§15. Denjoy's Theorem

This section will prove a basic result due to Arnaud Denjoy (1884-1974). We will state his result in three equivalent forms.

Theorem 15.1. *Let f be a C^2-smooth orientation preserving circle diffeomorphism with irrational rotation number. Then:*

(1) *f is topologically conjugate to a rotation; furthermore*

(2) *every orbit is dense, and*

(3) *for any non-trivial interval $I \subset \mathbb{R}/\mathbb{Z}$ there exists a forward image $f^{\circ q}(I)$, $q \geq 1$, which intersects I. (In other words, the non-wandering set $\Omega(f)$ is the entire circle.)*

Here most of the hypotheses are essential. Clearly the hypothesis of irrational rotation number is essential, and we will see in 15.5 that C^1-smoothness would not be enough. (However, the hypothesis that $f'(\xi) > 0$ everywhere can be weakened. Compare [Yoccoz, 1984], [Swiatek, 1991].) Note that we must assume C^2-differentiability for f, but obtain only a continuous conjugacy. It took almost fifty years until Michel Herman (1942-2000) was able to solve the more difficult problem of obtaining a smooth conjugacy under suitable hypotheses. Compare the discussion in §16D.

The proof begins as follows. According to 14.13, there is a monotone map g, semi-conjugating f to an irrational rotation. Thus each pre-image $g^{-1}(\eta) \subset \mathbb{R}/\mathbb{Z}$ is either a point or a closed interval. If all of these pre-images are points, then g is a homeomorphism, and f is actually topologically conjugate to an irrational rotation. In this case, it follows from 14.14 that every forward orbit under f is dense. On the other hand, if some $g^{-1}(\eta)$ is a non-trivial interval $\bar{I}$, then evidently $\bar{I}$ is a *wandering interval*. That is, the successive images $f^{\circ k}(\bar{I})$ are pairwise disjoint.

Thus, if we can prove (3), then (1) and (2) will follow. To complete the proof of Denjoy's Theorem, we must show that a C^2-diffeomorphism with irrational rotation number cannot have any wandering interval. The proof will be based on the following ideas. (Compare [de Melo and van Strien].)

§15A. Distortion Estimates. Let I be a closed interval of real numbers, and suppose that the real valued function F is defined and C^1-smooth on I, with derivative $F'(x) > 0$. By the *non-linearity* (or the *distortion*) of F on I we will mean the non-negative real number

$$\text{nonlin}(F, I) = \log \max_x F'(x) - \log \min_x F'(x) = \log \frac{\max_x F'(x)}{\min_x F'(x)} ,$$

where x varies over I. Note that $\text{nonlin}(F, I) = 0$ if and only if F is linear on I.

Lemma 15.2. *If $F : I_0 \to I_1$ and $G : I_1 \to I_2$ are C^1-smooth with positive derivative, then*

$$\text{nonlin}(G \circ F, I_0) \leq \text{nonlin}(F, I_0) + \text{nonlin}(G, I_1) .$$

Proof. This follows easily from the chain rule:

$$\log(G \circ F)'(x) = \log F'(x) + \log G'(y) ,$$

where $y = F(x)$. $\square$

Now suppose that F is smooth of class C^2. If F' on the interval I takes its maximum at $x_{\rm max}$ and its minimum at $x_{\rm min}$, then evidently

$$\begin{aligned} \mathrm{nonlin}(F\,,\,I) &= \log F'(x_{\rm max}) - \log F'(x_{\rm min}) = \int_{x_{\rm min}}^{x_{\rm max}} (\log F')'(x)\,dx \\ &= \int_{x_{\rm min}}^{x_{\rm max}} \frac{F''(x)}{F'(x)}\,dx \le \int_I \left|\frac{F''(x)}{F'(x)}\right| dx\,. \end{aligned} \tag{15:1}$$

Next consider an orientation preserving circle diffeomorphism f which is C^2-smooth. Let $I_0 \subset \mathbb{R}/\mathbb{Z}$ be an interval, and let $I_n = f^{\circ n}(I_0)$ be its image under n-fold iteration of f. Since the map f is invertible, this notation makes sense not only for $n \ge 0$ but also for $n < 0$. Note that the first derivative f' is a well defined continuous function from $\mathbb{R}/\mathbb{Z}$ to the positive reals. Hence the non-linearity $\mathrm{nonlin}(f\,,\,I_0) \ge 0$ is defined.

Lemma 15.3. *There exists a constant* $K = \int_{\mathbb{R}/\mathbb{Z}} |f''(\xi)/f'(\xi)|\,d\xi$ *with the following property. For any interval* $I_0 \subset \mathbb{R}/\mathbb{Z}$ *and any* $n > 0$, *if the first* n *forward images* $I_0\,, I_1\,, \ldots\,, I_{n-1}$ *are pairwise disjoint, then the non-linearity of the* n*-fold iterate satisfies*

$$\mathrm{nonlin}(f^{\circ n}\,,\,I_0) \le K < \infty\,.$$

Proof. By Lemma 15.2 this non-linearity is less than or equal to the n-fold sum $\mathrm{nonlin}(f,\,,I_0) + \cdots + \mathrm{nonlin}(f\,,\,I_{n-1})$, and by inequality (15: 1) this is less than or equal to the integral over $I_0 \cup I_1 \cup \cdots \cup I_{n-1}$ of $|f''/f'|$, which is less than or equal to K. □

Next let f be any orientation preserving circle homeomorphism with irrational rotation number, and let $f: \cdots \mapsto \xi_{-1} \mapsto \xi_0 \mapsto \xi_1 \mapsto \xi_2 \mapsto \cdots$ be any orbit.

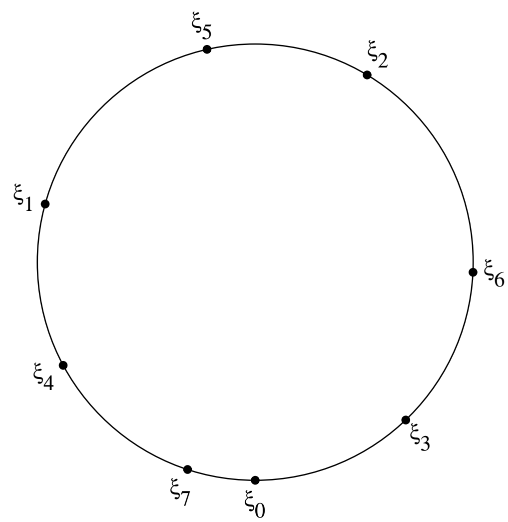

Figure 48. The points $\xi_1\,, \xi_3\,, \xi_7\,, \ldots$ *are closest returns to* ξ_0 *for the irrational rotation* $\xi \mapsto \xi + \sqrt{2}/2 \pmod 1$, *while* $\xi_2\,, \xi_4\,, \xi_5\,, \xi_6$ *are not.*

Definition: We will say that ξ_q is a *closest return* to ξ_0 if all of the points ξ_j with $0 < |j| < q$ belong to the same component of the complement $\mathbb{R}/\mathbb{Z} \smallsetminus (\{\xi_0\} \cup \{\xi_q\})$.

It is easy to see that there are infinitely many closest returns, as long as the rotation number is irrational. (According to 14.11, it suffices to consider the case of a rigid rotation.

In the case of a rotation, one can check that ξ_q is a closest return if and only if the distance $\mathbf{d}(\xi_0\,,\,\xi_q)$ is strictly less than $\mathbf{d}(\xi_0\,,\,\xi_n)$ for all $0 < n < q$.)

Proof of Theorem 15.1. Suppose that the circle homeomorphism f has a "wandering interval", that is an interval $I_0 \subset \mathbb{R}/\mathbb{Z}$ such that the iterated images $I_n = f^{\circ n}(I_0)$ satisfy $I_0 \cap I_n = \emptyset$ for $n > 0$, and therefore $I_m \cap I_{m+n} = \emptyset$ for all $m \in \mathbb{Z}$ and all $n \neq 0$. Choose a base point $\xi_0 \in I_0$ and let $\xi_n = f^{\circ n}(\xi_0) \in I_n$. Evidently the cyclic ordering of the various disjoint intervals $I_n \subset \mathbb{R}/\mathbb{Z}$ is precisely the same as the cyclic ordering of these representative points ξ_n . If the rotation number $\mathbf{rn}(f)$ is irrational, then we have shown that there exist infinitely many integers q so that ξ_q is a closest return to ξ_0 . Since all of the intervals I_k are pairwise disjoint, with total length less than or equal to one, it follows that we can choose such a closest return so that the lengths $\ell(I_q)$ and $\ell(I_{-q})$ are arbitrarily close to zero.

With this choice of q , we will enlarge the intervals $I_0 \mapsto \ldots \mapsto I_{q-1}$ so as to obtain disjoint intervals $\hat{I}_0 \mapsto \ldots \mapsto \hat{I}_{q-1}$ such that $f|\hat{I}_0$ is extremely nonlinear. To construct $\hat{I}_0$, consider the two points ξ_{-q} and ξ_0 . These points cut the circle into one interval J which contains all of the intervals I_j with $0 < |j| < q$ and a complementary interval J' which is disjoint from all of them. Let $\hat{I}_0$ be the interval $I_{-q} \cup J' \cup I_0$. Then it is not difficult to check that $\hat{I}_0$ is disjoint from its forward images $\hat{I}_j = f^{\circ j}(\hat{I}_0)$ for $0 < j < q$.

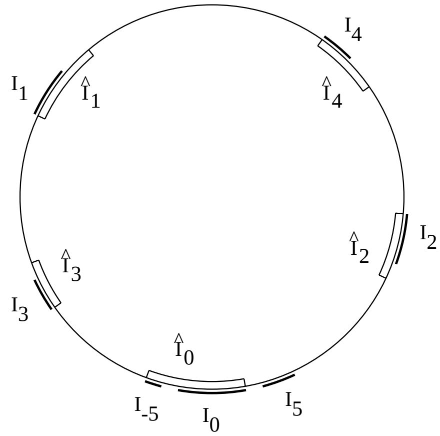

Figure 49. Construction of the interval $\hat{I}_0$ *. Here* $q = 5$ *, and the rotation number is* $(\sqrt{5} - 1)/2 \pmod{\mathbb{Z}}$ *.*

Let us apply the Mean Value Theorem to the smooth mapping $h = f^{\circ q}$, which carries I_{-q} onto I_0 , and carries I_0 onto I_q . According to this theorem, there exists a point $\alpha \in I_{-q}$ so that $h'(\alpha) = \ell(I_0)/\ell(I_{-q})$ and a point $\beta \in I_0$ so that $h'(\beta) = \ell(I_q)/\ell(I_0)$. Since $I_{-q} \cup I_0 \subset \hat{I}_0$, it follows that the non-linearity of h on $\hat{I}_0$ satisfies

$$\mathrm{nonlin}(h\,,\,\hat{I}_0) \quad = \quad \max_{\xi\in\hat{I}_0} \log h'(\xi) - \min_{\xi\in\hat{I}_0} \log h'(\xi) \quad \geq \quad \log h'(\alpha) - \log h'(\beta)\ .$$

Now, as the choice of closest return q tends to infinity, the length $\ell(I_0)$ remains fixed but $\ell(I_{-q})$ and $\ell(I_q)$ tend to zero. Hence $h'(\alpha) \to \infty$, $h'(\beta) \to 0$, and it follows that $\mathrm{nonlin}(f^{\circ q}\,,\,\hat{I}_0) \to \infty$. This is impossible; for by Lemma 15.3 there is a fixed upper bound $K < \infty$ for the non-linearities of all such compositions. This completes the proof that a

C^2-circle diffeomorphism with irrational rotation number has no wandering interval, and hence completes the proof of Theorem 15.1. □

Here is a interesting consequence.

Corollary 15.4. *If a non-monotone circle map f is C^2-smooth, with only finitely many critical points, and if the first derivative $f'(\xi)$ changes sign as we pass through any critical point, then f must have at least one periodic orbit.*

Proof. If the degree of f is different from $+1$, then the conclusion is clear since f must have a fixed point. In the degree one case, it follows by 14.13 that f is semiconjugate to an irrational rotation under some degree one monotone semiconjugacy g. Clearly any critical point of f, or any interval on which $f'(x) \le 0$, must be contained in the interior of some wandering interval $I = g^{-1}(\eta)$. There can be only finitely many such wandering intervals I on which f is non-monotone. Evidently f must carry the left [respectively right] endpoint of each such I to the left [respectively right] endpoint of $f(I)$. Hence we can choose a new map $\widehat{f}$ which coincides with f outside of these special wandering intervals, and is a C^2-diffeomorphism of the circle. Now the same map g will semiconjugate $\widehat{f}$ to an irrational rotation. This contradicts Denjoy's Theorem, and completes the proof of 15.4. □

§**15B. Denjoy Counterexamples.** Denjoy also proved the following result, which shows that the hypothesis of C^2-smoothness is essential in 15.1

Theorem 15.5. *For any irrational number α there exists a C^1-circle diffeomorphism f which has a wandering interval, and which has rotation number equal to α (modulo $\mathbb{Z}$).*

Proof. We will first construct a circle *homeomorphism*, and then show how to make it a C^1-diffeomorphism. Start with the rotation $\eta \mapsto \eta + \alpha$, and consider the orbit

$$\cdots \;\mapsto\; \eta_{-1} \;\mapsto\; \eta_0 \;\mapsto\; \eta_1 \;\mapsto\; \cdots$$

where $\eta_k \equiv k\,\alpha$. We will thicken this orbit, replacing each point η_n by an interval $I_n = [a_n\,,\,b_n]$ of length $\ell_n = b_n - a_n$. Evidently these lengths $\ell_n > 0$ must satisfy $\sum_{-\infty}^{\infty} \ell_n \le 1$. In fact, to simplify the construction, let us choose lengths $\ell_n > 0$ with sum precisely equal to 1. For each $n \in \mathbb{Z}$ let x_n be the unique number in the half-open interval $[0,1)$ which is congruent to $n\,\alpha \pmod{\mathbb{Z}}$. Now define the endpoints of the required intervals $I_n = [a_n\,,\,b_n]$ by the formula $a_n = a(x_n)$, $b_n = b(x_n)$, where

$$a(x) \;=\; \sum\{\ell_k \;\;;\;\; x_k < x\}\;, \qquad b(x) \;=\; \sum\{\ell_k \;\;;\;\; x_k \le x\}$$

for every $x \in [0,1)$. Thus the interval $[a(x), b(x)]$ has length $\ell_n > 0$ if $x = x_n$, but is degenerate with length zero if x is not one of the x_n. These intervals are disjoint, with union $[0,1)$. Then it is easy to see that there is one and only one continuous monotone map $G : \mathbb{R} \to \mathbb{R}$ which satisfies $G([a(x)\,,\,b(x)]) = x$ for every $x \in [0,1)$, and with $G(x+1) = G(x) + 1$.

In particular, these intervals $[a(x)\,,\,b(x)] \subset [0,1)$ are ordered in the same way as the points x. For each $x \in [0,1)$ with image ξ in the circle $\mathbb{R}/\mathbb{Z}$, let $I(\xi) \subset \mathbb{R}/\mathbb{Z}$ be the image of the corresponding interval $[a(x)\,,\,b(x)]$. Evidently these images are disjoint, with union $\mathbb{R}/\mathbb{Z}$, and are arranged in the same cyclic order as the points ξ themselves. In fact the map G is the lift of a monotone degree one circle map g with $g(I(\xi)) = \xi$. Thus

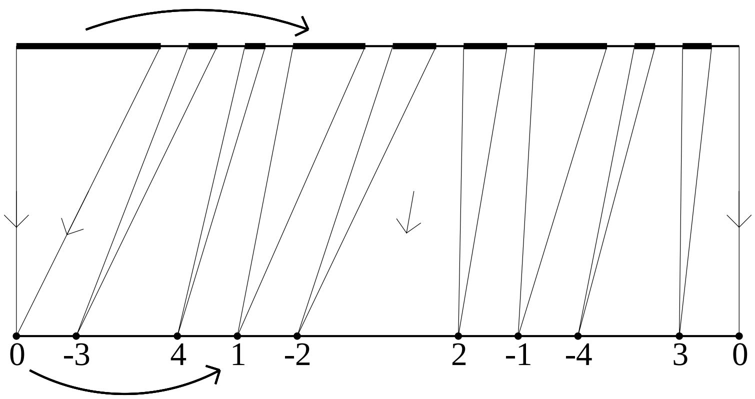

Figure 50. The bottom line, representing the circle $\mathbb{R}/\mathbb{Z}$, is mapped to itself by the irrational rigid rotation $t \mapsto t+\alpha \pmod{\mathbb{Z}}$. Each label n on the bottom line, standing for the point $n\alpha \pmod{\mathbb{Z}}$, corresponds to an entire interval $I(n\alpha)$ of length ℓ_n in the top line, which also represents the circle $\mathbb{R}/\mathbb{Z}$. Here $\sum_n \ell_n = 1$ so that the union of these intervals has full measure. The map f on the top circle, which carries $I(n\alpha)$ onto $I((n+1)\alpha)$, is monotonely semiconjugate to the rigid rotation on the bottom circle.

we can construct a degree one circle homeomorphism f which maps each $I(\xi) = g^{-1}(\xi)$ to $I(\xi+\alpha)$. In fact, if $I(\xi)$ reduces to a single point, then $f(I(\xi))$ is uniquely defined, while if $I(\xi)$ is a non-degenerate interval then we can choose any orientation preserving homeomorphism from $I(\xi)$ onto $I(\xi+\alpha)$. This f is the required map, which is monotonely semiconjugate to rotation by α.

If we want f to be a C^1-diffeomorphism, then we must be a little more careful with this construction. First, we must choose the lengths $\ell_n > 0$ so that

$$\lim_{|n|\to\infty} \ell_{n+1}/\ell_n \;=\; 1 \;.$$

For example let $\ell_n = c/(n^2+1)$, where the constant c is chosen so that $\sum_{-\infty}^{+\infty} \ell_n = 1$. Next we must choose each homeomorphism from $I_n = [a_n\,,\, b_n]$ to I_{n+1} to be a diffeomorphism, with derivative equal to $+1$ at the two endpoints, and with derivative converging uniformly to $+1$ as $|n| \to \infty$. As an example, we can define this diffeomorphism by the formula

$$a_n + x \quad\mapsto\quad a_{n+1} + \int_0^x \exp\Big(c_n\, t\,(\ell_n - t)\Big)\, dt \;.$$

It is not difficult to check that there is a unique value of the parameter $c_n \in \mathbb{R}$ so that the image of I_n will have the required length

$$\int_0^{\ell_n} \exp\Big(c_n\, t\,(\ell_n - t)\Big)\, dt \;=\; \ell_{n+1} \;.$$

Furthermore, c_n converges to zero, and hence the derivative converges uniformly to 1 as $\ell_{n+1}/\ell_n \to 1$. It is now reasonably straightforward to check that the map f which is constructed using these diffeomorphisms is C^1-smooth with derivative identically equal to $+1$, except within the interiors of the non-degenerate intervals $I(\xi)$. □

Remark 15.6. Instead of mapping $I_n = [a_n\,,\, b_n]$ to I_{n+1} by a homeomorphism, we could also choose some arbitrary continuous map $I_n \to I_{n+1}$ which carries left endpoint to left endpoint and right endpoint to right endpoint. In this way, we could construct a degree one circle map f which is not monotone, even though it has a well defined irrational rotation number. (Compare 14.13.) This shows also that the hypothesis of C^2-smoothness is essential in 15.4.

§**15C. Differential equations on the torus.** The title of Denjoy's original paper referred to differential equations on the torus, rather than to circle maps. In fact there is a very close relationship between these two subjects. (Compare §1C.) Suppose that we are given a doubly periodic C^1-smooth real valued function of two variables.

$$s(t,x) \;=\; s(t+1\,,\,x) \;=\; s(t\,,\,x+1)\;.$$

Solving the differential equation

$$dx/dt \;=\; s(t,x)$$

we obtain a family of smooth functions $x = \Phi_h(t)$, parametrized by the initial height $\Phi_h(0) = h \in \mathbb{R}$, and satisfying the required equation $d\,\Phi_h(t)/dt = s(t\,,\,\Phi_h(t))$ for all $t \in \mathbb{R}$. In particular, if we follow the solution from time $t = 0$ to $t = 1$, then we obtain a C^1-smooth diffeomorphism of $\mathbb{R}$,

$$x \;\mapsto\; F(x) \;=\; \Phi_x(1)\;,$$

which commutes with integer translations, and therefore corresponds to a circle diffeomorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. (If we followed the solution from time t_0 to t_0+1, then we would obtain a topologically conjugate circle map.)

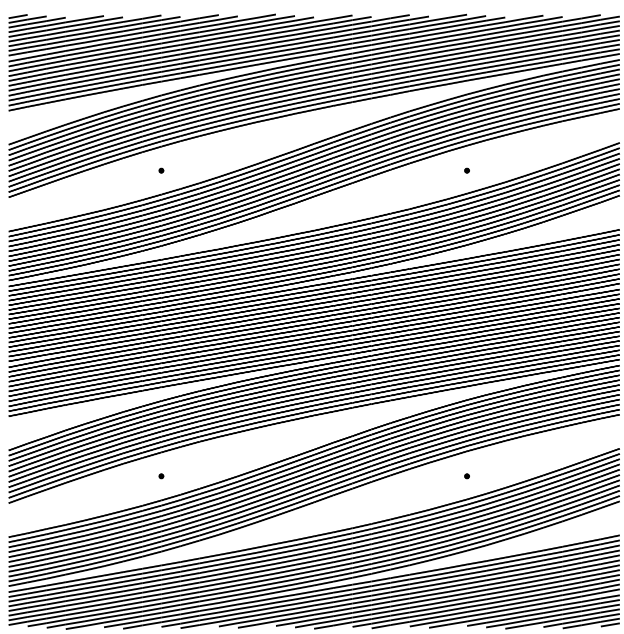

Figure 51. Trajectory curves for a flow on the (universal covering of the) torus $\mathbb{R}^2/\mathbb{Z}^2$. Here the integer lattice points are marked by heavy dots. The corresponding circle map has a wandering interval containing the origin, hence this example cannot be made C^2-smooth.

Here are some basic properties of this construction:

(1) The k-fold iterate of the map f (or F) corresponds to the map obtained by following the solutions from time $t = 0$ to $t = k$. That is, $F^{\circ k}(x) = \Phi_x(k)$.

(2) If we think of these curves as lying on the torus

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) ,$$

then a solution curve is closed (ie., comes back to itself after finite time) if and only if the corresponding orbit under the circle map f is periodic.

(3) The translation number $\mathbf{tn}(F)$ can be described as the "average slope" $\lim_{t\to\infty} (\Phi_h(t) - \Phi_h(0))/t$ associated with any solution curve $x = \Phi_h(t)$.

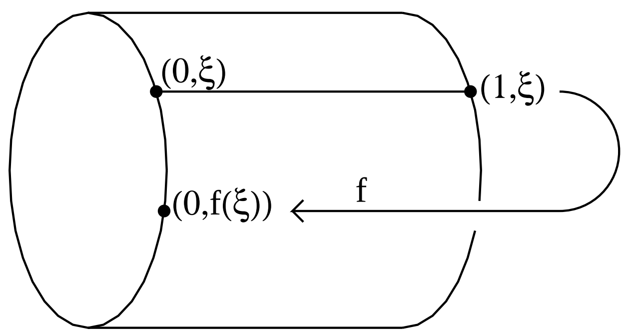

Figure 52. Construction of the mapping torus of f.

Any orientation preserving circle diffeomorphism can be obtained by this construction. Given a circle diffeomorphism f, form the *mapping torus* Torus(f) as follows. Start with the cylinder $[0\,,\,1] \times (\mathbb{R}/\mathbb{Z})$, and glue the right hand boundary $1 \times (\mathbb{R}/\mathbb{Z})$ onto the left hand boundary $0 \times (\mathbb{R}/\mathbb{Z})$ by the diffeomorphism

$$(1\,,\,\xi) \;\leftrightarrow\; (0\,,\,f(\xi)) .$$

The resulting identification space is the required mapping torus. The horizontal curves $\xi =$ constant in this mapping torus play the same role as the family of curves $x = \Phi_h(t)$ in the discussion above. For if we follow such a curve to the right from the point $(0\,,\,\xi)$ then we arrive at $(1\,,\,\xi)$, which is identified with $(0\,,\,f(\xi))$.

We can identify this mapping torus with the standard torus $\mathbb{T}^2$ as follows. Let F be a lift of f, and let $\tau(t)$ be a C^∞ function which takes the value zero for t close to zero and the value one for t close to one. Then the required diffeomorphism Torus$(f) \to \mathbb{T}^2$ is constructed by mapping each point $(t, x) \in [0, 1] \times \mathbb{R}$ to the point

$$(t\;,\;(1 - \tau(t))\,x + \tau(t)\,F(x)) \quad\in\quad [0, 1] \times \mathbb{R} .$$

Evidently two points $(1, x)$ and $(0, F(x))$ which are identified in $\mathbb{T}^2$ correspond to two points $(1\,,\,F(x))$ and $(0\,,\,F(x))$ which are identified in $\mathbb{T}^2$. Now the family of horizontal curves in the mapping torus corresponds to a suitable family of curves in $\mathbb{T}^2$, and differentiating these curves we obtain a corresponding differential equation $dx/dt = s(t, x)$ on $\mathbb{T}^2$. If f is C^r-smooth, then the resulting function $s(t, x)$ is also.