CHAPTER IV. DIFFERENTIABLE DYNAMICS

§11. Geodesics and Classical Mechanics.

The simplest and best known examples of measure preserving transformations arise from ordinary differential equations, and in particular from the Hamiltonian differential equations which are studied in the field of Classical Mechanics, and also in Riemannian Geometry. This section will outline some of the basic ideas. For more detailed presentations of Classical Mechanics, see for example [Arnold, 1978], [Abraham and Marsden], or [McDuff and Salamon]; and for Riemannian Geometry see for example [Boothby].

§11A. Volume Preserving Flows. Recall that a smooth map $\mathbf{y} = f(\mathbf{x})$, defined on an open subset of the Euclidean space $\mathbb{R}^n$, is *volume preserving* if and only if the Jacobean determinant

$$Jf(\mathbf{x}) \;=\; \det\,[\partial y_i/\partial x_j]$$

is identically equal to ± 1. Now consider the smooth flow $\{f_t\}$ generated by a differential equation of the form

$$\frac{dx_i}{dt} \;=\; v_i(x_1\,,\,\cdots\,x_n)\;,$$

or briefly $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x})$, where the velocity vector field $\mathbf{v}$ is a map from some region in $\mathbb{R}^n$ to $\mathbb{R}^n$ which is sufficiently smooth that local solutions exist, are unique, and depend differentiably on the initial point. The solution curves, locally at least, give rise to a smooth flow of the form

$$\mathbf{x}(t) \;=\; f_t(\mathbf{x}(0)) \qquad \text{with} \qquad f_{u+t} = f_u \circ f_t \qquad \text{and} \qquad f_0(\mathbf{x}) \equiv \mathbf{x}\;.$$

The following statement is well known.

> **Lemma 11.1.** *Every f_t is volume preserving if and only if the divergence $\nabla\cdot\mathbf{v} \;=\; \sum_i \partial v_i/\partial x_i$ is identically zero.*

Outline Proof. If we consider the determinant of a square matrix $A = [a_{ij}]$ as a function of the entries a_{ij}, then the partial derivative $\partial \det A/\partial a_{ij}$ evaluated at the identity matrx $A = I$ is equal to $+1$ if $i = j$, and is zero otherwise. It follows that the derivative of the function $t \mapsto Jf_t(\mathbf{x})$, evaluated at $t = 0$, is given by

$$\left.\frac{\partial Jf_t(\mathbf{x})}{\partial t}\right|_{t=0} \;=\; \left.\frac{\partial \det[\partial y_i/\partial x_j]}{\partial t}\right|_{t=0} \;=\; \sum_i \frac{\partial}{\partial t}\frac{\partial y_i}{\partial x_i} \;=\; \sum_i \frac{\partial}{\partial x_i}\frac{\partial y_i}{\partial t} \;=\; \sum_i \frac{\partial v_i}{\partial x_i} \;=\; \nabla\cdot\mathbf{v}\;.$$

In the volume preserving case, it follows that $\nabla\cdot\mathbf{v}$ is identically zero. Conversely, if $\nabla\cdot\mathbf{v} \equiv 0$, then taking the derivative of the identity

$$\log Jf_{u+t}(\mathbf{x}) \;=\; \log Jf_t(\mathbf{x}) + \log Jf_u(\mathbf{y})$$

with respect to u at $u = 0$, where $\mathbf{y} = f_t(\mathbf{x})$, we see easily that $d\log Jf_t(\mathbf{x})/dt$ is identically zero. This proves that the flow is volume preserving. □

Similarly, if the divergence is negative, then $Jf_t(\mathbf{x})$ decreases monotonically as a function of t.

In applications, one often wants to work on a more general manifold, where volumes are computed in local coordinates by an integral of the form

$$\mathrm{Vol}_n(U) \;=\; \int_U w(\mathbf{x})\; dx_1 \cdots dx_n \;,$$

and where $w(\mathbf{x}) > 0$ is a smooth weighting function. As an example, in Riemannian geometry, with a metric of the form $ds^2 = \sum g_{ij} dx^i dx^j$, one uses a weighting factor of $w(\mathbf{x}) = \sqrt{\det[g_{ij}]}$, in order to obtain an n-dimensional volume which is independent of the choice of local coordinates. The equation for a volume preserving flow in this more general case is discussed in Problem 11-a, and takes the form

$$\mathbf{v}\cdot\nabla w \;+\; w\,\nabla\!\cdot\mathbf{v} \;=\; 0 \;.$$

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§**11B. Classical Mechanics.** We consider physical systems, without friction, which are described by second order differential equations.

Example 11.2. An Elementary Example. Consider a roller-coaster, traveling without friction along a track which has equation

$$x = x(s)\;,\quad y = y(s)\;,\quad z = z(s)\;,$$

with horizontal coordinates $x\,,\,y$ and height z . Here s is to be the arclength along the track, so that the speed $\dot s = ds/dt$ is equal to $\sqrt{\dot x^2 + \dot y^2 + \dot z^2}$. Assuming the law of conservation of energy, we can derive the associated Hamiltonian differential equation as follows. Let m be the mass and γ the gravitational field strength. The potential energy is given by $\Phi(s) = \gamma\, m\, z(s)$, and the kinetic energy is equal to $m\,\dot s^2/2$. Thus the total energy is

$$\Phi(s) \;+\; \tfrac{1}{2} m\, \dot s^2 \;=\; \text{constant}\;.$$

Differentiating with respect to time, and then dividing by $\dot s$, this yields the differential equation

$$d\Phi/ds \;+\; m\,\ddot s \;=\; 0\;,$$

where $\ddot s = d^2 s/dt^2$. If we introduce the 2-dimensional *phase space*, consisting of all pairs (s, p) where s is the *position coordinate* and $p = m\,\dot s$ is the *momentum coordinate*, and if we express the total energy as a function

$$H(s, p) \;=\; \Phi(s) + \frac{1}{2m}\,p^2$$

of these coordinates, then this equation can be written in the "Hamiltonian normal form":

$$\dot s \;=\; \frac{\partial H}{\partial p}\;,\quad \dot p \;=\; -\frac{\partial H}{\partial s}\;. \qquad (11:1)$$

By definition, this total energy, expressed as a function $H(s, p)$ in these coordinates, is called the *Hamiltonian* function.

Example 11.3. Newtonian Gravitation. Now consider a system of k particles in euclidean 3-space, where the i-th particle has position $\vec{x}_i$ and mass m_i . As in §3C, it is convenient to combine the k 3-dimensional vectors $\vec{x}_i = (x_{i1}\,,\,x_{i2}\,,\,x_{i3})$ into a single $3k$-dimensional vector $\mathbf{x} = (\vec{x}_1\,,\,\ldots\,,\,\vec{x}_k)$. If these particles interact by a conservative force field, associated with the potential energy $\Phi(\mathbf{x}) = \Phi(\vec{x}_1, \ldots, \vec{x}_k)$ then the appropriate system of differential equations is

$$m_i \frac{d^2 x_{ij}}{dt^2} = -\frac{\partial \Phi}{\partial x_{ij}} \qquad (11:2)$$

for $1 \le i \le k$ and $1 \le j \le 3$. One important example is provided by the Newtonian k-body problem of §1A, with potential function

$$\Phi(\mathbf{x}) = -G \sum_{i<j} \frac{m_i m_j}{\| \vec{x}_i - \vec{x}_j \|} . \qquad (11:3)$$

If we introduce the *conjugate momentum coordinates* $\vec{p}_i = m_i \, d\vec{x}_i / dt$ and the total energy function

$$H(\mathbf{x}, \mathbf{p}) = \Phi(\mathbf{x}) + \sum_{i=1}^{k} \frac{1}{2m_i} \| \vec{p}_i \|^2 ,$$

then Equation (2) can be written in the Hamiltonian form

$$\dot{x}_{ij} = \frac{\partial H}{\partial p_{ij}} , \quad \dot{p}_{ij} = -\frac{\partial H}{\partial x_{ij}} . \qquad (11:4)$$

Example 11.4. Mechanical Systems. More generally, consider any mechanical system with finitely many degrees of freedom. We assume that the set of all *configurations* or *positions* of the system forms a smooth manifold M. The *phase space*, consisting of all pairs (*position, velocity*) or (*position, momentum*) can then be identified with the tangent bundle TM, or with the cotangent bundle T^*M. As an example, suppose that we are studying the motion of a rigid body, which might be a football, an airplane, or a tumbling asteroid. Then the configuration space M has dimension $n = 6$, since we need three coordinates to specify position and three further rotation coordinates. (In fact M can be identified with the six dimensional group consisting of all rigid motions of Euclidean 3-space.) Note that it is not possible to cover this M with just one smooth coordinate system. For this example, the phase space TM is twelve dimensional, since we need three linear velocity coordinates and three angular velocity coordinates in order to specify a tangent vector to M.

Near a point ξ of this manifold M, we can choose local coordinates $(x^1, \ldots, x^n)$, or briefly $\mathbf{x}$. (Here we follow conventions which are often used in Riemannian geometry, distinguishing between upper and lower indices.) A point $(\xi, \dot{\xi})$ in the tangent bundle over ξ in then specified by a pair of n-component vectors, which we may write as $(\mathbf{x}, \mathbf{v})$. Here $\mathbf{v}$ stands for a velocity $\mathbf{v} = \dot{\mathbf{x}} = d\mathbf{x}/dt$. Now assume that there are two kinds of energy associated with each point of phase space, namely a *potential energy* which depends only on the position ξ, and a *kinetic energy* which is a homogeneous quadratic function of the velocity $\dot{\xi}$ for each fixed position. Using the local coordinates $(\mathbf{x}, \mathbf{v})$, we will consider the potential energy as a function $\Phi = \Phi(\mathbf{x})$, and the kinetic energy as a function

$$\tfrac{1}{2} \sum_{i,j=1}^{n} g_{ij}(\mathbf{x}) \, v^i v^j .$$

Here $[g_{ij}]$ is to be a positive definite symmetric matrix which depends smoothly on the position $\mathbf{x}$. *In other words, we assume that the kinetic energy of a mechanical system is described by a Riemannian metric on its configuration space* M*, while potential energy is* *described by a smooth function* $\Phi : M \to$.

Using this kinetic energy metric $[g_{ij}(\mathbf{x})]$, it is often convenient to describe a velocity $\mathbf{v} = d\mathbf{x}/dt$ rather by the coordinates $\mathbf{p} = (p_1 , \ldots , p_n)$ where

$$p_i = \sum_{j=1}^{n} g_{ij} v^j .$$

A Riemannian geometer would describe the p_i as the *covariant components* of the velocity vector, or as the coefficients of the differential 1-form or *covector* $\sum_i p_i \, dx^i$ which is associated to this velocity vector, using the Riemannian metric to pass between tangent *vectors*, belonging to the n-dimensional vector space $T_\xi M$, and *covectors*, belonging to the dual vector space $T^*_\xi M = \mathrm{Hom}(T_\xi M , \;)$.

Physicists describe these coordinates p_i as *conjugate momentum coordinates* associated with the local coordinates $(x^1 , \ldots , x^n)$. As an example, for the Newtonian k-body problem of Example 11.3, the configuration space M has dimension $n = 3k$. If we use the *cgs* system of units, then we might measure the coordinates x^i in centimeters, the v^i in centimeters per second, and the g_{ij} in grams. The p_i would then be measured in gram centimeters per second, or briefly $\mathbf{gm{\cdot}cm/sec}$, as is appropriate for a momentum vector. On the other hand, to study a rigid body with center of gravity fixed at the origin, we might use three dimensionless angular coordinates x^i to describe a point in the configuration space M . In this case the v^i would be angular velocities, with units of $\mathbf{sec^{-1}}$, the g_{ij} would be moments of inertia with units of $\mathbf{gm \cdot cm^2}$, and the p_i would be angular momenta, with units of $\mathbf{gm{\cdot}cm^2/sec}$. However, for a more general problem, some of the x^i might be measured in centimeters and some in radians, so that it would be completely confusing to try to assign units to the various coordinates — the mathematical advantage of being allowed to work with a more or less arbitrary coordinate system would be lost if we insisted in keeping track of the physical meanings of individual coordinates. Note however that the energy functions $\Phi(\mathbf{x})$ and $\frac{1}{2} \sum g_{ij} v^i v^j$ do have an invariant physical meaning, and are measured in $\mathbf{gm{\cdot}cm^2/sec^2}$ in all cases.

Let $[g^{ij}(\mathbf{x})]$ be the inverse of the matrix $[g_{ij}(\mathbf{x})]$. Then the total energy H can be expressed as a function

$$H(\mathbf{x}, \mathbf{p}) = \tfrac{1}{2} \sum g^{ij} p_i p_j + \Phi(\mathbf{x})$$

of these coordinates x^i and p_i . The time evolution of such a mechanical system is then described by the Hamiltonian equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i} , \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} . \qquad (11:5)$$

One way of deriving these equations, based on a variational principle, will be described in §11C.

Example 11.5. Geodesic Flow. The *geodesic flow* on the tangent or cotangent bundle of a Riemannian manifold provides an important special case. Mathematically, this is just a "mechanical system" as described above, with a potential energy function $\Phi(\mathbf{x})$ which is identically zero. The corresponding solution curves, say $t \mapsto f_t(\mathbf{x} , \mathbf{v})$ in TM , project to geodesics $\mathbf{x} = \mathbf{x}(t)$ in the manifold M , with initial velocity vector $\dot{\mathbf{x}}(0)$ equal to $\mathbf{v}$. As one example, the physical problem of understanding the motion of a rigid body which is freely tumbling in empty space, reduces to the mathematical problem of describing the geodesic flow on the 3-dimensional manifold $M = \mathrm{SO}(3)$ of configurations, provided with the left invariant Riemannian metric which measures its angular kinetic energy.

§11C. Calculus of Variations. Let TM denote the tangent vector bundle for a smooth manifold M. Thus a point of TM can be thought of as a point of M together with a "velocity vector" at that point. If we have local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ in an open subset $U \subset M$, then we have corresponding local coordinates $(\mathbf{x}, \mathbf{v}) = (x^1, \ldots, x^n, v^1, \ldots, v^n)$ in the subset $TU \subset TM$, where we think of $\mathbf{v} = (v^1, \ldots, v^n)$ as the possible velocity vector $\mathbf{v} = d\mathbf{x}/dt$ of a smooth curve $t \mapsto \mathbf{x}(t)$ in M.

Now consider such a smooth curve $t \mapsto \mathbf{x}(t) \in M$ for $0 \le t \le 1$, and consider an integral along the curve of the form

$$\mathcal{A} \;=\; \int_0^1 L(\mathbf{x}, d\mathbf{x}/dt)\, dt\ , \qquad (11:6)$$

where $L(\mathbf{x}, \mathbf{v})$ is some prescribed smooth function $L : TM \to$ called the *Lagrangian*. This integral $\mathcal{A}$ is conventionally called the *action* associated with the path. A basic problem in the Calculus of Variations is to choose a path between specified endpoints in M which minimizes this action integral. More generally, we can consider a smooth variation in the path depending on an extra parameter u, but fixing the endpoints, and require that the derivative of the action with respect to u should be zero. The path, for such a value of u, is said to be *stationary* for this action integral.

As examples, if we want to study geodesics on a Riemannian manifold, the appropriate Lagrangian is just the "kinetic energy" $L(\mathbf{x}, \mathbf{v}) = \frac{1}{2}\sum g_{ij}(\mathbf{x}) v^i v^j$. For the more general mechanical system of Example 11.4 above, the appropriate Lagrangian is the *difference* of kinetic and potential energy

$$L(\mathbf{x}, \mathbf{v}) \;=\; \tfrac{1}{2}\sum g_{ij}(\mathbf{x}) v^i v^j \;-\; \Phi(\mathbf{x})\ .$$

However, for the following basic result we can use any smooth function of position and velocity.

Theorem 11.6 (Euler and Lagrange). *A smooth path $t \mapsto \mathbf{x}(t) \in M$, defined for $0 \le t \le 1$, is stationary for this integral* (11 : 6) *with respect to any smooth variation which keeps the endpoints $\mathbf{x}(0)$ and $\mathbf{x}(1)$ fixed if and only if it satisfies the system of differential equations*

$$\frac{\partial L}{\partial x^j} \;=\; \frac{d}{dt}\frac{\partial L}{\partial v^j} \qquad (11:7)$$

or more explicitly

$$\frac{\partial L}{\partial x^j} \;=\; \sum_k \left(\frac{\partial^2 L}{\partial v^j \partial x^k}\, v^k \;+\; \frac{\partial^2 L}{\partial v^j \partial v^k}\, \frac{dv^i}{dt} \right)\ , \qquad (11:7')$$

where v^k is identified with dx^k/dt.

Proof. Consider a one-parameter family of paths $\mathbf{f}_u : [0,1] \to M$, where u ranges over a neighborhood of 0 in . We assume that $\mathbf{f}_u(t)$ is smooth as a function of two variables, and that the end points $\mathbf{f}_u(0)$ and $\mathbf{f}_u(1)$ are independent of u. The action

$$\mathcal{A}(u) \;=\; \int_0^1 L\left(\mathbf{f}_u(t)\,,\;\frac{d\mathbf{f}_u(t)}{dt}\right)dt$$

is then a smooth function of u. We want to compute the derivative $d\mathcal{A}(u)/du$ at $u=0$. Define the *variation vector field* $\mathbf{w}(t)$ along the path $\mathbf{x}=\mathbf{f}_0(t)$ to be the first derivative

$$\mathbf{w}(t) \;=\; \frac{\partial \mathbf{f}_u(t)}{\partial u}\Big|_{u=0}\,,$$

or in other words $w^i(t)=\partial f^i_u(t)/\partial u|_{u=0}$. Now differentiating under the integral sign and then integrating by parts, since $\mathbf{w}(0)=\mathbf{w}(1)=0$, we see that

$$\begin{aligned}\frac{d\mathcal{A}(u)}{du}\Big|_{u=0} &= \int_0^1 \sum_{i=1}^n \left(\frac{\partial L}{\partial x^i}\,w^i(t) \;+\; \frac{\partial L}{\partial v^i}\,\frac{dw^i(t)}{dt}\right)dt\\ &= \int_0^1 \sum_{i=1}^n w^i(t)\left(\frac{\partial L}{\partial x^i} \;-\; \frac{d}{dt}\,\frac{\partial L}{\partial v^i}\right)dt\,.\end{aligned}$$

Evidently this vanishes for every choice of variation vector field $\mathbf{w}$ if and only if equation (11 : 7) is satisfied. □

Hypothesis. Assume now that the matrix of second derivatives

$$g_{jk}(\mathbf{x}\,,\,\mathbf{v}) \;=\; \frac{\partial^2 L}{\partial v^j\partial v^k}$$

is non-singular everywhere; and let $g^{ij}(\mathbf{x}\,,\,\mathbf{v})$ be the inverse matrix. Then we can multiply equation (11 : 7′) on the left with g^{ij} and sum over j, to form the following.

Lemma 11.7. *With g^{ij} as above, the Euler-Lagrange equation* (11 : 7) *is completely equivalent to the system of equations*

$$\frac{dx^i}{dt} \;=\; v^i\,, \qquad \frac{dv^i}{dt} \;=\; \sum_j g^{ij}\left(\frac{\partial L}{\partial x^j} \;-\; \sum_k \frac{\partial^2 L}{\partial v^j\partial x^k}\,v^k\right). \tag{11 : 8}$$

In this last form, we see that the solutions to this differential equation define, locally at least, a smooth flow on the tangent bundle TM. By definition, this is the *Euler-Lagrange flow*, associated with the Lagrangian function $L(\mathbf{x},\mathbf{v})$ on TM. Alternatively, we could also eliminate the v^j, and write (11 : 8) as a second order differential equation of the form

$$d^2\mathbf{x}/dt^2 \;=\; \mathbf{a}(\mathbf{x}\,,\,d\mathbf{x}/dt)\,.$$

Another consequence of 11.6 is a law of conservation of energy. To see this, let us compute the derivative of the function $L(\mathbf{x},\mathbf{v})$ as we follow a solution curve $t\mapsto(\mathbf{x}(t)\,,\,\dot{\mathbf{x}}(t))$ for the Euler-Lagrange flow. We have

$$\frac{dL}{dt} \;=\; \sum_i \left(\frac{\partial L}{\partial x_i}\,v^i \;+\; \frac{\partial L}{\partial v^i}\,\frac{dv^i}{dt}\right).$$

Making use of (11 : 7), we can write this as

$$\frac{dL}{dt} \;=\; \frac{d}{dt}\sum_i v^i\,\frac{\partial L}{\partial v^i}\;.$$

In other words, we have proved the following.

Lemma 11.8 (Conservation of Energy). *The quantity*

$$H \;=\; \sum_i v^i\,\frac{\partial L}{\partial v^i}(\mathbf{x},\mathbf{v}) \;-\; L(\mathbf{x},\mathbf{v})$$

remains constant as we follow the Euler-Lagrange flow.

By definition, this function H is called the ***Hamiltonian***, or the ***total energy function*** associated with the Lagrangian L on the tangent bundle TM . The numbers

$$p_i \;=\; \partial L/\partial v_i \tag{11 : 9}$$

are called the ***conjugate momentum coordinates*** associated with the Lagrangian L . In fact the hypothesis that the matrix $[g_{ij}] = [\partial^2 L/\partial v^i \partial v^j]$ is non-singular guarantees that the transformation

$$(x^1\,,\,\ldots\,,\,x^n\,,\,v^1\,,\,\ldots\,,\,v^n) \;\mapsto\; (x^1\,,\,\ldots\,,\,x^n\,,\,p_1\,,\,\ldots\,,\,p_n)$$

is locally a diffeomorphism. For each fixed $\mathbf{x}$, we can think of $(p_1\,,\,\ldots\,,\,p_n)$ as the coordinates of a point

$$dL(\mathbf{x}\,,\,\mathbf{v})|_{T_{\mathbf{x}}M} \;=\; \sum_i p_i\,dx^i$$

in the cotangent space $T^*_{\mathbf{x}}M$. Thus the transformation

$$\mathbf{v} \;\mapsto\; dL(\mathbf{x}\,,\,\mathbf{v})|_{T_{\mathbf{x}}M} \;=\; \sum_i p_i\,dx^i$$

maps the tangent space $T_{\mathbf{x}}M$ into the cotangent space $T^*_{\mathbf{x}}M$ by a local diffeomorphism.

We will think of $H = H(\mathbf{x}\,,\,\mathbf{p})$ as a locally defined function on the cotangent space T^*M ; but will continue to think of $L(\mathbf{x}\,,\,\mathbf{v})$ as a function on TM . In particular, when we write $\partial H/\partial x^i$ it is understood that $\mathbf{p}$ is to be kept fixed; but for $\partial L/\partial x^i$ it is $\mathbf{v}$ that is fixed. The total differential of H can be computed as:

$$\begin{aligned} dH \;&=\; d\Big(\sum v^i\,p_i\Big) \;-\; dL \\ &=\; \sum_i \Big(v^i\,dp_i + p_i\,dv^i - \frac{\partial L}{\partial x^i}\,dx^i - \frac{\partial L}{\partial v^i}\,dv^i\Big)\;, \end{aligned}$$

where the dv^i terms cancel by (11 : 9). Comparing this with the equation

$$dH \;=\; \sum\Big(\frac{\partial H}{\partial p_i}\,dp_i \;+\; \frac{\partial H}{\partial x^i}\,dx^i\Big)\;,$$

we see that

$$v^i \;=\; \frac{\partial H}{\partial p_i}\;, \qquad \frac{\partial L}{\partial x^i} \;=\; -\frac{\partial H}{\partial x^i}\;.$$

Since $v^i = dx^i/dt$, and since $dp_i/dt = \partial L/\partial x^i$ by (11 : 7), we have reduced our equations to the Hamiltonian form (11 : 5):

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}\,.$$

§**11D. Hamiltonian Systems.** Consider then any system of first order differential equations in $2n$ variables which can be written in this form (11 : 5). These generate, locally at least, a flow of the form

$$(\mathbf{x}(t)\,,\,\mathbf{p}(t)) = f_t(\mathbf{x}(0)\,,\,\mathbf{p}(0))$$

on the $2n$-dimensional phase space. One immediate consequence of (11 : 5) is the conservation law for energy

$$\frac{dH(\mathbf{x},\mathbf{p})}{dt} = \sum_1^n \Big(\frac{\partial H}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt}\Big) = 0\,.$$

Thus the flow f_t on phase space takes each constant energy hypersurface $H =$ constant into itself.

Another immediate property is the conservation law for $2n$-dimensional volume. In fact, the divergence of the generating vector field

$$(x^1\,,\,\ldots\,,\,x^n\,,\,p_1\,,\,\ldots\,,\,p_n) \mapsto \Big(\frac{\partial H}{\partial p_1}\,,\,\ldots\,,\,\frac{\partial H}{\partial p_n}\,,\,-\frac{\partial H}{\partial x^1}\,,\,\ldots\,,\,-\frac{\partial H}{\partial x^n}\Big)$$

is equal to

$$\sum_1^n \Big(\frac{\partial}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial x^i}\Big)\,,$$

which is identically zero. In other words, the $2n$-dimensional volume element $d^n\mathbf{x}\,d^n\mathbf{p}$ is preserved by the flow f_t.

Symplectic Forms. In the case of the cotangent manifold T^*M of an arbitrary n-dimensional smooth manifold M, a more invariant description of this preferred $2n$-dimensional volume element can be given as follows. A point of T^*M consists of a point $\xi \in M$ together with a 1-form or *covector* η at the point ξ. Given local coordinates $x^1\,,\,\ldots\,,\,x^n$ near the point ξ, we have a corresponding basis $dx^1\,,\,\ldots\,,\,dx^n$ for covectors at points near ξ. Using the projection $\pi : T^*M \to M$, we have corresponding covectors $\pi^*(dx^i) = d(x^i \circ \pi)$ on the smooth manifold T^*M. Now at any point (ξ,η) we can write $\eta = p_1\,dx^1 + \cdots + p_n\,dx^n$, and form a corresponding covector

$$\alpha(\xi,\eta) = \pi^*\eta = p_1\,\pi^*(dx^1) + \cdots + p_n\,\pi^*(dx^n)$$

at this point (ξ,η) on the smooth manifold T^*M. By definition, α is the *tautological 1-form* on T^*M. If we think of $(x^1\,,\,\ldots\,,\,x^n\,,\,p_1\,,\,\ldots\,,\,p_n)$ as local coordinates on the manifold T^*M, then we can write this tautological 1-form simply as $\alpha = \sum p_i\,dx^i$. By definition, the exterior derivative

$$d\alpha = \sum_{i=1}^n dp_i \wedge dx^i$$

is then called the canonical *symplectic form* ω on the manifold T^*M. More generally:

Definition. Given any $2n$-dimensional smooth manifold with local coordinates $u^1, \ldots, u^{2n}$, any differential 2-form can be expressed uniquely as

$$\omega = \frac{1}{2}\sum_{r,s=1}^{2n} w_{rs}(u^1, \ldots, u^{2n})\, du^r \wedge du^s \qquad (11:10)$$

with $w_{rs} = -w_{sr}$. Such a form is *symplectic* if the $2n \times 2n$ skew matrix $[w_{rs}]$ is everywhere non-singular, and if the exterior derivative $d\omega$ is identically zero. Clearly the canonical symplectic form on T^*M satisfies these conditions. In fact the corresponding skew matrix has the form

$$[w_{rs}] = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad (11:11)$$

with determinant $+1$, and with $d\omega = d(d\alpha) = 0$ since $d \circ d = 0$.

By the *symplectic gradient* of a smooth function $H(\mathbf{u})$ on a $2n$-manifold with symplectic form ω is meant the vector field $\mathbf{v} = \mathbf{grad}_\omega(H)$ whose components $(v^1, \ldots, v^{2n})$ satisfy the identity

$$\sum_s w_{rs} v^s = \frac{\partial H}{\partial u^r} .$$

The solution curves for the corresponding system of differential equations

$$\sum_s w_{rs} \frac{du^s}{dt} = \frac{\partial H}{\partial u^r} \qquad (11:12)$$

generate the *symplectic flow* f_t associated with this Hamiltonian function H. In particular, if we identify $(u^1, \ldots, u^{2n})$ with the local coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ on T^*M, and use the symplectic form $\omega = \sum dp_i \wedge dx^i = \frac{1}{2}\sum w_{rs} du^r \wedge du^s$ with $[w_{rs}]$ as in (11 : 11), then the differential equation (11 : 12) clearly reduces to the Hamiltonian equation (11 : 5).

One advantage of this more general formulation is that we are free to use more general coordinate systems. As an example, about any point where the symplectic gradient $\mathbf{grad}_\omega(H)$ is non-zero, we can choose new local coordinates $(u^1, \ldots, u^{2n})$ so that this symplectic gradient vector field is given simply by $\mathbf{grad}_\omega(H) = (1, 0 \ldots, 0)$, or in other words so that the associated flow is given by

$$f_t(u^1, \ldots, u^{2n}) = (u^1 + t, u^2, \ldots, u^{2n}) .$$

Equation (11 : 12) then takes the form

$$w_{r1} = \partial H / \partial u^r .$$

For example since $[w_{rs}]$ is skew, it follows that $\partial H / u^1 = w_{11} = 0$. Now the exterior derivative of H is given by $dH = \sum w_{r1}\, du^r$, and it follows that the symplectic form ω can be written as

$$\omega = dH \wedge du^1 + \frac{1}{2}\sum_{r,s=2}^{2n} w_{rs}\, du^r \wedge du^s .$$

The equation $d\omega = 0$ now clearly implies that

$$\frac{\partial w_{rs}}{\partial u^1} = 0 .$$

Thus the symplectic form $\omega = \frac{1}{2}\sum w_{rs} du^r du^s$ *is invariant under the Hamiltonian flow* f_t . Although we have proved this statement only for one very special local coordinate system, it follows that it is true in any smooth coordinate system.

Just as a Riemannian metric $ds^2 = \sum g_{ij}\, dx^i\, dx^j$ on an n-dimensional manifold gives rise to an n-dimensional volume form $\sqrt{\det[g_{ij}]}\; dx^1 \cdots dx^n$, so the symplectic form $(11:10)$ gives rise to a preferred $2n$-dimensional volume form

$$\Lambda \;=\; \sqrt{\det[w_{ij}]}\; du^1 \cdots du^{2n}$$

on phase space. By definition, this is the *Liouville volume form*.

An alternative formulation, which gives not only a preferred volume form but also a preferred orientation, is obtained by forming the n-fold exterior product of the 2-form ω with itself. We will write

$$\omega^{\wedge n}/n! \;=\; (\omega\wedge\cdots\wedge\omega)/n! \;=\; \mathrm{Pf}\,[w_{rs}]\; du^1\wedge\cdots\wedge du^{2n} \;.$$

Here $\mathrm{Pf}\,[w_{ij}]$ is a certain real valued polynomial function of degree n in the skew symmetric variables w_{ij} called the *Pfaffian*. (See for example [Milnor and Stasheff, p. 309].) The determinant of any skew matrix is equal to the square of its Pfaffian. The volume form Λ can be described as the "absolute value" $|\,\mathrm{Pf}\,[w_{rs}]\,|\; du^1 \cdots du^{2n}$ of this exterior $2n$-form $\omega^{\wedge n}/n!$.

In the special case of canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$, the matrix $[w_{rs}]$ takes the special form $(11:11)$ with determinant $+1$. Hence the Liouville volume form becomes

$$\Lambda \;=\; dx^1\cdots dx^n\, dp_1\cdots dp_n \;,$$

as in §3C and §11B.

As in §3C, this invariant $2n$-dimensional volume element Λ on phase space gives rise to an invariant $(2n-1)$-dimensional volume element Λ/dH on each constant energy hypersurface. Of course, to get much use out of this invariant volume or measure, we need to know that the total measure of the constant energy hypersurface is finite. Compare Problem 11-d.

§11E. Three Problems.

Problem 11-a. Volume Preserving Flows. Show that the flow generated by a differential equation $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x})$ preserves the volume form $w(\mathbf{x})\, dx_1\cdots dx_n$ if and only if

$$w(f_t(\mathbf{x}))\, Jf_t(\mathbf{x}) \;=\; w(\mathbf{x})$$

everywhere, or if and only if

$$\mathbf{v}\cdot\nabla w \;+\; w\,\nabla\!\cdot\mathbf{v} \;=\; 0$$

everywhere.

Problem 11-b. The Action Principle on a Riemannian manifold. With position and momentum coordinates $(\mathbf{x},\mathbf{p})$ as above, suppose that we are given a metric $\sum g_{ij}\, dx^i\, dx^j$ and a potential function $\Phi(\mathbf{x})$. Define the *action* associated with a smooth path $\mathbf{x} = \mathbf{x}(t)$ between two points to be the integral $\mathcal{A} \;=\; \int L(\mathbf{x}\,,\,\dot{\mathbf{x}})\, dt$ along this path, where

$$L(\mathbf{x}\,,\,\dot{\mathbf{x}}) \;=\; \Big(\tfrac{1}{2}\sum g_{ij}\,\dot{x}^i\,\dot{x}^j \;-\; \Phi(\mathbf{x})\Big)$$

(Thus the integrand is the *difference* between kinetic and potential energy.) Proceeding as in §11C, show that the Euler-lagrange equation for a stationary point of this action integral takes the form

$$\sum_i \frac{d}{dt}\left(g_{ih}\,\dot{x}^i\right) \;=\; \tfrac{1}{2}\sum_{ij}\frac{\partial g_{ij}}{x^h}\,\dot{x}^i\,\dot{x}^j \;-\; \frac{\partial\Phi}{x^h}\;. \qquad (11:13)$$

Remark. In terms of covariant differentiation on a Riemannian manifold, this equation (11 : 13) says that the covariant derivative with respect to t of the velocity vector $\dot{\mathbf{x}} = d\mathbf{x}/dt$ is equal to the negative of the Riemannian gradient $\mathbf{grad}(\Phi)$. In particular, in the Euclidean case where $[g_{ij}]$ is the identity matrix, this equation reduces to the simple form $\ddot{x}^h = -\partial\Phi/\partial x^h$. (Compare (11 : 2).) About any specified point of a Riemannian manifold one can always choose a very special "normal coordinate system", which has the properties that $[g_{ij}]$ is the identity matrix and $\partial g_{ij}/\partial x^h = 0$, at the specified point only. In such a coordinate system, equation (11 : 13) reduces to the simple form $\ddot{x}^h = -\partial\Phi/\partial x^h$, at the given point.

Now set $p_h = \sum_i g_{hi}\,\dot{x}^i$. Show that $\dot{x}^i = \sum_j g^{ij}\,p_j$ and $\sum g_{ij}\dot{x}^i\dot{x}^j = \sum g^{ij}p_ip_j$, but that

$$\sum_{ij}\frac{\partial g_{ij}}{\partial x^h}\,\dot{x}^i\,\dot{x}^j \;=\; -\sum_{ij}\frac{\partial g^{ij}}{\partial x^h}\,p_i\,p_j$$

since $[g_{ij}]$ and $[g^{ij}]$ are inverse matrices. Setting

$$H(\mathbf{x},\mathbf{p}) \;=\; \Phi(\mathbf{x}) + \tfrac{1}{2}\sum g^{ij}(\mathbf{x})\,p_i\,p_j\;,$$

conclude that equation (11 : 13) can be written in the Hamiltonian form

$$\dot{x}^i \;=\; \frac{\partial H}{\partial p_i}\,,\quad \dot{p}_i \;=\; -\frac{\partial H}{\partial x^i}\;.$$

(A more conceptual description of this derivation would involve the Legendre transform. Compare [Arnold] .)

Problem 11-c. A Roller-Coaster with Friction. Given a potential function $\Phi = \Phi(s)$ and a friction coefficient $c > 0$, consider the differential equation

$$m\,\ddot{s} + c\,\dot{s} + d\Phi/ds \;=\; 0\;.$$

For any non-constant solution, show that the total energy $H = \Phi + m\dot{s}^2/2$ decreases as the time t increases. Furthermore, for any bounded region U in phase space, show that the area of the image $f_t(U)$ decreases as t increases.

Problem 11-d. Finite Invariant Volume. Let M be a Riemannian manifold, for example an open subset of Euclidean space, and suppose that the potential function $\Phi : M \to$. is proper and bounded from below, so that each $\{\xi \in M\;;\;\Phi(\xi) \le \text{constant}\}$ is compact. (This condition is always satisfied in the case of a compact Riemannian manifold.) Show then that the constant energy hypersurface $H = \text{constant}$ is compact, and hence has finite volume with respect to the invariant measure Λ/dH.

Remark: This compactness condition is definitely not satisfied for the potential (11 : 3) associated with the Newtonian n-body problem. Thus, even though the Liouville volume element is preserved by the Newtonian flow, tools such as the Poincaré Recurrence Theorem are not available.