Review sheet for MAT511, Fall 2017

Define the converse, inverse and contrapositive of an implication. Prove that the converse is true iff the inverse is true.

Define quaternions. Define the conjugate and absolute value of a quaternion. Prove that the absolute value of the product of two quaternions is the product of their absolute values.

Prove that the sum of the squares of two positive odd numbers is not divisible by four.

State and prove the Pythagorean theorem.

Define the distance between two points in the plane if the points are given as (a, b) and (c, d). Consider the unit circle centered at $(0, 0)$ in the plane and let each point on it be $(\cos\alpha, \sin\alpha)$ for some angle α. Using the distance between points prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Define the union and intersection of two sets. Note that a non-empty set S is finite iff for some natural number n, there exists a bijection from S to the set $\{1, 2, 3, \ldots, n\}$. An infinite set a set that is not empty or finite. Prove that if S is infinite, then there exists an injection of the set of natural numbers into S.

Define an equivalence relation, and equivalence classes. Explain how an equivalence relation on a set S gives a decomposition of a S into disjoint subsets – a partition of S.

Define the cardinality of a set S. Call it $|S|$. Define $|S| \leq |T|$ where S and T are sets. Define the power set of a set S. Prove that the natural numbers and the rational numbers have the same cardinality.

State the Schroeder-Bernstein theorem about cardinality of sets.

Explain the principle of mathematical induction and use it to prove formulas for the sum of the first n natural numbers and the sum of the squares of the first n natural numbers. Let M be a map made by lines (not line segments) in the plane. Use induction to prove that M can be colored by two colors. Use induction to show that that any non-empty set of natural numbers has a least element. Show by example that this is not true for real numbers.

State and prove (by induction) the binomial theorem which gives a formula for the nth power of x + y.

Define a sequence of real numbers. Define a Cauchy (or cozy) sequence. State the least upper bound property for real numbers. Show by example that it does not hold for rational numbers.

Show that for any positive number ϵ and any real number r, there exists a natural number n, such that $n\epsilon > r$. Also show that for any positive number ϵ, there exists a natural number n such that $1/n < \epsilon$. Let l be the least upper bound of the set of all rational numbers whose square is less than 2. Prove that $l^2 = 2$.

Define the limit of a sequence and define

$$\lim_{x->a} f(x)$$

for a real-valued function f.

Find

$$\lim_{x->0} \left(\frac{\sin x}{x} \right)$$

and prove your answer.

Find

$$\lim_{x->0} \sin(1/x)$$

or prove that the limit does not exist.

Define a field, an ordered field and a complete ordered field.

Define open and closed sets of real numbers. Define interior points, exterior points and boundary points of a set of real numbers S. Define an open cover and a finite subcover of a set S. State and prove that Heine-Borel theorem.

Define a sequence and subsequence. State and prove the Bolzano-Weierstrass theorem.

Define the Euler characteristic of a polyhedron. Prove that the Euler characteristic of a convex polyhedron is 2. Prove that there are exactly five regular polyhedra.