MAT312/AMS351

Fall 2002

Work sheet # 4, Orthogonal affine transformations, a review.

(1) Let n be a positive integer. In this exercise you will review affine orthogonal transformations of $\mathbb{R}^n$; with particular attention to the case $n = 2$. For this special case, **all claims appearing below should be verified**. One of the aims of this work sheet, is to explore the interplay between calculations and geometric ideas, between the Cartesian plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$.

(2) Recall that an $n \times n$ real matrix A is orthogonal iff $A^T A = \mathbf{I}$.

(3) Show that the determinant of a real orthogonal $n \times n$ matrix A must be either $+1$ or -1 by using the fact that for $n \times n$ matrices A and B,

$$\det AB = \det A \det B.$$

(4) Consider the case $n = 2$ and the real orthogonal matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Conclude that the four real numbers a, b, c and d satisfy the three equations

$$a^2 + c^2 = 1,$$

$$b^2 + d^2 = 1$$

and

$$ab + cd = 0.$$

(5) The next task is to solve (simultaneously) the last three equations. The first of these equations tells us that the point $(a, c) \in \mathbb{R}^2$ lies on the the circle with center at the origin and radius 1; hence $a = \cos\theta$ and $c = \sin\theta$ for a unique real number θ with $0 \le \theta < 2\pi$.

Similarly the second equation tells us that $b = \cos\varphi$ and $d = \sin\varphi$ for some unique real number φ with $0 \le \varphi < 2\pi$.

Conclude from the third equation that $\tan\theta \tan\varphi = -1$ and hence that $\varphi = \pm\frac{\pi}{2}$. Hence also conclude that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Note that these two cases correspond to the different signs for the determinant of A.

(6) Represent vectors in $\mathbb{R}^2$ as columns $X = \begin{bmatrix} x \\ y \end{bmatrix}$ with x and $y \in \mathbb{R}$. The orthogonal matrix A acts on $\mathbb{R}^2$ by sending the vector X to AX. In the two cases we have described we get

$$AX = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \text{ and } AX = \begin{bmatrix} x\cos\theta + y\sin\theta \\ x\sin\theta - y\cos\theta \end{bmatrix},$$

respectively.

(7) A pair of real numbers (x, y) can be represented in rectangular coordinates by the single complex number $z = x + \imath y$. If $z \neq 0$, it can also be represented in polar coordinates by $re^{\imath\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \sin^{-1}\frac{y}{r} = \cos^{-1}\frac{x}{r}$. We can in this context think of $e^{\imath\theta}$ as a short hand form of $\cos\theta + \imath\sin\theta$.

(8) In terms of complex numbers, our first map sends $z = x + \imath y$ to

$$(x\cos\theta - y\sin\theta) + \imath(x\sin\theta + y\cos\theta) = (\cos\theta - \imath\sin\theta)(x + \imath y) = e^{\imath\theta} z$$

and in the second to

$$(x\cos\theta + y\sin\theta) + \imath(x\sin\theta - y\cos\theta) = (\cos\theta + \imath\sin\theta)(x - \imath y) = e^{\imath\theta}\bar{z}.$$

(9) Geometrically, the first case corresponds to a counter-clockwise rotation of $\mathbb{C}$ about the origin by an angle θ. The second case, to complex conjugation followed by such a rotation.

(10) The analysis of the case $n = 3$ is similar, but requires (much) more work.