CHAPTER III. MEASURE DYNAMICS.

§8. Ergodic Theory: Introduction.

The study of dynamics in a measure space is traditionally called *Ergodic Theory* (even when no ergodicity is involved), since the earliest work in this area centered around the problem of understanding the concept of ergodicity. This preliminary section will give some of the basic definitions and easier results. (For a more detailed presentation of measure theory, see for example [Rudin], and for a more thorough presentation of ergodic theory, see [Walters].)

§8A. Basic Measure Theory. By definition, a collection $\mathcal{A}$ of subsets $S \subset X$ is called a *σ-algebra* if

(1) $\mathcal{A}$ is closed under finite or countable unions: $S_i \in \mathcal{A} \Rightarrow \bigcup S_i \in \mathcal{A}$,

(2) $\mathcal{A}$ is closed with respect to taking complements: $S \in \mathcal{A} \Rightarrow (X \smallsetminus S) \in \mathcal{A}$, and

(3) the empty set $\emptyset$ belongs to $\mathcal{A}$.

(Here the letter σ is used as an abbreviation for "countable union".) The pair $(X, \mathcal{A})$ is then called a *measurable space*. A *measure* μ on $(X, \mathcal{A})$ is a function

$$\mu \;:\; \mathcal{A} \;\to\; [0\,,\,\infty]$$

which assigns to each $S \in \mathcal{A}$ a number $0 \le \mu(S) \le \infty$ satisfying the condition that

$$\mu(\,\bigcup S_i\,) \;=\; \sum \mu(S_i)$$

for any finite or countably infinite union of sets $S_i \in \mathcal{A}$ which are pairwise disjoint. We will be mainly interested in *finite measures*, which satisfy the condition that $\mu(S) < \infty$ for all S. The triple $(X, \mathcal{A}, \mu)$ is then called a *finite measure space*. In practice, we will usually normalize a finite measure by assuming that $\mu(X) = 1$. With this normalization, μ is called a *probability measure* on $(X, \mathcal{A})$, and $(X, \mathcal{A}, \mu)$ is called a *probability space*. For a probability measure, note that $0 \le \mu(S) \le 1$ for all $S \in \mathcal{A}$.

The Dirac Measure. Here is a trivial but important example. Let $\mathcal{A}$ be any σ-algebra of subsets of X. (As an example, we can take the biggest possible σ-algebra 2^X consisting of *all* subsets of X.) Choose some point $z \in X$. The associated *Dirac measure* δ_z is defined by the condition that

$$\delta_z(S) \;=\; \begin{cases} 1 & \text{if } z \in S\,, \\ 0 & \text{if } z \notin S\,. \end{cases}$$

More generally, given a finite or countably infinite sequence of points $z_i \in X$ and given weights $w_i > 0$ with sum $\sum w_i = 1$, we can form the probability measure $\mu \;=\; \sum w_i\,\delta_{z_i}$, defined by the formula

$$\mu(S) \;=\; \sum w_i\,\delta_{z_i}(S) \;=\; \sum\{w_i\;;\;z_i \in S\}\,.$$

Invariant Measures. A function $f : X \to X'$ from a measurable space $(X, \mathcal{A})$ to a measurable space $(X', \mathcal{A}')$ is called *measurable* if for every $S' \in \mathcal{A}'$ the pre-image

$$f^{-1}(S') \;=\; \{x \in X \;:\; f(x) \in S'\}$$

belongs to $\mathcal{A}$. We will write $f : (X, \mathcal{A}) \to (X', \mathcal{A}')$ in this case. (Caution: We do not assume that the forward image $f(S)$ of a set in $\mathcal{A}$ necessarily belongs to $\mathcal{A}'$.)

Given a measure μ on $(X, \mathcal{A})$ and given a measurable transformation $f : (X, \mathcal{A}) \to (X', \mathcal{A}')$, the ***push-forward*** $f_*(\mu)$ is a measure on $(X', \mathcal{A}')$ defined by the formula

$$f_*(\mu)\,(S') \;=\; \mu(f^{-1}(S'))$$

for every $S' \in \mathcal{A}'$. As an example, note that $f_*(\delta_z) = \delta_{f(z)}$.

Now consider a measurable transformation f from $(X, \mathcal{A})$ to itself. By definition, the measure μ on $(X, \mathcal{A})$ is f*-invariant*, or f is a *measure preserving* transformation, if $f_*(\mu) = \mu$, or in other words if

$$\mu(S) \;=\; \mu(f^{-1}(S))$$

for every $S \in \mathcal{A}$. As examples, the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ is invariant under the angle double map m_2 (even though $\mu(m_2(I)) = 2\mu(I)$ for any short interval $I \subset \mathbb{R}/\mathbb{Z}$). If $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x_0$ is a periodic orbit under f, then the average $(\delta_{x_1} + \cdots \delta_{x_n})/n$ is clearly an invariant probability measure on $(X, \mathcal{A})$.

Sets of Measure Zero. We often want to ignore sets with measure $\mu(S) = 0$. In fact, sometimes we don't really care about the measure itself, but only about which sets have measure zero.

Definitions. Two measures μ and ν on $(X, \mathcal{A})$ belong to the same *measure class*, if they have the same sets of measure zero. I will use the non-standard notation $[\mu] \subset \mathcal{A}$ for the collection of all sets $S \in \mathcal{A}$ which have measure $\mu(S) = 0$. Thus two measures μ and ν belong to the same measure class if and only if $[\mu] = [\nu]$.

A set $S \in \mathcal{A}$ is said to have *full measure* if its complement $X \smallsetminus S$ has measure zero; or equivalently if $S \cap T \neq \emptyset$ for *every* set T with $\mu(T) > 0$. (If $\mu(X)$ is finite, then S has full measure if and only if $\mu(S) = \mu(X)$, but if μ is σ-finite with $\mu(X) = \infty$ then there are *many* subsets of infinite measure which do not have full measure.)

A property is true for $[\mu]$-*almost every* point of X, or is true $[\mu]$-*almost everywhere*, if it is satisfied for all points of some set which has full measure in X.

We next give some application of these ideas, and then continue the discussion of basic measure theory in §8D below.

---

§**8B. Recurrence.** The following classical result helps to demonstrate the importance of invariant measures in dynamics. **Definition:** A measurable transformation $f : X \to X$ is *recurrent*,[1] for the measure class of μ, if for every set S with $\mu(S) > 0$ there exists at least one point $x \in S$ and one integer $n > 0$ so that $f^{\circ n}(x)$ also belongs to S.

[1] Caution: This is not standard terminology, although it seems reasonable to me. The term *conservative* is frequently used with this meaning, and *incompressible* is also used, although both words have other meanings as well. (Compare P. Halmos, "Ergodic Theory", Chelsea 1956; as well as U. Krengel, "Ergodic Theorems", de Gruyter 1985.) For that matter, the word *recurrent* has also been used with other meanings. For example Ghys, Goldberg and Sullivan show that the complex exponential map $\exp : \mathbb{C} \to \mathbb{C}$ generates a "recurrent relation", although this map is not recurrent in our sense. (See E. Ghys, L. Goldberg and D. Sullivan, On the measurable dynamics of $z \mapsto e^z$, Erg. Th. & Dy. Sys. **5** (1985) 329-335; together with M. Lyubich, Measurable dynamics of the exponential, Sov. Math. Dokl. **35** (1987) 223-226.)

Theorem 8.1 (Poincaré Recurrence Theorem). *Suppose that* μ *is a finite measure and that* $f : (X, \mathcal{A}) \to (X, \mathcal{A})$ *is a measure preserving transformation,* $f_*(\mu) = \mu$ *. Then* f *is recurrent for the measure class of* μ *.*

Proof. Given S with $\mu(S) > 0$, consider the sequence of measurable sets

$$S\,,\; f^{-1}(S)\,,\; f^{-2}(S)\,,\; \dots\;.$$

Since μ is an invariant measure, these all have measure equal to $\mu(S) > 0$. Hence these sets cannot be disjoint. For if they were disjoint, their union would have infinite measure, which is impossible. Therefore, we can find integers $0 \le m < n$ so that

$$f^{-m}(S) \cap f^{-n}(S) \neq \emptyset\;.$$

Choosing x_0 in this intersection, it follows that the point $x_m = f^{\circ m}(x_0)$ belongs to S , and that its iterated image $x_n = f^{\circ(n-m)}(x_m)$ also belongs to S . Thus we have produced at least one point $x_m \in S$ whose forward orbit returns to S . □

In fact recurrence easily implies a much sharper statement.

Lemma 8.2. *If* f *is recurrent and if* $\mu(S) > 0$ *, then for* $[\mu]$*-almost every point* $x \in S$ *the forward orbit of* x *returns to* S *infinitely often.*

Proof. Decompose S as a countable union

$$S \;=\; S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_\infty$$

of disjoint measurable subsets as follows. Let S_n be the set of points $x \in S$ such that the forward orbit of x returns to S exactly n times. If $x \in S_n$ with n finite, then clearly the forward orbit of x can never return to S_n . Hence every S_n with $n < \infty$ must have measure $\mu(S_n) = 0$. This proves that almost every point of S actually belongs to the subset S_∞ , as required. □

In the context of measure dynamics, it is convenient to say briefly that a set S is *invariant* under f whenever it is backward invariant, $S = f^{-1}(S)$. If f maps X onto itself, then any invariant set S is necessarily equal to its forward image $f(S)$, but in general we can only conclude that $f(S) \subset S$. If S is invariant, note that the transformation $f : S \to S$ is also measure preserving (where S is provided with the measure μ restricted to measurable subsets of S).

§**8C. Ergodicity and Transitivity.** Let μ be a measure on $(X, \mathcal{A})$. A measurable transformation $f : (X, \mathcal{A}) \to (X, \mathcal{A})$ is said to be *ergodic*, with respect to the measure class of μ , if it is not possible to express X as the union of two disjoint set of positive measure,

$$X = S \cup T \quad \text{with} \quad S \cap T = \emptyset\,, \quad \mu(S) > 0\,, \quad \text{and} \quad \mu(T) > 0\,,$$

where $f^{-1}(S) = S$ or equivalently $f^{-1}(T) = T$, so that S and T are f-invariant.

Closely related is the concept of measure-transitivity. By definition, f is *measure-transitive* if for any $S, T \in \mathcal{A}$ with $\mu(S) > 0$ and $\mu(T) > 0$ there exists $n > 0$ such that $f^{\circ n}(S) \cap T \neq \emptyset$, or equivalently

$$S \cap f^{-n}(T) \neq \emptyset\;.$$

A completely equivalent formulation would be that if $\mu(T) > 0$ then the union

$$f^{-1}(T) \cup f^{-2}(T) \cup f^{-3}(T) \cup \cdots$$

is a set of full measure, so that it must intersect every set of positive measure.

Lemma 8.3. *The measurable transformation f is transitive for the measure class of μ if and only if it is both recurrent and ergodic.*[2]

Proof. If f is both recurrent and ergodic, then for any S with $\mu(S) > 0$ let S_∞ be the subset of points whose orbits return to S infinitely often, as in 8.2. Then the infinite union

$$f^{-1}S_\infty \cup f^{-2}S_\infty \cup f^{-3}S_\infty \cup \cdots \;=\; S_\infty \cup f^{-1}S_\infty \cup f^{-2}S_\infty \cup f^{-3}S_\infty \cup \cdots$$

is invariant under f, with measure at least equal to $\mu(S) > 0$. Since f is ergodic, this proves that this union has full measure; which proves transitivity.

To prove that a transitive f is recurrent, simply take $S = T$ in the definition of transitivity. The proof of ergodicity is similarly trivial. $\square$

Combining 8.1 and 8.3, we clearly obtain the following.

Corollary 8.4. *A measure preserving transformation on a finite measure space is ergodic if and only if it is transitive.*

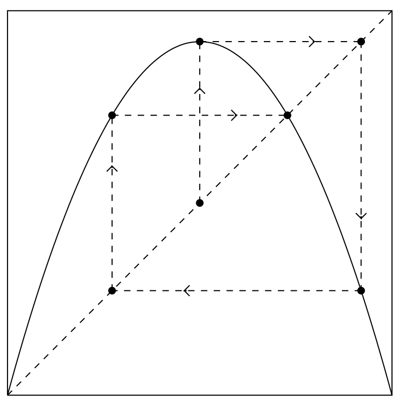

Figure 41. Graph of the map $f(x) = cx(1-x)$, where $c = 3.67857\cdots$ is chosen so that the third forward image of the critical point is the interior fixed point.

Remark 8.5. Some Examples. Any rotation of the unit disk provides an example of a Lebesgue measure preserving transformation which is recurrent but not ergodic or transitive. For both 8.1 and 8.4, it is essential that the measure μ be both finite and invariant. Here is an example: If $f(n) = n + 1$ on the space   of integers, then the measure $\mu(S) =$ cardinality(S) (on the σ-algebra of all subsets of) is invariant but not finite, while the measure $\mu(S) = \sum_{n\in S} 1/(n^2+1)$ is finite but not invariant. Evidently this map f is ergodic but not recurrent or transitive. A much less trivial example is the interval map shown in

[2] The proofs of 8.2, and 8.3 involve only the measure class of μ. They do not make any essential use of the actual measure. If we defined a "generalized measure class" to be simply a collection $\mathcal{N} \subset \mathcal{A}$ of measurable sets, closed under countable unions, such that any measurable subset of a set in $\mathcal{N}$ also belongs to $\mathcal{N}$, and with $X \notin \mathcal{N}$, then the definitions and proofs would go through without essential change. Simply substitute $S \in \mathcal{N}$ in place of $\mu(S) = 0$, and $S \in \mathcal{A} \smallsetminus \mathcal{N}$ in place of $\mu(S) > 0$.

Figure 41. Since the orbit of the critical point eventually lands on a repelling periodic point, it follows from classical results of Fatou that f has no attracting or one-sided-attracting periodic orbits. (See for example [Milnor 1999, 8.6 and 10.11].) It then follows from a theorem of [Blokh and Lyubich] that f is ergodic with respect to Lebesgue measure. But f is certainly not transitive or recurrent since its image $f([0,1])$ omits an entire interval.

Remark 8.6. Topological Transitivity and Intermingled Basins. If X is a topological space with a measure such that every non-empty open set has strictly positive measure, and if $f : X \to X$ is not only measurable but also continuous, then it is easy to see that measure-transitivity implies topological transitivity. (Compare §4C.) The converse statement is false. For example the complex exponential map $\exp : \mathbb{C} \to \mathbb{C}$ is topologically transitive but not ergodic. (See [Misiurewicz], together with [Lyubich 1987].) Here is an example due to Ittai Kan, which is even more surprising, and also somewhat easier to understand. Let X be the cylinder $(\mathbb{R}/\mathbb{Z}) \times [0,1]$, and define a diffeomorphism from X to itself by the formula

$$f(t\,,\,x) \;=\; \Big(3t\,,\; x + x(1-x)cos(2\pi t)/32\Big)$$

(where it is understood that t and $3t$ are real numbers modulo $\mathbb{Z}$). Thus every vertical line segment $t \equiv \text{constant}$ maps to a vertical line segment, but interior points of these line segments are moved up or down according as $\cos(2\pi t)$ is positive or negative. Kan shows that this map is topologically transitive. Now note that the open line segment

$$t \equiv (0 \bmod \mathbb{Z})\,, \qquad 0 < x < 1$$

maps onto itself, with all orbits converging towards the top endpoint $((0 \bmod \mathbb{Z})\,,\,1)$. Similarly, the open line segment with $t \equiv 1/2 \bmod \mathbb{Z}$ also maps into itself, with all orbits converging towards the bottom endpoint $((1/2 \bmod \mathbb{Z})\,,\,0)$. It follows easily that each of these two limit points has a dense set of vertical line segments as its "basin" of attraction. However, Kan proves much more: Let B_0 be the larger basin consisting of all points whose forward orbit converges to the bottom circle $(\mathbb{R}/\mathbb{Z}) \times \{0\}$, and let B_1 be the corresponding basin for the top circle. Then he shows that both B_0 and B_1 have Lebesgue measure $1/2$, so that the union $B_0 \cup B_1$ has full measure. *Thus f is not ergodic.* Furthermore, these two basins are intricately intermingled: For any non-empty open set U, he shows that both intersections $U \cap B_0$ and $U \cap B_1$ have positive measure. Now let S be the set of points $(t_0\,,\,x_0) \in B_0$ which have $x_0 \geq 1/2$, but such that the orbit $(t_0, x_0) \mapsto (t_1, x_1) \mapsto \cdots$ satisfies $x_n < 1/2$ for $n > 0$. Then S is a set of positive measure such that no orbit ever returns to S. Thus f is also not recurrent for the Lebesgue measure class.

Here is a quite different characterization of ergodicity for the case of finite f-invariant measures. Clearly the set $\mathcal{M} = \mathcal{M}(X,\mathcal{A})$ consisting of all probability measures on $(X,\mathcal{A})$ is a convex set in the vector space consisting of all real valued functions $\mathcal{A} \to \mathbb{R}$, and the set of f-invariant probability measures is a convex subset of $\mathcal{M}$. We will need the following.

Definition. Let K be a convex subset of a real vector space. A point $\kappa \in K$ is called an *extreme point* if it cannot be expressed as a weighted average, $\kappa = (1-t)\,\kappa_0 + t\,\kappa_1$ with $0 < t < 1$, where κ_0 and κ_1 are distinct elements of K.

Lemma 8.7. *Let $f : (X,\mathcal{A}) \to (X,\mathcal{A})$ be a measurable transformation, and let $\mu : \mathcal{A} \to \mathbb{R}$ be an invariant probability measure on $(X,\mathcal{A})$. Then μ is* *ergodic* ($\Leftrightarrow$ *transitive*) *if and only if it is an extreme point of the convex set* $\mathcal{M}_f = \mathcal{M}_f(X, \mathcal{A})$ *consisting of all invariant probability measures on* $(X, \mathcal{A})$ *.*

Proof. We will need the statement that there can be at most one invariant probability measure in a given ergodic measure class. (Compare the discussion following 3.9.) One proof, based on the Birkhoff Ergodic Theorem, will be given in 9.5. Another, based on the Radon-Nikodým Theorem proceeds as follows. If μ and v belong to the same measure class, then we can express $\mu(S)$ as the integral $\int_S \rho\, d\nu$ of an appropriate non-negative density function. Now if both μ and nu are f-invariant, then the function ρ must also be f-invariant almost everywhere. In the ergodic case, this implies that ρ is constant almost everywhere, hence $\mu = \nu$.

First suppose that μ is not an extreme point of $\mathcal{M}_t$. Then we can express μ as a weighted average $\mu_t = t\mu_1 + (1-t)\mu_0$ of two distinct measures μ_0 and μ_1 , with $0 < t < 1$. But all such measures μ_t are f-invariant, and clearly all belong to the same measure class. Hence this class can not be ergodic.

Conversely, if μ is not ergodic, then we can express X as the disjoint union $S_0 \cup S_1$ of two fully invariant measurable sets of strictly positive measure. Let $\mu_i(S) = \mu(S \cap S_i)/\mu(S_i)$. Then evidently μ_0 and μ_1 are distinct invariant probability measures with $\mu = \mu(S_0)\,\mu_0 + \mu(S_1)\,\mu_1$. Thus μ is not an extreme point of $\mathcal{M}_f$. $\square$

Another important condition, much stronger than transitivity, is "*mixing*". This will be described in Problem 8-d below.

---

§**8D. Borel Sets and Real Valued Functions.** This will be a brief review of standard material. For any topological space Y , define the σ-algebra $\mathcal{B}_Y$ of *Borel sets* as the smallest σ-algebra which contains all open subsets of Y , or equivalently contains all closed subsets of Y . In particular, for the topological space of real numbers, we can form the σ-algebra $\mathcal{B}$. Alternatively, $\mathcal{B}$ can be defined as the smallest σ-algebra of subsets of which contains all open or closed intervals $I \subset$.

A real valued function $\varphi : X \to$ on a measurable space $(X, \mathcal{A})$ is said to be *measurable* if it is measurable as a function from $(X, \mathcal{A})$ to (, $\mathcal{B}$) . An equivalent condition would be that for every interval $I \subset$ the pre-image $\varphi^{-1}(I)$ must belong to $\mathcal{A}$. (For our purposes, we do not allow the improper values $\varphi(x) = \pm\infty$.) We will need the following elementary observation.

Lemma 8.8. *The sum and product, and also the maximum of two measurable real valued functions are again measurable. Furthermore the supremum of any countable collection of measurable real valued functions is measurable, provided that it takes finite values. Similarly, the lim sup of any countable sequence of measurable real valued functions is again measurable if it takes finite values.*

Proof. If $\varphi + \psi = \eta$ with φ and ψ measurable, then the set $\{x : \eta(x) > c\}$ can be expressed as a countable union of sets $\{x\,;\, \varphi(x) > a\} \cap \{x : \psi(x) > b\} \in \mathcal{A}$, where a and b range over pairs of rational numbers with sum $a + b > c$. It follows easily that η is measurable. The proofs for a product, maximum, or minimum are similar. Note next that the pointwise limit of a sequence $\varphi_1 \le \varphi_2 \le \varphi_3 \le \cdots$ of measurable real valued function is clearly measurable, provided that this limit takes finite values. More generally, we get the corresponding statement for the supremum of measurable function $\{\psi_n\}$ since

$$\sup\{\psi_n(x)\}_{n\geq 1} \;=\; \lim_{n\to\infty} \max\{\psi_1(x)\,,\,\ldots\,,\,\psi_n(x)\}\;.$$

The corresponding statement for a lim sup now follows from the identity

$$\limsup_{n\to\infty}\;\psi_n \;=\; \lim_{n\to\infty}\;(\,\sup\{\psi_n\,,\,\psi_{n+1}\,,\,\psi_{n+2}\,,\,\ldots\})\;. \qquad \square$$

Integration. For any measurable space $(X, \mathcal{A})$, we will be interested in measurable real valued functions $\varphi : (X, \mathcal{A}) \to (\;\;, \mathcal{B}\;)$. If $\varphi : (X, \mathcal{A}) \to (\;\;, \mathcal{B}\;)$ is any *bounded* measurable function, and if μ is a finite measure on $(X, \mathcal{A})$, then the *integral*

$$\int \varphi\, d\mu \;=\; \int_X \varphi(x)\, d\mu(x)$$

is a real number which can be defined as follows. (Compare any text on measure theory.) For any $\epsilon > 0$, choose a partition $X = \bigcup S_i$ into finitely many disjoint sets $S_i \in A$ which are small enough so that the diameter of $\varphi(S_i)$ is less than ϵ , and let $x_i \in S_i$ be representative points. Then the limit of $\sum \varphi(x_i)\,\mu(S_i)$ as $\epsilon \to 0$ is the required integral.

More generally, if μ is *σ-finite*, that is if X is a countable union of sets of finite measure, then φ is called an *integrable* function, or an element of $L^1(X, \mathcal{A}, \mu)$, if there is a uniform upper bound for the integrals $\int_S |\varphi|$, as S ranges over all sets of finite measure on which $|\varphi|$ is bounded. If φ is integrable, then the integral $\int_X \varphi \in$ can be defined as a limit of such finite integrals $\int_S \varphi$. There is a completely analogous theory of measurable (or integrable) complex valued functions.

If $f : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is a measurable transformation, and if $\mu' = f_*(\mu)$, then for any bounded measurable $\varphi : X' \to$ note the identity

$$\int_X \varphi(f(x))\, d\mu(x) \;=\; \int_{X'} \varphi(x')\, d\mu'(x')\;.$$

In particular, for an invariant measure $f_*(\mu) = \mu$, where $f : (X, \mathcal{A}) \to (X, \mathcal{A})$, we have

$$\int_X (\varphi \circ f)\; d\mu \;=\; \int_X \varphi\; d\mu\;. \qquad (8:1)$$

It is often convenient to introduce the Hilbert space $L^2 = L^2(X, \mathcal{A}, \mu)$ consisting of all square integrable functions $\varphi : X \to$, with Hermitian inner product

$$(\varphi\,,\,\psi) \;=\; \int \varphi(x)\,\overline{\psi(x)}\, d\mu(x) \;\in$$

for $\varphi\,,\psi \in L^2$, where $\overline{\psi(x)}$ denotes the complex conjugate. For our purposes, L^2 can simply be identified with the completion of the space of bounded measurable complex valued functions on X under the norm

$$\|\varphi\|_2 \;=\; \sqrt{(\varphi, \varphi)} \;=\; \sqrt{\int |\varphi|^2\, d\mu}\;.$$

(Here functions which are equal almost everywhere must be identified with each other.) Note the triangle inequality $\|\varphi + \psi\|_2 \leq \|\varphi\|_2 + \|\psi\|_2$ and the Schwarz inequality

$$|(\varphi\,,\,\psi)| \;\leq \|\varphi\|_2\,\|\psi\|_2\;. \qquad (8:2)$$

Note also that any measurable transformation $f : (X, \mathcal{A}) \to (X', \mathcal{A}')$ gives rise to a norm preserving embedding $\varphi' \mapsto \varphi' \circ f$ of the Hilbert space $L^2(X', \mathcal{A}', f_*(\mu))$ into $L^2(X, \mathcal{A}, \mu)$.

Perhaps the simplest example is the Hilbert space $L^2(\ /\ , \mathcal{B}_{\ /\ }, \text{Lebesgue})$, or briefly $L^2(\ /\ )$. Here a fundamental result, due in some sense to Fourier, asserts that the exponential functions

$$\mathbf{e}_n(x) \;=\; e^{2\pi i n x} \;, \qquad \mathbf{e}_n : \ /\ \to \qquad (8:3)$$

form a maximal collection of orthonormal vectors in $L^2(\ /\ )$. Any function which is orthogonal to $\mathbf{e}_n$ for every n must be zero almost everywhere.

---

§8E. Some Problems.

Problem 8-a. Rotations. Let $f : \ /\ \to \ /\$ be the rotation $f(x) = x + c$, and let $S \subset \ /\$ be any measurable f-invariant set, $S = S + c$. Let $\varphi(x) = \mathbf{1}_S(x)$ be the characteristic function of S , equal to 1 if $x \in S$ and zero otherwise. Note that $\varphi(x+c) = \varphi(x)$. Using the notation (8 : 3), show that the coefficients of the Fourier series

$$\sum_{n=-\infty}^{\infty} a_n \, \mathbf{e}_n(x)$$

for $\varphi(x)$, given by

$$a_n \;=\; \int_{\ /\ } \varphi(x)\, \mathbf{e}_{-n}(x)\, dx \;,$$

must satisfy $a_n = \mathbf{e}_n(c)\, a_n$. Conclude that f is ergodic if and only if the constant c is irrational. (It will follow from Corollary 9.3 that this statement is completely equivalent to Weyl's Theorem 3.7.)

Problem 8-b. Angle doubling. Now consider the doubling map $m_2(x) = 2x$ on $\ /\$, if φ is the characteristic function of a measurable invariant set $S = m_2^{-1}(S)$, so that $\varphi(m_2(x)) = \varphi(x)$, show that the Fourier coefficients must satisfy

$$a_{2n} \;=\; a_n \;, \quad a_{2n+1} = 0 \;.$$

Conclude that m_2 is ergodic. (It will follow from 9.3 that this is equivalent to Borel's Theorem 3.8.)

Problem 8-c. Torus maps. More generally, let X be a torus $\ ^d/\ ^d$, with Lebesgue measure λ . A maximal set of orthonormal vectors in the Hilbert space $L^2(X, \lambda)$ is provided by the products

$$\mathbf{e}_h(\mathbf{x}) \;=\; e^{2\pi i h(\mathbf{x})} \;=\; \mathbf{e}_{m_1}(x_1) \cdots \mathbf{e}_{m_d}(x_d) \;,$$

where $h(\mathbf{x}) = \sum m_i x_i$ is any linear map from $\ ^d$ to $\ $ with integer coefficients. If $f : \ ^d/\ ^d \to \ ^d/\ ^d$ is induced from a non-singular linear map $\ell : \ ^d \to \ ^d$ with integer coefficients, show that f is measure preserving, and that the induced linear map $\varphi \mapsto \varphi \circ f$ on the Hilbert space $L^2(\ ^d/\ ^d, \lambda)$ is given by

$$\mathbf{e}_h \;\mapsto\; \mathbf{e}_h \circ f \;=\; \mathbf{e}_{h \circ \ell} \;.$$

Show that f is ergodic if and only if no eigenvalue of ℓ is a root of unity. For example the first two of the following matrices correspond to ergodic transformations of the 2-dimensional torus, but the third does not:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} , \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} , \quad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} .$$

Problem 8-d. Mixing transformations. By definition, a measure preserving transformation f on a probability space is *mixing* if, for every pair of measurable subsets S and T, the measure $\mu(S \cap f^{-n}(T))$ converges to the product $\mu(S)\,\mu(T)$ as $n \to \infty$. Roughly speaking, this means that the information that $x \in S$ gives no information at all about the probable location of $f^{\circ n}(x)$ for large n.

Let f be a measure preserving transformation of a probability space. Show that f is mixing if and only if

$$\lim_{n \to \infty} \int \varphi(x)\, \psi(f^{\circ n}(x))\, d\mu(x) \;=\; \Big(\int \varphi\, d\mu\Big)\, \Big(\int \psi\, d\mu\Big)$$

for every pair of functions φ and ψ in $L^2(X, \mu)$. Show in fact that it suffices to check that

$$\int \varphi(x)\, \psi(f^{\circ n}(x))\, d\mu(x) \;\to\; 0$$

as $n \to \infty$ for φ and ψ in the subspace consisting of square integrable functions with $\int \varphi\, d\mu = 0$, or even for φ and ψ belonging to some maximal orthonormal subset of this subspace.

As examples, show that the angle doubling map on the circle is mixing, while an irrational rotation of the circle is not mixing (although it is ergodic by 8-a). For $f : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ induced by a non-singular homogeneous linear map ℓ as in 8-c, show that f is mixing if and only if no eigenvalue of ℓ is a root of unity (or if and only if f is ergodic).

Problem 8-e. Iterations. If f is transitive, note that the composition $f \circ f$ may not even be ergodic. (Compare Figure 22.) However, if f is recurrent, show that $f \circ f$ is also recurrent. (If a point of S returns to S after m and n iterations, where $m < n$, then at least one of the three numbers m, n, $n - m$ must be even.) Show more generally that every $f^{\circ n}$ is also recurrent.

Problem 8-f. No Non-Trivial Attracting Sets. (Compare §5A.) Let μ be an f-invariant measure on a locally compact space, and suppose that all compact subsets have finite measure. If A is an attracting set with basin B, show that the difference set $B \smallsetminus A$ has measure zero. In classical mechanics (or for Hamiltonian dynamical systems) one encounters such invariant measures with the additional property that every non-empty open sets has strictly positive measure. In such cases, show that there cannot be any compact attracting sets, except in the trivial sense of a compact invariant set which is also open and invariant, so that the basin B is equal to A.

Problem 8-g. Hénon maps. Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a real constant $\delta \neq 0$, construct an associated map $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane by setting

$$F(x\,,\, y) \;=\; (y\,,\, f(y) - \delta\, x)\;.$$

Show that F maps $\mathbb{R}^2$ diffeomorphically onto itself, and in fact that any pair $(x_0\,,\, x_1)$ in $\mathbb{R}^2$ gives rise to a bi-infinite sequence

$$\cdots\,,\; x_{-1}\,,\; x_0\,,\; x_1\,,\; x_2\,,\; \cdots$$

where $F(x_{i-1}\,,\, x_i) = (x_i\,,\, x_{i+1})$ or equivalently

$$\delta\, x_{i-1} + x_{i+1} \;=\; f(x_i)\;.$$

Show that F has Jacobian determinant identically equal to δ. Hence for any bounded measurable set S the Lebesgue area $\mu(F(S))$ is equal to $|\delta|\,\mu(S)$.

Now suppose that the ratio $|f(x)/x|$ tends to infinity as $|x| \to \infty$. (For example this condition will always be satisfied if f is a non-linear polynomial.) Then there exists a constant $c > 0$ so that

$$|f(x)| \;>\; \Big(1 + |\delta|\Big)|x| \qquad \text{whenever} \qquad |x| \;\geq\; c\,.$$

Let I be the interval $[-c\,,\,c]$, and let $I^2 \subset \mathbb{R}^2$ be the corresponding square of area $4c^2$. If $|x_0| \geq c$, show that either

$$|x_0| \;<\; |x_1| \;<\; |x_2| \;<\; \cdots \qquad \text{tending to} \quad +\infty\,,$$

or

$$|x_0| \;<\; |x_{-1}| \;<\; |x_{-2}| \;<\; \cdots \qquad \text{tending to} \quad +\infty\,,$$

or both. In particular, if the orbit of a point in I^2 ever leaves this square, then it can never return. Show that a point of $\mathbb{R}^2$ has bounded forward orbit if and only if it eventually hits the square I^2 and then never leaves it. Show that the set K consisting of points with bounded backward and forward orbits is a compact subset of I^2.

If $|\delta| > 1$ so that F increases areas, show that the set $K^+ \subset \mathbb{R}^2$ consisting of points with bounded forward orbit has Lebesgue measure zero. (Thus there can be no compact attracting sets.) Similarly, if $|\delta| < 1$ show that the set K^- of points with bounded backward orbit has measure zero. Whenever $|\delta| \neq 1$, show that both sets $K^\pm$ have measure either zero or infinity, and show that the set $K = K^+ \cap K^-$ has measure zero.

On the other hand, in the area preserving case $\delta = \pm 1$, show that both difference sets $K^\pm \smallsetminus K$ have measure zero. Thus, for x outside some set of measure zero, the forward orbit of x is bounded if and only if the backward orbit of x is bounded, and $\mu(K^\pm) = \mu(K) < 4c^2$.