Elliptic Curves as Attractors in $\mathbb{P}^2$

Part 1: Dynamics

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Abstract

A study of the extent to which an elliptic curve in the real or complex projective plane can be an attractor under some rational map. We describe a rich family of examples, with widely varying dynamical behaviors. In some cases we provide proofs, but in other cases the discussion is empirical, based on numerical computation.

1 Introduction.

In the paper “Self-maps of $\mathbb{P}^2$ with invariant elliptic curves”, Bonifant and Dabija constructed a number of examples of rational maps $f$ of the real or complex projective plane with an elliptic curve $C = f(C)$ as invariant subset. (See [BD] in the list of references.) This case of a curve of genus one is of particular interest since genus zero examples are easy to construct, while higher genus examples cannot exist. The present paper studies the extent to which such an invariant elliptic curve $C \subseteq \mathbb{P}^2$ can be an “attractor”. Here we must distinguish several possible degrees of attraction. By definition, $C$ will be called:

- a measure-theoretic attractor if its attracting basin, consisting of all points whose orbits converge to $C$, has positive Lebesgue measure;
- a trapped attractor if there it has a compact trapping neighborhood $N$ such that $f(N) \subseteq N$ and $C = \bigcap_n f^{\circ n}(N)$;
- strongly attracting if there is a neighborhood $N$ and numbers $\lambda < 1$ and $c > 0$ such that the Riemannian distance $r(p)$ from $p$ to the curve satisfies $r\left(f^{\circ n}(p)\right) \leq c \lambda^n r(p)$ for all $p \in N$.

In both the real and complex cases, we provide examples in which $C$ is a measure theoretic attractor. In fact there are examples in which there are two distinct smooth measure theoretic attractors whose attracting basins are so intermingled that they have the same topological closure. We provide an example of a singular real elliptic curve which is strongly attracting. However, we prove that a nonsingular complex elliptic curve cannot be strongly attracting, and conjecture that it cannot have a trapping neighborhood. In fact, it seems likely that the attracting basin of a nonsingular complex elliptic curve cannot have interior points, so that the set of points not attracted to $C$ must be everywhere dense.

We describe examples in which it seems possible that the elliptic curve is a “global attractor” so that its attracting basin has full measure. We also provide a family of examples where there appears to be a pair of Herman rings as attractor, with an open neighborhood as its attracting basin. However,
in many cases the dynamical behavior is sufficiently confusing that we are not even sure what to conjecture.

(Some References: For concepts of attractor, see [ABS], [M1], [GI], and for exotic examples see [AKYY], [Ka]. For dynamics in \( \mathbb{P}^2 \) see [FS], [FW], [JW], [S], [U].)

Here is a brief outline. Section 2 describes the \textit{transverse Lyapunov exponent} along an invariant elliptic curve. This is a primary indicator of whether or not the curve is attracting. Methods for actually computing this transverse exponent will be described in Part 2, the sequel to this paper, although the conclusions of such computation will often be quoted below. Section 3 describes a three parameter family of maps of particular interest, and Section 4 describes six explicit examples, based on numerical computation. Three of these suggest that the real or complex Fermat curve can be a measure-theoretic global attractor, with almost all orbits converging towards it; while Example 4.4 suggests that a cycle of two Herman rings can be an attractor in the complex case. (Possibly it can even be a global attractor?) So far, the discussion has been empirical. However, Sections 5 and 6 provide explicit proofs. Example 5.1 describes maps with three different attractors with thoroughly intermingled basins, all of positive measure. (Compare [AKYY].) Two of these basins are dense in the Julia set, while the third basin, which is everywhere dense, is the Fatou set. Theorem 6.3 provides examples of singular real elliptic curves which are strongly attracting under suitable rational maps; while Theorem 6.4 shows that a smooth complex elliptic curve can never be strongly attracting. (We don’t know whether singular complex curves or non-singular real curves can be strongly attracting.) The last section provides a brief description of open problems.

The discussion will usually concentrate on the complex case, although the color illustrations (Figures 3 through 10) will necessarily illustrate the real case.

2 \textbf{The Transverse Lyapunov Exponent.}

One particular virtue of an elliptic curve \( C \subset \mathbb{P}^2(\mathbb{C}) \) is that any self-map must be linear in terms of suitable coordinates. In fact such a curve is conformally isomorphic to some flat torus \( \mathbb{C}/\Omega \), where \( \Omega \) is a lattice. More precisely, there is a holomorphic uniformizing map

\[ v : \mathbb{C}/\Omega \to C \]

which is biholomorphic in the case of a smooth elliptic curve, and is one-to-one except on finitely many singular points in the case of a singular curve. Any holomorphic self-map of \( C \) lifts to a holomorphic self-map of \( \mathbb{C}/\Omega \) which is necessarily linear, \( t \mapsto at + b \). The coefficient \( a \) is called the \textit{multiplier} of \( f \). Note that the product \( a \Omega \) is a sublattice of index \( d \) in \( \Omega \), and that \( |a|^2 = d \), where \( d \) is the degree of the homogeneous polynomials which define the map \( f \). The topological degree of \( f \) restricted to \( C \) is \( d \), while the topological degree on \( \mathbb{P}^2(\mathbb{C}) \) is \( d^2 \). The algebraic number of fixed points is \( d^2 + d + 1 \).

It follows easily that the normalized Lebesgue measure on \( \mathbb{C}/\Omega \) pushes forward to a canonical smooth probability measure \( \mu \) on \( C \) which is invariant under every non-constant self-map. We will always assume that \( d \geq 2 \), so that this canonical measure \( \mu \) is ergodic.

In the case of an elliptic curve defined by equations with real coefficients, the real curve

\[ C_\mathbb{R} = C \cap \mathbb{P}^2(\mathbb{R}) \]

has either one or two connected components. If \( f \) is a rational map of \( \mathbb{P}^2(\mathbb{R}) \) with \( f(C_\mathbb{R}) \subset C_\mathbb{R} \), then the image \( C_\mathbb{R}^0 = f(C_\mathbb{R}) \) is necessarily just one connected component. In this case we have a uniformizing map
$\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}_p^0$, such that $f$ corresponds to a map on $\mathbb{R}/\mathbb{Z}$ which is linear with constant integer multiplier. Again the invariant curve $\mathbb{C}_p^0$ has a canonical invariant probability measure.

To fix ideas we will concentrate on the complex case, but constructions in the real case are completely analogous. The notation $T\mathbb{P}^2|_\mathcal{C}$ will be used for the complex 2-plane bundle of vectors tangent to $\mathbb{P}^2(\mathbb{C})$ at points of the submanifold $\mathcal{C}$, and the abbreviated notation $T(\mathbb{P}^2/\mathcal{C})$ will be used for the complex line bundle over $\mathcal{C}$ having the quotient vector space $T_p(\mathbb{P}^2/\mathcal{C}) = T_p\mathbb{P}^2/T_p\mathcal{C}$ as typical fiber. In other words, there is a short exact sequence

$$0 \rightarrow T\mathcal{C} \rightarrow T\mathbb{P}^2|_\mathcal{C} \rightarrow T(\mathbb{P}^2/\mathcal{C}) \rightarrow 0$$

of complex vector bundles over $\mathcal{C}$. It will be convenient to refer to $T(\mathbb{P}^2/\mathcal{C})$ as the “normal bundle” of $\mathcal{C}$, although that designation isn’t strictly correct. If $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ with $f(\mathcal{C}) \subset \mathcal{C}$, then $f$ induces linear maps

$$f'_n(p) : T_p(\mathbb{P}^2/\mathcal{C}) \rightarrow T_{f(p)}(\mathbb{P}^2/\mathcal{C}),$$

and these linear maps collectively form a fiberwise linear self-map $f'_n : T(\mathbb{P}^2/\mathcal{C}) \rightarrow T(\mathbb{P}^2/\mathcal{C})$.

Now choose a metric on this complex normal bundle. That is, choose a norm $||\tilde{v}||_n$ on each quotient vector space $T_p(\mathbb{P}^2/\mathcal{C})$ which depends continuously on $\tilde{v}$, vanishes only on zero vectors, and satisfies $||tv||_n = |t||||\tilde{v}||_n$. (The subscript $n$ stands for “normal”.) Then the linear map $f'_n$ of (1) has an operator norm

$$||f'_n|| = ||f'_n||_n/||\tilde{v}||_n$$

which is well defined and satisfies the chain rule. Here $\tilde{v}$ can be any non-zero vector in the fiber $T_p(\mathbb{P}^2/\mathcal{C})$ over $p$. By definition, the average value

$$\text{Lyap}_\mathcal{C}(f) = \int_{\mathcal{C}} \log ||f'_n|| d\mu(p)$$

is described as the transverse Lyapunov exponent along the invariant curve $\mathcal{C}$. Using the fact that the metric $\mu$ is invariant under $f$, it is not hard to check that this average value is independent of the choice of metric. By the Birkhoff Ergodic Theorem, it coincides with the rate of exponential growth

$$\text{Lyap}_\mathcal{C} = \lim_{k \to \infty} (1/k) \log |(f^\circ k)'_n(p)|$$

for almost every choice of initial point $p \in \mathcal{C}$.

Thus a negative value of $\text{Lyap}_\mathcal{C}$ means that under iteration of $f$ almost any point which is “infinitesimally close” to $\mathcal{C}$ will converge towards $\mathcal{C}$. A key role in this case is played by the stable sets of the various points $p \in \mathcal{C}$. By definition, the stable set of $p$ is the union of all connected sets containing $p$ for which the diameter of the $n$-th forward image tends to zero as $n \to \infty$. Many such stable sets are smooth curves. With a little imagination, some of these are clearly visible in Figures 3 through 10. It is natural to expect that negative values of $\text{Lyap}_\mathcal{C}$ will imply that $\mathcal{C}$ is a measure-theoretic attractor. (A sketch of a proof is given in [AKYY]; and a very special case will be proved in Theorem 5.2 below.)

On the other hand, if $\text{Lyap}_\mathcal{C} > 0$ then almost any “infinitesimally close” point will be pushed away from $\mathcal{C}$. It also seems natural to conjecture that positive values should imply that the attracting basin of $\mathcal{C}$ has measure zero. (However, compare Remark 5.5.)
Remark 2.1. Numerical computations are extremely delicate near $C$. Thus it is essential to work with multiple precision arithmetic; but even so, numerical simulation of the dynamics must be understood as a hint of the true state of affairs, rather than a definitive answer. One surprising aspect of these maps is that in some cases orbits tend to spend quite a bit of time extremely close to $C$ even when the transverse exponent is positive. (Compare Figures 5 and 7.)

3 The Desboves family.

Let $\Phi(x, y, z)$ be the homogeneous polynomial $x^3 + y^3 + z^3$. The Fermat curve $F$ is defined as the locus of zeros $\Phi(x, y, z) = 0$ in the projective plane $\mathbb{P}^2$. (Here we can work either over the real numbers or over the complex numbers.) All of the examples in §4 will belong to a family of 4th-degree rational maps of $\mathbb{P}^2$ which carry this Fermat curve into itself, as introduced in [BD] §6.3. We will call these Desboves maps, since they arise from a simple perturbation of a map originally described by A. Desboves in [De]. The construction starts with the family of cubic curves $\Phi_k = 0$, where

$$\Phi_k(x, y, z) = x^3 + y^3 + z^3 - 3kxyz .$$

These are smooth elliptic curves, except for the special cases $k^3 = 1$ and the limiting case $k = \infty$. These curves provide a smooth foliation of the complex projective plane, except at a set $S_0 \cup S_1 \cup S_2$ of 21 singular points. Here the subscript on $S_j$ indicates the number of coordinates $x$, $y$, $z$ which are zero. The set $S_0$ consists of the nine points of $\mathbb{P}^2(\mathbb{C})$ which satisfy the equations $x^3 = y^3 = z^3$, while $S_1$ consists of the nine points which satisfy the equations

$$xyz = x^3 + y^3 + z^3 = 0 ,$$

and $S_2$ consists of the three points where two out of the three coordinates $x$, $y$, $z$ are zero. (Seven of the points of $S_0 \cup S_1 \cup S_2$ lie in the real plane, and are visible in Figure 1.)

The classical Desboves map

$$F_0(x, y, z) = \left( x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3) \right)$$

from $\mathbb{C}^3$ to itself induces a self-map $f_0$ of $\mathbb{P}^2(\mathbb{C})$ which maps each of these elliptic curves onto itself with multiplier $-2$. Geometrically, it is defined by the property that the line from $p$ to $f_0(p)$ is tangent to the elliptic curve through $p$. Its set of fixed points on each curve $\Phi_k = 0$ coincides with the set $S_1$ described above, and can also be identified with the set of points of inflection on any one of these curves, or as the set of points where all of these curves intersect. This map $f_0$ is not everywhere defined: it has a twelve point set $S_0 \cup S_2$ of points of indeterminacy. However, for any specified $k$, if we replace $F_0$ by the sum

$$F_L(x, y, z) = F_0(x, y, z) + L(x, y, z) \Phi_k(x, y, z)$$

where $L$ is any linear map from $\mathbb{C}^3$ to itself, then we obtain a new map $f_L$ of $\mathbb{P}^2(\mathbb{C})$ which coincides with $f_0$ on the particular curve $\Phi_k(x, y, z) = 0$. For a generic choice of $L$, the resulting map $f_L$ of $\mathbb{P}^2(\mathbb{C})$ is well defined everywhere. The set of fixed points of $f_L$ on the curve $\Phi_k = 0$ is precisely the set $S_1$ mentioned above.

To simplify the discussion, we will restrict attention to the case $k = 0$, taking

$$\Phi(x, y, z) = \Phi_0(x, y, z) = x^3 + y^3 + z^3 ,$$

and will take a linear map $L$ which is described by a diagonal matrix, $L(x, y, z) = (ax, by, cz)$.
Figure 1: “Foliation” of the real projective plane by the family of elliptic curves $\Phi_k = 0$, which intersect only at their three common inflection points. Here $\mathbb{P}^2(\mathbb{R})$ is represented as a 2-sphere with antipodal points identified. In the limiting case as $k \to \infty$, the curve degenerates to the union $xyz = 0$ of the three coordinate lines. These intersect at the points $(-1,0,0), (0,1,0)$ and $(0,0,1)$, which are respectively near the left, top, and center of the figure.

**Definition 3.1.** The resulting 3-parameter family of maps of the real or complex projective plane will be called the family of *Desboves maps*. If we use the notation $(x : y : z)$ for a point of $\mathbb{P}^2$ then these maps $f = f_{a,b,c}$ are given by the formula

$$f(x : y : z) = \left( x(y^3 - z^3 + a\Phi) : y(z^3 - x^3 + b\Phi) : z(x^3 - y^3 + c\Phi) \right), \tag{3}$$

where $a, b, c$ are the parameters. Each such $f$ maps the Fermat curve $F$, defined by the equation $\Phi(x, y, z) = 0$, into itself. Furthermore, each $f$ maps each of the coordinate lines $x = 0$ or $y = 0$ or $z = 0$ into itself.

**Remark 3.2. Symmetries.** If we permute the three parameters $(a, b, c)$ cyclically, then clearly we obtain a new map $f_{b,c,a}$ which is holomorphically conjugate to $f_{a,b,c}$. We can generalize this construction very slightly by allowing odd permutations also, but changing signs. If $S_3$ is the symmetric group consisting of all permutations $i \mapsto \sigma_i$ of the three symbols $\{1, 2, 3\}$, then $S_3$ acts as a group of rotations of $\mathbb{R}^3$ or $\mathbb{C}^3$ as follows. For each $\sigma \in S_3$ let

$$\sigma(z_1, z_2, z_3) = \text{sgn}(\sigma) \left( z_{\sigma_1}, z_{\sigma_2}, z_{\sigma_3} \right). \tag{4}$$

This action will be called a *sign-corrected permutation of coordinates*. Now a brief computation shows that the homogeneous map $F_{a,b,c}$ of $\mathbb{R}^3$ or $\mathbb{C}^3$ is linearly conjugate to the map

$$\sigma \circ F_{a,b,c} \circ \sigma^{-1} = F_{\sigma(a,b,c)}.$$

It follows that the associated map $f_{a,b,c}$ of the projective plane is holomorphically conjugate to the map $f_{\sigma(a,b,c)}$. One can check that these are the only holomorphic conjugacies between Desboves maps (for example, by making use of the eigenvalues of the first derivative of $f$ at the 21 fixed points).
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In the complex case, note also that each Desboves map \( f \) commutes with a finite group \( G \cong \mathbb{Z}/3 \times \mathbb{Z}/3 \) of symmetries of the projective plane. In fact \( f \circ g = g \circ f \) for each \( g \) in the group \( G \) consisting of all automorphisms

\[
g(x : y : z) = (\alpha x : \beta y : z) \quad \text{with} \quad \alpha^3 = \beta^3 = 1 .
\]  

**Remark 3.3. A closely related map.** It is sometimes convenient to pass to the quotient space \( \mathbb{P}^2/\mathbb{G} \) which is isomorphic to \( \mathbb{P}^2 \) itself, but with coordinates \((x^3 : y^3 : z^3)\). If we introduce variables \( \xi = x^3, \eta = y^3, \) and \( \zeta = z^3, \) and set \( \sigma = \xi + \eta + \zeta, \) then the map \( (x : y : z) \mapsto (\xi : \eta : \zeta) \) transforms the Fermat curve \( \mathcal{F} \) to a line \( \sigma = 0 \). Under this transformation, the Desboves map (3) is semiconjugate to a different rational map

\[
(\xi : \eta : \zeta) \mapsto \left( (\xi(\eta - \zeta + a\sigma)^3 : \eta(\xi - \zeta + b\sigma)^3 : \zeta(\xi - \eta + c\sigma)^3) \right),
\]
also of degree four. Evidently this new map carries the line \( \sigma = 0 \) into itself by a Lattès map, that is the image of a rigid torus map under a holomorphic semiconjugacy. (Compare [M2].)

**Remark 3.4. Fixed Points.** Generically, each Desboves map has 21 fixed points. However, there are two kinds of exception:

- If the product \( abc(a+b+c) \) is zero, then one or more of the fixed points will be replaced by an indeterminacy point.
- If one or more of the differences \( b-a, c-b, \) or \( a-c \) is equal to +1, then there is an entire line of fixed points.

There are always nine fixed points on the Fermat curve \( \mathcal{F} \), forming the set \( S_1 \subset \mathcal{F} \) described above (that is, the intersection of \( \mathcal{F} \) with the locus \( xyz = 0 \)). Consider for example the three points \((0 : \sqrt[3]{-1} : 1)\) obtained by intersecting the curve \( \mathcal{F} \) with the invariant line \( x = 0 \). If we introduce the coordinate \( Y = y/z \in \mathbb{C} \) on this line, then the restriction of the map \( f \) to this line is a rational map given by the formula

\[
Y \mapsto Y \frac{bY^3 + (b+1)}{(c-1)Y^3 + c},
\]
with fixed points at \( Y \) equal to zero, infinity, and at the cube roots of \(-1\). A brief computation shows that the derivative of this one variable map at these fixed points is respectively

\[
\frac{(b+1)/c, (c-1)/b, \text{and } 3(c-b) - 2 \text{ counted three times for the three intersections with } \mathcal{F}}{.}
\]  

(Something very special occurs when \( c = b+1 \). In that case, all five derivatives are +1, and in fact it is easy to check that every point on the line \( x = 0 \) is fixed under \( f \).) Similarly, permuting the coordinates cyclically, we obtain corresponding formulas for the invariant lines \( y = 0 \) and \( z = 0 \).

**Remark 3.5. The Attracting Basin.** According to [BD] Theorem 5.4, if \( \mathcal{C} = f(\mathcal{C}) \) is any invariant elliptic curve, then the set of iterated preimages of any point of \( \mathcal{C} \) is everywhere dense in the Julia set \( J(f) \). It seems likely that the following further statements are true:
Conjecture. The entire attracting basin $B(\mathcal{C})$, consisting of all points whose forward orbits converge to $\mathcal{C}$, is contained in the Julia set $J(f)$; hence the closure $\overline{B(\mathcal{C})}$ is precisely equal to $J(f)$. Furthermore, $B(\mathcal{C})$ has no interior points, so the closure of the complementary set $\mathbb{P}^2(\mathbb{C}) \setminus B(\mathcal{C})$ consisting of points which are not attracted to $\mathcal{C}$, is the entire plane $\mathbb{P}^2(\mathbb{C})$.

In the Desboves case, the second statement would follow from the first, as follows. We assume that points outside of the Julia set are not in the basin $B = B(F)$, so we need only show that every point of $J$ can be approximated by points outside of $B$. Since the average of the differences $c-b$, $b-a$, $a-c$ is zero, it follows from (6) that the average of the transverse derivatives at the fixed points of $f$ in $F$ is $-2$. Hence at least one of these fixed points is strictly repelling. Suppose for example that the point $(0: -1: 1)$ is repelling within the line $x = 0$. Then the intersection of the basin $B$ with this line consists only of countably many iterated preimages of $(0: -1: 1)$. Therefore there are points $(0: y: z)$ arbitrarily close to $(0: -1: 1)$ which are not in this basin. Since every point of $J$ can be approximated by iterated preimages of $(0: -1: 1)$, it can also be approximated by iterated preimages of such points $(0: y: z)$, as required.

On the other hand, some of these fixed points on $F$ may be attracting in the transverse direction. For example, if $|3(c-b) - 2| < 1$ then each of the three fixed points where $F$ intersects the line $x = 0$ is a saddle, repelling along the Fermat curve, but attracting along this line, which intersects it transversally. The stable manifold for such a saddle point can be identified with its immediate attracting basin within the line $x = 0$. It is not hard to check that this stable manifold is contained in the Julia set. Hence its iterated preimages must be dense in the Julia set.

The attraction will be particularly strong if $c-b = \frac{2}{3}$, so that the transverse derivative $3(c-b) - 2$ is zero, or in other words so that the associated fixed points are transversally superattracting. Similarly, the transverse derivative at the three points where $y = 0$ (or where $z = 0$) is zero if and only if $a-c = \frac{2}{3}$ (or respectively $b-a = \frac{2}{3}$).

Definition 3.6. We will say that $f$ belongs to the two-thirds family if two of the three differences $b-a$, $c-b$ and $a-c$ are equal to $\frac{2}{3}$, or equivalently if two out of three of the fixed points on the Fermat curve are transversally superattracting. Since the average of the three values of the transverse derivative is always $-2$, if two out of three values are zero then it follows that the remaining one is $-6$ (rather strongly repelling). To fix our ideas, let us suppose that

$$ (a,b,c) = (b - \frac{2}{3}, b, b + \frac{2}{3}) , $$

so that the transverse derivative is zero when $x = 0$ or $z = 0$. The associated transverse Lyapunov exponent, plotted as a function of $b$, is shown in Figure 2. In the real case, this transverse exponent is negative (that is attracting) if and only if $|b| < 0.901\ldots$, while in the complex case is is negative if and only if $b < 0.274\ldots$. See Part 2 of this paper for such computations.

4 Empirical Examples.

This section will provide empirical discussions of six examples from the Desboves family of §3. Four of these six examples belong to the “two-thirds” subfamily of Definition 3.6.

Example 4.1. The Fermat curve as a global attractor? If we choose Desboves parameters $(b - \frac{2}{3}, b, b + \frac{2}{3})$ with $|b|$ small, then the transverse Lyapunov exponent is negative in both the real
Figure 2: Graph of the transverse exponents along the real and complex Fermat curves as functions of the middle parameter $b$ for the "two-thirds family" of Definition 3.6, with Desboves parameters $(b - \frac{2}{3}, b, b + \frac{2}{3})$. The lower graph represents the real case, with a transverse exponent which is strictly smaller (more attracting). In both cases the function is even, with a sharp minimum at $b = \pm 1/9$. The box encloses the region $-1 \leq b \leq 1$ with $-2.1 \leq \text{Lyap}_F \leq 1$.

and complex cases. Numerical computation suggests that nearly all orbits actually converge to the Fermat curve. (Perhaps even all but a set of measure zero?) As an example, consider the case $(a, b, c) = \left(-\frac{2}{3}, 0, \frac{2}{3}\right)$. Using the Gnu multiple precision arithmetic package, and starting with several thousand randomly chosen points on the real or complex projective plane, one can check that all orbits land on the curve, to the specified accuracy, within a few hundred iterations. Of course, even if we could work with infinite precision arithmetic, such a computation could not prove that a given orbit converges to the curve, and also could not rule out the possibility of other attractors with extremely small basins. In fact it seems possible that periodic attractors with high period and small basin exist for a dense open set of parameter values. This case $b = 0$ is rather special in one way, since the map $f_{-2/3, 0, 2/3}$ has points of indeterminacy; namely those points where $(x^3 : y^3 : z^3)$ is equal to either $(1 : 7 : 1)$ or $(0 : 1 : 0)$. However, the behavior for small non-zero values of $b$ seems qualitatively similar.

**Example 4.2. An even stronger attractor.** The case $b = \pm \frac{1}{9}$ yields a even more strongly attracting Fermat curve, as illustrated in Figure 3. The transverse derivative has a simple zero at the point $(-1, 1, 0)/\sqrt{2}$ to the upper left of the figure, and a double zero at the point $(0, -1, 1)/\sqrt{2}$ near the bottom. A numerical search suggests that this is the "most attracting" example, in the sense that the transverse exponent takes its most negative value of $-2.0404\ldots$ for the real map (or $-0.6801\ldots$ for the complex map). Certainly these are the extreme values for real parameters within the two-thirds family, as graphed in Figure 2.

**Note on Pictorial Conventions.** Each of the color pictures which follow shows the real projective plane, represented as a unit 2-sphere with antipodal points identified, as in Figure 1. The $x$-axis (towards the right) and the $y$-axis (pointing almost vertically) are close to the plane of the paper, with
Figure 3: (Example 4.2.) On the left: dynamics on the real projective plane for the Desboves map in the two-thirds family with parameters \((a, b, c) = \left(-\frac{5}{9}, \frac{1}{9}, \frac{7}{9}\right)\). On the right: plot of \(|x|^2, |y|^2, |z|^2\) and \(|\Phi|\) as functions of the number of iterations for a typical randomly chosen complex orbit. Here each orbit point \((x : y : z)\) has been normalized so that \(|x|^2 + |y|^2 + |z|^2 = 1\). In this run, it took 23 iterations to come close enough to the Fermat curve so that \(|\Phi|\) appears to be zero on the graph.

Figure 4: (See Example 4.3.) For the map with Desboves parameters \((\frac{1}{3}, 0, -\frac{1}{3})\), the Fermat curve again seems to attract all or nearly all orbits in both the real and complex cases.
Figure 5: (Example 4.4.) Dynamics for the parameters $(-\frac{1}{5}, \frac{7}{15}, \frac{17}{15})$. Left: In the real case there are two attractors. The basin of the Fermat curve is colored as in Figures 3, 4. However, the two small white circles also form an attractor. The corresponding basin is shown in dark grey. Right: A typical randomly chosen orbit for the complex map. This orbit often comes very close to the Fermat curve during the first 4000 iterations, but then seems to converge to a cycle of two Herman rings.

Figure 6: (Example 4.5.) Plots for the map with Desboves parameters $(-1.4, -8, 1.4)$. Here the coloring is as in the previous figures except that it describes convergence to the “equator” $y = 0$, rather than to the invariant Fermat curve. For this map, the “north pole” $(0 : 1 : 0)$ also attracts many orbits.
Figure 7: (Example 4.6.) On the left: Corresponding figure for the real Desboves map with parameters \(\left(\frac{1}{3}, 1, \frac{5}{3}\right)\), again describing convergence to (or at least coming close to) the Fermat curve. On the right: One randomly chosen orbit for the complex map through 10000 iterations.

Figure 8: (Example 5.1.) Plots for the “elementary map” with parameters \((a, b, c) = (-1, \frac{1}{3}, 1)\). In this case, every great circle through the north pole \((0 : 1 : 0)\) maps to a great circle through the north pole. There are three attractors: the Fermat curve \(\mathcal{F}\), the equator \(\{y = 0\}\), and the north pole, each marked in white. The corresponding attracting basins are colored red, blue, and grey respectively. (However, the closely intermingled blue and red yield a purple effect.) The graphs on the right show an orbit which nearly converges to \(\{y = 0\}\) but then escapes towards \(\mathcal{F}\).
Figure 9:  (Example 6.1.) The Cassini quartic with parameter $k = \frac{1}{8}$, shown in black, consists of an outer circle $C_0^{0}$ with two self-intersections and a much smaller inner circle $C_1^{0}$. Here the warmer colors describe convergence towards $C_0^{0}$ for the rational map $f_1$ with parameter $a = 1$. The blue region is the basin of a superattracting fixed point at $(0 : 0 : 1)$, while the grey region is the basin of another attracting fixed point directly above it at $(0 : 1 : 2)$.

Figure 10:  Corresponding picture for the same Cassini curve with $k = \frac{1}{8}$, but using the map with parameter $a = \frac{2}{5}$. 
the $z$-axis pointing up out of the paper. The Fermat curve $x^3 + y^3 + z^3 = 0$ is traced out in white. In Figures 3 and 4, other points are colored from red to blue according as their orbits converge more rapidly or more slowly towards this Fermat curve, and subsequent figures use various modifications of this scheme. As an example, in Figure 3 the “equator” $y = 0$ is blue, since orbits with $y = 0$ do not converge to $\mathcal{F}$, so orbits near the equator cannot converge rapidly towards $\mathcal{F}$.

The graphs to the right of Figures 3 through 8 illustrate some more or less typical randomly chosen orbit for the associated complex map. Here each orbit point $(x : y : z)$ has been normalized so that $|x|^2 + |y|^2 + |z|^2 = 1$. The horizontal coordinate measures the number of iterations, while the vertical coordinates in each of the four stacked graphs represent respectively $|x|^2$, $|y|^2$, $|z|^2$, and $|\Phi(x, y, z)|$.

**Example 4.3. Another global attractor?** If we take Desboves coordinates $(\frac{1}{3}, 0, -\frac{1}{3})$, then again the Fermat curve seems to attract nearly all orbits. Compare Figure 4. Here the transverse derivative has a double zero at the fixed point $(-1 : 0 : 1)$ in the middle of the large red region. It is a curious fact that the transverse exponents in this case are precisely the same as those for Example 4.1, namely $-1.456 \cdots$ for the real map, or $-0.549 \cdots$ for the complex map.

**Example 4.4. A cycle of Herman rings?** Now suppose that we choose Desboves parameters in the two-thirds family, with $(a, b, c)$ equal to $(-\frac{1}{5}, \frac{7}{15}, \frac{17}{30})$. Here the transverse exponent is $-0.509 \cdots$ for the real map, but $+0.402 \cdots$ for the complex map. Thus we can expect the Fermat curve to be an attractor in the real case, but not in the complex case. The left half of Figure 5 illustrates the dynamics in the real case. Numerical computation suggests that some 83% of the orbits converge to the Fermat curve, while the remaining 17% converge to a pair of small circle. (The attractive basin for this pair of circles is conjecturally a dense open subset of $\mathbb{P}^2(\mathbb{C})$.) The map $f = f_{a,b,c}$ carries each of these circles to the other, reversing orientation, while $f \circ f$ carries each circle to itself with rotation number on the order of $\pm 0.185 \cdots$. Of course such a phenomenon can be expected to be highly sensitive to small changes in the parameters, and it difficult to distinguish a rotation circle with irrational rotation number from a periodic orbit which lies on a circle and has high period.

In the complex case, the Fermat curve is no longer an attractor. In fact, almost all orbits seem to eventual land near this cycle of circles and then to behave just like an orbit on these circles. This suggests that most orbits converge to a cycle of two Herman rings in $\mathbb{P}^2(\mathbb{C})$, with the pair of real circles as their central circles. Again we must be cautious, since such a phenomenon must be highly sensitive to perturbations. Furthermore, the convergence is very slow, and there may be much more chaotic attractors. An empirical plot of the rotation number as a function of the parameter $c$, keeping $a$ and $b$ fixed, is shown in Figure 11.

**Example 4.5. The line $z = 0$ as a measure theoretic attractor?** (Compare Figure 6.) For the parameter values $(a, b, c) = (-1.4, -0.8, 1.4)$, the Lyapunov exponent turns out to be strictly positive, equal to $0.247 \cdots$ in the real case, or to $0.352 \cdots$ in the complex case. The invariant Fermat curve does not seem to play any significant dynamical role in this case. On the other hand, the equator $y = 0$ seems to be at least a measure-theoretic attractor; and there is also an attracting fixed point at the north pole $(0 : 1 : 0)$. In fact many randomly chosen real or complex orbits converge to the north pole $(0 : 1 : 0)$, but even more seem to converge to the equator.

**Example 4.6. A composite “almost” attractor.** For the real or complex map with Desboves parameters $(\frac{1}{3}, 1, \frac{5}{3})$ as illustrated in Figure 7, typical orbits seem to spend a great deal of time quite close to the Fermat curve $\mathcal{F}$, even though the transverse exponent is strictly positive, equal to
Figure 11: An empirical plot of the rotation number as a function of the parameter \( c \) in Example 4.4, keeping \( a \) and \( b \) fixed. The computation is carried out by (1) starting at a random initial point and iterating many times in the hope of getting close to an attractor; then (2) looking at the sequence of closest return times to see if they are compatible with a circle rotation; and if not then (3) repeating the entire process many times. We get essentially the same graph whether using random real or complex initial points. The precise form of the gaps in this graph around each rotation number which is rational with small denominator depends somewhat on the random choices.

0.081\ldots \) in the real case or to \( 1.032\ldots \) in the complex case. This curve is not an attractor by itself, since nearby orbits eventually get kicked away from it. However, the union 

\[ A = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\} \cup F, \tag{8} \]

or in other words the variety \( xyz\Phi(x, y, z) = 0, \) does seem to behave like an attractor, at least in a statistical sense. (Compare [GI].) Typical orbits seem to spend most of the time extremely close to the Fermat curve or else almost stationary, very close to one of the four saddle fixed points which are emphasized in the figure. However, they do not stay in any one of its four irreducible components, but sometimes jump from one component to another. Furthermore, it seems likely that typical orbits will escape completely from a neighborhood of this variety, very infrequently but infinitely often.

Here is a more detailed description, as illustrated in Figure 12. To fix ideas, we will refer to the real case; but the complex case is not essentially different. A randomly chosen orbit seems to spend most of the time either wandering chaotically very close to the Fermat curve or else almost stationary, very close to one of the four saddle fixed points which are emphasized in the figure. However, such an orbit does not seem to stay close to any one of the four components of this variety forever. For example, it is likely to escape from the neighborhood of the Fermat curve \( F \) when it comes very close to the strongly repelling point \( F \cap \{y = 0\} \) which is circled in Figure 12. It will then shadow the coordinate line \( y = 0, \) jumping quickly to a small neighborhood of the saddle point \( x = y = 0, \) and then slowly coming closer to this point for thousands of iterates. Again it must eventually escape, now shadowing the line \( x = 0, \) and jumping quickly either towards the saddle point \( F \cap \{x = 0\} \) or towards the
saddle point \( x = z = 0 \). In either case it again spends a long time approaching this saddle point, but then escapes. In the first case, it is now very close to the Fermat curve and shadows it for a long time with a highly chaotic orbit before starting the cycle again. In the second case where it escapes near the saddle point \( x = z = 0 \), it then shadows the line \( z = 0 \) as it quickly converges towards the saddle point \( \mathcal{F} \cap \{ z = 0 \} \), where it again remains for a long time before repeating the cycle.

5 Intermingled Basins

Example 5.1. “Elementary” Maps. (Compare [BD] §6.4.) Finally we come to examples where we can provide complete proofs. Among our Desboves maps, there is a one-parameter family of examples which are easier to understand since we can separate the variables to simplify the discussion. Suppose that we choose parameters \((a, b, c)\) with \(a = -1\) and \(c = 1\). Then the map \( f(x : y : z) = (x' : y' : z') \) of formula (1), which now depends on a single parameter \( b \), satisfies

\[
x' = x(-x^3 - 2z^3) \quad \text{and} \quad z' = z(2x^3 + z^3).
\]

It follows that each line \((x : z) = \text{constant}\) through the north pole \((0 : 1 : 0)\) maps to another line \((x' : z') = \text{constant'}\) through the north pole. If we set \(X = x/z\) and \(X' = x'/z'\), then the correspondence

\[
F : X \mapsto X' = -X \frac{X^3 + 2}{2X^3 + 1}
\]

does not depend on the choice of parameter. This rational map (9) is described as a Lattès map, since it is the image of a rigid map on the torus \( \mathcal{F} \cong \mathbb{C}/\Omega \) under the semiconjugacy \((x : y : z) \mapsto (x : z)\). (Compare Remark 3.3.) Over the real numbers, \( F \) is a covering map from the circle \( \mathbb{P}^1(\mathbb{R}) \) to itself of topological degree \(-2\), with a smooth ergodic invariant measure.
Figure 13: On the left: Plot of the transverse Lyapunov exponent along the Fermat curve as a function of the parameter $b$ for the elementary family of Example 5.1, with Desboves parameters $(-1, b, 1)$. On the right: Corresponding plot for the transverse exponent along the line $y = 0$. In both cases, the graph for the complex map lies above the graph for the real map.

Over either the real or complex numbers, if we think of $\mathbb{P}^2 \setminus \{(0 : 1 : 0)\}$ as a (real or complex) line-bundle over the projective line $\mathbb{P}^1$ with projection $(x : y : z) \mapsto (x : z)$, then we have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^2 \setminus \{(0 : 1 : 0)\} & \xrightarrow{f} & \mathbb{P}^2 \setminus \{(0 : 1 : 0)\} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1
\end{array}
$$ (10)

where $f$ carries each fiber into a fiber by a polynomial map

$$
y \mapsto y' = by^4 + ((b + 1)z^3 + (b - 1)x^3)y,
$$ (11)

with coefficients which vary with the fiber. If we exclude the degenerate case $b = 0$ (compare Remark 5.6), then these polynomial maps have degree four. Furthermore, the distinguished point $(0 : 1 : 0)$ is superattracting, and serves as the point at infinity for each one. In the real case, these maps are all unimodal.

Since the rational map $F$ of the base space has no attracting cycles, it follows that an elementary map can have no attracting cycles other than the exceptional fixed point $(0 : 1 : 0)$.

In this special case of an elementary map, we can give a relatively easy proof that a negative transverse exponent implies that the Fermat curve is a measure-theoretic attractor. (The most attracting case $b = \frac{1}{3}$ is illustrated in Figure 8.) However, we get a surprising bonus. The invariant line $\{y = 0\}$ is also carried into itself by the Lattès map (9). Hence it also has a canonical invariant measure, and a well defined transverse Lyapunov exponent. According to Figure 13, for real values of $b$ both of these transverse exponents are strictly negative provided that $|b|$ is fairly small and non-zero.

**Theorem 5.2.** In the case of a real or complex elementary map, if the transverse Lyapunov exponent for the Fermat curve $F$ is strictly negative, then the attracting basin $B(F)$, consisting of points whose orbit converges to $F$, has strictly positive measure. In fact any
neighborhood of a point of \( F \) intersects \( B(F) \) in a set of positive measure. Similarly, if the transverse Lyapunov exponent for the coordinate line \( y = 0 \) is strictly negative, then the attracting basin for this line has positive measure, and intersects any neighborhood of a point of this line in a set of positive measure.

For the real map illustrated in Figure 8, a very rough estimate suggests that about 66% of the points in \( \mathbb{P}^2 \) are attracted to \( (0 : 1 : 0) \), about 17% to \( \{ y = 0 \} \), and about 17% to the Fermat curve. For the associated complex mapping, the figures are 81%, 13%, and 6%. (However, the computation is highly sensitive, and these estimates may well be quite inaccurate.) It may be conjectured that every point outside of a set of measure zero lies in the union of these three attracting basins.

In the complex case we can give a much more precise picture using results from [BD]. As usual, define the Fatou set to be the largest open set on which the sequence of iterates of \( f \) forms a normal family, and define the Julia set \( J \) to be its complement in \( \mathbb{P}^2(\mathbb{C}) \). If \( p \) is any point of an invariant elliptic curve, then they show in Theorem 5.4 that the iterated preimages of \( p \) are everywhere dense in the Julia set. Furthermore, according to [BD] Proposition 6.16 we have:

**Proposition 5.3.** If \( f \) is a complex elementary map, then the attracting basin for the superattracting point \( (0 : 1 : 0) \) coincides with the Fatou set, and is connected and everywhere dense in the projective plane. Furthermore, if \( U \) is a small neighborhood of a point of the Julia set, then the union of the forward images of \( U \) is the entire space \( \mathbb{P}^2 \setminus \{ (0 : 1 : 0) \} \).

In particular, it follows that the attracting basins for the Fermat curve \( F \) and for the invariant line \( y = 0 \) have no interior points. Furthermore, it follows that \( f \) is topologically transitive, so that the forward orbit of a generic point of \( J \) is everywhere dense in \( J \). In particular, there is an uncountably infinite set of points which do not belong to the attracting basin for either the Fermat curve, the line \( y = 0 \), or the exceptional point \( (0 : 1 : 0) \).

**Corollary 5.4.** Let \( A \) denote either the Fermat curve or the line \( y = 0 \). Then, for a complex elementary map, the closure \( \overline{B(A)} \) of the basin of \( A \) is precisely equal to the Julia set. Furthermore, if the transverse exponent \( \text{Lyap}_A(f) \) is negative, then for every neighborhood \( U \) of a point of the Julia set, the intersection \( U \cap B(A) \) has positive Lebesgue measure.

Thus, in the case where the transverse exponents \( \text{Lyap}_F(f) \) and \( \text{Lyap}_{\{y=0\}}(f) \) are both negative, it follows that the basins for these two attractors are intimately intermingled.\(^1\)

**Proof of Theorem 5.2.** Each fiber \( (x : z) = \text{constant} \) of the fibration \( (x : y : z) \mapsto (x : z) \) has a canonical flat metric

\[
|dy|/\sqrt{|dx|^2 + |dz|^2} ,
\]

which gives rise to a norm \( \|\vec{v}\|_t \) for vectors tangent to the fiber. Let

\[
\|f'_t(p)\| = \|f'_t\vec{v}\|/\|\vec{v}\|_t
\]

\(^1\)Compare [AKYY], [Ka]. Such examples of intermingled basins seem to be known only in cases where the attractors themselves are quite smooth. We don’t know whether there can be two fractal attractors whose basins have the same closure.
be the norm of the partial derivative along the fiber; where \( \vec{v} \) can be any non-zero vector tangent to the fiber at \( p \). (Note that vectors tangent to the fiber map to vectors tangent to the fiber.) This is well defined, depending only on the base point \( p \) of \( \vec{v} \). At points of the curve \( \mathcal{F} \), we want to compare \( \|\vec{v}\|_t \) with the semi-definite norm \( \|\vec{v}\|_n \) which is obtained by first projecting \( \vec{v} \) to the quotient vector space \( T_p\mathbb{P}^2/T_p\mathcal{F} \) and then using a positive definite norm in this quotient space. It is not hard to check that the fiber direction is transverse to the tangent space of \( \mathcal{F} \) except at the three inflection points where \( \mathcal{F} \) intersects the line \( y = 0 \). Therefore the ratio \( \|\vec{v}\|_n/\|\vec{v}\|_t \geq 0 \) is a continuous function on \( \mathcal{F} \) which vanishes only at these three points. Furthermore, the logarithm \( \lambda(p) \) of this ratio has only logarithmic singularities, and hence is an integrable function on \( \mathcal{F} \). Since the measure \( d\mu \) is \( f \)-invariant, it follows that the difference

\[
\int_{\mathcal{F}} \log \|f'_n\| \, d\mu - \int_{\mathcal{F}} \log \|f'_t\| \, d\mu = \int_{\mathcal{F}} \lambda \circ f \, d\mu - \int_{\mathcal{F}} \lambda \, d\mu
\]

is zero. In other words, the average value \( \int_{\mathcal{F}} \log \|f'_n\| \, d\mu \) coincides with the transverse exponent \( \text{L} \text{yap}_{\mathcal{F}} \) of \( \S \).

For any \( p \) and \( q \) belonging to the same fiber, let \( \delta(q, p) \geq 0 \) be the distance of \( q \) from \( p \), using the flat metric (12) on this fiber. Then

\[
\delta(f(q), f(p)) = \|f'_t(p)\| \delta(q, p) + o(\delta(q, p))
\]

as \( q \) tends to \( p \). This estimate holds uniformly throughout a neighborhood of \( \mathcal{F} \). Hence, given any \( \epsilon > 0 \), we can choose \( \delta_0 \) so that

\[
\delta(f(q), f(p)) \leq (\|f'_t(p)\| + \epsilon) \delta(q, p) \quad \text{whenever} \quad p \in \mathcal{F} \quad \text{and} \quad \delta(q, p) < \delta_0.
\]

Choose \( \epsilon \) small enough so that

\[
\int_{\mathcal{F}} \log (\|f'_t(p)\| + \epsilon) \, d\mu(p) < 0.
\]

Let \( p_0 \mapsto p_1 \mapsto \cdots \) be the orbit of \( p_0 \) under \( f \). By the Birkhoff Ergodic Theorem, the averages

\[
\frac{1}{n} \left( \log (\|f'_t(p_0)\| + \epsilon) + \log (\|f'_t(p_1)\| + \epsilon) + \cdots + \log (\|f'_t(p_{n-1})\| + \epsilon) \right)
\]

converge to the integral (14) for almost all \( p_0 \in \mathcal{F} \). In particular, it follows that the number

\[
\log (\|f'_t(p_0)\| + \epsilon) + \log (\|f'_t(p_1)\| + \epsilon) + \cdots + \log (\|f'_t(p_{n-1})\| + \epsilon)
\]

\[
= \log (\|f'_t(p_1)\| + \epsilon) \cdots (\|f'_t(p_{n-1})\| + \epsilon)
\]

is negative for large \( n \). Hence the maximum

\[
\sigma(p_0) = \max_{n \geq 0} \left( (\|f'_t(p_0)\| + \epsilon) (\|f'_t(p_1)\| + \epsilon) \cdots (\|f'_t(p_{n-1})\| + \epsilon) \right) \geq 1
\]

is well defined, measurable, and finite almost everywhere. If \( \delta(q, p_0) \leq \delta_0/\sigma(p_0) \), then it follows from (13) that \( \delta(f^{on}(q), f^{on}(p_0)) \leq \delta_0 \) for all \( n \), and also that this distance converges to zero as \( n \to \infty \).

Now let \( S \) be the set of positive measure consisting of all \( q \) with \( \delta(q, p) \leq \delta_0/\sigma(p) \) for some \( p \in \mathcal{F} \). Then for all \( q \in S \) it follows that the orbit of \( q \) converges to \( \mathcal{F} \). Evidently, \( S \) intersects every neighborhood of a point of \( \mathcal{F} \) in a set of positive measure. The proof for the coordinate line \( y = 0 \) is completely analogous. □
Remark 5.5. Conversely, it seems natural to conjecture that the basin of $F$ has measure zero whenever the transverse Lyapunov exponent is positive. However, this cannot be proved simply by reversing the inequalities in the argument above. The problem is that $\log \left( \| f'_t \| - \epsilon \right)$ is not a meaningful approximation to $\log \| f'_t \|$, since $\| f'_t \|$ must sometimes be smaller than any given $\epsilon$. In fact, almost every orbit on $F$ must pass arbitrarily close to the critical locus of $f$. But whenever $p$ is very close to the critical locus, there is a real possibility that $f(p)$ will be much closer to $F$ than would have been predicted from the differential inequality. Could this effect be strong enough to make $F$ a measure theoretic attractor even in some cases where the transverse Lyapunov exponent is positive?

Proof of Corollary 5.4. It follows immediately from 5.3 that the basins of $F$ and $\{ y = 0 \}$ are contained in the Julia set. On the other hand, if $p \in F \cap \{ y = 0 \}$ then the iterated preimages of $p$ are contained in both basins, and are dense in $J$. Therefore, the closure of either basin is equal to $J$.

Now if the open set $U$ intersects the Julia set, then it contains an iterated preimage of $p$. Since $f$ is an open mapping, it follows that some forward image $f^{on}(U)$ is an open neighborhood of $p$. By Theorem 5.2 this image intersects each basin $B(A)$ in a set of positive measure. Choosing a regular value of $f^{on}$ which is a point of density for this intersection, and choosing choosing a point $q \in U$ which maps to this regular value, it follows easily that any neighborhood of $q$ intersects $B(A)$ in a set of positive measure.

Remark 5.6. The above discussion has assumed that the parameter $b$ is non-zero. For the elementary map with $b = 0$, we have a much simpler situation. The exceptional point $(0 : 1 : 0)$ becomes a point of indeterminacy. Each fiber of our line bundle maps linearly to a fiber, carrying the point with $y = 0$ on the first fiber to the corresponding point on the second fiber. In fact it is not hard to see that the complement $\mathbb{P}^2 \setminus \{(0 : 1 : 0)\}$ is “foliated” by $f$-invariant copies of the Fermat curve, which intersect only along the locus $F \cap \{ y = 0 \}$.

6 Strong Attraction.

Recall that an invariant curve $C = f(C)$ is strongly attracting under the rational map $f$ if there are constants $c > 0$ and $0 < \lambda < 1$ so that the Riemannian distance $r(p)$ from $p$ to $C$ satisfies

$$r(f^{on}(p)) \leq c \lambda^n r(p)$$

for all $p$ in some neighborhood of $C$. A completely equivalent condition would be that we can choose $k > 0$ so that

$$r(f^{ok}(p)) \leq \frac{1}{2} r(p)$$

for all $p$ in some neighborhood. If $N$ is a compact neighborhood for which one of these inequalities holds, then it follows easily that the set $N'$ consisting of points $p$ with $f^{on}(p) \in N$ for all $n \geq 0$ is a trapping neighborhood of $C$.

This section will first describe a strongly attracting example in the case of a real singular curve, and then show that no such example is possible for a smooth complex curve. (In the complex singular case and in the real nonsingular case, we don’t know whether strong attraction is possible.)
Example 6.1. A real elliptic curve as strong attractor. This last example will study the case of a singular elliptic curve. As in [BD] §8.6, consider the Cassini quartic curve \( C \) with homogeneous equation \( \Phi(x,y,z) = 0 \), where\(^3\)

\[
\Phi(x,y,z) = \Phi_k(x,y,z) = x^2y^2 - (x^2 + y^2)z^2 + k z^4
\]
depends on a single parameter \( k \neq 0, 1 \). Over the complex numbers, this is an elliptic curve with nodes at the two points \((1 : 0 : 0)\) and \((0 : 1 : 0)\). That is, the uniformizing map \( C/\Omega \to C \subset \mathbb{P}^2(\mathbb{C}) \) has transverse self-intersections at these two points. Define a one-parameter family of homogeneous polynomial maps from \( \mathbb{C}^3 \) to itself by the formula \( F(x,y,z) = F_a(x,y,z) = (x',y',z') \), where

\[
x' = -2xy(x^2 + y^2 - 2k z^2), \quad y' = y^4 - x^4, \quad z' = -a \Phi(x,y,z) + 2xy(x^2 - y^2).
\] (17)

According to [BD], the curve \( C \) is invariant under the induced rational map \( f = f_a : \mathbb{P}^2 \to \mathbb{P}^2 \). It is not hard to check that the singular point \((0 : 1 : 0) \in C\) at the north pole, is a saddle fixed point of \( f_a \), with eigenvalues \(-2\) and \(0\), and that the point \((0 : 0 : 1)\), near the center of the figures, is a superattracting fixed point whenever \( a \neq 0 \).

Now suppose that the parameters \( k \) and \( a \) are real. The corresponding real curve \( C_\mathbb{R} = C \cap \mathbb{P}^2(\mathbb{R}) \) is connected when \( k < 0 \), but has two connected components when \( k > 0 \). Note the identities

\[
F(-x,-y,z) = F(x,y,z) \quad \text{and} \quad F^{o2}(-y,x,z) = F^{o2}(x,y,z),
\]

which imply that the Julia set of \( f_a \) has \(90^\circ\)-rotational symmetry about the point \((0 : 0 : 1)\). These maps are illustrated in Figures 9 and 10 for the case \( k = 1/8 \). The two branches of \( C \) through the singular point \((1 : 0 : 0)\) map to one branch through \((0 : 1 : 0)\) while the two branches through \((0 : 1 : 0)\) map to the other branch through this point.

Let \( C^0_\mathbb{R} \) be the connected component of the real curve which contains this fixed point, and hence maps onto itself. We will show in Theorem 6.3 that \( C^0_\mathbb{R} \) is strongly attracting under the map \( f_a \), provided that the parameters \( k \) and \( a \) are sufficiently small. We first prove the following result which yields somewhat less than strong attraction.

**Lemma 6.2.** If \( 0 < |k| < 1/4 \), and if \( a \) is sufficiently small, then the curve \( C^0_\mathbb{R} \) is a trapped attracting set for the map \( f_a \) on the real projective plane.

**Proof.** Let \( (x',y',z') = F(x,y,z) \). The quotient

\[
\Phi_F(x,y,z) = \Phi(x',y',z')/\Phi(x,y,z)
\]
is a polynomial of degree 12 in \( x, y, z \), depending on the parameter \( a \). In general this polynomial seems rather complicated, but in the special case \( a = 0 \) computation shows that it takes the simple form

\[
\Phi_F(x,y,z) = 16 k x^2 y^2 (x^2 - y^2)^4.
\] (18)

As a convenient measure of the distance of a point of \( \mathbb{P}^2 \) from the curve \( \Phi = 0 \) we take the ratio

\[
s(x:y:z) = |\Phi(x,y,z)|/(x^2 + y^2)^2.
\]

\(^3\)This expression yields curves which are equivalent, under a complex linear change of coordinates, to quartic curves introduced in 1680 by the French-Italian astronomer Giovanni Domenico Cassini, in connection with planetary orbits.
This ratio is finite except at the value \( s(0 : 0 : 1) = +\infty \), and it vanishes only on the Cassini curve. We want to prove an inequality of the form

\[
s(x' : y' : z') \leq \lambda s(x : y : z)
\]

whenever \( s(x : y : z) \) is sufficiently small, where \( \lambda < 1 \) is constant. To do this, we consider the ratio

\[
s_f(x, y, z) = \frac{s(x' : y' : z')}{s(x : y : z)} = \frac{\Phi(x', y', z')}{\Phi(x, y, z)} \left( \frac{x^2 + y^2}{x^2 + y^2} \right) \left( \frac{x^2 + y^2}{x^2 + y^2} \right).
\]

In the special case \( a = 0 \), using the identity (18) and the inequality

\[
x^2 + y^2 \geq \frac{y^2}{2} = (y^4 - x^4)^2 = (x^2 + y^2)(x^2 - y^2)^2,
\]

together with \(|2xy| \leq x^2 + y^2\), we see that

\[
\frac{s(x' : y' : z')}{s(x : y : z)} \leq \frac{\Phi(x', y', z')}{\Phi(x, y, z)} \left( \frac{x^2 + y^2}{x^2 + y^2} \right) \left( \frac{x^2 + y^2}{x^2 + y^2} \right) = \frac{16|k|^2y^2}{(x^2 + y^2)^2} \leq 4|k|.
\]

If \( 0 < |k| < 1/4 \), then we can choose \( \lambda \) so that \( 4|k| < \lambda < 1 \). If \( N \) is any compact subset of \( \mathbb{P}^2(\mathbb{R}) \setminus \{(0 : 0 : 1)\} \), then for any \( a \) which is sufficiently close to zero it then follows by continuity that the required inequality (19) holds uniformly throughout \( N \). Thus all orbits of \( f_a \) in \( N \) converge uniformly towards the subset \( C_\mathbb{R} \). In the case where there are two components, the image \( f_a(C_\mathbb{R}) \) is necessarily equal to the component \( C_\mathbb{R}^0 \subset C_\mathbb{R} \), so it follows that all orbits in \( N \) converge uniformly to \( C_\mathbb{R}^0 \).

(In the limiting case \( a = 0 \), there is no superattracting point, and in fact \( (0 : 0 : 1) \) becomes a point of indeterminacy. The proof shows that the basin of \( C_\mathbb{R}^0 \) under the map \( f_0 \) is the entire domain of definition \( \mathbb{P}^2(\mathbb{R}) \setminus \{(0 : 0 : 1)\} \) for this map when \( 0 < |k| < 1/4 \).)

The statement of 6.2 is certainly good enough for most purposes. The proof can even be used to check the sharper statement that the Riemannian distance of \( f^n(x : y : z) \) from \( C_\mathbb{R}^0 \) is less than \( \lambda^{n/2} \) for all \( n \geq 0 \) and for all \((x : y : z)\) in some sufficiently small neighborhood of the curve. However, the argument does not prove that \( C_\mathbb{R}^0 \) is strongly attracting, since our measure \( s(x : y : z) \) of distance from \( C_\mathbb{R}^0 \) does not have the same order of magnitude as the Riemannian distance \( r(x : y : z) \). In fact the ratio \( s/r \) takes values arbitrarily close to zero as we approach one of the two singular points. Thus the following sharper result will require more work.

**Theorem 6.3.** If \(-1/2 < k < 1/3\) and if \( a \) is sufficiently small, then the curve \( C_\mathbb{R}^0 \) is strongly attracting.

To prove this statement, we must construct a better measure of distance. This can be done by lifting to the two-fold orientable covering space of the projective plane \( \mathbb{P}^2(\mathbb{R}) \), which can be identified with the unit sphere \( S^2 \). If \( k < 1 \), then the Cassini curve \( C_\mathbb{R}^0 \) lifts to a curve in \( S^2 \) which is the union \( \widetilde{C} = \widetilde{C}_1 \cup \widetilde{C}_2 \) of two smooth real analytic curves which cross each other transversally. To see this, it will be convenient to remove the two poles \((0,0,\pm1)\) from \( S^2 \), and then divide all three coordinates by \( \sqrt{x^2 + y^2} \). In other words we project radially from the 2-sphere to the cylinder.
\[ x^2 + y^2 = 1. \] Setting \( x = \sin \theta \) and \( y = \cos \theta \), so that \( 2xy = \sin(2\theta) \), the defining equation \( \Phi(x, y, z) = x^2y^2 - z^2(x^2 + y^2 - kz^2) = 0 \) for the curve \( \mathcal{C} \) on the cylinder can be written as
\[
\sin^2(2\theta) - 4z^2(1 - kz^2) = 0 \quad \text{or} \quad \sin(2\theta) = \pm 2z\sqrt{1 - kz^2}.
\]
If \( k < 1 \), then it is not hard to check that the correspondence \( z \mapsto \varphi(z) = 2z\sqrt{1 - kz^2} \) maps the interval \( |z| \leq 1/\sqrt{2} \) diffeomorphically onto an interval \( |\varphi(z)| \leq \sqrt{2 - k} \), where \( \sqrt{2 - k} > 1 \). Thus whenever \( k < 1 \) we can solve the equation \( \sin(2\theta) = 2z\sqrt{1 - kz^2} \) uniquely for \( z \) as a smooth function \( z = \varphi^{-1} \circ \sin(2\theta) \) with \( z^2 < \frac{1}{2} \). Equivalently, we can factor the polynomial \( \Phi(x, y, z) \) as a real analytic product \( \Phi = \Phi_1 \Phi_2 \) throughout some neighborhood of \( \mathcal{C} \), where
\[
\Phi_1(x, y, z) = xy - z\sqrt{x^2 + y^2 - kz^2}, \quad \Phi_2(x, y, z) = xy + z\sqrt{x^2 + y^2 - kz^2}.
\]
We can then define \( \mathcal{C}_j \) to be the zero set of \( \Phi_j \), or in other words the locus
\[
\sin(2\theta) + (-1)^j 2z\sqrt{1 - kz^2} = 0
\]
on the cylinder. With the normalization \( x^2 + y^2 = 1 \), note that the ratio \( xy/z = \pm \sqrt{1 - kz^2} \) is bounded away from zero on \( \mathcal{C} \), being positive on \( \mathcal{C}_1 \) and negative on \( \mathcal{C}_2 \). (In the special case \( z = 0 \), this ratio must be interpreted as a limit.)

Similarly, the map \( f \) lifts to a map \( \tilde{f} \) on the cylinder which maps both components of the curve \( \mathcal{C} \) onto the single component \( \mathcal{C}_1 \). In fact, in the special case \( a = 0 \) inspection of Equation (17) shows that the ratio \( x'/y' \) is strictly positive when \( z \neq 0 \), so that \( \tilde{f} = (\mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{C}_1 \), and the same conclusion for small values of \( a \) follows easily, using continuity and analytic continuation.

The quotient \( \Phi_F(x, y, z) = \Phi(x', y', z')/\Phi(x, y, z) \) , lifted to the cylinder, also splits as a product \( \Phi_1 \Phi_2 \). Since \( \Phi_2(x, y, z) = 2xy \) whenever \( \Phi_1(x, y, z) = 0 \), it follows from the equations (17) that \( \Phi_2 = (x^2 + y^2 - 2kz^2)(x^2 - y^2) \) everywhere along the curve \( \mathcal{C}_1 \). Combining this with equation (18), we see that
\[
\Phi_{1F}(x, y, z) = \frac{16k x^2 y^2(x^2 - y^2)^3}{1 - 2kz^2}
\]
on \( \mathcal{C}_1 \), when \( x^2 + y^2 = 1 \) and when \( a = 0 \). Therefore, again using the inequality (20), we see that
\[
\frac{\Phi_{1F}}{x^2 + y^2} \leq \frac{16k x^2 y^2(x^2 - y^2)}{1 - 2kz^2} = \frac{4k \sin^2(2\theta) \cos(2\theta)}{1 - 2kz^2} \leq \frac{2|k|}{1 - 2kz^2}
\]
when \( x^2 + y^2 = 1 \) and \( a = 0 \). Now if \( -\frac{1}{2} < k \leq 0 \) it follows immediately that \( 2|k|/(1 - 2kz^2) < 1 \). Similarly, if \( 0 < k < \frac{1}{3} \), then using the fact that \( z^2 < \frac{1}{2} \), it follows easily that \( 2|k|/(1 - 2kz^2) < 1 \). In fact in both cases it is easy to check the more precise statement that \( 2|k|/(1 - 2kz^2) < \lambda \) for some constant \( \lambda < 1 \).

We can use the ratio \( s_1(x, y, z) = \Phi_1(x, y, z) / (x^2 + y^2) \) as a measure of distance from the curve \( \mathcal{C}_1 \). In fact it is not hard to check that the ratio of \( s_1(x, y, z) \) to the Riemannian distance \( r_1(x : y : z) \) is bounded and bounded away from zero throughout a neighborhood of \( \mathcal{C}_1 \). But it follows from (21) that \( s_1(x', y', z') \leq \lambda s_1(x, y, z) \) for all \( (x, y, z) \) in some neighborhood of the curve, where \( \lambda < 1 \). Again, this inequality will continue to hold throughout a compact neighborhood of \( \mathcal{C} \) whenever the parameter \( a \) is sufficiently close to zero. This completes the proof that \( \mathcal{C} \) is strongly attracting, and hence that the original curve \( \mathcal{C}_0 \) is strongly attracting.

**Smooth Complex Curves.** Now consider the case of a smooth elliptic curve in \( \mathbb{P}^2(\mathbb{C}) \). We will prove the following.
**THEOREM 6.4.** Let $\mathcal{C} \subset \mathbb{P}^2(\mathbb{C})$ be a smoothly embedded elliptic curve, and let $f$ be a rational map from $\mathbb{P}^2(\mathbb{C})$ to itself with $f(\mathcal{C}) = \mathcal{C}$. Then $\mathcal{C}$ cannot be strongly attracting.

The proof will be based on the fact that a smooth elliptic curve in $\mathbb{P}^2(\mathbb{C})$ can be approximated arbitrarily closely by other elliptic curves which are not conformally isomorphic to it. If such a curve were strongly attracting, then we will show that one could construct a sequence of real analytic retractions of the form

$$g_k = f^{-k} \circ \pi \circ f^k,$$

from a neighborhood $N$ onto $\mathcal{C}$, where $\pi : N \to \mathcal{C}$ is the orthogonal projection which carries each $p \in N$ to the closest point of $\mathcal{C}$. More precisely, since $f^{-k}$ is a many valued map, this means that we must construct $g_k$ so that

$$f^k \circ g_k = \pi \circ f^k,$$

with $g_k$ equal to the identity map on $\mathcal{C}$. Here $f^k$ is holomorphic, and we will see that the projection $\pi$ becomes a better and better approximation to a holomorphic map as we approach $\mathcal{C}$, so that $g_k$ is also a good approximation to a holomorphic map for large $k$. In fact we claim that $g_k$ restricted to any nearby curve $\mathcal{C}'$ is a quasiconformal diffeomorphism, with a dilatation which converges to one as $k \to \infty$. However, to carry out this argument, we must be sure that the maps $g_k|_{\mathcal{C}'}$ have no critical points. This will require several preliminary steps.

We first show that the property of strong attraction depends only on the first derivatives of $f$ along the curve $\mathcal{C}$. Let $\|f'_n(p)\|$ be the norm of the linear map $f'_n(p) : T_p(\mathbb{P}^2/\mathcal{C}) \to T_{f(p)}(\mathbb{P}^2/\mathcal{C})$ as described in §2.

**LEMMA 6.5.** The curve $\mathcal{C}$ is strongly attracting if and only if there is an integer $k > 0$ so that the norm of the $k$-th iterate of $f'_n$ satisfies the condition that

$$\|f'^{ok}_n(p)\| < 1 \quad (22)$$

for all $p \in \mathcal{C}$. 

Figure 14:
Proof. If \( \vec{v} \) is a unit vector orthogonal to \( C \) at \( p \), then there is a unique geodesic \( t \mapsto \eta(t) \) with \( \eta(0) = p \) and with tangent vector \( \eta'(0) \) equal to \( \vec{v} \). Note that \( r(\eta(t)) = t \) for small \( t \geq 0 \), where \( r(q) \) denotes the Riemannian distance of \( q \) from \( C \). Then the derivative of \( r \circ f^{\circ k} \circ \eta \) at \( t = 0 \) can be identified with the norm of \( f_n^k(\vec{v}) \), or in other words with \( \|f_n^{\circ k}(p)\| \). If (16) holds, so that
\[
\|f_n^{\circ k}(p)\| \leq \frac{1}{2} r(\eta(t)) = t/2 ,
\]
then differentiating this inequality at \( t = 0 \), we conclude that \( \|f_n^{\circ k}(p)\| \leq 1/2 \), as required.

Conversely, if \( \|f_n^{\circ k}(p)\| < 1 \) for all \( p \in C \), then by compactness it is bounded away from \( 1 \). Therefore, after replacing \( k \) by a suitable multiple, we may assume that \( \|f_n^{\circ k}(p)\| \leq \text{constant} < 1/2 \). Now expressing \( r \circ f^{\circ k} \circ \eta(t) \) as a linear Taylor series plus a remainder term which depends continuously on the unit normal vector \( \vec{v} \), we conclude that (16) holds throughout some neighborhood.

It is often convenient to choose a smoothed out norm on the normal line bundle of \( C \), defining the new norm of a vector \( \vec{v} \in T(\mathbb{P}^2/C) \) to be the sum
\[
\|\vec{v}\|_n + \|f_n'(\vec{v})\|_n + \cdots + \|f_n^{\circ (k-1)}(\vec{v})\|_n .
\]
(23)
If we use this new norm in place of \( \|\vec{v}\|_n \), then the differential inequality (22) clearly takes the simpler form \( \|f_n'(p)\| < 1 \) for all \( p \in C \). In fact it is not hard to check that
\[
\|f_n'(p)\| \leq \lambda < 1
\]
(24)
for some uniform constant \( \lambda \).

Cone Fields and Pseudometrics. The condition of strong attraction gives us a very tight control of the dynamics in a neighborhood of \( C \). We can express this in terms of an invariant cone field, or equivalently in terms of a controlling pseudometric.

A Riemannian pseudometric on a smooth manifold \( M \) is defined classically, in terms of local coordinates \( u^1, \ldots, u^n \), as an expression
\[
\psi = \sum_{ij} \psi_{ij} \, du^i du^j ,
\]
where \( [\psi_{ij}] \) is a symmetric matrix of smooth functions on the coordinate neighborhood. It can be defined in a more invariant manner as a smooth function from the tangent vector bundle \( TM \) to the real numbers such that the restriction of \( \psi \) to each tangent vector space \( T_pM \) is homogeneous and quadratic. In terms of the local coordinates \( v^1, \ldots, v^n \) of a vector \( \vec{v} \in T_pM \), we write \( \psi(\vec{v}) = \sum_{ij} \psi_{ij}(p)v^i v^j \).
Such a pseudometric is nonsingular\(^4\) if each matrix \( [\psi_{ij}] \) is nonsingular. The pull-back \( f^*\psi \) under a smooth map \( f \) is defined by
\[
(f^*\psi)(\vec{v}) = \psi(f'(\vec{v})
\]
for every vector \( \vec{v} \). (If \( f \) has critical points, then the pull-back \( f^*\psi \) may well be singular even when \( \psi \) is nonsingular; but that will not cause any difficulties.)

Definition 6.6. The pseudometric \( \psi \) is called a Riemannian metric if it is positive definite, in the sense that \( \psi(\vec{v}) > 0 \) for every non-zero vector \( \vec{v} \). The notation \( \psi(\vec{v}) = \|\vec{v}\|^2 \) will often be used for such a Riemannian metric. However, we will be primarily interested in pseudometrics which are indefinite, in the sense that \( \psi(\vec{v}) \) takes both positive and negative values.

\(^4\)Nonsingularity is often required as part of the definition of “pseudometric”; but this would be awkward for our purposes since, for example, we need to be able to add or subtract any two pseudometrics.
To every pseudometric there is associated its field of positive cones, consisting of all vectors \( \vec{v} \in TM \) with \( \psi(\vec{v}) > 0 \). In the indefinite case, this cone field determines \( \psi \) up to multiplication by a positive real valued function. However, if \( \psi \) is positive or negative definite then the cone field gives no further information.

**Definition 6.7.** We will say that \( \psi \) controls the smooth map \( f : M \to M \) if the difference \( f^*\psi - \psi \) is positive definite everywhere, or in other words if

\[
\psi(f^*\vec{v}) > \psi(\vec{v})
\]

for every \( p \in M \) and every non-zero \( \vec{v} \in T_p M \).

If \( \psi \) controls \( f \), then it follows immediately that \( f' \) maps the positive cone \( \psi(\vec{v}) > 0 \) at \( p \) into the positive cone at \( f(p) \). This condition is particularly useful in the study of hyperbolic dynamical systems. If \( \psi \) itself is positive definite, then control means that \( f \) is an expanding map, while if \( \psi \) is negative definite it means that \( f \) is a contracting map.

If \( M \) is compact, note that this is an open condition. That is, if \( \psi_0 \) controls \( f_0 \) then \( \psi \) controls \( f \) whenever \( \psi \) is close enough to \( \psi_0 \) in the \( C^0 \)-topology and \( f \) is close enough to \( f_0 \) in the \( C^1 \)-topology.

**Lemma 6.8.** If \( C \) is strongly attracting under \( f \), then there exist a controlling pseudometric for \( f \) throughout a neighborhood of \( C \). This pseudometric \( \psi \) is positive definite on the tangent bundle of \( C \), and negative definite on the normal bundle.

**Proof.** Choose a Riemannian metric \( \vec{v} \mapsto \|\vec{v}\|^2 \) throughout a neighborhood of \( C \) which agrees with the flat conformal metric on the curve itself. Thus the derivative \( f' : T_p C \to T_{f(p)} C \) will satisfy the inequality

\[
\|f'\vec{v}\| > \|\vec{v}\|
\]

for every vector \( \vec{v} \neq \vec{0} \) which is tangent to \( C \). We will also need a semi-definite norm as follows. For each \( p \in C \) and each \( \vec{v} \in T_p \mathbb{P}^2 \), let \( \|\vec{v}\|_n \) be the norm of the image of \( \vec{v} \) in \( T_p \mathbb{P}^2 / T_p C \), using a norm which satisfies the inequality

\[
\|f'\vec{v}\|_n \leq \lambda \|\vec{v}\|_n \quad \text{for} \quad \lambda < 1 ,
\]

as in (23) and (24). We can now define the pseudometric \( \psi \) on the tangent space of \( \mathbb{P}^2 \) restricted to \( C \) by the formula

\[
\psi(\vec{v}) = \epsilon \|\vec{v}\|^2 - \|\vec{v}\|^2 ,
\]

where \( \epsilon \) is a small constant. We must show that the difference \( f^*\psi - \psi \) is positive definite everywhere on \( C \). It will be convenient to write this difference as \( f^*\psi - \psi = \epsilon \alpha + \beta \), where

\[
\alpha(\vec{v}) = \|f'\vec{v}\|^2 - \|\vec{v}\|^2
\]

is clearly positive definite throughout the subbundle \( T C \), and where

\[
\beta(\vec{v}) = \|\vec{v}\|_n^2 - \|f'\vec{v}\|_n^2
\]

is positive definite except on the subbundle \( T C \). In order to show that \( f^*\psi - \psi \) is positive definite throughout \( T_{\mathbb{P}^2} | C \), it suffices to show that \( \psi(f'\vec{v}) > \psi(\vec{v}) \) for all \( \vec{v} \) in the unit sphere bundle
consisting of those \( \vec{v} \in T\mathbb{P}^2|_{C'} \) with \( \|\vec{v}\| = 1 \). Let \( K \) be the compact set consisting of all such unit vectors with \( \alpha(\vec{v}) \leq 0 \). Evidently \( |\alpha(\vec{v})| \) is bounded on \( K \), with \( |\alpha(\vec{v})| \leq |\alpha|_{\text{max}} \). Since \( K \) is disjoint from \( TC \), it follows that \( \beta(\vec{v}) \) is bounded away from zero on \( K \). Thus we can choose \( \epsilon > 0 \) so that

\[
\beta(\vec{v}) > \epsilon |\alpha|_{\text{max}} \quad \text{for} \quad \vec{v} \in K.
\]

It then follows easily that \( \epsilon \alpha(\vec{v}) + \beta(\vec{v}) \geq 0 \) for all unit vectors, and hence for all non-zero vectors at points of \( C \).

Finally, choosing some arbitrary extension of \( \psi \) to a neighborhood of \( C \), it follows by continuity that \( f^*\psi - \psi \) is positive definite throughout some sufficiently small neighborhood of \( C \). \( \square \)

**Approximately Holomorphic Maps.** The proof of Theorem 6.4 continues as follows. Suppose that \( M' \subset M \) are compact complex manifolds. and that \( f : M \to M' \) is a holomorphic map with \( f(M') = M' \). The complex structure on \( M \) can be specified by the fiberwise linear map \( J : TM \to TM \), where \( J(\vec{v}) = i\vec{v} \) so that \( J \circ J = -I \). Since \( f \) is holomorphic, it satisfies the Cauchy-Riemann equation \( f' \circ J = J \circ f' \).

Choosing some real analytic Hermitian metric on \( M \), the \( \epsilon \)-neighborhood \( N \) of \( M' \) will fiber as an \( \epsilon \)-disk bundle over \( M' \) for small \( \epsilon \). This fibration is only real analytic; in general it cannot be made complex analytic. (Certainly in the case which interests us, where \( M' \) is an elliptic curve which is not isomorphic to nearby elliptic curves, the projection map \( \pi : N \to M' \) cannot be made holomorphic.) However, both \( T_pM' \) and its orthogonal complement \( T^\perp_p \), the kernel of \( \pi_* : T_pM \to T_pM' \), are complex vector spaces, so the Cauchy-Riemann equation \( \pi_* \circ J = J \circ \pi_* \) will be satisfied at points of \( M' \). It follows by continuity that it is approximately satisfied close to \( M' \), and this will be enough for the argument.

**Definition 6.9.** Let \( g : M'' \to M' \) be a diffeomorphism between complex manifolds, and suppose that \( M' \) is equipped with a Riemannian metric which satisfies the Hermitian condition \( \|i\vec{v}\| = \|J\vec{v}\| \). Then \( g \) will be called **\( \delta \)-approximately holomorphic** if

\[
\|g_* \circ J \circ g_*^{-1} - J\| < \delta \quad \text{or equivalently if} \quad \|g_* \circ J \circ g_*^{-1} \circ J^{-1} - I\| < \delta
\]

at every point of \( M' \), using the usual norm \( \|L\| = \sup_{\vec{v} \neq 0} \|L(\vec{v})\|/\|\vec{v}\| \) for linear operators.

In general, this condition definitely depends on the choice of metric on \( M' \). However, in the special case where \( M' \) and \( M'' \) are complex curves we have the following. Recall that the **dilatation** of an orientation preserving diffeomorphism \( g : M'' \to M' \) between Riemann surfaces at a point \( p \) can be defined as the ratio \( \beta/\alpha \) where

\[
\beta = \sup_{\|\vec{v}\| = 1} \|g_*\vec{v}\|, \quad \alpha = \inf_{\|\vec{v}\| = 1} \|g_*\vec{v}\|,
\]

and where \( \vec{v} \) ranges over unit vectors in \( TM'_p \). This ratio is independent of the choice of conformal (= Hermitian) metrics.

**Lemma 6.10.** This dilatation is related to the operator \( g_* \circ J \circ g_*^{-1} \circ J^{-1} - I \) by the equation

\[
\beta/\alpha - 1 = \|g_* \circ J \circ g_*^{-1} \circ J^{-1} - I\|.
\]
Proof. It suffices to consider the map \( g(x + iy) = x\alpha + iy\beta \) in one complex variable, with \( \beta \geq \alpha \). Evidently the dilatation is equal to \( \beta/\alpha \), and a brief computation shows that
\[
(g_* \circ J \circ g_*^{-1} \circ J^{-1})(x + iy) = x \frac{\alpha}{\beta} + iy \frac{\beta}{\alpha}.
\]
Subtracting \( x + iy \) and maximizing the absolute value subject to the condition that \( x^2 + y^2 = 1 \), we obtain \( \beta/\alpha - 1 \), as required.

The proof of Theorem 6.4 will be based on the following statement.

**Theorem 6.11.** Suppose that the complex curve \( C \subset M \) is strongly attracting under \( f \). If the submanifold \( C' \) is sufficiently \( C^1 \)-close to \( C \), then for any \( \delta > 0 \) there exists a diffeomorphism \( g_k : C' \to C \) which is \( \delta \)-approximately holomorphic.

**Proof.** As described at the beginning of §6, whenever \( f \) maps \( N \) into itself, we can form the retraction map \( g_k = f^{-k} \circ \pi \circ f^k \) from \( N \) to \( C \). To see that this is well defined, pass to the universal covering spaces
\[
\tilde{C} \subset \tilde{N} \xrightarrow{\tilde{f}} \tilde{N} \xrightarrow{\tilde{\pi}} \tilde{C}.
\]
Since \( f \) is expanding on \( C \), it certainly follows that \( \tilde{f} \) maps \( \tilde{C} \) diffeomorphically onto itself. Therefore the map
\[
\tilde{g}_k = \tilde{f}^{-k} \circ \tilde{\pi} \circ \tilde{f}^k : \tilde{N} \to \tilde{C}
\]
is well defined; and since \( \tilde{\pi} \) reduces to the identity map on \( \tilde{C} \), it follows that \( \tilde{g}_k \) does also. Finally, since \( \tilde{g}_k \) commutes with the group of deck transformations\(^5\) of \( \tilde{N} \) over \( N \), it follows that \( \tilde{g}_k \) gives rise to a corresponding retraction \( g_k : N \to C \).

Again let \( J : TM \to TM \) be the fiberwise linear map \( J\tilde{v} = i\tilde{v} \) satisfying \( J \circ J = -I \). If \( u^1, \ldots, u^{2n} \) are local coordinates and \( \{ \partial_j = \partial/\partial u^j \} \) is the associated basis for tangent vectors, then we can write \( J\partial_j = \sum b_j^h \partial_h \) where \([a_j^h]\) is a matrix of smooth functions with square equal to \(-I\). This \( J \) reduces to the standard form \( J_0 \), where
\[
J_0(\partial_{2k-1}) = \partial_{2k} , \quad J_0(\partial_{2k}) = -\partial_{2k-1}
\]
if and only if the \( u^{2k-1} + iu^{2k} \) are holomorphic coordinates. To simplify the notation, let me specialize to the case where \( M \) is a complex surface. We can choose a local holomorphic coordinate \( u^1 + iu^2 \) in the curve \( C \). Using a Hermitian metric, each normal vector space \( T_p \) will also be a complex vector space. It is not difficult to choose local coordinates \( u^1, \ldots, u^4 \) near a point of \( C \) so that \( C \) is the locus \( u^3 = u^4 = 0 \), and so that the complex structure \( J \) reduces to \( J_0 \) on \( C \). It then follows that \( J \) is close to \( J_0 \) at any point \( p \) which is close to \( C \). More precisely, if \( r \) is the distance from \( p \) to \( C \) we can write
\[
J\partial_j = J_0\partial_j + \sum b_j^h \partial_h \quad \text{with} \quad b_j^h = O(r).
\]

Now consider a complex curve \( C' \subset M \) which is close to \( C \), and projects diffeomorphically to \( C \). Then at any point \( p \in C' \) we can choose a basis \( \{ \tilde{v}, J\tilde{v} \} \) for \( T_p C' \) so that \( \pi_* (\tilde{v}) = \partial_1 \), hence
\[
\tilde{v} = \partial_1 + v^3 \partial_3 + v^4 \partial_4 , \quad \text{and} \quad J\tilde{v} = \partial_2 - v^4 \partial_3 + v^3 \partial_4 + O(r).
\]

---

\(^5\)The situation for the map \( \tilde{f} \) is somewhat more complicated. In fact \( f \) induces an embedding \( f' \) from the group \( \Pi \) of deck transformations into itself, with \( \tilde{f}(\sigma \tilde{p}) = f'(\sigma) \tilde{f}(\tilde{p}) \) for each \( \sigma \in \Pi \).
Under the projection \( \pi : \mathcal{C}' \subset N \to \mathcal{C} \), the vector \( J\bar{v} \) projects to \( \pi_*(J\bar{v}) = \partial_2 + O(r) \), or more precisely
\[
\pi_*(J\bar{v}) = \partial_2 + (v^3b_3^1 + v^4b_4^1)\partial_1 + (v^3b_3^2 + v^4b_4^2)\partial_2,
\]
where \( b_j^h = O(r) \). If the components \( v^3 \) and \( v^4 \) remain bounded, then it follows that the linear map \( \pi_* \circ J \circ \pi_*^{-1} \) from \( T_p\mathcal{C} \) to itself satisfies \( \left( \pi_* \circ J \circ \pi_*^{-1} - J \right)\partial_1 = O(r) \), and a similar argument shows the same for \( \partial_2 \). Thus
\[
\|\pi_* \circ J \circ \pi_*^{-1} - J\| = O(r),
\]
hence the dilatation of this map \( \pi : \mathcal{C}' \to \mathcal{C} \) tends to one as the distance of \( \mathcal{C}' \) from \( \mathcal{C} \) tends to zero.

Now, choosing some fixed curve \( \mathcal{C}_0' \) which is \( C^1 \)-close to \( \mathcal{C} \), we want to apply this argument to the curve \( \mathcal{C}' = f^{ok}(\mathcal{C}_0') \), where \( k \) is large so that \( \mathcal{C}' \) is very close to \( \mathcal{C} \). Here the hyperbolic estimates become crucial. By Lemmas 6.5 and 6.8, we can choose a pseudometric \( \psi \) so that \( f^s\psi - \psi \) is positive definite throughout a neighborhood of \( \mathcal{C} \), and so that \( \psi \) itself is positive definite on \( T_p\mathcal{C} \) and negative definite on \( T_p^\perp \) for each \( p \in \mathcal{C} \). Since we assume that \( \mathcal{C}_0' \) is \( C^1 \)-close to \( \mathcal{C} \), it follows that the tangent space \( T_p\mathcal{C}_0' \) is contained in the closed positive cone \( \psi(\bar{v}) \geq 0 \). Hence the tangent space \( f^{ok}_*(T_p\mathcal{C}') \) of the \( k \)-th forward image is also contained in this positive cone. This implies that the coefficients \( b_j^h \) in equation (27) for the image \( f^{ok}(\mathcal{C}_0') \) remain bounded. Therefore, the dilatation of \( \pi \circ f^{ok}|_{\mathcal{C}_0'} \) tends to one as \( k \to \infty \), and it follows immediately that the same is true of \( g_k \). This completes the proof of Theorem 6.11.

On the other hand, we have the following statement from Teichmüller theory.

**Lemma 6.12.** Suppose that there exist quasiconformal homeomorphisms from the elliptic curve \( \mathcal{C}_1 \) to \( \mathcal{C}_2 \) with dilatation arbitrarily close to one. Then \( \mathcal{C}_1 \) must be conformally isomorphic to \( \mathcal{C}_2 \).

**Proof.** This is an immediate consequence of compactness of the space of quasiconformal homeomorphisms with bounded dilatation. On a more elementary level, if \( \mathcal{C}_1 \cong \mathcal{C}/\Lambda_1 \) and \( \mathcal{C}_2 \cong \mathcal{C}/\Lambda_2 \) where \( \Lambda_1 \) and \( \Lambda_2 \) are unimodular lattices, then the optimal quasiconformal map in any homotopy class is given by a real-linear map, corresponding to a linear transformation \( L \in \text{SL}(2, \mathbb{R}) \) with \( L(\Lambda_1) = \Lambda_2 \). (Compare [Kr], p. 101.) Such a linear transformation has dilatation one only if \( L \) is a rotation. Similarly if a sequence of elements of \( \text{SL}(2, \mathbb{R}) \) has dilatation converging to one, then some subsequence must converge to a rotation. The conclusion follows easily.

**Proof of Theorem 6.4.** Any smooth elliptic curve in \( \mathbb{P}^2(\mathbb{C}) \) belongs to a smooth family of such curves, for example defined by equations of the form \( x^3 + y^3 + z^3 = 3kxyz \), where nearby curves are not conformally isomorphic. Combining this statement with Theorem 6.11 and Lemma 6.12, we obtain a contradiction which completes the proof of Theorem 6.4.

**7 Open Problems.**

The results of this note leave a number of open questions. Here is a brief list of some key questions.

- Can the basin of attraction, consisting of all points whose forward orbit converges to the smooth complex elliptic curve \( \mathcal{C} \), contain an interior point? In particular, can this basin intersect the Fatou set of \( f \)? (See Remark 3.5.)
In the space of all degree $d$ maps preserving a given elliptic curve, do those with an attracting periodic orbit form a dense open subset?

To what extent can a singular complex elliptic curve, or a smooth real elliptic curve be an attractor? In particular, can the complex Cassini quartic be a trapped attractor?

Does a nonlinear rational function with $f(C) = C$ necessarily have a repelling fixed point on the complex elliptic curve $C$? In all known examples, there is such a repelling point. Any repelling periodic point would provide an explicit obstruction to the existence of a trapping neighborhood.

We conjecture that there is a robust set of real parameter values such that the complex map has a cycle of Herman rings, with a dense open set as basin of attraction. (Example 4.4.) How can one prove such a statement?

What can one say about the dynamics when the elliptic curve has positive transverse Lyapunov exponent? Could such a map have an absolutely continuous invariant measure? Is it true that an elliptic curve can never be a measure-theoretic attractor when its transverse exponent is positive? (Compare Remark 5.5.)

What is the explanation for the apparent phenomenon that typical orbits often spend much time extremely close to a repelling elliptic curve? (For example compare Figure 5.)

References


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