical values \( \pm 1 \) and \( \pm 1/k \), and hence that \( \text{sn}^2(u) \) has critical values \( 1, 1/k^2, \infty, \) and \( 0 \). On the other hand the map \( F \) has three critical values \( 1, 1/k^2, \) and \( \infty, \) which satisfy the equation
\[
F(1) = F(1/k^2) = F(\infty) = 0 = F(0).
\]
Each of these critical values is the image under \( F \) of two distinct simple critical points.
Lucyan Böttcher in 1904 cited this same example (with a different version of the misprint). He was perhaps the first to think of this example from a dynamical viewpoint, and to use the term “chaotic” to describe the behavior of the sequence of iterates of \( F \). In fact he described an orbit \( z_0 \mapsto z_1 \mapsto \cdots \) as chaotic if for every convergent subsequence \( \{z_n\} \) the differences \( n_{i+1} - n_i \) are unbounded. (Note that this includes examples such as irrational rotations which are not chaotic in the modern sense.)

Böttcher actually cited a much earlier paper, written by Charles Babbage in 1815, for a fundamental property of semiconjugacies. Here is a loose translation from the Russian, using more modern notations:

**Babbage Principle.** If a function \( \phi(z) \) satisfies the condition \( \phi \circ F = f \circ \phi \) and if we know how to iterate \( f \), then we also know how to iterate \( F \) by means of the identity \( \phi \circ F^{\circ n} = f^{\circ n} \circ \phi \).

(Actually, in the application \( \phi \) is usually many-to-one, so it would be more accurate to say that if we can iterate \( F \) then we can iterate \( f \).) This principle is the basis for all such examples, as was clearly recognized by both Schröder and Böttcher. I have been unable to locate such a clear statement in Babbage’s paper, but Babbage certainly used the idea, for example to find periodic points of a semiconjugate map. (See Problem XI in [Ba, page 412].)

J. F. Ritt in the 1920’s carried out many further developments of these ideas.

**§8. Further Remarkable Properties.** This concluding section will outline a number of special properties shared by some or all quotients of affine maps. Recall that a Lattès map

\[
L/G_n : T/G_n \to T/G_n \quad \text{with} \quad T = \mathbb{C}/\Lambda
\]

is completely determined by the integer \( n \) together with the lattice \( \Lambda \) and the affine map \( L(\tau) = a\tau + b \), which must satisfy the conditions that

\[
G_n\Lambda = \Lambda, \quad \text{that} \quad a\Lambda \subset \Lambda, \quad \text{and that} \quad b \in \text{Fix}(G_n, T).
\]

**Definition:** Such a Lattès map will be called flexible if we can vary \( \Lambda \) and \( L \) continuously so as to obtain other Lattès maps which are not conformally conjugate to it.

**Lemma 8.1.** A Lattès map \( L/G_n : T/G_n \to T/G_n \) is flexible if and only if \( n = 2 \), and the affine map \( L(\tau) = a\tau + b \) has integer derivative, \( L' = a \in \mathbb{Z} \).

We can easily classify such maps into two kinds of connected one parameter families. In each case, the coefficients \( a \) and \( b \) will remain constant but \( T \) will vary through all possible conformal diffeomorphism classes.

- A flexible Lattès map with no postcritical fixed point has a unique normal form as follows. Let \( \mathbb{H} \) be the upper half-plane, and choose any \( \gamma \in \mathbb{H}/\mathbb{Z} \). Take \( T = \mathbb{C}/\Lambda \) where \( \Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z} \) is the lattice generated by \( 1 \) and \( \gamma \), and take \( L(\tau) = a\tau + 1/2 \) where \( a \geq 3 \) is any odd integer.

- For a flexible Lattès map which does have a postcritical fixed point, we can take \( L(\tau) = a\tau \) with \( a \geq 2 \). Such a map is determined up to holomorphic conjugacy by this integer \( a \), together with a specification of the lattice \( \Lambda \). In fact we can again take \( \Lambda \) to be the lattice \( \mathbb{Z} \oplus \gamma\mathbb{Z} \) spanned by \( 1 \) and \( \gamma \), but now require \( \gamma \) to satisfy the Siegel inequalities (10). Such a \( \gamma \) is uniquely determined by the conformal equivalence class of
$T = \mathbb{C}/\Lambda$, except for the identification $\gamma = x + iy \mapsto -\gamma = -x + iy$ for $\gamma$ on the boundary of the Siegel region.

These statements, as well as the statement of 8.1, follow easily from the classification theorems in §4. □

The class of flexible Lattès maps has several interesting properties. They are the only known rational maps which admit a continuous family of deformations preserving the topological conjugacy class. In fact the $C^\infty$ conjugacy class remains almost unchanged as we deform the torus. Differentiability fails only at the postcritical points, and the multipliers of periodic orbits remain unchanged even at postcritical points.

Closely related is the following:

**Fundamental Conjecture.** The flexible Lattès maps are the only rational maps which admit an “invariant line field” on their Julia set.

By definition $f$ has an **invariant line field** if its Julia set $J$ has positive Lebesgue measure, and if there is a measurable $f$-invariant field of real one-dimensional subspaces of the tangent bundle of $\hat{\mathbb{C}}$ restricted to $J$. The importance of this conjecture is demonstrated by the following. (See [MSS], and compare the discussion in [Mc2].)

**Theorem of Manè, Sad and Sullivan.** If this Fundamental Conjecture is true, then hyperbolicity is dense among rational maps. That is, every rational map can be approximated by a hyperbolic map.

To see that every flexible Lattès map has such an invariant line field, note that any torus $\mathbb{C}/(\mathbb{Z} + \gamma \mathbb{Z})$ is foliated by an $L$-invariant family of circles $\mathcal{S}(t) = \text{constant}$.

If $f$ is the associated Lattès map $L/G_2$, then this circle foliation maps to an $f$-invariant foliation of $J(f) = \hat{\mathbb{C}}$ which is not only measurable but actually smooth, except at the four postcritical points.

Here is another characterization.

**Lemma 8.2.** A Lattès map is flexible if and only if the multiplier for every periodic orbit is an integer.

**Proof.** This follows from Corollary 2.6. If $n > 2$ or if $a \notin \mathbb{Z}$, then we can find infinitely many integers $p > 0$ so that $\omega a^p \not\in \mathbb{Z}$ for some $\omega \in G_n$. The number of fixed points of the map $\omega L^p$ grows exponentially with $p$ (the precise number is $|\omega a^p - 1|^2$), and each of these maps to a periodic point of the associated Lattès map $f$. If we exclude the three or four postcritical points, then the derivative of $f^p$ at such a point will be $\omega a^p$, so that the multiplier of this periodic orbit cannot be an integer. □

It seems very likely that Chebyshev-like maps and flexible Lattès maps are the only rational maps such that the multiplier of every periodic orbit is an integer.

Let us define the **multiplier spectrum** of a degree $d$ rational map $f$ to be the function which assigns to each $p \geq 1$ the unordered list of multipliers at the $d^p + 1$ (not necessarily distinct) fixed points of the iterate $f^p$. Call two maps **isospectral** if they have the same multiplier spectrum.

**Theorem of McMullen.** The flexible Lattès maps are the only rational maps which admit non-trivial isospectral deformations. The conjugacy class of any rational map which is not flexible Lattès is determined, up to finitely many choices,
by its multiplier spectrum.

This is proved in [Mc1, §2]. McMullen points out that the Lattès maps $L/G_2$ associated with imaginary quadratic number fields provide a rich source of isospectral examples. First note the following.

**Lemma 8.3.** Two Lattès maps $L/G_2 : \mathbb{T}/G_2 \to \mathbb{T}/G_2$ are isospectral if and only if they have the same derivative $L' = a$, up to sign, and the same numbers of periodic orbits of various periods within the postcritical set $P_f$.

**Proof Outline.** The number of fixed points of the map $\pm L^p$ on the torus can be computed as $| \pm a^p - 1 |^2$. Each pair $\{ \pm t \}$ of such fixed points corresponds to a single fixed point of the corresponding iterate $f^p$, where $f \cong L/G_2$. Whenever $t \neq -t$ on the torus, the multiplier of $f^p$ at this fixed point is also equal to $\pm a^p$. For the exceptional points which belong to the subset Fix($G_2, \mathbb{T}$) and correspond to postcritical points of $f$, the multiplier of $f^p$ is equal to $a^{2p}$. Thus, to determine the multiplier spectrum completely, we need only know how many points of various periods there are in the postcritical set. \[\Box\]

**Examples.** Let $\xi = i \sqrt{k}$ where $k > 0$ is a squarefree integer, and let $a = m\xi + n$. Then for each divisor $d$ of $m$ the lattice $\mathbb{Z}[\xi] \subset \mathbb{C}$ is a $\mathbb{Z}[a]$-module. Hence the linear map $L(\tau) = a\tau$ acts on the associated torus $\mathbb{T}/\mathbb{Z}[\xi]$. If $m$ is highly divisible, then there are many possible choices for $d$. Suppose, to simplify the discussion, that $mk$ and $n$ are both even, so that $a^2$ is divisible by two in $\mathbb{Z}[a]$. Then multiplication by $a^2$ acts as the zero map on the group Fix($G_2, \mathbb{T}$) consisting of elements of order two in $\mathbb{T}$. Thus $0 = L(0)$ is the only periodic point under this action, hence the image of $0$ in $\mathbb{T}/G_2$ is the only postcritical periodic point of $L/G_2$. It then follows from 8.3 that these examples are all isospectral.

Anna Zdunik [Z] has given a completely different characterization of Lattès maps, using only measure theoretic properties. It is not hard to see that the standard probability measure on the flat torus pushes forward under $\Theta$ to an ergodic $f$-invariant probability measure on the Riemann sphere. This measure is smooth and in fact real analytic, except for mild singularities at the postcritical points. Furthermore, it is balanced, in the sense that the preimage $f^{-1}(U)$ of any simply connected subset of $\hat{\mathbb{C}} \setminus P_f$ is a union of $n$ disjoint sets of equal measure. Hence it coincides with the unique measure of maximal entropy, as constructed by Lyubich [Ly], or by Freire, Lopez and Mané [FLM]. The converse theorem is much more difficult:

**Theorem of Zdunik.** The Lattès maps are the only rational maps for which the measure of maximal entropy is absolutely continuous with respect to Lebesgue measure.

We can think of the maximal entropy measure $\mu_{\text{max}}$ as describing the asymptotic distribution of random backward orbits. That is, if we start with any non-exceptional initial point $z_0$, and then use a fair $d$-sided coin or die to iteratively choose a backward orbit

$$\ldots \mapsto z_{-2} \mapsto z_{-1} \mapsto z_0,$$

then $\{z_n\}$ will be equidistributed with respect to $\mu_{\text{max}}$.

This measure $\mu_{\text{max}}$ always exists. An absolutely continuous invariant measure is much harder to find, and an invariant measure which is ergodic and belongs to the same measure
class as Lebesgue measure is even rarer. However Lattès maps are not the only examples: Mary Rees [Re] has shown that for every degree \( d \geq 2 \) the moduli space of degree \( d \) rational maps has a subset of positive measure consisting of maps \( f \) which have an ergodic invariant measure \( \mu \) in the same measure class as Lebesgue measure. Such a measure is clearly unique, since Lebesgue almost every forward orbit \( z_0 \mapsto z_1 \mapsto z_2 \cdots \) must be equidistributed with respect to \( \mu \).

Using these ideas, an equivalent statement of Zdunik’s Theorem would be the following.

**Corollary.** Lattès maps are the only rational maps for which Lebesgue randomly chosen forward orbits have the same asymptotic distribution as randomly chosen backward orbits.

In general, different rational maps have different invariant measures (except that every invariant measure for \( f \) is also an invariant measure for its iterates \( f^{op} \)). However, every Lattès map \( L/G_n \) shares its measure \( \mu_{\text{max}} \) with a rich collection of Lattès maps \( \tilde{L}/G_n \) where \( \tilde{L} \) ranges over all affine maps of the torus which commute with the action of \( G_n \). This collection forms a semigroup which is not finitely generated. (If we consider only the linear torus maps \( \tilde{L}(\tau) = \alpha \tau \), then this semigroup is commutative.) I don’t know any other examples, outside of the Chebyshev-like maps, of a semigroup of rational maps which is not finitely generated, and which shares a common non-atomic invariant measure.

Closely related is the study of commuting rational maps. Here we have the following result which was proved by Ritt [R2], and later by Eremenko [E].

**Theorem of Ritt and Eremenko.** If \( f \circ g = g \circ f \) where \( f \) and \( g \) are rational maps of degree at least two, and if no iterate of \( f \) is equal to an iterate of \( g \), then they are both quotients of affine maps, and have a common orbifold metric. In particular, they are either both Lattès maps, both Chebyshev maps, or both power maps.

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**References.** (For historical information see [A]; for general mathematical background see for example [M3]; and for higher dimensional examples see [BL] or [Du].)

On Lattès Maps


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