MAT 211 Summer 2015 Homework 4
Solution Guide

Due in class: June 22^{nd}.

Problem 1. Which of the following subsets of $\mathbf{P_2}$ are subspaces of $\mathbf{P_2}$? If it is not a subspace, explain your reason; if it is a subspace, please find a basis for it.

a) $\{p(t) : p(0) = 2\}$;

Answer. No, because the zero polynomial is not in the set. □

b) $\{p(t) : p(2) = 0\}$.

Answer. Yes. (You check the properties yourself.) Let $p(t) = a + bt + ct^2$, then $p(t) = a + 2b + 4c = 0$. Hence, $a = -(2b + 4c)$. Then

$$p(t) = -(2b + 4c) + bt + ct^2 = b(t - 2) + c(t^2 - 4).$$

The subspace is spanned by $(t - 2,\ t^2 - 4)$, and they are clearly linearly independent, hence $(t - 2,\ t^2 - 4)$ form a basis of the subspace. □

Problem 2. Which of the following subsets of $\mathbf{Mat_{3\times3}}(\mathbb{R})$ are subspaces of $\mathbf{Mat_{3\times3}}(\mathbb{R})$? If it is not a subspace, explain your reason; if it is a subspace, please find a basis for it.

a) The set of all diagonal 3×3 matrices;

b) The set of all 3×3 matrices whose entries are all greater than or equal to 0.

Answer. No, because it is not closed under scalar multiplication (take a matrix in the set then time it by -1). □

Problem 3. Find a basis for each of the following spaces and determine its dimension.

a) The space of all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ such that $a + d = 0$;

Answer. This is Question 3 in the midterm 2. Check the solution for midterm 2. □

b) The space of all 2×2 matrices A such that commute with $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

c) The space of all 3×3 matrices A such that commute with

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and A commutes with B. Then $AB = BA$, i.e.,

$$\begin{bmatrix} 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we have

$$\begin{cases} ca_{11} = a_{22} = a_{33} \\ a_{12} = a_{23} \\ a_{21} = a_{31} = a_{32} = 0 \end{cases}.$$

Hence,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{11} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The basis is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

and the dimension is 3. □

Problem 4. Find out which of the following transformations are linear. For those that are linear, determine whether they are isomorphisms. Explain your reason.

a) $T(M) = M + I_2$ from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$, where I_2 is the 2×2 identity matrix;

Answer. No, because

$$T(M_1 + M_2) = (M_1 + M_2) + I_2$$

while

$$\begin{aligned} T(M_1) + T(M_2) =& (M_1 + I_2) + (M_2 + I_2) \\ =& (M_1 + M_2) + 2I_2. \end{aligned}$$

They are not equal. □

b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbb{R}$;

Answer. No, because

$$\begin{aligned} T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) \\ &= (ka)(kd) - (kb)(kc) \\ &= k^2(ad - bc) \\ &= k^2 T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \end{aligned}$$

For any $k \neq 0, 1$, the linearity does not hold. □

c) $T(f(t)) = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$ from $\mathbf{P_3}$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$, where $\mathbf{P}_3$ is the space of all polynomials with degree smaller than or equal to 3.

Proof. Yes, it is a linear transformation. (Check the authenticity!) T is an isomorphism because

A) $\dim(\mathbf{P_3}) = 4 = \dim \mathbf{Mat_{2\times 2}}(\mathbb{R})$;

B) $ker(T) = \{0\}$. If $T(f(t))$ is the zero matrix in $\mathbf{Mat_{2\times 2}}(\mathbb{R})$, then $f(0) = f(1) = f(2) = f(3) = 0$. Since ant nonzero polynomial in $\mathbf{P}_3$ has at most 3 zeros, f can only be the zero function.

□

Problem 5. For which constants k is the linear transformation

$$T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$$

an isomorphism from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$?

Answer. Given the standard basis $\mathfrak{B}$ on $\mathbf{Mat_{2\times 2}}(\mathbb{R})$. Suppose

$$[M]_{\mathfrak{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

then $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{aligned} T(M) &= \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} 2a + 3c & 2b + 3d \\ 4c & 4d \end{bmatrix} - \begin{bmatrix} 3a & kb \\ 0 & kd \end{bmatrix} \\ &= \begin{bmatrix} -a + 3c & (2 - k)b + 3d \\ 4c & (4 - k)d \end{bmatrix}. \end{aligned}$$

□

Hence,

$$[T(M)]_{\mathfrak{B}} = \begin{bmatrix} -a+3c \\ (2-k)b+3d \\ 4c \\ (4-k)d \end{bmatrix}$$
$$= a\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2-k \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 3 \\ 0 \\ 4-k \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 2-k & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4-k \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

The matrix of T under the standard basis $\mathfrak{B}$ is

$$A = \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 2-k & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4-k \end{bmatrix}.$$

Then T is an isomorphism if and only if the matrix A is invertible, if and only if the rank of the matrix is 4. Hence, $k \neq 2$ and $k \neq 4$.

Problem 6. Find the image and the kernel of the following linear transformations. Write your answer as a span of "vectors". What is the rank and nullity of each linear transformation? Use your result to test the Rank-Nullity Theorem.

a) $T(M) = M\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$;

b) $T(f(t)) = f(7)$ from $\mathbf{P_2}$ to $\mathbb{R}$;

Answer. Given $\mathbf{P_2}$ the basis $\mathfrak{B} = (1, x, x^2)$, then for any polynomial $a + bx + cx^2$, then

$$T(f(X)) = f(7) = a + 7b + 49c = \begin{bmatrix} 1 & 7 & 49 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Hence, the matrix of T under the basis is

$$A = \begin{bmatrix} 1 & 7 & 49 \end{bmatrix}.$$

The image is $span\{1\}$, which is indeed all the real numbers. (Note that T maps $\mathbf{P_2}$ to $\mathbb{R}$.) The kernel of A is

$$ker(A) = span\left(\begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -49 \\ 0 \\ 1 \end{bmatrix}\right),$$

namely,

$$ker(T) = span\left(-7 + x, \ -49 + x^2\right).$$

It is clear that $\dim(im(T)) = 1$, $\dim(ker(T)) = 2$ and

$$\dim(im(T)) + \dim(ker(T)) = 3 = \dim(\mathbf{P_2}).$$

□

c) $T(M) = M\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}M$ from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$.

Problem 7. Determine whether the following "vectors" in the given linear space are linearly independent or not. Explain your answer and show all your work.

a) The polynomials $f(t) = 7+3t+t^2$, $g(t) = 9+9t+4t^2$ and $h(t) = 3+2t+t^2$ in the linear space $\mathbf{P_2}$;

Answer. Given $\mathbf{P_2}$ the standard basis $\mathfrak{B} = (1, x, x^2)$, then

$$[f]_{\mathfrak{B}} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}, \ [g]_{\mathfrak{B}} = \begin{bmatrix} 9 \\ 9 \\ 4 \end{bmatrix}, \ [h]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Since

$$rref\left(\begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix}\right) = I_3,$$

The coordinate vectors are linearly independent. Hence, f, g and h are linearly independent in $\mathbf{P_2}$. □

b) The matrices

$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\\5 & 7\end{bmatrix}, \begin{bmatrix}1 & 4\\6 & 8\end{bmatrix}$$

in the linear space $\mathbf{Mat_{2\times 2}}(\mathbb{R})$.

(Hint: you can first give a basis to the linear space, then you just need to show whether the coordinate vectors are linearly independent or not.)

Answer. The same idea as a). The matrices turn out to be linearly dependent. □

Problem 8. Let T be a linear transformation from $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ to $\mathbf{Mat_{2\times 2}}(\mathbb{R})$ defined by

$$T(M) = M\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}.$$

Find the matrix of T with respect to the following basis of $\mathbf{Mat_{2\times 2}}(\mathbb{R})$

$$\mathfrak{B} = \left(\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}\right).$$

Answer. (Note that the basis here is not the standard basis.) given any $M \in \mathbf{Mat_{2\times 2}}(\mathbb{R})$, suppose $[M]_{\mathfrak{B}} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$, then

$$M = a\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix} + c\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix} + d\begin{bmatrix}0 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}a+c & b+d\\2c-a & 2d-b\end{bmatrix}.$$

Then

$$\begin{aligned}
T(M) &= \begin{bmatrix}a+c & b+d\\2c-a & 2d-b\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\\
&= \begin{bmatrix}(a+c)+3(b+d) & 2(a+c)+6(b+d)\\(2c-a)+3(2d-b) & 2(2c-a)+6(2d-b)\end{bmatrix}\\
&= \begin{bmatrix}(a+3b)+(c+3d) & (2a+6b)+(2c+6d)\\2(c+3d)-(a+3b) & 2(2c+6d)-(2a+6b)\end{bmatrix}\\
&= (a+3b)\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix} + (c+3d)\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix} + (2a+6b)\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix} + (2c+6d)\begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}.
\end{aligned}$$

Hence,

$$[T(M)]_{\mathfrak{B}} = \begin{bmatrix} a+3b \\ 2a+6b \\ c+3d \\ 2c+3d \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix} [M]_{\mathfrak{B}}.$$

Therefore, the matrix of T with respect to the given basis $\mathfrak{B}$ is

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}.$$

□

Problem 9. Let V be the space of all upper triangular 2×2 matrices. Consider the linear transformation

$$T \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aI_2 + bP + cP^2$$

from V to V, where I_2 is the 2×2 identity matrix, $P = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and P^2 represents the product of P with itself.

a) Find the matrix A of T with respect to the basis

$$\mathfrak{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

Answer. First, we have

$$[I_2]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ [P]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and $P^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$ hence

$$[P^2]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 8 \\ 9 \end{bmatrix}.$$

Let $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $[M]_{\mathfrak{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$\begin{aligned} [T(M)]_{\mathfrak{B}} &= a[I_2]_{\mathfrak{B}} + b[P]_{\mathfrak{B}} + c[P^2]_{\mathfrak{B}} \\ &= a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c \begin{bmatrix} 1 \\ 8 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix} [M]_{\mathfrak{B}}. \end{aligned}$$

Hence, the matrix of T under the basis $\mathfrak{B}$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix}.$$

□

b) Find bases of the image and kernel of T, and thus determine the rank of T.

Proof.

$$im(A) = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right),$$

hence,

$$im(T) = span\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \right).$$

$Rank(T) = dim(im(T)) = 2.$

$$ker(A) = span\left(\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \right),$$

hence,

$$ker(T) = span\left(\begin{bmatrix} 3 & -4 \\ 0 & 1 \end{bmatrix} \right).$$

□

Problem 10. In the plane V defined by the equation $2x_1 + x_2 - 2x_3 = 0$, consider the bases

$$\mathfrak{U} = (\vec{a}_1, \vec{a}_2) = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right)$$

and

$$\mathfrak{B} = (\vec{b}_1, \vec{b}_2) = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right).$$

a) Find the change of basis matrix S from $\mathfrak{B}$ to $\mathfrak{U}$;

Answer. Since $\vec{b}_1 = \vec{a}_1$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2$, we have

$$[\vec{b}_1]_{\mathfrak{U}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ [\vec{b}_2]_{\mathfrak{U}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence, the change of basis matrix S from $\mathfrak{B}$ to $\mathfrak{U}$ is

$$S = \begin{bmatrix} [\vec{b}_1]_{\mathfrak{U}} & [\vec{b}_2]_{\mathfrak{U}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

□

b) Find the change of basis matrix from $\mathfrak{U}$ to $\mathfrak{B}$;

Answer. The matrix is

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

□

c) Write an equation relating the matrices $[\vec{a}_1 \quad \vec{a}_2]$, $[\vec{b}_1 \quad \vec{b}_2]$, and $S = S_{\mathfrak{B}\to\mathfrak{U}}$.

Answer.

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = [\vec{a}_1 \quad \vec{a}_2] \, S.$$

□