Solution Guide for Homework 2

Due in Class: June 8^{th}.

Reading: reading section 2.3 and 2.4.

Problem 1. Does the following product of a matrix and a vector make sense? If so, write down your result.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Problem 2. Consider an $n \times m$ matrix A, a vector $\vec{x}$ in $\mathbb{R}^m$, and a scalar k. Show that

$$A(k\vec{x}) = k(A\vec{x}).$$

Answer. Let $A = \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix}$, where $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_3$ are the column vectors of A. For any scalar k, any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$\begin{aligned} A(k\vec{x}) &= \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix} \\ &= (kx_1)\vec{c}_1 + (kx_2)\vec{c}_2 + (kx_3)\vec{c}_3 \\ &= k(x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3) \\ &= k\left(\begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = k(A\vec{x}). \end{aligned}$$

□

Problem 3. (Linear Combination of Vectors)
Recall that a vector $\vec{v}$ is said to be a linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there are scalars $x_1, x_2, \ldots, x_k$ such that

$$\vec{v} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

a) Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$? If so, find one solution for the scalars x_1 and x_2.

Answer. By the definition of *linear combination*, the question actually asks that whether there exist scalars x_1 and x_2 such that

$$\begin{bmatrix}7\\8\\9\end{bmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}. \qquad (*)$$

Namely, to solve the following linear system:

$$\begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_1\end{bmatrix} = \begin{bmatrix}7\\8\\9\end{bmatrix}.$$

By applying Gauss-Jordan elimination to the augmented matrix $\begin{bmatrix}1 & 4 & | & 7\\2 & 5 & | & 8\\3 & 6 & | & 9\end{bmatrix}$, we get its reduced row echelon form as $\begin{bmatrix}1 & 0 & | & -1\\0 & 1 & | & 2\\0 & 0 & | & 0\end{bmatrix}$. Hence, the linear system is consistent, so it has solution. Also, note that $rank = 2 = \{\#\text{ of variables}\}$, there is no free variable, hence there is a unique solution

$$\begin{cases} x_1 &= -1\\ x_2 &= 2.\end{cases}$$

You can check the answer by pulling it back to equation $(*)$. □

b) Let A be a $n \times m$ matrix and $\vec{b}$ be a column vector in $\mathbb{R}^n$. Show that the linear system $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of the matrix A. (hint: interpret the product of a matrix and a vector as a linear combination of the column vectors of the matrix.)

Answer. ($\Longrightarrow$) if $\vec{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_m\end{bmatrix}$ is a solution to the linear system $A\vec{x} = \vec{b}$, then

$$\vec{b} = A\vec{v} = \begin{bmatrix}| & | & \cdots & |\\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_m\\ | & | & \cdots & |\end{bmatrix}\begin{bmatrix}v_1\\v_2\\\vdots\\v_m\end{bmatrix} = v_1\vec{c}_1 + v_2\vec{c}_2 + \cdots + v_m\vec{c}_m$$

is a linear combination of the column vectors of A, with coefficients being the components of $\vec{v}$.

($\Longleftarrow$) Conversely, suppose $\vec{b} = v_1\vec{c}_1 + v_2\vec{c}_2 + \cdots + v_m\vec{c}_m$ is a linear combination of the column vectors of A, where $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m$ are the m column vectors of A. Then

$$\vec{b} = v_1\vec{c}_1 + v_2\vec{c}_2 + \cdots + v_m\vec{c}_m = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

Hence, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ is a solution of the linear system $A\vec{x} = \vec{b}$. □

Remark. To show that something is *equivalent* to something else (usually stated in this way: a statement P holds *if and only if* a statement Q holds), we need to show the both directions are correct, namely ($\Longrightarrow$) — P implies Q, and ($\Longleftarrow$) — Q implies P.

c) For which values of the constants b and c is the vector $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$? What condition b and c should satisfy?

Answer. By the consequence of b), the vector $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ if and only if the following linear system has a solution

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ b \\ c \end{bmatrix}. \qquad (**)$$

By the theorem of the number of solutions to linear system, $(**)$ has a solution if and only if the linear system is consistent. Using the Gauss-Jordan elimination, we can get the rref of the augmented matrix as follows:

$$\begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 1 & 3 & 4 & | & b \\ 1 & 2 & 3 & | & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & 6-c \\ 0 & 1 & 1 & | & c-3 \\ 0 & 0 & 0 & | & b-2c+3 \end{bmatrix}.$$

The linear system is consistent if and only if

$$b - 2c + 3 = 0.$$

Therefore, the vector $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ if and only if $b - 2c + 3 = 0$. □

Problem 4.

a) Consider the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Is the dot product $T(\vec{x}) = \vec{v} \cdot \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.

b) Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T (in terms of the components of $\vec{v}$).

c) Conversely, consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$. Show that there is a vector $\vec{v}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{v} \cdot \vec{x}$, for all $\vec{x}$ in $\mathbb{R}^3$.

Answer. For any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$, then

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + x_3T(\vec{e}_3).$$

Note that T is a linear transformation to $\mathbb{R}$, so $T(\vec{e}_1)$, $T(\vec{e}_2)$ and $T(\vec{e}_3)$ are all numbers (not vectors). Let $\vec{v} = \begin{bmatrix} T(\vec{e}_1) \\ T(\vec{e}_2) \\ T(\vec{e}_3) \end{bmatrix}$, then $T(\vec{x}) = \vec{v} \cdot \vec{x}$. □

Problem 5. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Let T be the transformation defined by $T(\vec{x}) = A(B\vec{x})$ for any $\vec{x}$ in $\mathbb{R}^2$. Show that T is a linear transformation. What is the matrix of T? (hint: check the two definitive properties of linear transformation.)

Problem 6. Lemma *Let $\vec{v}$ and $\vec{w}$ be two nonzero vectors in the plain $\mathbb{R}^2$, then $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.*

Let $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Prove the lemma by answering the questions below:

a) Show that if $\vec{v} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$ for some scalar k, then $\vec{v} \cdot \vec{w} = 0$;

Answer. If $\vec{v} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$, then

$$\vec{v} \cdot \vec{w} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = k(-w_2 w_1 + w_1 w_2) = 0.$$

□

b) Show that the reverse is true: if $\vec{v} \cdot \vec{w} = 0$, then there exists some scalar k such that $\vec{v} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$;

Answer. If $\vec{v} \cdot \vec{w} = 0$, then

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 = 0. \tag{***}$$

Since $\vec{w} \neq \vec{0}$, then either $w_1 \neq 0$ or $w_2 \neq 0$. If $w_1 \neq 0$, then by equation $(***)$, $v_1 = -v_2 w_2 / w_1$, hence

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 w_2 / w_1 \\ v_2 \end{bmatrix} = \left(\frac{v_2}{w_1} \right) \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix},$$

where $k = v_2 / w_1$; if $w_2 \neq 0$, then by equation $(***)$, then $v_2 = -v_1 w_1 / w_2$, hence

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 w_1 / w_2 \end{bmatrix} = \left(-\frac{v_1}{w_2} \right) \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix},$$

where $k = -v_1 / w_2$. No matter what case it is, we always have $\vec{v} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$ for some scalar k. □

c) Note that the vector $\begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$ is the vector obtained by rotating the vector $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ counterclockwise by $90°$. (We showed this in the class. You can take it for granted.) Use this fact to show the lemma is right.

Answer. By the consequence of *a)* and *b)*, $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v} = k \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$ for some scalar k. The latter shows that $\vec{v}$ is parallel to $\begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$, hence is perpendicular to $\vec{w}$, as the vector $\begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$ is obtained by rotating $\vec{w}$ with $90°$ counterclockwise. Hence, $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v} \perp \vec{w}$. □

Problem 7. Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find the unit vector that is parallel to $\vec{v}$ and of the same direction.

Problem 8. (To answer the following question, please review your notes.)

a) What is the matrix of the counterclockwise rotation by an angle θ (in radians)?

Answer.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

□

b) Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit vector and L be line through origin and the point (u_1, u_2). What is the matrix of the projection to L?

Answer.

$$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}.$$

□

c) Given $\vec{u}$ and L the same as in *b*). What is the matrix of the reflection about L?

Answer.

$$\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix} = 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Remark. Note that the formulas in *b*) and *c*) only apply when $\vec{u}$ is a unit vector.

Problem 9. Compute the following matrix productions.

a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

c) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. Show that $(AB)C = A(BC)$ by computing $(AB)C$ and $A(BC)$ respectively.

d) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute AB and BA respectively. Is the matrix product commutative in general?

Problem 10. Decide whether the matrices below are invertible. If they are, find the inverse. Show all your work.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$