§2. The Simplest Chaotic Systems.

This section will describe several closely related examples which help to illustrate sensitive dependence on initial conditions and chaotic dynamics.

§2A. Two Maps of the Interval. Let J be the closed interval $[-1, 1]$ of real numbers and let q be the quadratic map $q(x) = 2x^2 - 1$ from J onto itself. This $q(x)$ can be described as the degree two *Chebyshev polynomial*. (Compare Problem 2-a.) An equivalent map was studied briefly from a dynamical point of view by S. Ulam and J. von Neumann in 1947.

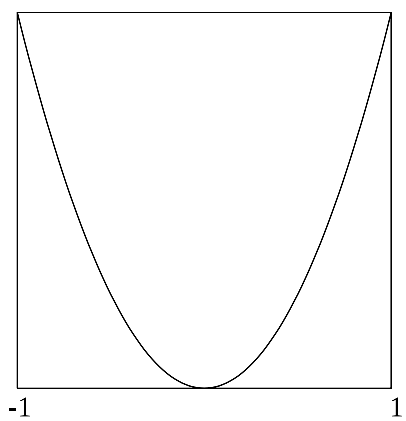

Figure 7. Graph of the quadratic map $q(x) = 2x^2 - 1$.

The following closely related example has been studied by many authors. Let I be the unit interval $[0, 1]$. The piecewise linear map

$$T(\xi) = \begin{cases} 2\xi & \text{for} \quad 0 \le \xi \le 1/2\,, \\ 2 - 2\xi & \text{for} \quad 1/2 \le \xi \le 1 \end{cases}$$

from I onto itself is called a *tent map*.

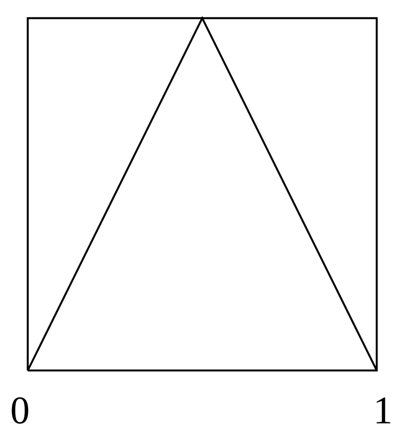

Figure 8. Graph of the tent map T .

These two maps are actually the same, up to a change of coordinate. More precisely, there is a homeomorphism

$$x = h(\xi) = \cos(\pi\xi)$$

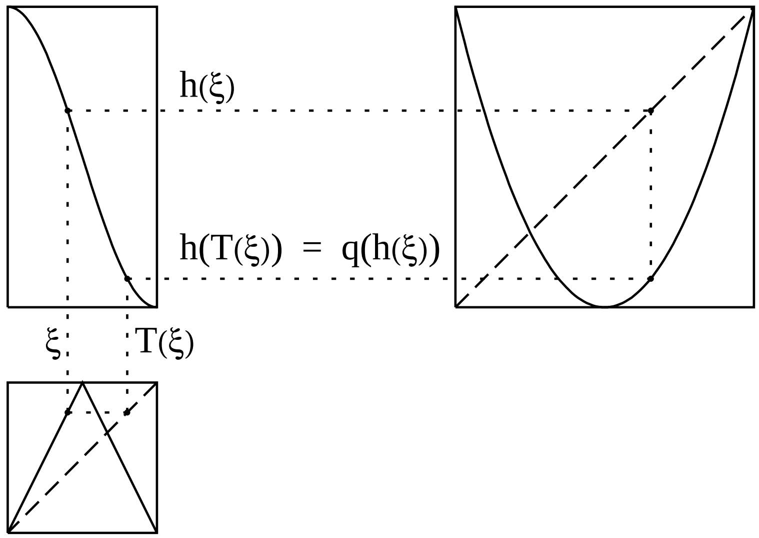

Figure 9. Topological conjugacy h from T onto q .

from the interval I onto the interval J so that $h(T(\xi)) = q(h(\xi))$ for every $\xi \in I$. In other words, the following diagram is *commutative*,

$$\begin{array}{ccc} I & \xrightarrow{T} & I \\ \downarrow h & & \downarrow h \\ J & \xrightarrow{q} & J \end{array} \qquad (2:1)$$

so that we can write $q = h \circ T \circ h^{-1}$. We say that T is *topologically conjugate* to q , or that h *conjugates* T to q . (Compare §4.) The proof that this diagram $(2:1)$ commutes is an easy exercise based on the identity

$$\cos(2\theta) \;=\; \cos(2\pi - 2\theta) \;=\; \cos^2(\theta) - \sin^2(\theta) \;=\; 2\cos^2(\theta) - 1 \;.$$

Remarks: The two maps q and T seem rather different. For example one is smooth while the other is not smooth at the origin. It is remarkable that the conjugating homeomorphism h is smooth, with smooth inverse except at the two endpoints. Note that h reverses orientation. Hence it takes an interior maximum point for T to an interior minimum point for q . (Similarly, q is conjugate under a *linear* change of coordinates to the logistic map $x \mapsto 4\,x\,(1-x)$, with an interior maximum point. Compare (1: 4) in §1.)

---

§2B. Angle Doubling: the Squaring Map on the Circle. Let $S^1 \subset \mathbf{C}$ be the *unit circle*, consisting of all complex numbers $z = x + iy$ with $|z|^2 = x^2 + y^2$ equal to one. The *squaring map* $P_2(z) = z^2$ carries each point $z = e^{i\theta}$ on the circle to the point $e^{2i\theta}$. In other words, it doubles the angle θ , which is well defined modulo 2π . In practice, it is often more convenient to set $\theta = 2\pi\tau$, where τ is a real number which is well defined modulo one. We will write

$$z \;=\; e^{2\pi i \tau} \qquad \text{with} \qquad \tau \in \ . \ / \ . \ .$$

Thus the squaring map $P_2 : S^1 \to S^1$ is topologically conjugate to the *doubling map*

$$m_2(\tau) \equiv 2\,\tau \qquad (\text{mod }\ )$$

on the circle of real numbers modulo one.

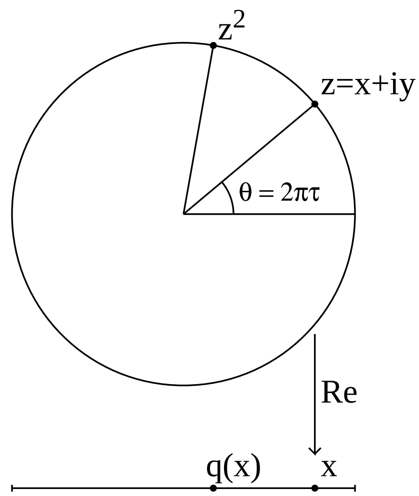

Figure 10. Angle doubling and the quadratic map q *.*

This is closely related to the quadratic map q . In fact the projection map

$$\mathrm{Re} : S^1 \to [-1\,,\,1]$$

which carries each $z = x + iy \in S^1$ to its real part $\mathrm{Re}(z) = x$ satisfies

$$\mathrm{Re}((x+iy)^2) \;=\; x^2 - y^2 \;=\; 2x^2 - 1 \;=\; q(x)\ .$$

We will say briefly that the projection map $\mathrm{Re}(x + iy) = x$ *semi-conjugates* the squaring map P_2 to the quadratic map q .

Newton's Method: a Chaotic Case. Here is a curious variant of the angle doubling example. Recall that Newton's method for solving a polynomial equation $f(x) = 0$ involves iterating an associated rational map

$$N(x) \;=\; x - f(x)/f'(x)$$

where f' is the first derivative of f . Note that the fixed points $N(x) = x$ are precisely the roots $f(x) = 0$. If we start with any point x_0 which is sufficiently close to a root $\hat{x}$ of f , then the orbit $N :\ x_0 \mapsto x_1 \mapsto x_2 \mapsto \ldots$, where $x_{k+1} = N(x_k)$, will always converge to $\hat{x}$. Suppose however that we choose $f(x) = x^2 + 1$, with no real roots. Then

$$N(x) \;=\; x - \frac{x^2+1}{2x} \;=\; (x - x^{-1})/2\ . \qquad (2:2)$$

If we start with any real x_0 , then the sequence $N :\ x_0 \mapsto x_1 \mapsto \cdots$ of real numbers cannot converge, since N has no real fixed point. Thus Newton's algorithm cannot converge; instead it seems to have a nervous breakdown, and jumps about chaotically.

To understand this example properly, since $N(0) = \infty$, it is necessary to compactify the real numbers by adding a point at infinity, thus forming the *real projective line*. It is not difficult to check that the resulting map N from $\ \cup\infty$ to itself is topologically conjugate to the squaring map $P_2 : S^1 \to S^1$. (See Problem 2-e below.)

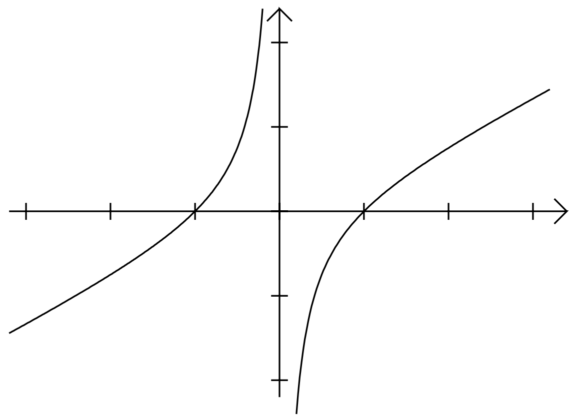

Figure 11. The map $N(x) = (x - x^{-1})/2$.

Remark. Still another topologically conjugate map is illustrated in Figure 6 (left).

---

§**2C. Sensitive Dependence.** This angle doubling example displays sensitive dependence[1] on initial conditions in a very transparent form. Consider an orbit

$$m_2 \;:\; \tau_0 \;\mapsto\; \tau_1 \;\mapsto\; \tau_2 \;\mapsto\; \cdots \;.$$

In other words, suppose that the elements $\tau_0 , \tau_1 , \tau_2 , \ldots$ of . /. satisfy the precise congruence $\tau_{k+1} \equiv 2\tau_k \pmod{\ }$ for every $k \geq 0$. Suppose however that we can measure the initial value τ_0 only to within an accuracy of ϵ . Then we can predict the value of:

τ_1	to an accuracy of	$2\,\epsilon$
τ_2	to an accuracy of	$4\,\epsilon$
τ_3	to an accuracy of	$8\,\epsilon$

$$\cdots$$

and so on; we can predict the value of τ_{10} to an accuracy of $1024\,\epsilon$, and τ_{20} to an accuracy of $1048576\,\epsilon$. *If we choose* n *large enough so that* $2^n\epsilon > 1$ *, then our approximate measurement of* τ_0 *will provide no information at all about* τ_n *.*

Similar remarks apply to the quadratic map q , the tent map T , and to the Newton map N . For example, suppose that

$$q \;:\; x_0 \;\mapsto\; x_1 \;\mapsto\; x_2 \;\mapsto\; \cdots$$

is a precise orbit for the quadratic map q . An approximate measurement for x_0 will show, for example, that it lies in some interval $[\alpha, \beta]$, and hence that a corresponding angle $\tau = \arccos(x)/2\pi$ lies in an interval between $\arccos(\beta)/2\pi$ and $\arccos(\alpha)/2\pi$. If ϵ is the length of this last interval, and if $2^n\epsilon > 1$, then similarly we have no information at all about the correct value for x_n .

---

§**2D. Symbolic Dynamics: The One-Sided 2-Shift.** We can represent each of these examples by a symbolic coding. The case of the doubling map $m_2 :$. /. $\rightarrow$. /. is easiest

[1] For a precise definition, see §4D.

to see. Represent each $\tau_0 \in [0,1]$ by its base two expansion

$$\tau_0 \;=\; 0\,.\,b_0\,b_1\,b_2\,\cdots\ \text{(base 2)} \;=\; b_0/2 + b_1/4 + b_2/8 + \cdots\ , \qquad (2:3)$$

where each ***bit*** b_n is either zero or one. Then

$$2\,\tau_0 \;=\; b_0\,.\,b_1\,b_2\,b_3\,\cdots\ \text{(base 2)}\ ,$$

and reducing modulo one we get

$$\tau_1 \;=\; 0\,.\,b_1\,b_2\,b_3\,\cdots\ \text{(base 2)}\ ,$$

with $\tau_1 \equiv 2\,\tau_0$ (mod) . *Thus the doubling map modulo one corresponds to the* ***shift map***

$$\sigma(b_0\,,\,b_1\,,\,b_2\,,\,\ldots) \;=\; (b_1\,,\,b_2\,,\,b_3\,,\,\ldots)$$

on the space of all one-sided infinite sequences of zeros and ones. We can give this space of sequences the structure of a totally disconnected compact metric space, for example by defining the distance function:

$$\mathbf{d}((b_0\,,\,b_1\,,\,\ldots)\,,\,(b'_0\,,\,b'_1\,,\,\ldots)) \;=\; \sum_0^\infty |b'_n - b_n|/2^n\ .$$

We will use the notation $\{0,1\}$ for the resulting space, where $= \{0,1,2,\cdots\}$ stands for the set of natural numbers.

Topologically, the space $\{0,1\}$ is a ***Cantor set***. It is not difficult to check that the shift map σ on $\{0,1\}$ exhibits sensitive dependence on initial conditions. (For a dynamical system embedded in Euclidean space which is topologically conjugate to this shift map, see Figure 6 right.) Evidently the formula (6) describes a mapping from $\{0,1\}$ onto / which semi-conjugates the shift map $\sigma : \{0,1\} \to \{0,1\}$ to the doubling map m_2 . Compare Problem 2-c.

---

§**2E. The Solenoid.** (Compare [Shub].) First consider a rather trivial modification of the squaring map of §2B. Consider the product manifold $S^1 \times \mathbf{C}$ embedded in $\mathbf{C} \times \mathbf{C}$, and the map

$$F_0(z,w) \;=\; (z^2\,,\,\lambda w)$$

from $S^1 \times \mathbf{C}$ to itself, where λ is some constant with $0 < |\lambda| < 1$. Then clearly the w-coordinate tends to zero as we iterate F_0 . Thus all orbits converge towards the circle $S^1 \times \{0\}$, and F_0 maps this circle onto itself by the squaring map. *This provides a simple example of a smooth "attractor" with chaotic dynamics.*

Now suppose we perturb this mapping to

$$F_\epsilon(z,w) \;=\; (z^2\,,\ \epsilon\,z + \lambda w)\ ,$$

where ϵ is some non-zero constant which may be very small. Then we will see that the geometry changes drastically. The precise choice of ϵ doesn't matter here, since every such F_ϵ is smoothly conjugate to the map

$$F_1(z,w) \;=\; (z^2\,,\ z + \lambda w)$$

under the conjugating transformation $(z,w) \mapsto (z\,,\,w/\epsilon)$. Thus it suffices to study F_1 .

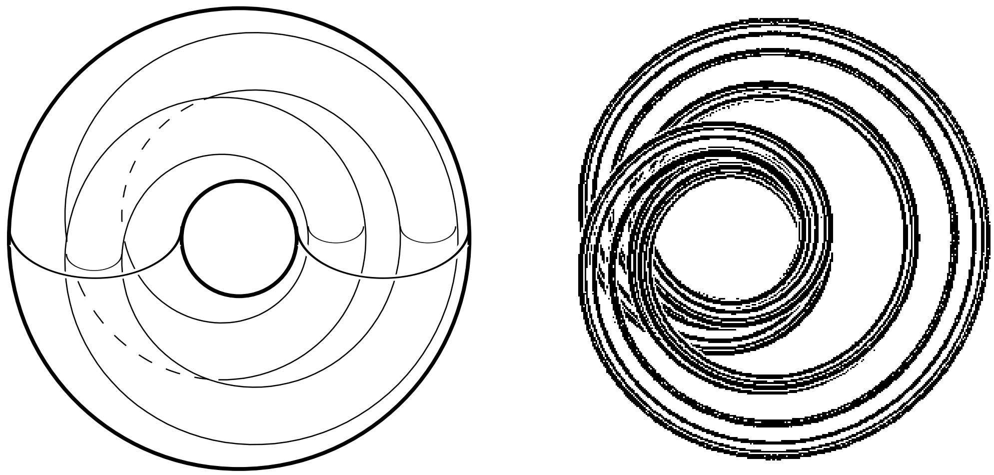

Figure 12. Left: Picture of the solid torus T (embedded in 3-space), and of the sub-torus $F_1(T)$ which winds twice around it. Right: The solenoid $\bigcap f^{\circ n}(T)$.

Let $T \subset S^1 \times \mathbf{C}$ be the solid torus consisting of all (z,w) with $|w| \le c$, where the constant $c \ge 2$ is to be chosen as follows. If $|\lambda| < 1/2$, take $c = 2$; but otherwise choose any c which is large enough so that $1 + c|\lambda| < c$. First note the following:

Assertion. *Every orbit of F_1 in $S^1 \times \mathbf{C}$ eventually lands in this solid torus T , and the image $F_1(T)$ is strictly contained in the interior of T . Furthermore, if $|\lambda| < 1/2$, then T maps diffeomorphically onto its image, which is therefore a solid torus smoothly embedded in the interior of T .*

Proof. Let $k = 1/c + |\lambda| < 1$. Note that the second coordinate of $F_1(z,w) = (z^2\,,\ z + \lambda w)$ satisfies

$$|z + \lambda w| \ \le\ 1 + |\lambda w|\ .$$

If $|w| \le c$, then the right hand side is $\le 1 + c|\lambda| < c$. On the other hand, if $|w| > c$, then this right hand side is $< |w|/c + |\lambda w| = k\,|w|$. Since the factor $k = 1/c + |\lambda|$ is strictly less than 1 , the successive values of $|w|$ must decrease geometrically until $|w| \le c$.

Finally, in the case $|\lambda| < 1/2$, if $F_1(z,w) = F_1(z',w')$ with $(z,w) \ne (z',w')$, then it is easy to check that z' must be equal to $-z$. The equation $z + \lambda w = -z + \lambda w'$ with $|w|\,,\ |w'| \le c = 2$ then leads to a contradiction, since $2\,z = \lambda(w' - w)$ with $|\lambda(w' - w)| < 2$. Thus F_1 restricted to T is one-to-one. This map is clearly smooth with smooth inverse. □

It follows that the successive images

$$T\ \supset\ f(T)\ \supset\ f^{\circ 2}(T)\ \supset\ \cdots$$

form a nested sequence of compact sets. Let

$$A\ =\ \bigcap f^{\circ n}(T)$$

be their intersection. *It follows that every orbit of F_1 converges towards this compact set A .* There is an obvious semi-conjugacy

$$(z,w)\ \mapsto z$$

which carries this compact set A onto the circle. Thus the dynamics of F_1 restricted to this "attractor" A is at least as complicated as the dynamics of the squaring map on S^1.

If $|\lambda| < 1/2$, then we can describe the geometry more precisely. The image $F_1(T)$ is a solid torus which wraps twice around T, and shown in Figure 12. Similarly, the second image $f^{\circ 2}(T)$ wraps four times around T, and so on. Although the original mapping F_1 from $S^1 \times \mathbf{C}$ is two-to-one, and the squaring map on S^1 is also two-to-one, it is interesting to note that F_1 maps the attractor $A = \bigcap f^{\circ n}(T)$ homeomorphically onto itself (still assuming that $|\lambda| < 1/2$). In order to understand this passage from a many-to-one map to a homeomorphism, it is convenient to consider the following construction.

Definition. Start with a mapping $f : X \to X$ from a compact space onto itself. By a *full orbit* for f we will mean a function $n \mapsto x_n$ from the set of all integers to X satisfying the identity $f(x_n) = x_{n+1}$, so that

$$\cdots \mapsto x_{-2} \mapsto x_{-1} \mapsto x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots .$$

The space $\hat{X}$ of all such full orbits is to be topologized as a subset of the infinite cartesian product X, that is the cartesian product $\cdots \times X \times X \times X \times \cdots$, indexed by the positive *and* negative integers. This space $\hat{X}$, together with the shift homeomorphism

$$\hat{f} \; : \; \{x_n\} \;\mapsto\; \{f(x_n)\} = \{x_{n+1}\}$$

from $\hat{X}$ to itself, will be called the *natural homeomorphic extension* of the dynamical system (X, f). Note that there is a natural semi-conjugacy

$$\{x_n\}_{n\in} \;\mapsto\; x_0$$

from the dynamical system $(\hat{X}, \hat{f})$ onto (X, f).

If f is a homeomorphism from X onto itself, then such a full orbit is uniquely determined by the single point x_0, hence $\hat{X} \cong X$. However, we are rather interested in the case when f is many-to-one. Then, to specify a point of $\hat{X}$, we must choose not only a point $x_0 \in X$ but also a point $x_{-1} \in f^{-1}(x_0)$, then a point $x_{-2} \in f^{-1}(x_{-1})$, and so on. Thus this space of full orbits can also be described as the *inverse limit* (also called *projective limit*) of the *inverse sequence* of mappings

$$X \overset{f}{\leftarrow} X \overset{f}{\leftarrow} X \overset{f}{\leftarrow} \cdots .$$

In the special case of the squaring map on S^1, the corresponding space $\hat{S}^1$ of full orbits is called the dyadic *solenoid*. Now suppose that we start with any such full orbit

$$\cdots \mapsto z_{-2} \mapsto z_{-1} \mapsto z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$$

for the squaring map on S^1. If we start with any $w \in \mathbf{C}$ with $|w| \le c$ and apply the n-fold iterate of F_1 to the point (z_{-n}, w) in the solid torus T, then a brief computation shows that

$$F_1^{\circ n}(z_{-n}, w) \;=\; (z_0,\; \lambda^n w + \sum_{k=1}^{n} \lambda^{k-1} z_{-k}) .$$

Evidently, as $n \to \infty$, this image converges to the limit $(z_0, \sum_{k=1}^{\infty} \lambda^{k-1} z_{-k})$.

Lemma 2.1 *The attractor $A = \bigcap f^{\circ n}(T)$ for the map F_1 is precisely the image of the natural homeomorphic extension $\hat{S}^1$ of the squaring map under the* *correspondence*

$$\{z_n\}_{n\in} \quad \mapsto \quad (z_0\,,\ \sum_{k\geq 1}\lambda^{k-1}z_{-k})\ .$$

This map $\hat{S}^1 \to A$ *is always a semi-conjugacy from* $\hat{f}$ *onto* $F_1|A$ *. In the case* $|\lambda| < 1/2$ *, it is a topological conjugacy, which maps the solenoid* $\hat{S}^1$ *homeomorphically onto the attractor* A *.*

The proof is easily supplied. □

---

§2F. Problems for the Reader.

Problem 2-a. Chebyshev polynomials. Define the degree n *Chebyshev map*

$$Ч_n : [-1\,,\,1] \to [-1\,,\,1]$$

inductively by the formula

$$Ч_{n+1}(x) + Ч_{n-1}(x) \;=\; 2x\,Ч(x)\ ,$$

starting with

$$Ч_0(x) = 1\,, \quad Ч_1(x) = x\,, \quad Ч_2(x) = 2x^2 - 1\,, \quad \ldots\ .$$

Show inductively that

$$Ч_n\Big(\frac{z+z^{-1}}{2}\Big) \;=\; \frac{z^n + z^{-n}}{2}\ .$$

Taking $z = x + iy \in S^1$, with $x = \cos(\theta)$, $y = \sin(\theta)$, conclude that $Ч_n(x) = \mathrm{Re}(z^n)$, or equivalently

$$Ч_n(\cos\,\theta) = \cos(n\theta)\ .$$

Thus the projection $\mathrm{Re} : S^1 \to [-1\,,\,1]$ *semi-conjugates the* n*-th power map* P_n *to the* n*-th Chebyshev map.*

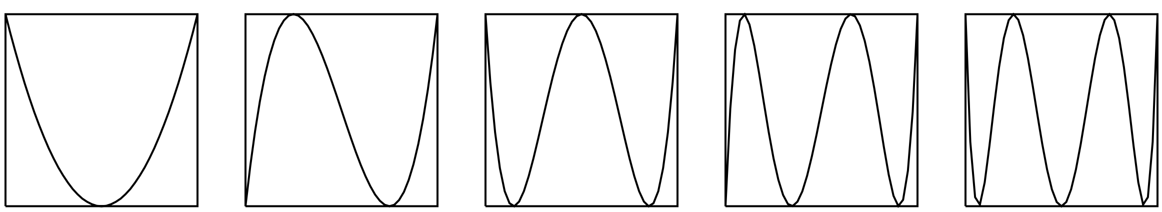

Figure 13. The Chebyshev maps of degree 2 through 6.

Show also that the composition $Ч_n \circ Ч_k$ is equal to $Ч_{nk}$. (For example, taking $k = 0$, it follows that $Ч_n(1) = 1$.)

Remark: These polynomials, introduced by P. L. Chebyshev in 1854, satisfy the orthogonality condition

$$\int_{-1}^{1} Ч_n(x)\,Ч_k(x)\,dx/\sqrt{1-x^2} \;=\; 0 \qquad \text{for} \qquad n \neq k\ .$$

Problem 2-b. Sawtooth maps. With $h(\xi) = \cos(\pi\xi)$ as in 2.2 above, show that the topologically conjugate map $S_n(\xi) = h^{-1} \circ Ч_n \circ h$ has slope $\pm n$ everywhere, with

$$S_n(0) = 0\,, \quad S_n(1/n) = 1\,, \quad S_n(2/n) = 0\,, \quad \ldots\ .$$

In particular, S_2 is equal to the tent map T . If $Ч_n^{\circ k}(\hat{x}) = \hat{x}$ is a periodic point, with $\hat{x} < 1$, show that the derivative $d\,Ч_n^{\circ k}(x)/dx$ at $\hat{x}$ is equal to $\pm\, n^k$.

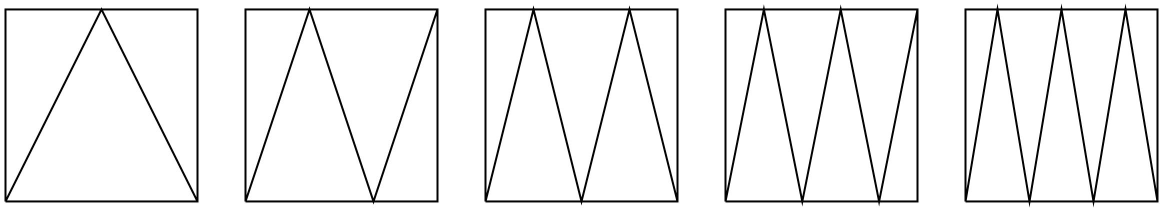

Figure 14. The sawtooth maps S_n for $n = 2$ through 6 .

Problem 2-c. Symbolic dynamics for the n-tupling map. For any $n \ge 2$, let $m_n : \;/\; \to \;/\;$ be the map

$$m_n(x) \equiv n\,x \pmod{\;} .$$

Cover the circle $\;/\;$ by n non-overlapping closed intervals I_α of length $1/n$, where

$$I_\alpha \;=\; \Big[\frac{\alpha}{n}\,,\,\frac{\alpha+1}{n}\Big] \;\subset\; \;/\;$$

for $0 \le \alpha < n$. Given any sequence $\alpha_0\,,\,\alpha_1\,,\,\alpha_2\,,\,\ldots$ with $\alpha_i \in \{0, 1, \ldots, n-1\}$, show that there is one and only one number

$$x_0 \;=\; 0\,.\,\alpha_0\,\alpha_1\,\alpha_2\,\cdots\ (\text{base } n) \;=\; \sum \alpha_i/n^{i+1} \qquad (2:4)$$

in $\;/\;$ with the property that the orbit

$$m_n \;:\; x_0 \;\mapsto\; x_1 \;\mapsto\; x_2 \;\mapsto\; \cdots$$

of x_0 under m_n satisfies

$$x_k \;\in\; I_{\alpha_k} \quad \text{for every} \quad k \ge 0\ .$$

Show that this formula (7) defines a mapping from the space $\{0, 1, \ldots, n-1\}$ of sequences onto the circle $\;/\;$ which semi-conjugates the shift map

$$\sigma_n \;:\; \{0, 1, \ldots, n-1\} \;\to\; \{0, 1, \ldots, n-1\}$$

to the n-tupling map m_n .

Show that a point $x \in \;/\;$ has just one pre-image in $\{0, 1, \ldots, n-1\}$ unless it can be expressed as a fraction of the form i/n^k , in which case it has exactly two pre-images.

Problem 2-d. Symbolic dynamics for a sawtooth map. Similarly, given any sequence $\alpha_0\,,\,\alpha_1\,,\,\alpha_2\,,\,\ldots$ with $\alpha_i \in \{0, 1, \ldots, n-1\}$, show that there is one and only one number $\xi_0 \in [0, 1]$ so that the orbit

$$S_n \;:\; \xi_0 \;\mapsto\; \xi_1 \;\mapsto\; \xi_2 \;\mapsto\; \cdots$$

satisfies $\xi_k \in I_{\alpha_k}$ for every k . Thus there also exists a map from $\{0, 1, \ldots, n-1\}$ onto $[0, 1]$ which semi-conjugates the n-shift onto the sawtooth map S_n . (Note that this is not at all the same as the map used in Problem 2-c.)

Problem 2-e. "Newton's method". Show that the fractional linear transformation

$$h(x) \;=\; \frac{x+i}{x-i}$$

(where $i = \sqrt{-1}$) carries the real projective line $\cup\,\infty$ diffeomorphically onto the unit circle $S^1 \subset \mathbf{C}$, and show that it satisfies the identity

$$h(x)^2 \;=\; \Big(\frac{x+i}{x-i}\Big)^2 \;=\; h(N(x))\ ,$$

with $N(x) = (x - 1/x)/2$ as in Equation (5). Thus the rational map N on $\cup\,\infty$ is *smoothly conjugate* to the squaring map P_2 on the unit circle.

Problem 2-f. The two sided 2-shift. Show that the natural homeomorphic extension of the one sided shift on two symbols is topologically conjugate to the *two sided shift*, that is the space $\{0,1\}$, together with the shift map on this space. Show that the correspondence $\{\alpha_n\} \mapsto \{z_n\}$, where

$$z_n \;=\; \exp(2\pi i \tau_n)\,, \qquad \tau_n \;=\; \sum_{k=0}^{\infty} \alpha_k/2^{k+1}\ ,$$

is a semi-conjugacy from this two sided shift onto the dyadic solenoid.