Problems on Complex Hénon Maps 1

Fatou components: Volume preserving case

We suppose throughout that $f : \mathbb{C} \to \mathbb{C}$ is a volume-preserving Hénon map. Let Ω be a component of the interior of K such that $f(\Omega) = \Omega$.

Problem 1. We define

$$\mathfrak{G} = \{ \text{ limits of subsequences } g := lim_{j\to\infty} f^{n_j} \}$$

Is $\mathfrak{G}$ connected?

We know that $\mathfrak{G}$ is a compact, abelian Lie group, so the connected component of the identity is a torus $\mathbb{T}^\rho$. Is the connected component of the identity actually equal to $\mathfrak{G}$?

Problem 2. Suppose that the restriction $f|_\Omega$ has a fixed point. Conjugating by a translation, we may assume that the fixed point is the origin. Let $A := Df(0,0)$ be the differential. Since the set of all powers A^n, $n \in \mathbb{Z}$, is bounded, we know that A is conjugate to a diagonal matrix $\mathrm{diag}(\lambda, \mu)$, where $|\lambda| = |\mu| = 1$. Define

$$\Psi_n := \frac{1}{n} \sum_{j=0}^{n-1} A^{-j} \circ f^j$$

It follows that any limit $\Psi = \lim_{j\to\infty} \Psi_{n_j}$ gives a conjugacy: $A^{-1} \circ \Psi \circ f = \Psi$.

Is it true that $\Psi : \Omega \to \mathbb{C}^2$ is one-to-one?

Suggestion: show first that Ψ must be unbranched, since the branch locus must be invariant under the group action.

Problem 3. Let Ψ and Ω be as above, and define $\mathcal{D} := \Psi(\Omega)$.

Show that it is not possible for $\mathcal{D}$ to be the bidisk $\Delta \times \Delta = \{(x, y) \in \mathbb{C}^2 : |x|, |y| < 1\}$.

One thing that may be helpful is the observation that the torus $\{|x| = |y| = 1\}$ is the *distinguished boundary* of the bidisk, in the sense that the polynomial hull of the torus is the closure of the bidisk. If $\Psi : \Omega \to \Delta^2$ is one-to-one and onto, then the preimage of the torus (whatever that means) must be a special subset of $\partial\Omega$. It may be useful (or not) to break up this question into the separate cases of rank 1 and rank 2.

Problem 4. *Show that the interior of K cannot be (biholomorphically equivalent to) the bidisk.* This is stronger than the assumption that Ω is equivalent to a bidisk: it assumes that Ω is also the entire interior of K.

Problem 5. *Show that the interior of K cannot be (biholomorphically equivalent to) the ball* $\mathbb{B}^2 := \{|x|^2 + |y|^2 < 1\}$. The ball, in some sense, is the opposite of the bidisk: the ball is smooth and everywhere strictly convex, whereas the bidisk is everywhere "flat", except at the "corner" formed by the distinguished boundary. However, both of these domains have the common feature that their automorphism groups are large enough to be transitive.