MAT 211 Summer 2015 Homework 3

Due in Class: June 15^{th}.

Problem 1. Given the following matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & -1 & 1 & 1 \\ 9 & 0 & 1 & 5 \end{bmatrix}.$$

What is the image of A? What is the kernel of A? Express them as a span of vectors.

Answer. The reduced row-echelon form of A is

$$rref(A) = \begin{bmatrix} 1 & 0 & 1/9 & 5/9 \\ 0 & 1 & -5/9 & 11/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the reduced form, we can tell that the first and second column vectors of A are linearly independent, and the others are redundant. Hence,

$$im(A) = span\left\{ \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Remark. The image of A is spanned by the columns of A, not the columns of the reduced form.

The kernel of A is, by definition, the set of all solutions to the linear system $A\vec{x} = \vec{0}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$. From the reduced form of A, we can see that x_3 and x_4 are free variables. So first let $x_3 = 1$ and $x_4 = 0$, we get one solution $\begin{bmatrix} -1/9 \\ 5/9 \\ 1 \\ 0 \end{bmatrix}$; then let $x_3 = 0$ and $x_4 = 1$, then get another solution $\begin{bmatrix} -5/9 \\ -11/9 \\ 0 \\ 1 \end{bmatrix}$. These two vectors are linearly independent by the way we assign values for x_3 and x_4, so they form a basis of the kernel. Namely,

$$ker(A) = span\left\{ \begin{bmatrix} -1/9 \\ 5/9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/9 \\ -11/9 \\ 0 \\ 1 \end{bmatrix} \right\}$$

□

Problem 2. Let T be the orthogonal projection onto the plane $x+2y+3z=0$. What is the image of T? What is the kernel of T? (You can just describe the image and kernel geometrically.)

Answer. The image of T is the set of all vectors that is in the plane; the kernel of T is the set of all vectors that are perpendicular to the plane. □

Problem 3. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 9 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

a) Can you find a matrix A such that $im(A) = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

Answer.

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \\ 3 & 1 & 5 \end{bmatrix}.$$

□

b) Is the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in the span of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$? Explain your reason.

Answer. If $\vec{e}_1$ is in the span of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$, then there are scalars k_1, k_2 and k_3 such that $\vec{e}_1 = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$, namely, $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ is the solution to the linear system

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the augmented matrix of the above linear system

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 1 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right].$$

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 1 & | & 1/9 \\ 0 & 1 & 2 & | & 2/9 \\ 0 & 0 & 0 & | & -1/45 \\ 0 & 0 & 0 & | & 5/99 \end{bmatrix}$$

is inconsistent, hence the linear system has no solutions. Therefore, $\vec{e}_1$ is NOT in the span of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. □

Problem 4. Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Show that the kernel of T satisfies the following properties, hence is a subspace of the domain $\mathbb{R}^m$:

a) The zero vector $\vec{0}$ is in the kernel T;

Answer. Since $T(\vec{0}) = \vec{0}$, $\vec{0}$ is in the kernel. □

b) The kernel is closed under addition;

Answer. If $\vec{v}$ and $\vec{w}$ are in the kernel, then $T(\vec{v}) = \vec{0}$ and $T(\vec{w}) = \vec{0}$. By the linearity of T, we have

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}.$$

Hence, if $\vec{v}$ and $\vec{w}$ are in the kernel, then $\vec{v} + \vec{w}$ is also in the kernel. □

c) The kernel is closed under scalar multiplication.

Answer. If $\vec{v}$ is in the kernel, then for any scalar k, we have

$$T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}.$$

Hence, If $\vec{v}$ is in the kernel, then for any scalar k, $k\vec{v}$ is also in the kernel. □

Problem 5. Which of the following sets are subspaces of $\mathbb{R}^3$? If it is a subspace, find a basis for it; if it is not a subspace, explain your reason.

a) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$;

Answer. NOT a subspace because the zero vector is not in the subset. □

b) $W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq y \right\}$;

Answer. NOT a subspace. Since $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is in the subset but $\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is not in the subset, so the subset is not closed under scalar multiplication. □

c) $W_3 = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ are arbitrary constants} \right\}$.

Answer. IS a subset. You can either check that the three definitive properties of subspace is satisfied, or observe that W_3 is actually the image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

□

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the first and second columns are linear independent and the third column is redundant. Hence,

$$W_3 = im(A) = span \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}.$$

Problem 6. Consider the plane V in $\mathbb{R}^3$ given by the equation

$$x_1 + x_2 + x_3 = 0.$$

Namely, $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 + x_3 = 0 \right\}$.

a) Show that V is a linear subspace of $\mathbb{R}^3$;

b) Find a basis for V;

Answer. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in V, then $a+b+c=0$, then $c=-(a+b)$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ -(a+b) \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

One can observe that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are in V. (The basis must be vectors in the space!) V is spanned by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and they are linearly independent. Hence, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for V. (You can also find other basis using your own idea.) □

c) Find a matrix A such that $V = ker(A)$;

Answer.

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 + x_3 = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}.$$

Hence, $V = ker(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})$. □

d) Find a matrix B such that $V = im(B)$. (hint: V is a span of its basis.)

Answer.

$$V = span\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Hence,

$$V = im\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \right).$$

□

Problem 7. Are the following vectors linearly independent? Explain your answer.

a) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\6\end{bmatrix};$

Answer. Let $A = \begin{bmatrix}1&1&1\\1&2&3\\1&3&6\end{bmatrix}$, $rref(A) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $rank(A) = 3 =$ # of vectors . Hence, the vectors are linearly independent. □

b) $\begin{bmatrix}1\\2\\-1\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}9\\0\\1\\5\end{bmatrix}.$

Answer. Let $B = \begin{bmatrix}1&4&9\\2&-1&0\\-1&1&1\\3&1&5\end{bmatrix}$, then $rref(B) = \begin{bmatrix}1&0&1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$, $rank(B) = 2 <$ # of vectors. Hence, the vectors are NOT linearly independent. □

Problem 8. Find a basis of the kernel of the matrix

$$\begin{bmatrix}1&2&0&3&5\\0&0&1&4&6\end{bmatrix}.$$

Justify your answer carefully; that is, explain how you know that the vectors you found are linearly independent and span the kernel.

Problem 9. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\4\end{bmatrix}, \begin{bmatrix}2\\3\\4\\k\end{bmatrix}$$

Answer. Let

$$A = \begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\2&3&4&k\end{bmatrix},$$

then

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix}.$$

The vectors form a basis for $\mathbb{R}^4$ if and only if they are linearly independent, if and only if $rank(A) = 4$, if and only if $k - 29 \neq 0$, namely, $k \neq 29$. □

Problem 10. Can you find a 3×3 matrix A such that $im(A) = ker(A)$? Explain. (Hint: use the Rank-Nullity Theorem.)

Answer. By the Rank-Nullity Theorem, $\dim(ker(A)) + \dim(im(A)) = 3$. If $im(A) = ker(A)$, then $\dim(ker(A)) = \dim(im(A)) = 1.5$. This contradicts to the fact that the dimension must be integer. Hence, we can NOT find a 3×3 matrix A such that $ker(A) = im(A)$. □

Problem 11. Suppose A and B are two matrices such that the product AB makes sense.

a) Show that $im(AB)$ is contained in $im(A)$. What can you say about the quantities of $rank(A)$ and $rank(AB)$?

   Answer. For any $\vec{y} \in im(AB)$, then there exists an vector $\vec{x}$ in the domain of B such that $\vec{y} = AB\vec{x}$. Then $\vec{y} = A(B\vec{x})$ is in the image of A. Hence, $im(AB)$ is contained in $im(A)$. An immediate corollary is that $\dim(im(AB)) \leq \dim(im(A))$. Since $rank(AB) = \dim(im(AB))$ and $rank(A) = \dim(im(A))$, we have $rank(AB) \leq rank(A)$. □

b) Show that $ker(B)$ is contained in $ker(AB)$, then further show that $rank(AB) \leq rank(B)$ using the Rank-Nullity Theorem;

   Answer. For any $\vec{x}$ in the kernel of B, then $B\vec{x} = \vec{0}$, then $AB\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$. Hence, $\vec{x}$ in also in the kernel of AB. Hence, $ker(B)$ is contained in $ker(AB)$. We have $\dim(ker(B)) \leq \dim(ker(AB))$. By the Rank-Nullity Theorem, we have

$$\begin{aligned} & \dim(ker(B)) + \dim(im(B)) \\ =& \# \text{ of columns of } B \\ =& \# \text{ of columns of } AB \\ =& \dim(ker(AB)) + \dim(im(AB)) \end{aligned}.$$

   Hence,

$$rank(AB) = \dim(im(AB)) \leq \dim(im(B)) = rank(B).$$

□

c) Conclude that $rank(AB) \leq \min\{rank(A), rank(B)\}$.

Answer. Just follows from a) and b). $\square$

Problem 12. We are told that a certain 5×5 matrix A can be written as

$$A = BC$$

where B is 5×4 matrix and C is a 4×5 matrix. Explain why A is not invertible. (Hint: use the consequence of Problem 11.)

Answer. By the consequence of Problem 11,

$$rank(A) \leq \min\{rank(B), rank(C)\} \leq 4 < 5.$$

Hence, A is not invertible. $\square$

Problem 13. Determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v}_1, \ldots, \vec{v}_m$. If $\vec{x}$ is in V, find the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}_{\mathcal{B}}]$.

a) $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Answer. $\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$. The $\mathcal{B}$-coordinate of $\vec{x}$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. $\square$

b) $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Answer. The linear system

$$\begin{bmatrix} 1 & 2 & | & 2 \\ 1 & 0 & | & 3 \\ 0 & 1 & | & 4 \end{bmatrix}$$

is inconsistent, hence $\vec{x}$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$, namely, $\vec{x}$ is NOT in the span of $\vec{v}_1$ and $\vec{v}_2$. $\square$

c) $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Answer. Solve the linear system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right],$$

we get a solution $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Hence, $\vec{x} = \vec{v_1} - \vec{v_2}$. The $\mathcal{B}$-coordinate of $\vec{x}$ is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. □

Problem 14. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Answer. Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}\}$. Then

$$\begin{cases} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\vec{v_1} + 5\vec{v_2} \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\vec{v_1} + 3\vec{v_2} \end{cases}$$

You can just use elimination to solve the above system. Here we prefer to write the system in the matrix product form.

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}.$$

Therefore, the basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix} \right\}.$$

□