§3. Probabilistic Methods.

The central paradox in the study of chaotic dynamics is the dichotomy between *determinism* and *chance*. We are studying completely deterministic problems, in which the present state of a system uniquely determines its future behavior. And yet it turns out that the most useful description of behavior must often be in terms of probabilities.

Let us first review some basic ideas from probability theory:

§3A. Bernoulli and Borel: the Law of Large Numbers. Jakob Bernoulli, almost 300 years ago, studied the random process of repeatedly flipping a coin. Assume that the coin will always land either with the "head" side up or with the "tail" side up. It is convenient to label the two sides of the coin by 1 (for "heads") and 0 (for "tails"). Thus each outcome of a coin toss is now represented by a *bit* α_i , which is equal to zero or one. With this convention, the outcome of an entire sequence of coin tosses is represented by a sequence $\alpha_1, \alpha_2, \ldots$ of such bits.

For each individual coin toss, suppose that $p = p(1)$ is the probability of heads and $1 - p = p(0)$ is the probability of tails. Here p is to be some fixed real number, strictly between zero and one. (In the special case of a "fair coin", where heads and tails are equally probable, we take $p = 1/2$.) We assume also that there is absolutely no correlation between the results of different coin tosses. By definition, this means that the *probability* $\mathrm{Prob}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ associated with each possible sequence of length n is equal to the product $p(\alpha_1)\, p(\alpha_1) \cdots p(\alpha_n)$. One then speaks of *independent Bernoulli trials*. Evidently these probabilities are positive real numbers with sum equal to one. In the special case of a fair coin, with $p = 1/2$, note that each of these probabilities $\mathrm{Prob}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is equal to $1/2^n$.

The sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ can be described as the number of heads which occur during the first n coin tosses; and the average $(\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ can be described as the *proportion* of heads which occur. Bernoulli showed that this proportion $(\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ of heads is likely to be very close to p , provided that n is large:

> **Lemma 3.1. Law of Large Numbers.** *For any fixed* $\epsilon > 0$ *, the probability* $P_n(\epsilon)$ *that the average* $\frac{1}{n}(\alpha_1 + \cdots + \alpha_n)$ *differs from* p *by more than* ϵ *is less than* $C(\epsilon)/n$ *for some constant* $C(\epsilon)$ *which is independent of* n *. Hence this probability* $P_n(\epsilon)$ *tends to zero as* $n \to \infty$ *.*

Definitions. The proof will be based on the following ideas. Let X be a metric space, and let $\mathcal{B}$ be the collection of *Borel subsets* of X , that is the smallest collection of subsets which contains all open subsets, all closed subsets, and all countable unions or intersections of sets from $\mathcal{B}$. By a *probability measure* μ on X we mean a countably additive function $\mu : \mathcal{B} \to [0, 1]$, satisfying the condition that $\mu(X) = 1$. (See Appendix A for a more detailed explanation.) By definition, the measure $\mu(S)$ of a subset $S \in \mathcal{B}$ is described as the *probability* that a "μ*-randomly chosen*" point of X will belong to this subset S . For the proof of 3.1 it would suffice to consider the case of a finite space $X = \{0, 1\} \times \cdots \times \{0, 1\}$, but it is useful to have the more general concept.

Suppose now that we fix some space X with a probability measure μ . By a real *random variable* we will mean simply a bounded measurable function $\phi : X \to$. (By definition, a real valued function ϕ is *measurable* if, for each interval I of real numbers, the subset $\phi^{-1}(I) \subset X$ belongs to this preferred collection $\mathcal{B}$ of subsets.) The *expected value* or *mean*

$\mathcal{E}$, and the *variance* $\mathcal{V}$, of such a random variable ϕ are defined by the equations

$$\mathcal{E}(\phi) \;=\; \int_X \phi(x)\, d\mu\,, \qquad \mathcal{V}(\phi) \;=\; \int_X \left(\phi(x) - \mathcal{E}(\phi)\right)^2 d\mu\ .$$

The identity

$$\mathcal{V}(\phi) \;=\; \mathcal{E}(\phi^2) - \mathcal{E}(\phi)^2$$

is easily verified.

Lemma 3.2. Chebyshev's Inequality. *For any* $\epsilon > 0$ *, let* $P(\epsilon)$ *be the probability that a* μ*-randomly chosen point* $x \in X$ *will satisfy* $|\phi(x) - \mathcal{E}(\phi)| > \epsilon$ *. Then* $P(\epsilon) < \mathcal{V}(\phi)/\epsilon^2$ *.*

Proof. Let $S(\phi, \epsilon)$ denote the set of points $x \in X$ for which

$$|\phi(x) - \mathcal{E}(\phi)| \;>\; \epsilon\ .$$

Then

$$\mathcal{V}(\phi) \;=\; \int_X (\phi - \mathcal{E}(\phi))^2\, d\mu \;\geq\; \int_{S(\phi,\epsilon)} (\phi - \mathcal{E}(\phi))^2\, d\mu \;>\; \epsilon^2\, \mu(S(\phi, \epsilon))\ ,$$

hence $\mu(S(\phi, \epsilon)) < \mathcal{V}(\phi)/\epsilon^2$ as required. □

Two such real valued functions ϕ and ψ are called *independent random variables* if for any pair of intervals I , $J \subset$ the measure of the intersection $\phi^{-1}(I) \cap \psi^{-1}(J)$ is equal to the product $\mu(\phi^{-1}(I))\, \mu(\psi^{-1}(J))$. Intuitively this means that information as to whether $\phi(x)$ is (or is not) in some specified interval gives no information at all about $\psi(x)$.

Lemma 3.3. *For any two random variables* ϕ *and* ψ *we have*

$$\mathcal{E}(\phi + \psi) = \mathcal{E}(\phi) + \mathcal{E}(\psi)\ .$$

Furthermore, if ϕ *and* ψ *are independent random variables, then*

$$\mathcal{E}(\phi \cdot \psi) = \mathcal{E}(\phi)\, \mathcal{E}(\psi) \quad \textit{and} \quad \mathcal{V}(\phi + \psi) \;=\; \mathcal{V}(\phi) + \mathcal{V}(\psi)\ .$$

Here $\phi \cdot \psi$ stands for the pointwise product function $x \mapsto \phi(x)\, \psi(x)$. The proof of 3.3 is easily supplied. □

Note that the hypothesis that ϕ and ψ are independent is essential. For example, for the case $\phi = \psi$ we have $\mathcal{V}(2\phi) = 4\, \mathcal{V}(\phi)$.

Proof of 3.1. To apply these ideas to independent Bernoulli trials, we introduce the compact metric space $X = \{0, 1\}$ consisting of all infinite sequences of zeros and ones. (For this section, it will be convenient to redefine as the set of strictly positive integers.) Bernoulli's probability distribution gives rise to a probability measure $\mu = \mu_p$ on the collection $\mathcal{B}$ of Borel subsets of X . (See Chapter IV and Appendix A for details.) By definition, the measure

$$\mu((\alpha_1\, ,\, \ldots\, ,\, \alpha_n) \times \{0, 1\} \times \{0, 1\} \times \cdots)$$

of a *cylinder set* in which the first n coordinates are prescribed is given by the product $p(\alpha_1) \cdots p(\alpha_n)$, as above. It will be convenient to identify the symbol α_i with the projection map

$$(\alpha_1\, ,\, \alpha_2\, ,\, \ldots) \;\mapsto\; \alpha_i \;\in\; \quad ,$$

and hence to consider the $\alpha_i : X \to$. as independent random variables. A brief computation shows that the mean and variance are given by

$$\mathcal{E}(\alpha_i) \;=\; p\,, \qquad \mathcal{V}(\alpha_i) \;=\; p(1-p)\ .$$

Let

$$\phi_n(\alpha_1\,,\,\alpha_2\,,\,\ldots) = (\alpha_1 + \cdots + \alpha_n)/n$$

be the average of the first n of these α_i . Then

$$\mathcal{E}(\phi_n) = p\ , \qquad \mathcal{V}(\phi_n) \;=\; \sum_1^n \mathcal{V}(\alpha_i/n) \;=\; \sum_1^n p(1-p)/n^2 \;=\; p(1-p)/n\ .$$

By 3.2 it follows that

$$\mu(S(\phi_n,\epsilon)) \;<\; C/n \qquad \text{with} \qquad C \;=\; p(1-p)/\epsilon^2\ ,$$

as required. □

The statement of 3.1 has the defect that the series $\sum_n C/n$ is divergent. Thus, it seems to allow the possibility that $|\frac{1}{n}(\alpha_1+\cdots+\alpha_n)-p| > \epsilon$ for *infinitely many* values of n , with strictly positive probability, even though the probability $P_n(\epsilon)$ that $|\frac{1}{n}(\alpha_1+\cdots+\alpha_n)-p| > \epsilon$ for any particular large n is very small. This defect was corrected by Emile Borel, two hundred years after Bernoulli's work:

Lemma 3.4. *The probability* $P_n = P_n(\epsilon)$ *that* $|\frac{1}{n}(\alpha_1+\cdots+\alpha_n)-p| > \epsilon$ *is actually less than some constant divided by* n^2 *. Hence the sum*

$$P_n + P_{n+1} + P_{n+2} + \cdots$$

converges to zero as $n \to \infty$ *.*

The proof is similar to the proof of 3.1, but involves a computation of the expected value of the 4-th power $(\phi - \mathcal{E}(\phi))^4$. See Problem 3-a for details. □

Corollary 3.5. The Strong Law of Large Numbers. *For a sequence* $\alpha_1\,,\,\alpha_2\ \ldots$ *of independent Bernoulli trials, chosen according to the probability distribution* $\mu = \mu_p$ *, the sequence of averages* $(\alpha_1+\cdots+\alpha_n)/n$ *converges to* p *with probability one.*

Proof. Again let $S(\phi_n,\epsilon) \subset \{0,1\}$ denote the set of sequences for which

$$|\tfrac{1}{n}(\alpha_1+\cdots+\alpha_n)-p| > \epsilon\ .$$

Then the probability $\mu(S(\phi_n,\epsilon))$ is less than C/n^2 by 3.4. Let $T(n,\epsilon)$ denote the infinite union $S(\phi_n,\epsilon) \cup S(\phi_{n+1},\epsilon) \cup \cdots$. It follows that

$$\mu(T(n,\epsilon)) \;<\; \sum_{i=n}^{\infty} C/i^2 \;<\; C/(n-1)\ ,$$

which tends to zero as $n \to \infty$. Since

$$T(1,\epsilon) \supset T(2,\epsilon) \supset \cdot$$

are sets with measure tending to zero, it follows that the intersection $\bigcap_{n\ge 1} T(n,\epsilon)$ is a set of measure zero. *Thus, with probability one, a* μ*-randomly chosen sequence of independent Bernoulli trials will not belong to this intersection* $\bigcap_{n\ge1} T(n,\epsilon)$ *.* In other words, it must satisfy the condition that

$$\left|\tfrac{1}{n}(\alpha_1 + \cdots + \alpha_n) - p\right| \;\le\; \epsilon$$

for all sufficiently large n. Since this is true for *every* positive number ϵ, it follows that

$$\tfrac{1}{n}(\alpha_1 + \cdots + \alpha_n) \;\to\; p$$

with probability one, as required. (Here we make use of the property that a countable union of sets of measure zero must again have measure zero.) □

More generally, instead of sequences of zeros and ones, we can work with sequences from any finite alphabet A. Suppose that each symbol $a \in A$ is assigned a probability $p(a) > 0$, with $\sum_{a\in A} p(a) = 1$. Then there is a corresponding Bernoulli measure μ on the space A of sequences $(a_0\,, a_1\,, \ldots)$ so that each cylinder set $S = (a_0\,, a_k) \times A \times A \times \cdots$ has measure $\mu(S)$ equal to the product $p(a_0)\cdots p(a_k)$.

Corollary 3.6. *Let $a \in A$ be some fixed symbol in our finite alphabet. Then for μ-almost every sequence $(a_0\,, a_1\,, \ldots) \in A$, the proportion of the first n symbols $a_0\,, \ldots\,, a_{n-1}$ which are equal to this specified a converges to $p(a)$ as $n \to \infty$.*

In other words, this property is true except for sequences $(a_0\,, a_1\,, \ldots)$ belonging to a set $S \subset A$ with measure $\mu(S) = 0$.

Proof. Project A to $\{0,1\}$ by sending a to one and everything else to zero. then A maps to $\{0,1\}$, and a μ-randomly chosen sequence in A corresponds to a sequence in $\{0,1\}$ which is randomly chosen with respect to the Bernoulli measure $\mu_{p(a)}$. The conclusion then follows from 3.5. □

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§**3B. Natural Measures and Ergodic Measures.** A central problem of dynamics is to understand the behavior of the orbit

$$f \;:\; x_0 \;\mapsto\; x_1 \;\mapsto\; x_2 \;\mapsto\; \cdots$$

of a typical point $x_0 \in X$ under a mapping $f : X \to X$. We will need the following.

Definition. A sequence of points $x_0\,, x_1\,, x_2\,, \ldots$ in a topological space X is *evenly distributed* with respect to a probability measure μ on X if the following condition is satisfied. For any bounded continuous function $\phi : X \to$, the limit

$$\lim_{n\to\infty} \left(\phi(x_0) + \phi(x_1) + \cdots + \phi(x_{n-1})\right) / \, n \qquad (3{:}1)$$

should exist and be equal to the integral $\int_X \phi(x)\, d\mu$.

In the special case of an orbit $x_0 \mapsto x_1 \mapsto \cdots$, this limit $(3{:}1)$, if it exists, is called the *time average* of the function ϕ over the orbit of x_0. Thus an orbit is evenly distributed with respect to μ if and only if this time average is defined and equal to the *space average* $\int_X \phi(x)\, d\mu$ for every bounded continuous test function $\phi : X \to$.

Remark. It would often be convenient to replace the continuous function ϕ by the *characteristic function*

$$\mathbf{1}_S(x) \;=\; \begin{cases} 1 & \text{if } x \in S\,, \\ 0 & \text{if } x \notin S \end{cases} \qquad (3{:}2)$$

of some measurable subset $S \subset X$. The ratio

$$(\mathbf{1}_S(x_0) + \mathbf{1}_S(x_1) + \cdots + \mathbf{1}_S(x_{n-1}))\,/\,n$$

would then measure the proportion of the first n points in the sequence which belong to S . We would like to say that this proportion converges to the measure $\mu(S) = \int_S \mathbf{1}_S(x)d\mu$ as $n \to \infty$. In fact this is true provided that the subset S satisfies the following condition: The topological *boundary*

$$\partial S \;=\; \overline{S} \cap \overline{(X \smallsetminus S)}\;, \tag{3:3}$$

that is the set of all common limit points of S and of its complement, must have measure $\mu(\partial S)$ equal to zero. (Problem 3-b.)

Here are two examples, both using the standard Lebesgue measure λ on the circle $\mathbb{R}/\mathbb{Z}$. By an *irrational rotation* of this circle we will mean a transformation of the form

$$\rho_\alpha(t) \;\equiv\; t + \alpha \pmod{\mathbb{Z}}\;,$$

where α is any irrational number.

Theorem 3.7 (Weyl). *For any irrational rotation ρ_α and for any starting point $t_0 \in \mathbb{R}/\mathbb{Z}$, the orbit*

$$\rho_\alpha \;:\; t_0 \;\mapsto\; t_1 \;\mapsto\; t_2 \;\mapsto\; \cdots$$

is evenly distributed with respect to the Lebesgue measure λ on $\mathbb{R}/\mathbb{Z}$.

Now consider the doubling map $m_2(t) \equiv 2t \pmod{\mathbb{Z}}$. We say that a property is true for λ*-almost every* point of $\mathbb{R}/\mathbb{Z}$ if the set $F \subset \mathbb{R}/\mathbb{Z}$ of points for which it is *false* has measure $\lambda(F) = 0$.

Theorem 3.8 (Borel). *For λ-almost every point $t_0 \in \mathbb{R}/\mathbb{Z}$, the orbit*

$$m_2 \;:\; t_0 \;\mapsto\; t_1 \;\mapsto\; t_2 \;\mapsto\; \cdots$$

is evenly distributed with respect to the Lebesgue measure λ on $\mathbb{R}/\mathbb{Z}$.

Proofs will be given at the end of this section.

Note the difference between these two statements. *Every* orbit for an irrational rotation is evenly distributed. On the other hand, for the doubling map m_2 , it is easy to find orbits which are not evenly distributed. For example, if t_0 is rational, then its orbit under m_2 is finite, and hence certainly not evenly distributed. (In fact, even though "almost all" orbits are evenly distributed, it is difficult to find explicit and non-artificial examples of numbers t_0 for which the orbit is known to be evenly distributed.) The difference between these two examples seems to be related to the fact that an irrational rotation is a "non-chaotic" map in which nearby initial points lead to orbits which remain close to each other forever, while m_2 is chaotic.

Now let us discuss the corresponding property for the quadratic map $q(x) = 2x^2 - 1$ of §2A. Let $N \subset \mathbb{R}/\mathbb{Z}$ be the set of measure zero consisting of "bad" points for the doubling map m_2 , that is points whose orbit is not λ-evenly distributed. Clearly the projection map

$$h \;:\; t \;\mapsto\; \mathrm{Re}(e^{2\pi it}) \;=\; \cos(2\pi t)$$

from $\mathbb{R}/\mathbb{Z}$ to the interval $J = [-1\,,\,1]$, as illustrated in Figure 10, carries N to a set $h(N) \subset J$ of Lebesgue measure zero. Every $x_0 \in J \smallsetminus h(N)$ is the image of a point

$t_0 \in . / \smallsetminus N$. Thus the orbit of t_0 is evenly distributed around the circle. However, this does not mean that the orbit $\{x_j\}$ of x_0 is *Lebesgue* evenly distributed in the interval J . Rather, the proportion of x_j , with $0 \le j < n$, which lie in an subinterval $S \subset J$ must converge to the length of the set $h^{-1}(S) \subset . /$. Inspecting Figure 10, we see that a small interval near either end of J is likely to contain many more orbit points than an interval of the same length near the middle of J .

Formalizing this argument, we define the *push forward* $\nu = h_*(\lambda)$ of the given measure λ on $. /$ by the formula

$$\nu(S) \;=\; \lambda(h^{-1}(S)) \;.$$

As as an example, if S is the interval $[a\,,\,b] \subset J$, then it is not hard to check that

$$\nu[a\,,\,b] \;=\; \frac{\arcsin(b) - \arcsin(a)}{\pi} \;=\; \frac{1}{\pi}\int_a^b \frac{dx}{\sqrt{1-x^2}} \;. \qquad (3:4)$$

(Compare Problem 3-g.)

Corollary 3.9 (Ulam and von Neumann). *For Lebesgue almost every point x_0 in the interval $-1\,,\,1$ the orbit $q : x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$ is evenly distributed with respect to the measure $\nu = h_*(\lambda)$ described above.*

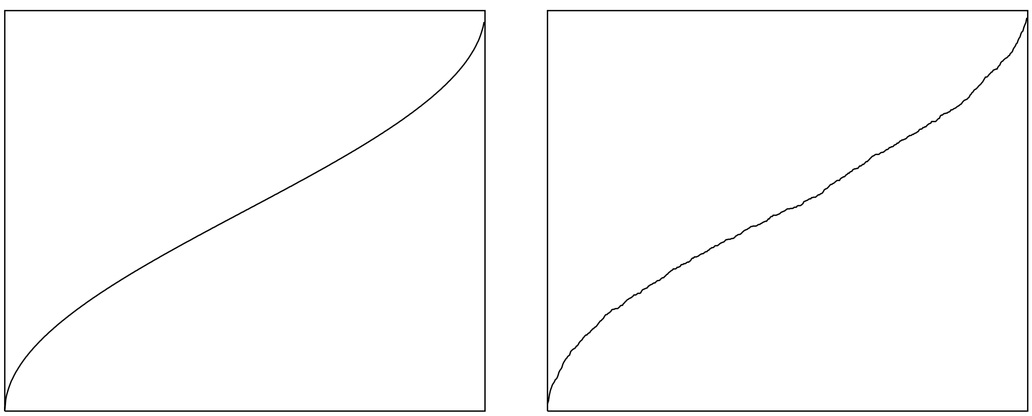

Figure 15. The theoretical distribution curve $x \mapsto \nu[-1\,,\,x]$ for typical orbits of the quadratic map q , and a plot of the actual distribution curve for one thousand iterates with a pseudo-randomly chosen x_0 .

We can illustrate this corollary by a computer experiment as follows. Suppose that we choose $x_0 \in J$ by some “pseudo-random” number generator, and follow its orbit $q : x_0 \mapsto x_1 \mapsto \cdots$ for say $n = 1000$ iterations. Then for each $x \in J$ we can plot the fraction of these x_i which lie in the interval $[-1\,,\,x]$ as a (necessarily monotone increasing) function of x . One such plot is shown in Figure 15b, and compared with the theoretical curve $x \mapsto (\,\arcsin(x) - \arcsin(-1))/\pi$ of Figure 15a. The match between these two figures is reasonably good.

Remark. The reader may well be suspicious at this point, since we have seen in §2C that such a computer generated orbit is quite unreliable. For this particular example, it is not difficult to show that the computer generated orbit is quite close to *some* true orbit. (Compare Problem 3-c.) However, the question as to whether a computer generated "random" orbit is close to a truly random orbit is more difficult, and not well understood, by me at least. By way of comparison, if we tried a corresponding computer experiment with the tent map, or with the map m_2, or with the squaring map P_2, we would get a completely wrong result. (Problem 3-d.) The prevailing belief is that these particular examples are anomalous, and that such computer drawn pictures are usually reasonably accurate; but caution is *always* appropriate. (Compare [Lorenz 1989].)

This example 3.9 should help to motivate the following.

Definition. Let X be a region in Euclidean space, and let $f : X \to X$ be a given function. We will say that a probability measure ν on X is a *natural measure* for f if, for Lebesgue almost every point $x_0 \in X$, the orbit of x_0 under f is evenly distributed with respect to ν. (More generally, X could be any smooth manifold. Compare Problem 3-e.)

Note. Such a ν is also called a *Bowen-Ruelle-Sinai measure*, or an *asymptotic measure* associated with the Lebesgue measure λ on X. In fact [Bowen 1975], [Ruelle 1976], and [Sinai 1972] constructed such natural measures for the important class of "hyperbolic" or "Axiom A" dynamical systems. More generally, instead of starting with Lebesgue measure, we could start with some arbitrarily prescribed measure μ_0 on X. By definition, a probability measure μ is called an *asymptotic measure* for μ_0 if μ_0-almost every orbit is evenly distributed with respect to μ. The case $\mu = \mu_0$ is particularly important. For the purposes of this section, we adopt the following.

Preliminary Definition. A probability measure μ on a topological space will be called *ergodic* for the map f if and only if μ-almost every orbit of f is evenly distributed with respect to μ.

(For the more customary definition, which does not involve any topology, see §8. For the proof that this definition is equivalent in the compact metric case, see §10.) The natural measures which occur in practice usually turn out to be ergodic also. However, this need not be the case. See Example 3.10 below.

As examples, the Lebesgue measure λ on $/$ is itself a natural ergodic measure for the irrational rotation ρ_α or for the doubling map m_2, while the measure ν of Equation (3:4) is natural and ergodic for the map $q : J \to J$.

Note that such a natural measure ν is uniquely determined, whenever it exists. For if ν' were another natural measure associated with the same f, then clearly we would have

$$\int_X \phi(x)\, d\nu \;=\; \int_X \phi(x)\, d\nu'$$

for every bounded continuous function $\phi : X \to$. But this implies that the two measures ν and ν' are the same. (Compare 10.4 or 10.8.)

Note also that a natural measure ν for f is necessarily *invariant* under f. That is, ν coincides with the push forward $f_*(\nu)$. Compare Problem 3-f.

As another example, it is not difficult to show that the rational map

$$N(x) \;=\; (x - x^{-1})/2$$

of §2B has a natural measure ν , with

$$\nu[a\,,\,b] \;=\; \frac{\arctan(b)-\arctan(a)}{\pi} \;=\; \frac{1}{\pi}\int_a^b \frac{dx}{1+x^2}\;. \tag{3:5}$$

See Problem 3-h.

Here is a more classical example. Gauss, in his study of continued fractions, studied the discontinuous function $\gamma:(0\,,\,1]\to(0\,,\,1]$ which assigns to each $0<x\le 1$ the *fractional part* of the reciprocal $1/x$. That is, we set

$$1/x = n+\gamma(x) \quad \text{with} \quad n\in \quad \text{and} \quad 0<\gamma(x)\le 1\;. \tag{3:6}$$

He showed that the measure

$$\nu[a,b] \;=\; \frac{\log(1+b)-\log(1+a)}{\log 2} \;=\; \frac{1}{\log 2}\int_a^b \frac{dx}{1+x}\;. \tag{3:7}$$

is invariant under γ . (Problem 3-h.) In fact this ν is actually a natural ergodic measure for f . (Compare [Cornfeld-Fomin-Sinai].)

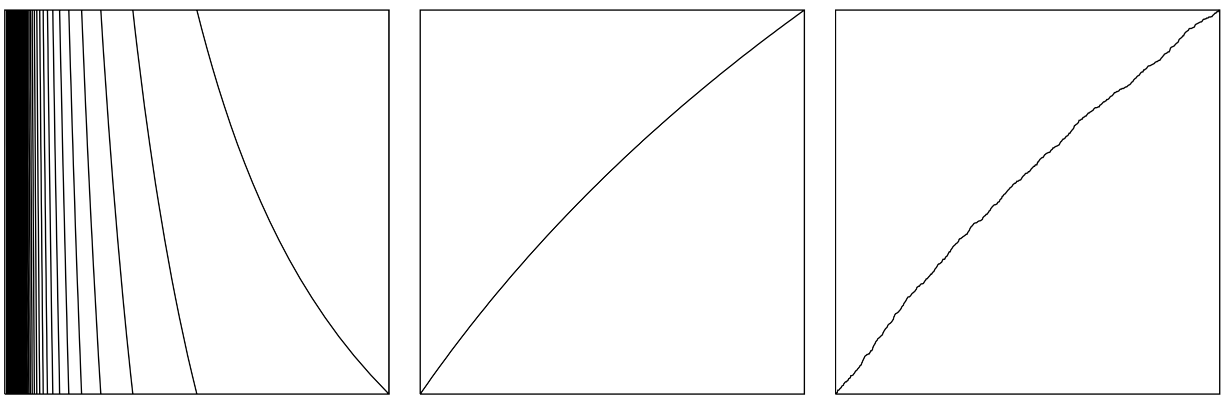

Figure 16. The Gauss map γ , together with a theoretical distribution curve for the associated natural measure, and an actual distribution curve for 1000 iterates with pseudo-random x_0 .

Two measures μ and μ' are said to belong to the same *measure class* if they have the same sets of measure zero. In the examples discussed above, the natural measure ν clearly belongs to the same measure class as Lebesgue measure. However, this need not be the case. Here is a trivial example. Map $J=[-1\,,\,1]$ to itself by the formula $f(x)=x/2$. Then every orbit converges to zero. More generally, let $f:X\to X$ be any map such that Lebesgue almost every orbit converges towards a fixed point $f(c)=c$. Then it follows easily that f has a natural measure, which coincides with the *Dirac measure* δ_c , defined by the condition that

$$\delta_c(S) \;=\; \begin{cases} 1 & \text{if} \quad c\in S\,, \\ 0 & \text{if} \quad c\notin S\,. \end{cases} \tag{3:8}$$

In fact, if $\{x_j\}$ is any sequence converging to c , then for every continuous function $\phi: X\to$, as $n\to\infty$ it is not hard to see that

$$(\phi(x_0)+\cdots+\phi(x_{n-1}))/n \quad\longrightarrow\quad \phi(c) \;=\; \int_X \phi(x)\,d\delta_c\;.$$

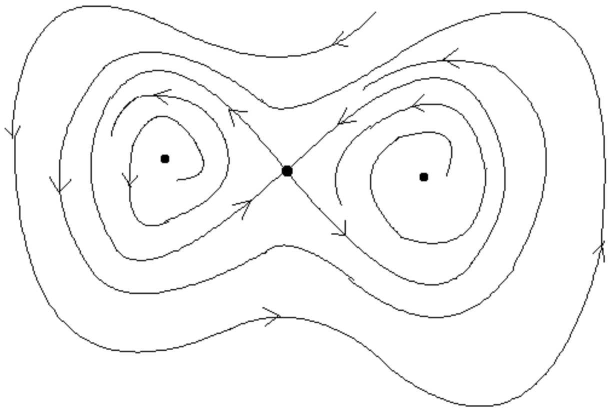

Figure 17. Solution curves for a differential equation in the plane with two homoclinic orbits.

Here is a similar but less obvious example. Consider the solution curves to a differential equation $dx/dt = v(x)$, as sketched in Figure 17, where $x \mapsto v(x)$ is a suitable smooth vector field on the plane. We assume that every solution curve which starts out far from the origin will spiral in towards a figure eight curve Γ, as illustrated, and that every non-constant solution curve which starts out inside of Γ will spiral out towards one lobe of Γ. Thus the vector field v is to have just three zeros, namely one *repelling point* inside each lobe of Γ, and one *saddle point* $\mathbf{s}$ where the two lobes of Γ come together. (Each of these lobes is called a *homoclinic curve*, meaning that this curve $x(t)$ converges to a common point s as $t \to +\infty$ or as $t \to -\infty$.)

It is not difficult to show that there is a unique natural measure for this system, and that it coincides with the Dirac measure $\delta_{\mathbf{s}}$. This statement is to be understood as follows: *For any neighborhood N of $\mathbf{s}$ and any solution curve $t \mapsto x(t)$, if we measure the proportion of values of t within the interval $[0, T]$ for which $x(t) \in N$, then this proportion tends to one as $T \to \infty$.* The intuitive explanation for this is that each solution curve moves rapidly past most points of Γ, but slows down drastically every time that it approches the point $\mathbf{s}$.

These example might lead the reader to believe that natural measures usually exist, and that they can usually be described by simple formulas. However, the first statement is debatable, while the second is definitely false. As noted earlier, Bowen, Ruelle and Sinai showed that a natural measure exists for any attracting basin in the class of *hyperbolic* or *Axiom A* dynamical systems. They seem to exist in much greater generality. For example [Jakobson] has shown that, with positive probability, a randomly chosen quadratic map of the interval has an absolutely continuous natural ergodic measure, although it is very hard to decide in specific examples. (Figure 18.) For analogous 2-dimensional results, see [Benedicks and Young]. On the other hand, [Hofbauer and Keller] have constructed quadratic interval maps with no natural measure.

Example 3.10. Perhaps the simplest example of an attractor with no natural measure is the following, which I understand is due to Mañé. (A somewhat similar example is described by [Zakharevich].) Consider a vector field in the plane with two saddle points $\mathbf{s}$ and $\mathbf{s}'$

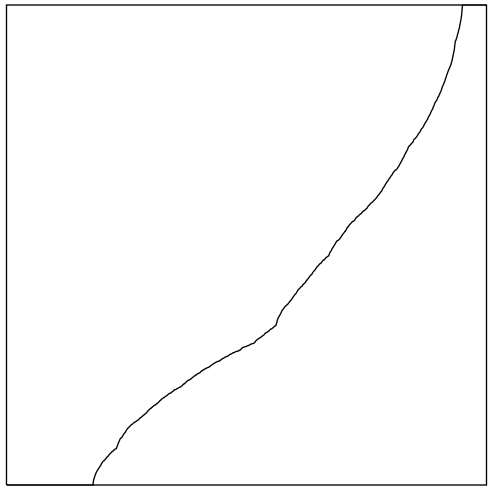

Figure 18. Empirical distribution curve for a randomly chosen orbit of the quadratic map $x \mapsto 3.8\,x\,(1-x)$ *of Figure 5, on the unit interval. This map appears to have an absolutely continuous natural measure, supported on the attracting sub-interval* $[0.1805\,,\,0.95]$ *, but a proof of this statement would be extremely difficult.*

which are joined by two trajectories, as illustrated, so that these two trajectories bound a region U which contains one repelling fixed point $\mathbf{r}$. Suppose that all orbits in $U \setminus \{\mathbf{r}\}$ spiral outwards so as to accumulate on the boundary ∂U . Choose neighborhoods N and N' of the two saddle points, so that each orbit, as it spirals out close to the boundary, will pass alternately through N and N' . Let T_i and T_i' be the time taken to pass through N and N' respectively, on the i-th pass. Then we will see that the sequences $\{T_i\}$ and $\{T_i'\}$ tend to infinity. However, the time needed to pass between N and N' clearly remains bounded as $i \to \infty$. Thus orbits spend most of their time in $N \cup N'$. Therefore a natural measure, if it exists, must assign full measure to $N \cup N'$. Since these neighborhoods can be arbitrarily small, this means that it assigns full measure to the finite set $\{\mathbf{s}\} \cup \{\mathbf{s}'\}$.

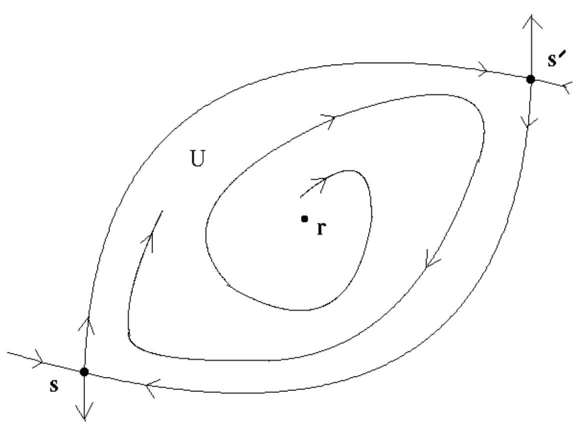

Figure 19. A pair of "heteroclinic" orbits. (The limit of one curve as the parameter t *tends to* $+\infty$ *is equal to the limit of the other as* $t \to -\infty$ *.)*

To make this discussion more explicit, let us suppose that we can choose local coordinates x, y near the saddle point $\mathbf{s}$ so that the associated differential equation has the form

$$dx/dt \;=\; -\alpha\, x\ , \qquad dy/dt \;=\; \beta\, y\ ,$$

with solution curves

$$x \;=\; x_0\, e^{-\alpha t}\ , \qquad y \;=\; y_0\, e^{\beta t}\ . \tag{3:9}$$

Here $-\alpha\,,\,\beta$ are invariants called the ***eigenvalues*** at $\mathbf{s}$. Similarly, suppose that there are corresponding local coordinates $x'\,,\,y'$ near $\mathbf{s}'$ with eigenvalues $-\alpha'\,,\beta'$. Then we will prove the following three assertions.

1. *The hypothesis that orbits spiral out to the boundary of* U *imposes the requirement that* $\alpha\alpha' \ge \beta\beta'$.

2. *If* $\alpha\alpha' > \beta\beta'$ *, then the two sequences* $\{T_i\}$ *and* $\{T_i'\}$ *grow exponentially as* $i \to \infty$ *, and it follows that there is no natural measure.*

3. *On the other hand, there exist examples with* $\alpha\alpha' = \beta\beta'$ *for which a natural measure* ν *does exist. This measure assigns positive weight to each of the two saddle fixed points,* $\nu(\{\mathbf{s}\})\,,\,\nu(\{\mathbf{s}'\}) > 0$ *, and hence is not ergodic.*

To begin the proof, let us study the solution curves (3:9) Note that the quantity

$$T \;=\; -\log\left(x^{1/\alpha} y^{1/\beta}\right) \;=\; -\frac{\log(x)}{\alpha} - \frac{\log(y)}{\beta}\ ,$$

remains constant along each solution curve. In the (x, y)-plane, consider the box B defined by

$$0 \;\le\; x\,,\; y \;\le\; 1\ ,$$

as illustrated.

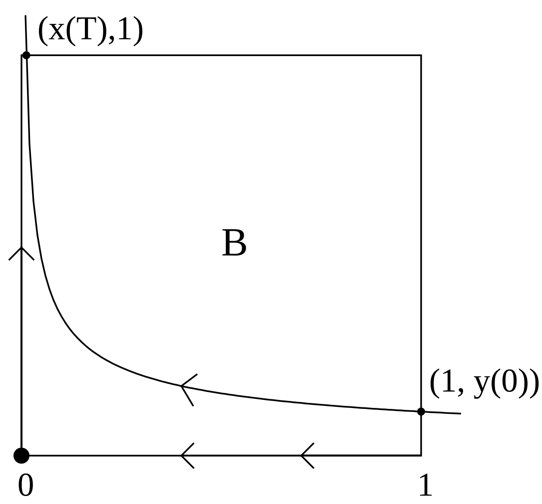

Figure 20. A solution curve, as described by (3 : 9), in the unit square.

A brief computation shows that this invariant T is precisely the time which it takes for a trajectory to traverse this box B. More precisely, a trajectory which enters B through the right hand edge $x = 1$ at time $t = 0$ and height $y(0) = e^{-\beta T}$ will exit through the top edge $y = 1$ at time T, with coordinate $x(T) = e^{-\alpha T}$. Thus the entrance and exit coordinates are related by the equation

$$y(0) \;\mapsto\; x(T) \;=\; y(0)^{\alpha/\beta} \;.$$

There is a similar equation representing entrance and exit coordinates for the analogous box B' near the other saddle point. Thus, if we make one complete loop near the outer boundary, then we have the transitions

$$y \;\;\mapsto\;\; x = y^{\alpha/\beta} \;\;\mapsto\;\; y' = \phi(x) \;\;\mapsto\;\; x' = (y')^{\alpha'/\beta'} \;\;\mapsto\;\; y = \psi(x')\;,$$

where $\phi(x) = bx + (\text{higher terms})$ and $\psi(x') = b'x' + (\text{higher terms})$ are smooth functions with leading coefficients $b\,,\,b' > 0$, obtained by following trajectories from one coordinate system to the other. It follows that the height y when entering the box B is replaced, after one loop, by a new height which is given by the formula

$$y \;\;\mapsto\;\; f(y) \;=\; c\,y^{\gamma} + (\text{higher terms})\;,$$

where $\gamma = (\alpha\alpha')/(\beta\beta')$ and $c = b^{\alpha'/\beta'}b' > 0$. The new height $f(y)$ must be strictly smaller than y. Otherwise orbits would not spiral out towards the boundary. It follows easily that $\gamma \ge 1$.

First consider the case where strict inequality holds, $\gamma > 1$. After a scale change, we may assume that $c = 1$. To simplify the discussion, let us assume also that the "higher terms" are zero, so that $f(y) = y^{\gamma}$ with $\gamma > 1$. Recall that $y = e^{-\beta T}$, where T is the time needed to pass through the box B. Since $f(y) = e^{-\gamma\beta T}$, it follows that the time needed to pass through the box on the next pass is γT. Thus these times grow exponentially. Similarly, the times needed to pass through B' grow exponentially with the same factor $\gamma > 1$. As we pass alternately through B and B' the passage times are $T\,,\ T'\,,\ \gamma T\,,\ \gamma T'\,,\ \gamma^2 T\,,\ \ldots$. After n passages through each, the total time spent in B' is

$$(1 + \gamma + \cdots + \gamma^{n-1})\,T' \;\sim\; \gamma^n T/(\gamma - 1)\;.$$

Since the total time between boxes grow only linearly with n, it follows similarly that the total time is asymptotic to $\gamma^n(T + T')/(\gamma - 1)$, hence the proportion of time spent in B' converges to $T'/(T + T')$, provided that we measure this proportion just after leaving B'. However, we could equally well measure this proportion just after leaving B. In this case, a similar computation shows that the proportion of time spent in B' converges to $T'/(\gamma T + T')$ where

$$\frac{T'}{\gamma T + T'} \;<\; \frac{T'}{T + T'}\;.$$

This discrepency shows that the time averages do not converge to a limit. Hence there cannot be any natural measure.

Now consider the situation when $\gamma = 1$, so that

$$f(y) \;=\; c\,y + (\text{higher terms})\;.$$

Again it is convenient to simplify the situation by constructing the example so that the higher terms are zero. Clearly we must have $0 < c < 1$ so that orbits will converge to ∂U. In this case, it is not difficult to check that the times to traverse B on successive passes form an arithmetic progression

$$T\,,\ T+k\,,\ T+2k\,,\ \ldots$$

with $k = -log(c)/\beta > 0$. Thus the total time spent in B during n passes is

$$nT + kn(n-1)/2 \ \sim\ kn^2/2\ .$$

Similarly, the time spent in B' is asymptotic to $k'n^2/2$, where it is not hard to show that $k' = \alpha k/\beta' = \beta k/\alpha'$. It follows easily that an asymptotic measure does exist, with

$$\nu(B) \ = \ \frac{k}{k+k'} \ = \ \frac{\beta'}{\alpha+\beta'} \ = \ \frac{\alpha'}{\alpha'+\beta}\ . \qquad \square$$

To conclude this section, let us prove Theorems 3.7 and 3.8.

Proof of 3.7. First consider the case of a function ϕ which has a finite Fourier series expansion,

$$\phi(t) \ = \ \sum_{k=-N}^{N} a_k\, e^{2\pi i k t}$$

with $a_k \in \mathbf{C}$. Evidently the integral of ϕ over $/$ is equal to the coefficient a_0 of the constant term. Since $t_j \equiv t_0 + j\alpha$ (mod) , we can write

$$\sum_{j=0}^{n-1} \phi(t_j) \ = \ \sum_k a_k \sum_{j=0}^{n-1} e^{2\pi i k t_j} \ = \ \sum_k a_k\, e^{2\pi i k t_0}(1 + u^k + u^{2k} + \cdots + u^{(n-1)k})$$

with $u = e^{2\pi i k\alpha}$. The geometric series in parentheses on the right sums to n when $k = 0$, but sums to $(u^{nk}-1)/(u^k-1)$ for $k \ne 0$. Since

$$\Big|\frac{u^{nk}-1}{u^k-1}\Big| \ < \ 2/|u^k-1|$$

is uniformly bounded as $n \to \infty$, it follows that $\sum_{j=0}^{n-1}\phi(t_j)/n$ converges to a_0 , as required.

To complete the proof, we need the standard result that any continuous real or complex valued function on the circle can be uniformly approximated by a function with finite Fourier series. (See for example [Dieudonné, p. 134].) But if $|\phi(t) - \psi(t)| < \epsilon$ uniformly, then

$$|\int \phi(t)\, d\mu - \int \psi(t)\, d\mu| \ < \ \epsilon$$

and

$$|(\phi(t_0) + \cdots + \phi(t_{n-1}))/n \ - \ (\psi(t_0) + \cdots + \psi(t_{n-1}))/n| \ < \ \epsilon\ ,$$

and the conclusion follows easily. $\square$

Proof of 3.8. This is a variant of the Strong Law of Large Numbers, as stated in 3.6. Suppose that we represent a number $t_0 \in$ by its base p expansion,

$$t_0 \ = \ 0\,.\,\tau_0\,\tau_1\,\tau_2\,\cdots \ \ (\text{base } p) \ \ = \ \sum \tau_i/p^{i+1}$$

with $\tau_i \in \{0, 1, \ldots, p-1\}$. This defines a projection map

$$h\ :\ \{0, 1, \ldots, p-1\} \ \ \to \ \ / \ \ .$$

Let β be the $(1/p\,,\ \ldots\ ,\ 1/p)$-Bernoulli measure on this space of sequences. In other words, we assume that the projections to $\tau_0, \tau_1, , \ldots$ are independent random variables, where each of the p symbols $\tau_i \in \{0, 1, \ldots, p-1\}$ is assigned the probability $1/p$. Pushing this measure forward to $/$ we obtain a probability measure which assigns to each interval $I \subset /$ the number $I \mapsto \beta(h^{-1}(I))$. Evidently this coincides with the Lebesgue measure λ on $/$. *In other words, we can construct a λ-randomly chosen point $t - 0 \in /$, by choosing the entries τ_i in the base p expansion as a sequence of independent Bernoulli trials, where each possible τ_i has probability $1/p$.*

Now consider the orbit $t_0 \mapsto t_1 \mapsto \cdots$ under multiplication by p in $/$. Evidently this corresponds to the shift map on $\{0, 1, \ldots, p-1\}$, so that

$$t_k = 0 \,.\, t_k\, t_{k+1}\, t_{k+2} \ldots \quad (\text{base } p)\,.$$

In other words, if we divide the circle $/$ into p equal intervals I_τ of length $1/p$, then the k-th forward image of t_0 under multiplication by p belongs to the subinterval I_{τ_k}. (There is some ambiguity if t_k belongs to the common endpoint of two consecutive intervals, but this possibility occurs with probability zero, and hence can be ignored.) Therefore, according to 3.6, for a λ-randomly chosen t_0, we have the following statement with probability one.

> *For each $\tau \in \{0, 1, \ldots, p-1\}$, the limit as $n \to \infty$ of the proportion of t_i, $0 \le i < n$, which lie in the subinterval I_τ is equal to $\lambda(I_\tau) = 1/p$.*

In other words, if $\mathbf{1}_\tau$ is the characteristic function of the subinterval I_τ, then the average $(\mathbf{1}_\tau(t_0) + \cdots + \mathbf{1}_\tau(t_{n-1}))\,/\,n$ converges to $1/p$ with probability one.

As an example, for the case $p = 2$, this statement says that for λ-almost every t_0 the proportion of time that the orbit of t_0 under the doubling map m_2 spends in each of the subintervals $[0, 1/2]$ and $[1/2, 1]$ converges to $1/2$ as the number of iterations tends to infinity.

We need to sharpen this statement, showing for example that the proportion of time spent in the interval $[0, 1/4]$ converges to $1/4$. However, we know that this statement is true for the map $m_4 : t \mapsto 4t \pmod{}$, which is just the composition $m_2 \circ m_2$ of the doubling map with itself. Thus, if $t_k \equiv 2^k t_0 \pmod{}$, then in the limit the orbit

$$m_2^{\circ 2} : t_0 \mapsto t_2 \mapsto t_4 \mapsto \ldots$$

spends $1/4$ of the time in each of the subintervals $[\tau/4\,,\,(\tau+1)/4]$. But the same is true for the orbit $m_2^{\circ 2} : t_1 \mapsto t_3 \mapsto t_5 \mapsto \cdots$ under multiplication by 4. Combining these two statements, we see that the same is true for the orbit under m_2.

Similarly, by studying the orbit of t_0 under multiplication by 2^q, we see that the proportion of time which this orbit, and hence also the orbit under m_2, spends in any interval $[\tau/2^q\,,\,(\tau+1)/2^q]$ converges to $1/2^q$ as $n \to \infty$. Since q can be arbitrarily large, it follows easily that the sequence of t_i is λ-uniformly distributed. □

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§**3C. Maxwell, Boltzmann and Jüttner: The Hard Sphere Gas.** This section is a digression from the main theme of these notes, and can be skipped on a first reading.

The *kinetic theory of heat*, developed by Maxwell and Boltzmann in the late 19-th century, provides the classical example of a mathematical theory which is completely deterministic, and yet can best be studied by probabilistic methods. This section gives a rough outline of this theory in its simplest case, the *hard sphere gas*.

It will also discuss the *relativistic* version of the theory, introduced some years later by Jüttner. This relativistic theory is significantly different only in the case of ultra-high temperatures, perhaps near the center of an extremely hot star, where the speeds of the individual particles may approach the speed of light.

Think of a “gas” which is made out of some enormous number n of tiny perfectly elastic spheres which are enclosed within a rigid box. Whenever two of these spheres collide with each other, or with the walls of the box, they rebound with no loss of energy. Each small sphere or *particle* has an associated *mass* m_α . At each time t it has a *position* 3-vector $\vec{x}_\alpha$ and (except at the precise time of a collision) a *velocity* vector $\vec{v}_\alpha = d\vec{x}_\alpha/dt$, a *momentum* vector $\vec{p}_\alpha$, and *energy* E_α . In the classical pre-relativistic theory, these are given by

$$\vec{p}_\alpha \;=\; m_\alpha \vec{v}_\alpha \,, \qquad E_\alpha \;=\; \tfrac{1}{2} m_\alpha \|\vec{v}_\alpha\|^2 \;=\; \tfrac{1}{2}\|\vec{p}_\alpha\|^2/m_\alpha \,. \tag{3:10}$$

Here E_α is known more precisely as the *kinetic energy* of the α-th particle. For the relativistic theory, we choose units of time and distance so that the speed of light is equal to one, and use the corrected formulas

$$(E_\alpha \,,\, \vec{p}_\alpha) \;=\; m_\alpha (1 \,,\, \vec{v}_\alpha)/\sqrt{1 - v_\alpha^2} \tag{3:10$'$}$$

where $v_\alpha = \|\vec{v}_\alpha\|$, so that

$$E_\alpha \;=\; m_\alpha (1 + \tfrac{1}{2}v_\alpha^2 + \tfrac{3}{8}v_\alpha^4 + \cdots) \;=\; \sqrt{m_\alpha^2 + \|\vec{p}_\alpha\|^2}\,, \qquad \vec{p}_\alpha/E_\alpha = \vec{v}_\alpha \,.$$

In this case, E_α can no longer be described as “kinetic” energy, since it takes on the non-zero value $E_\alpha = m_\alpha$ even when the velocity is zero. In either case, the energy E_α of the α-th particle can be expressed as a function of its mass and momentum, with partial derivatives

$$\partial E_\alpha/\partial p_{\alpha i} \;=\; v_{\alpha i} \tag{3:11}$$

for $i = 1, 2, 3$. Furthermore, in either case the *total energy* $H = \sum_{\alpha=1}^n E_\alpha$ remains constant through time. (Note that the total momentum $\sum \vec{p}_\alpha$ is not constant, since it changes every time a particle bounces off the boundary of our enclosure.)

It will be convenient to use the following notations. Let $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{v}$ be the $3n$-dimensional vectors

$$\begin{aligned}
\mathbf{x} \;&=\; (x_1 \,, \ldots \,, x_{3n}) \;=\; (\vec{x}_1 \,, \ldots \,, \vec{x}_n)\\
\mathbf{p} \;&=\; (p_1 \,, \ldots \,, p_{3n}) \;=\; (\vec{p}_1 \,, \ldots \,, \vec{p}_n)\\
\mathbf{v} \;&=\; (v_1 \,, \ldots \,, v_{3n}) \;=\; (\vec{v}_1 \,, \ldots \,, \vec{v}_n) \;=\; d\mathbf{x}/dt \,.
\end{aligned}$$

By definition, $\mathbf{x}$ varies over the *configuration space*, which is a bounded region $\mathcal{U} \subset\ ^{3n}$, defined by appropriate inequalities, and $\mathbf{p}$ varies over $3n$-dimensional *momentum space*. Evidently the total energy H can be expressed as a real valued function on momentum space, with

$$\partial H/\partial p_j \;=\; v_j \tag{3:12}$$

for each $1 \le j \le 3n$.

We will work in the $6n$-dimensional *phase space*, that is the cartesian product $\mathcal{U} \times\ ^3$ of configuration space and momentum space, with coordinates $(\mathbf{x}\,, \mathbf{p})$. Following Liouville, another, much more subtle invariant is the $6n$-dimensional Euclidean volume element $d\Lambda = d^{3n}(\mathbf{x})\, d^{3n}(\mathbf{p})$ in phase space: If we follow the time evolution of our system of particles from time $t = t_0$ to time $t = t_1$, then the resulting transformation from phase space to itself is *volume preserving*, with Jacobian determinant identically equal to one wherever this determinant is defined. This volume preserving property (and the more precise "symplectic invariance property") is a geometrical consequence of the fact that our system is time-reversible, without friction. Compare Problems 3-j, 3-k, 3-l. (For further discussion, see §11.)

This $6n$-dimensional Liouville volume element $d\Lambda$ gives rise to an invariant $(6n-1)$-dimensional volume element $d\Lambda/dH$ on each constant energy hypersurface $H = \text{constant}$ in phase space. This quotient volume element $d\Lambda/dH$ is characterized by the following property: *Any $6n$-dimensional integral $\int \phi\, d\Lambda$ over a region in phase space can be replaced by a double integral, first integrating ϕ with respect to $d\Lambda/dH$ over each hypersurface $H = \text{constant}$ and then integrating with respect to dH.*

To illustrate this construction, consider the function $r = \sqrt{x_1^2 + \cdots + x_k^2}$ on k-dimensional Euclidean space, and let $dx_1 \cdots dx_k$ be the usual k-dimensional Euclidean volume element. Then $(dx_1 \cdots dx_k)/dr$ is the usual $(k-1)$-dimensional volume element on the sphere $r = \text{constant}$.

In practice, we will want to multiply the measure $d\Lambda/dH$ by the *normalizing constant* c^{-1}, where

$$c \;=\; \int_{H=H_0} d\Lambda/dH \;, \qquad (3{:}13)$$

so as to obtain a *probability measure*, that is a measure with the property that the integral $\int_{H=H_0} c^{-1} d\Lambda/dH$ over the entire hypersurface is equal to one.

To proceed further, we will need the following.

3.11. Ergodic Hypothesis. This invariant probability measure $c^{-1}\, d\Lambda/dH$ on the constant energy hypersurface $H = H_0$ in phase space is an *ergodic measure*, as described in §3B.[1] In other words, for every starting point $(\mathbf{x}(0)\,,\,\mathbf{p}(0))$ outside of a set of measure zero in this hypersurface, the orbit $t \mapsto (\mathbf{x}(t)\,,\,\mathbf{p}(t))$ is *evenly distributed* throughout the hypersurface with respect to the probability measure $c^{-1}\, d\Lambda/dH$. That is, if we follow this orbit for time $0 \le t \le t_0$, then as $t_0 \to \infty$ the proportion of time which it spends within any reasonable subset W of the hypersurface converges towards the volume $c^{-1} \int_W d\Lambda/dH$. (Compare Problem 3-b.)

At this point we are in a rather awkward situation. In the classical non-relativistic case, we are describing an explicit mathematical property of an explicitly described mathematical model which has been studied exhaustively for over a hundred years, and lies at the very foundation of the applications of probabilistic methods to mechanics. It must be either true or false. In fact it is universally believed to be true, except in very degenerate situations such as a hard sphere gas consisting of only one particle, or a highly compressed gas in which the particles are so crowded that they can't get past each other. Rigorous mathematical work on this question began with [Sinai]. Until very recently this remained simply a hypothesis except in very special cases where the number n of particles is very small (See [Simányi].) However, [Simányi and Szász] have recently announced a proof for arbitrary n, so this anomalous situation may be resolved.

[1] See Chapter IV for further information about ergodic theory.

In the relativistic case the situation is even worse, since I am not even aware of any consistant mathematical model for elastic collision between particles of positive diameter. We will simply have to postulate that such a model exists, and has the required property.

One basic invariant of such a system of particles with total energy H_0 is the *thermodynamic entropy* $S(H_0)$. By definition, this is the logarithm of the volume of the region $H \le H_0$ in phase space.

$$S(H_0) \;=\; \log \int_{H \le H_0} d\Lambda \;. \tag{3:14}$$

Another even more fundamental invariant is the *temperature* $T = T(H)$, defined by the equation

$$dS(H)/dH \;=\; 1/T \;. \tag{3:15}$$

To help motivate these ideas, first consider the simplest case of a non-relativistic gas made up out of n identical particles of mass m. Then the total energy is given by

$$H \;=\; \sum_{j=1}^{3n} p_j^2/(2m) \;=\; \|\mathbf{p}\|^2/(2m) \;.$$

Hence the region $H \le H_0$ in phase space, defined by the equality $\|\mathbf{p}\|^2 \le 2mH_0$, is the cartesian product of the fixed region $\mathcal{U}$ with a $3n$-dimensional ball of radius $r = \sqrt{2mH_0}$. Its volume $e^{S(H_0)}$ is proportional to r^{3n}, or to $H_0^{3n/2}$, so that

$$S(H) \;=\; \frac{3\,n}{2}\log(H) + \text{constant} \;.$$

Using the definition (3:15), we can now compute temperature by the formula

$$\frac{1}{T} \;=\; \frac{dS(H)}{dH} \;=\; \frac{3\,n}{2H} \;,$$

or in other words

$$T \;=\; \tfrac{2}{3}\,H/n \;. \tag{3:16}$$

The temperature T is equal to H/n, the average energy per particle, multiplied by a constant factor of $2/3$. We will see presently that this same formula is true for a gas made of particles of assorted masses, at least in the non-relativistic case. (See 3.12 or Problem 3-i.) However, it is definitely false for a relativistic gas.

Similarly, the constant energy hypersurface $H = H_0$ can be described as the cartesian product of the fixed region $\mathcal{U}$ and the sphere $\mathcal{S}_r$ of radius $r = \sqrt{2mH}$ in momentum space. The probability measure

$$c^{-1}\, d^{3n}(\mathbf{x})\, d^{3n}(\mathbf{p})/dH$$

on this constant energy hypersurface $\mathcal{U}\times\mathcal{S}_r$ is simply the cartesian product of the Euclidean measure $d^{3n}(\mathbf{x})$ on configuration space and the standard $(3n-1)$-dimensional measure on the sphere $\mathcal{S}_r$, multiplied by a suitable normalizing constant. The ergodic hypothesis says roughly that if we start almost anywhere and follow the orbit for a long time then:

> (a) *The position vector* $\mathbf{x}$ *is as likely to be in any one region of* $\mathcal{U}$ *as in any other region of the same* $3n$*-dimensional volume.*

(*b*) *The total momentum vector* $\mathbf{p}$ *is as likely to be in any one region of the sphere* $\mathcal{S}_r$ *as in any other region of the same* $(3n-1)$*-dimensional volume.*

(*c*) *These position and momentum vectors are independent of each other as random variables. Information about position gives no information about momentum, and vice versa.*

In fact the positions $\vec{x}_\alpha$ of the individual particles are nearly independent of each other; the only qualification is that two particles cannot occupy the same space. Thus, in the limit as the ratio of particle size to box size tends to zero, the $\vec{x}_\alpha$ do become independent random variables.

If furthermore the number n of particles is very large, then this probability distribution for positions implies that the particles are more or less evenly distributed around the containing box with very high probability. For example if we divide the box into two equal halves, then approximately half of the particles must be in the left hand half; this is a straightforward consequence of the Law of Large Numbers, at least in the limiting case as the particle size tends to zero. On the other hand, for a system with only a few particles, our mathematical model is still perfectly valid, but the Law of Large Numbers does not apply.

We return to the general case, where the particles need not have the same mass, and where dynamics can be either relativistic or non-relativistic. For any j, consider the product $v_j\, p_j$ of velocity and momentum in the j-th coordinate direction. We will think of this product as a real valued function on the constant energy hypersurface $H = H_0$.

Theorem 3.12. Equipartition Principle. *For any* $1 \le j \le 3n$, *the expected value*

$$\mathcal{E}(v_j\, p_j) \;=\; c^{-1}\int_{H=H_0} v_j\, p_j\, d\Lambda/dH$$

of the product $v_j\, p_j$ *is equal to the temperature* T.

In particular, for the α-th particle, it follows that the expected value of the inner product $\vec{v}_\alpha \cdot \vec{p}_\alpha$ is given by

$$\mathcal{E}(\vec{v}_\alpha \cdot \vec{p}_\alpha) \;=\; \sum_{i=1}^{3} \mathcal{E}(v_{\alpha,i}\, p_{\alpha,i}) \;=\; 3\,T\,.$$

This expected value does not depend on the mass m_α.

In the non-relativistic case, since $\vec{p}_\alpha = m_\alpha \vec{v}_\alpha$, the inner product $\vec{v}_\alpha \cdot \vec{p}_\alpha$ is equal to $m_\alpha\, v_\alpha^2$, which is twice the kinetic energy E_α. Thus we obtain the following.

Corollary 3.13. Equipartition Principle for Energy. *In the non-relativistic case, the expected value* $\mathcal{E}(E_\alpha)$ *for the kinetic energy of the* α*-th particle is equal to* $\frac{3}{2}T$. *This expected energy does not depend on the mass* m_α. *Summing over all* n *particles, it follows that the product* $\frac{3}{2}nT$ *is equal to the total kinetic energy* H.

(The last statement is true since the sum of the random variables E_α on the constant energy hypersurface $H = H_0$ takes the constant value H.)

Thus heavier particles move more slowly on the average. More precisely, the root-mean-square speed for a particle of mass m at temperature T is

$$\sqrt{\mathcal{E}(v^2)} \;=\; \sqrt{3T/m}\,.$$

Remark. To relate such a statement to the real world, consider a pleasantly warm temperature of $80°F \approx 300°K$. Multiplying by *Boltzmann's constant*,

$$1°K = 1.4 \times 10^{-16}\,\mathrm{gm\,cm^2/sec^2}\ ,$$

we see that this corresponds to $T = 4.2\times 10^{-14}\,\mathrm{gm\,cm^2/sec^2}$. The most common particle in the atmosphere is a nitrogen molecule, with mass equal to 28.0 atomic mass units. Dividing by *Avogadro's number*, 6.0×10^{23} amu/gram , we obtain $m = 4.7\times 10^{-23}$ gm . Thus the root-mean-square speed for a nitrogen molecule is $\sqrt{3T/m} = 52,000\,\mathrm{cm/sec} = 1900\,\mathrm{km/hour}$, somewhat more than the speed of sound. (Compare [Feynman].)

Proof of 3.12. We will first prove the following equality:

$$\int_{H=H_0} v_j\, p_j\, d\Lambda/dH \;=\; \int_{H\le H_0} d\Lambda \;=\; e^{S(H_0)}\ . \qquad (3:17)$$

The $(3n-1)$-dimensional volume form $d^{3n}(\mathbf{p})/dH = dp_1\cdots dp_{3n}/dH$ can be written as a $(3n-1)$-fold product

$$dp_2\, dp_3\,\cdots\, dp_{3n}/|\partial H/\partial p_1| \;=\; p_2\cdots p_{3n}/|v_1|\ .$$

Hence, multiplying by $v_1\, p_1$ we see that

$$v_1\, p_1\, d^{3n}(\mathbf{p})/dH \;=\; |p_1|\, dp_2\, dp_3\,\cdots\, dp_{3n}\ .$$

An elementary argument now shows that the integral of this expression over the hypersurface $H = H_0$ yields precisely the enclosed volume, $\int_{H\le H_0} dp_1\, dp_2\,\cdots\, dp_{3n}$. Now multiplying by the constant $\int_{\mathcal{U}} d^{3n}(\mathbf{x})$ we obtain the required identity. Evidently a similar argument works for any $v_j p_j$.

Next we must compute the normalizing constant $c = \int_{\mathcal{U}} d^{3n}(\mathbf{x}) \int_{H=H_0} d^{3n}(\mathbf{p})/dH$. Almost by definition, this is equal to

$$\frac{d}{dH_0}\int_{H\le H_0} d^{3n}(\mathbf{x})\, d^{3n}(\mathbf{p}) \;=\; \frac{d}{dH_0} e^{S(H_0)} \;=\; e^{S(H_0)}\,\frac{dS}{dH_0} \;=\; \frac{e^{S(H_0)}}{T}\ . \qquad (3:18)$$

Dividing (22) by (23), we obtain the required formula $\mathcal{E}(v_j\, p_j) = T$. $\square$

Next let us discuss the pressure exerted by a gas on the walls of its enclosure. Whenever the α-th particle bounces off of (say) the right hand wall of our box, this wall is subject to an *impulse* (change in momentum) of $2\, p_{\alpha 1}$. By definition, the ***pressure*** P of the gas is the *expected impulse per unit time and unit area* which arises from the collisions of all of the particles. Let V be the 3-dimensional volume of our box, and let V_α be the volume of the α-th particle.

> **Corollary 3.14. Ideal Gas Law.** In the limit as the volume-density $\sum V_\alpha/V$ tends to zero, we have the asymptotic equality $P\,V \;=\; n\,T$.

(In the non-relativistic case, using 3.13, we could also write this equality as $P\,V = \frac{2}{3}\,H$.)

Proof. First consider the case of a single particle, with velocity $\vec{v}$ and momentum $\vec{p}$ which moves within the box $a_i \le x_i \le b_i$. This particle will hit the right hand wall during a very short time interval $t_0 \le t \le t_0 + \Delta t$ if and only if its position vector $\vec{x}$ at time t_0 satisfies $b_1 - v_1\,\Delta t \le x_1 \le b_1$. The probability of this is equal to the ratio $v_1\,\Delta t/(b_1 - a_1)$

if $v_1 > 0$, and is zero if $v_1 \le 0$. Thus the expected impulse per unit time on the right hand wall is equal to the integral of $2\,v_1\,p_1/(b_1 - a_1)$ over the half-space $v_1 > 0$ with respect to the probability distribution $c^{-1}d\Lambda/dH$. Using 3.12, we see easily that this integral is equal to $T/(b_1 - a_1)$. Hence the expected impulse per unit time per unit area for a single particle is equal to T/V . Now adding these expected impulses over the n different particles, we obtain the required formula $P = n\,T/V$. □

Remark. Again let us bring this formula into some contact with reality, by considering a room at a temperature of $300°K = 4.2 \times 10^{-14}\,\mathrm{gm\,cm^2/sec^2}$. Since the atmospheric pressure is about $P = 10^6\,\mathrm{gm/(cm\,sec^2)}$, it follows that the number of molecules per cubic centimeter is given by

$$n/V \;=\; P/T \;\approx\; 2.4 \times 10^{19}\,\mathrm{molecules/cm^3}\ .$$

Now let us consider the limiting case as the number n of particles tends to infinity. We will describe an intuitive proof of the following result, due to Maxwell and Boltzmann in the non-relativistic case, and to Jüttner in the relativistic case.

Theorem 3.15. *In the limit as the number of particles tends to infinity, the various momentum vectors $\vec{p}_\alpha$ become independent random variables. Each $\vec{p}_\alpha$ has probability distribution*

$$c^{-1}\; e^{-E_\alpha/T}\, d^3(\vec{p}_\alpha)\quad ,$$

where

$$c \;=\; c\,(m_\alpha\,,\,T) \;=\; \iiint e^{-E_\alpha/T}\, d^3(\vec{p}_\alpha)$$

is the appropriate normalizing constant.

Note: In the non-relativistic case, E_α is a homogeneous quadratic function of $\vec{p}_\alpha$, so this is a *Gaussian* distribution, and it is not hard to show that the normalizing constant is given by $c = (\pi/(2m_\alpha T))^{3/2}$. For the relativistic case, c can be expressed in terms of an appropriate Bessel function. Compare [Synge].

Outline Proof of 3.15. Let $d\tilde{\Lambda}$ be the $3(n-1)$-dimensional volume element obtained by taking the product of $d^3(\vec{p}_\beta)$ over all $\beta \ne \alpha$. Then the preferred volume element on the hypersurface $H = \text{constant}$ in momentum space can be written as a product $d^3(\vec{p}_\alpha) \cdot d\tilde{\Lambda}/dH$. If we ignore the α-th particle, note that the total energy for the remaining particles can be written as $\tilde{H} = H - E_\alpha$. Let $\tilde{S}(H - E_\alpha)$ be the thermodynamic entropy for this system of $n-1$ particles. In the limit as $n \to \infty$, the energy E_α of one single particle can be assumed to be very small compared with the total energy H , so it suffices to use the first order Taylor series

$$\tilde{S}(H - E_\alpha) \;=\; \tilde{S}(H) - E_\alpha\, d\tilde{S}/dH \;=\; \tilde{S}(H) - E_\alpha/T\ ,$$

by (4). The volume of the corresponding region $H - E_\alpha \le \text{constant}$ is, by definition, equal to the exponential

$$e^{\tilde{S}(H-E_\alpha)} \;=\; e^{\tilde{S}(H)} \cdot e^{-E_\alpha/T}$$

of this expression. To find the volume of its boundary $\tilde{H} = \text{constant}$ with respect to the volume element $d\tilde{\Lambda}/dH$ we simply differentiate with respect to H , thus replacing the constant factor of $e^{\tilde{S}(H)}$ by the constant factor of $d\,e^{\tilde{S}(H)}/dH$ (which is equal to $e^{\tilde{S}(H)}/T$). Thus the probability distribution for $\overrightarrow{p}_\alpha$ is proportional to $e^{-E_\alpha/T}\,d^3(\overrightarrow{p}_\alpha)$, as required. A similar argument shows that the joint probability distribution for two distinct momentum vectors $\overrightarrow{p}_\alpha$ and $\overrightarrow{p}_\beta$ is is proportional to $e^{-(E_\alpha+E_\beta)/T}\,d^3(\overrightarrow{p}_\alpha)\,d^3(\overrightarrow{p}_\beta)$, which splits as a product, as required. □

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§3D. Problems for the Reader.

Problem 3-a. The Strong Law of Large Numbers. If $\mu = \mu_p$ is the Bernoulli measure on the space $\{0,1\}$, with $p = p(1)$, and if $\phi_n = (\alpha_1 + \cdots + \alpha_n)/n$ is the proportion of ones in the first n trials, estimate the integral of

$$(\phi_n - p)^4 \;=\; ((\alpha_1 - p) + \cdots + (\alpha_n - p))^4/n^4$$

with respect to μ . Show that this integrand expands as a sum of n terms of the form $(\alpha_i - p)^4/n^4$ plus $n(n-1)/2$ terms of the form

$$6(\alpha_i - p)^2\,(\alpha_j - p)^2/n^4\ ,$$

plus many other terms which can be ignored since they integrate to zero. Conclude that this integral is bounded from above by an expression of the form

$$\frac{C_1\,n}{n^4} + \frac{C_2\,n(n-1)}{n^4} \;<\; \frac{C}{n^2}\ .$$

Proceeding as in the proof of 3.2, conclude that the measure of the set $S(\phi_n, \epsilon)$ where $|\phi_n(x) - p| > \epsilon$ is less than $C/(\epsilon^4 n^2)$, thus proving Lemma 3.4.

Problem 3-b. Even distribution. Let μ be a probability measure on the metric space X . If the sequence of points $x_0\,,\,x_1\,,\,\ldots \in X$ is μ-evenly distributed, as defined in §3B, show that it satisfies the following condition:

> $(*)$ *For any Borel subset* $S \subset X$ *such that the topological boundary* ∂S *has measure* $\mu(\partial S) = 0$ *, the limit*
>
> $$\lim_{n\to\infty}\,(\mathbf{1}_S(x_0) + \cdots + \mathbf{1}_S(x_{n-1}))/n$$
>
> *exists and is equal to* $\mu(S)$ *.*

Here $\mathbf{1}_S : X \to \{0,1\}$ denotes the characteristic function of S . Conversely, show that this property $(*)$ implies that the sequence is μ-evenly distributed. (For any bounded continuous $\phi : X \to$, cover X by sets of the form $\{x \in X \;:\; c_i \le \phi(x) < c_{i+1}\}$; and make use of the fact that $\phi^{-1}(c)$ has measure zero for all but countably many c .)

Problem 3-c. Shadowing. By an ϵ*-pseudo-orbit* for a map $f : X \to X$ on a metric space with distance $d(x,y)$ is meant a sequence of points $y_0\,,\,y_1\,,\,\ldots\,,\,y_n$ such that $d(f(y_i)\,,\,y_{i+1}) < \epsilon$ for $0 \le i < n$. The map f has the *shadowing property* if, for every such pseudo-orbit there exists a true orbit

$$f \;:\; x_0 \;\mapsto\; \cdots \;\mapsto x_n$$

with $d(x_i\,,\,y_i) < \epsilon'$ for all i , where ϵ' tends to zero as $\epsilon \to 0$. (Comapare [Bowen 1978].)

Show that the doubling map m_2 has this shadowing property. (Take $x_n = y_n$ and work backwards.) Conclude that the quadratic map $q(x) = 2x^2 - 1$ on the interval $[-1\,,\,1]$

does also. On the other hand, show that a rotation of the circle (or the map q on the full real line) does not have the shadowing property.

Problem 3-d. When computation goes wrong. In a computer, a non-zero real number is typically represented by the closest rational number of the form $\pm 2^k(1+r)$ where r is a dyadic fraction with $0 \le r < 1$, say of the form $r = n/2^{60}$, and where k is an integer with $|k|$ bounded. Show then that every computer generated orbit for the doubling map on $\;/\;$ must converge to the fixed point after finitely many iterations. Prove a corresponding statement for the tent map.

Now consider a computer generated orbit for the squaring map $P_2(z) = z^2$, where $|z_0| = 1$. Show that most orbits will diverge from the unit circle, converging either to zero or to infinity.

Remark. In both cases one could fudge the computation to get reasonable results. For the map m_2 or T, it is necessary to add some pseudo-random "noise". This is analogous to the process of generating a Lebesgue-random number in $[0,1]$ by inductively choosing the bits in its binary expansion, using some pseudo-random number generator. For the squaring map P_2, the appropriate procedure is to correct at each step by dividing each computer generated z_{j+1} by $|z_{j+1}|$. A similar correction procedure is appropriate for any dynamical system $f : X \to X$ where X is conveniently described as a submanifold of some Euclidean space. (For example X might be a constant energy hypersurface.)

Problem 3-e. Sets of measure zero on a manifold. Show that any C^1-smooth map from an open subset of R^k into R^k carries sets of Lebesgue measure zero to sets of Lebesgue measure zero. Now for any C^1-smooth manifold M^k of dimension k, define a subset $S \subset M^k$ to be a *Lebesgue null set* if its intersection with each coordinate neighborhood has Lebesgue measure zero. Conclude that this property does not depend on the choice of coordinate neighborhoods used to cover M^k.

Problem 3-f. Invariance of natural measures. For any continuous $f : X \to Y$ and any probability measure μ on X, show that the push forward $f_*(\mu)\,(S) = \mu(f^{-1}(S))$ satisfies

$$\int_Y \phi(y)\, df_*(\mu) \;=\; \int_x \phi \circ f(x)\, d\mu$$

for any bounded continuous $\phi : Y \to$. If ν is a natural measure for $f : X \to X$, prove that $f_*(\nu)$ is also a natural measure, and hence that $f_*(\nu) = \nu$.

Problem 3-g. The Perron-Frobenius operator. If μ is a measure on the line or circle given by $\mu(S) = \int_S w(x)\,dx$ with $w(x) \ge 0$, and if f is a C^1-smooth map whose derivative f' vanishes at most on a set of measure zero, show that

$$f_*(\mu)\,(S) \;=\; \int_S \hat{w}(y)\,dy \qquad \text{with} \qquad \hat{w}(y) \;=\; \sum_{f(x)=y} w(x)/|f'(x)|\;,$$

to be summed over all x with $f(x) = y$.

Problem 3-h. Two examples. Using Problem 2-5, show that the rational map N has a natural measure, given by formula $(3\!:\!4)$. For the Gauss map $x \mapsto \mathrm{frac}\,(1/x)$, show that formula $(3\!:\!6)$ defines an invariant probability measure.

Problem 3-i. Equipartition; a convenient change of coordinates. For a non-relativistic gas consisting of n particles, each having position $\vec{x}_\alpha$ and mass m_α, it is convenient to introduce the coordinates

$$\hat{\mathbf{x}} \;=\; (\sqrt{m_1}\,\vec{x}_1\,,\,\ldots\,,\,\sqrt{m_n}\,\vec{x}_n) \qquad \text{and} \qquad \hat{\mathbf{v}} = d\,\hat{\mathbf{x}}/dt\ .$$

Thus $\hat{\mathbf{x}}\,,\,\hat{\mathbf{v}} \in {}^{N}$ where $N = 3\,n$. The Liouville volume element can then be written as $d\Lambda = d^N(\hat{\mathbf{x}})\,d^N(\hat{\mathbf{v}})$, and the total energy $H = \sum_1^n E_\alpha$ is given by

$$2\,H \;=\; \|\hat{\mathbf{v}}\|^2 \;=\; \sum_1^N \hat{v}_j^2\ .$$

Thus each constant energy hypersurface in the N-dimensional velocity space is just a sphere of radius $r = \sqrt{2\,H}$. Show that the measure $d^N(\hat{\mathbf{v}})/dH$ is proportional to the standard measure on the sphere, and show that the expected value $\mathcal{E}(v_j^2)$ of the random variable $\hat{v}_j^2$ on this sphere is equal to $2\,H/N = T$. Conclude that the expected energy for the α-th particle is given by $\mathcal{E}(E_\alpha) \;=\; 3\,T/2$.

In order to study how the energy is distributed between different particles, for any $K = 3k < N$, set

$$\sum_1^K v_j^2 \;=\; 2\,H\,\cos^2(\theta)\,, \qquad \sum_{K+1}^N v_j^2 \;=\; 2\,H\,\sin^2(\theta)\,.$$

Show that the total energy $E_1 + \cdots + E_k$ of the first k particles is equal to $H\,\cos^2(\theta)$, with expected value

$$\mathcal{E}(E_1 + \cdots + E_k) \;=\; H\,k/n \;=\; H\,K/N\ .$$

Show that the volume element on the sphere corresponds to $\cos^{K-1}(\theta)\,\sin^{N-K-1}(\theta)\,d\theta$ multiplied by an appropriate constant. If

$$A(p,q) \;=\; \int_0^{\pi/2} \cos^p(\theta)\,\sin^q(\theta)\,d\theta\ ,$$

show that

$$\mathcal{E}(\cos^2(\theta)) \;=\; \frac{K}{N} \;=\; \frac{A(K+1\,,\,N-K-1)}{A(K-1\,,\,N-K-1)}\ ,$$

and that

$$\mathcal{V}(\cos^2(\theta)) \;=\; \frac{A(K+3\,,\,N-K-1)}{A(K-1\,,\,N-K-1)} - \mathcal{E}(\cos^2(\theta))^2 \;<\; \frac{2\,K}{N^2}\ .$$

Using 3.2, conclude that the probability that $\cos^2(\theta)$ differs from $K/N = k/n$ by more than $\epsilon\,k/n$ is less than $2/(k\,\epsilon^2)$, and hence tends to zero as $k \to \infty$. Equivalently, the probability that

$$|E_1 + \cdots + E_k - H\,k/n| \;>\; \epsilon\,H\,k/n$$

tends to zero as $k \to \infty$.

Problem 3-j. The action principle. In non-relativistic mechanics, the *action* of a particle with trajectory $\vec{x}(t)$, momentum $\vec{p}(t)$, and kinetic energy $E(t)$ over a time interval $t_0 \le t \le t_1$ is defined to be the integral

$$A \;=\; \int_{t_0}^{t_1} E(t)\,dt \;=\; \tfrac{1}{2}\int_{t_0}^{t_1} \vec{p}\cdot d\vec{x}\ .$$

The "principle of least action" asserts that the total action for a collection of particles over a time interval $t_0 \le t \le t_1$, with fixed endpoints at t_0 and t_1 , should always be minimized by the actual trajectories. A more correct statement is the *principle of stationary action*. This asserts that the derivative of action with respect to any variation of the orbits which keeps the endpoints fixed is necessarily zero.

If we use the special coordinates $\hat{\mathbf{x}}$ and $\hat{\mathbf{v}} = d\hat{\mathbf{x}}/dt$ for phase space, as in Problem 3-i, then the total action is given by $2A = \int_{t_0}^{t_1} \|\hat{\mathbf{v}}\|^2 \, dt$. Here $\hat{\mathbf{x}}$ varies over a bounded region $\mathcal{U}$ in . ${}^N =$. ^{3n} . The point $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t) \in \mathcal{U}$ moves in a straight line with constant velocity $\hat{\mathbf{v}}$, except at a point where it is reflected off the boundary of $\mathcal{U}$. This boundary $\partial\mathcal{U}$ is made up of two kinds of $(N-1)$-dimensional pieces, namely points where the α-th particle hits the boundary of the enclosure, and points where the α-th particle hits the β-th particle. It also contains a complicated set of points corresponding to multiple collisions, where two or more such piece come together. However, this is a set of $(N-1)$-dimensional measure zero, and can be ignored. This mathematical model is known in the dynamics literature as *billiards*.

Consider for example an orbit $\hat{\mathbf{x}} = \hat{\mathbf{x}}(t)$ which starts at $\hat{\mathbf{x}}^0$ with velocity $\hat{\mathbf{v}}^0$ at time t^0 , bounces off the boundary $\partial\mathcal{U}$ at time t^* and position $\hat{\mathbf{x}}^*$, and then continues with velocity $\hat{\mathbf{v}}^1$ to the point $\hat{\mathbf{x}}^1$ at time t^1 . Show that

$$2\,A \;=\; \hat{\mathbf{v}}^0 \cdot (\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^0) \;+\; \hat{\mathbf{v}}^1 \cdot (\hat{\mathbf{x}}^1 - \hat{\mathbf{x}}^*) \;,$$

and that under any variation of $\hat{\mathbf{x}}^0$, $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^* \in \partial\mathcal{U}$ and t^* we have

$$dA \;=\; (\|\hat{\mathbf{v}}^0\|^2 - \|\hat{\mathbf{v}}^1\|^2)\, dt^* \;+\; (\hat{\mathbf{v}}^0 - \hat{\mathbf{v}}^1) \cdot d\,\hat{\mathbf{x}}^* \;+\; (\hat{\mathbf{v}}^1 \cdot d\hat{\mathbf{x}}^1 - \hat{\mathbf{v}}^0 \cdot d\hat{\mathbf{x}}^0) \;. \qquad (3{:}19)$$

If $\hat{\mathbf{x}}^0$ and $\hat{\mathbf{x}}^1$ are kept fixed, then the principle of stationary action asserts that this expression vanishes identically. Setting the coefficient of $d\,t^*$ equal to zero, this yields the law of conservation of energy,

$$\|\hat{\mathbf{v}}^0\|^2 \;=\; \|\hat{\mathbf{v}}^1\|^2 \;.$$

We cannot simply set the coefficient of $d\,\hat{\mathbf{x}}^*$ equal to zero, since $\hat{\mathbf{x}}^*$ can vary only over the boundary $\partial\mathcal{U}$. However, since $(\hat{\mathbf{v}}^0 - \hat{\mathbf{v}}^1) \cdot d\,\hat{\mathbf{x}}^* = 0$, we see that the component of momentum *tangent to the boundary* of $\mathcal{U}$ is preserved.

Now suppose that we consider a variation of $\hat{\mathbf{x}}^0$, $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^* \in \partial\mathcal{U}$ and of t^* so that these conservation laws are satisfied at every stage. In other words, assume that all of the terms involving $d\hat{\mathbf{x}}^*$ and dt^* in the equality (3:19) vanish identically. Then (3:19) reduces to

$$dA \;=\; \hat{\mathbf{v}}^1 \cdot d\hat{\mathbf{x}}^1 - \hat{\mathbf{v}}^0 \cdot d\hat{\mathbf{x}}^0 \;. \qquad (3{:}20)$$

If we can use $\hat{\mathbf{x}}^0 = (x_1^0\,, \ldots\,, x_N^0)$ and $\hat{\mathbf{x}}^1 = (x_1^1\,, \ldots\,, x_N^1)$ as local coordinates for our $2N$-dimensional phase space, this means that

$$\partial A/\partial x_j^1 = v_j^1 \quad \text{and} \quad \partial A/\partial x_j^0 = -v_j^0$$

for $j = 1\,, \ldots\,, N$. Show that each of the two correspondences

$$(\hat{\mathbf{x}}^0\,, \hat{\mathbf{x}}^1) \mapsto (\hat{\mathbf{x}}^0\,, \hat{\mathbf{v}}^0) \quad \text{and} \quad (\hat{\mathbf{x}}^0\,, \hat{\mathbf{x}}^1) \mapsto (\hat{\mathbf{x}}^1\,, \hat{\mathbf{v}}^1)$$

has Jacobian determinant equal to the determinant of the $N \times N$ matrix

$$\left[\frac{\partial^2 A}{\partial x_j^0\,\partial x_k^1}\right] .$$

Conclude that the correspondence $(\hat{\mathbf{x}}^0\,, \hat{\mathbf{v}}^0) \;\mapsto\; (\hat{\mathbf{x}}^1\,, \hat{\mathbf{v}}^1)$ has Jacobian determinant identically equal to one.

Problem 3-k. More directly, using the calculus of differential forms, show that (3:20), together with the identity $d\,(d\,A) = 0$, implies that

$$\sum d\,v_i^0 \wedge d\,x_i^0 \;=\; \sum d\,v_i^1 \wedge d\,x_i^1 \;.$$

In other words, the transformation from phase space to itself associated with time evolution from time t^0 *to time* t^1 *preserves the symplectic* 2*-form* $\sum_{i=1}^N dv_i \wedge dx_i$. Multiplying this 2-form by itself N times, conclude that this time evolution transformation preserves the $2N$-dimensional Liouville volume element $d^N(\hat{\mathbf{x}})\,d^N(\hat{\mathbf{v}})$.

Problem 3-l. In the relativistic case, action is rather defined by the Lorentz invariant formula

$$A \;=\; \int m\sqrt{1-v^2}\,dt \;=\; \int (E\,dt - \overrightarrow{p}\cdot d\overrightarrow{x})\;.$$

For a collision between two point particles, show that the principle of stationary action implies the conservation law for energy-momentum

$$\sum_\alpha (E_\alpha(t^0)\,,\,\overrightarrow{p}_\alpha(t^0)) \;=\; \sum_\alpha (E_\alpha(t^1)\,,\,\overrightarrow{p}_\alpha(t^1))\;.$$

However, for a kinetic theory, we need to consider collisions between particles of positive size, and it is difficult to provide a good relativistically invariant mathematical model. Instead, let us consider a system of n particles enclosed in a rigid box over a time interval $t^0 \le t \le t^1$ and simply assume that, as in Problem 3-j, the various elastic collisions make no contribution to the variation dA . (Here we use coordinates for space-time so that the box has velocity zero.) Show then, in analogy with (25), that

$$dA \;=\; -\sum_\alpha (\overrightarrow{p}_\alpha(t^1)\,d\overrightarrow{x}_\alpha(t^1) - \overrightarrow{p}_\alpha(t^0)\,d\overrightarrow{x}_\alpha(t^0))\;.$$

Show, as above, that the time evolution for this system of relativistic particles preserves the $6n$-dimensional Liouville volume form, or more precisely that is preserves the symplectic 2-form

$$\sum_\alpha d\overrightarrow{p}_\alpha \wedge d\overrightarrow{x}_\alpha\;.$$