MAT312/AMS351

Fall 2,002

Worksheet # 1, Induction

(1) (Sums of integers.) Recall that we proved by induction that for all positive integers n,

$$1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

(2) (Sums of squares of integers.) Similarly (Exercise 1.1.1) prove by induction that for all positive integers n,

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(3) (Sums of cubes of integers – Exercise 1.1.2.) The aim of this exercise is to formulate and then prove a similar result for sums of cubes.

(4) Notice that the sum of the first n positive integers is a quadratic polynomial in n. The sum of the squares of the first n positive integers is a cubic polynomial in n. It is hence reasonable to expect that the sum of the cubes of the first n positive integers is a fourth degree polynomial in n; that is,

$$1^3 + 2^3 + ... + n^3 = an^4 + bn^3 + cn^2 + dn + e, \tag{1}$$

for some constants a, b, c, d and e that do not depend on the variable n. What are the corresponding constants for sums of integers and sums of squares of integers? Can you make some "educated guesses" about what the 5 constants should be?

(5) If we are not to rely on guesswork, then one of our tasks is to determine the 5 constants. If (1) is to hold for all integers n, it certainly should hold for for $n = 1$, 2, 3, 4 and 5, leading us to five equations

$$1 = a + b + c + d + e,$$

$$9 = 16a + 8b + 4c + 2d + e,$$

$$36 = 81a + 27b + 9c + 3d + e,$$

$$100 = 256a + 64b + 16c + 4d + e$$

and

$$225 = 625a + 125b + 25c + 5d + e.$$

(6) If our intuition is right, the above system of linear equations should have a unique solution. Recall from your linear algebra course that a necessary and sufficient condition is that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \\ 625 & 125 & 25 & 5 & 1 \end{bmatrix}$$

be nonsingular. One could certainly compute its determinant by hand and show that it is nonzero. Do it using MAPLE. You should get that the determinant equals 288.

(7) Now use MAPLE to solve the equation. You should have obtained a polynomial p with rational coefficients. You are trying to prove by induction that

$$1^3 + 2^3 + ... + n^3 = p(n)$$

for all positive integers n.

Let's make the polynomial look prettier. First write $p(n)$ as $\frac{P(n)}{N}$ where P is a polynomial with integer coefficients and N is a positive integer, chosen as small as possible. Now factor the polynomial P. The formula you now need to establish for sums of cubes should appear similar to the ones for sums of integers and sums of squares. Prove by induction that the formula you obtained is true finishing this exercise.

(8) To get used to work with symbolic manipulation programs you may want, after attempting by yourself the steps outlined above, to consult the attached MAPLE worksheet that outlines the commands needed to perform the calculations. There is very nontrivial initial investment of time in learning to use a program of this kind. But, it pays off in the long run.

(9) Were your "educated guesses" about what the values of the 5 constants close to the mark?

(10) Note that MAPLE has a command that evaluates $p(n)$ directly.

(11) Can you formulate and prove a similar result for sums of fourth powers of integers?