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Degree complexity of birational maps related to matrix inversion: symmetric case

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Abstract For $q \geq 3$, we let $\mathcal{S}_q$ denote the projectivization of the set of symmetric $q \times q$ matrices with coefficients in $\mathbb{C}$. We let $I(x) = (x_{i,j})^{-1}$ denote the matrix inverse, and we let $J(x) = (x_{i,j}^{-1})$ be the matrix whose entries are the reciprocals of the entries of x. We let $K|\mathcal{S}_q = I \circ J : \mathcal{S}_q \to \mathcal{S}_q$ denote the restriction of the composition $I \circ J$ to $\mathcal{S}_q$. This is a birational map whose properties have attracted some attention in statistical mechanics. In this paper we compute the degree complexity of $K|\mathcal{S}_q$, thus confirming a conjecture of Angles d'Auriac et al. (J Phys A Math Gen 39:3641–3654, 2006).

Keywords Birational mappings · Degree complexity · Matrix inversion · Symmetric matrices

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1 Introduction

Fix $q \geq 3$, let $\mathcal{M}_q$ denote the space of $q \times q$ matrices with coefficients in $\mathbb{C}$, and let $\mathbb{P}(\mathcal{M}_q)$ denote its projectivization. Then the mapping $K : \mathbb{P}(\mathcal{M}_q) \to \mathbb{P}(\mathcal{M}_q)$ is defined as follows: $K = I \circ J$, where $J(x) = (x_{i,j}^{-1})$ takes the reciprocal of each entry of the matrix $x = (x_{i,j})$, and $I(x) = (x_{i,j})^{-1}$ is the matrix inverse. The map K is of interest since it represents a basic symmetry in certain problems of lattice statistical mechanics, and has been studied in [1–8,12].

The degree complexity of K is the exponential rate of growth of the degrees of its iterates:

$$\delta(K) = \lim_{n\to\infty} (deg(K^n))^{1/n}. \tag{1.1}$$

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There are many K-invariant subspaces $\mathcal{T} \subset \mathbb{P}(\mathcal{M}_q)$. The first were considered are $\mathcal{S}_q$ (the space of symmetric matrices), $\mathcal{C}_q$ the cyclic (also called circulant) matrices, and $\mathcal{SC}_q = \mathcal{S}_q \cap \mathcal{C}_q$ (see [12] for more K-invariant subspaces of $\mathbb{P}(\mathcal{M}_q)$). In view of complex dynamics, as well as physical meaning, the map K as well as the restrictions of K to invariant spaces are of interest. One of the basic questions is to determine the degree complexities $\delta(K|\mathcal{T})$. The values $\delta(K|\mathcal{C}_q)$ were found in [4,7]; the values of $\delta(K|\mathcal{SC}_q)$ were found in [2] for prime q's, and in [4] for general q's. Based on extensive computations, [2] has conjectured that

$$\delta(K|\mathcal{C}_q) = \delta(K) = \delta(K|\mathcal{S}_q), \tag{1.2}$$

for all q. In [5], we proved that $\delta(K) = \delta(K|\mathcal{C}_q)$. In this paper we prove the remaining conjectured equality.

Theorem 1 $\delta(K|\mathcal{S}_q) = \delta(K) = \delta(K|\mathcal{C}_q)$ *is the largest modulus of the roots of the polynomial* $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$.

The proof of Theorem 1 is similar to the proofs for other cases (general matrices, $\mathcal{C}_q$, $\mathcal{SC}_q$) in that we repeatedly blowup subvarieties to construct a space $Z \to \mathbb{P}(\mathcal{S}_q)$, and we conclude by showing that $\delta(K)$ equals the spectral radius $sp(K_Z^*)$ of the pullback operator $K_Z^* : Pic(Z) \to Pic(Z)$ for the lifted map $K_Z : Z \to Z$. However, the behavior of singular orbits is much more complicated for the symmetric case that we consider here. Let us give a brief comparison of these proofs in the following.

The computations of $\delta(K|\mathcal{C}_q)$ and $\delta(K|\mathcal{SC}_q)$ can be reduced to computations of $\delta(F)$ where $F = L \circ J$ for appropriate linear maps L. It was shown in [3] (respectively [4]) that after a finite series of blowups $Z \to \mathcal{C}_q$ (respectively $Z \to \mathcal{SC}_q$), the induced maps F_Z on Z is algebraic stable, i.e. satisfy

$$(F_Z^n)^* = (F_Z^*)^n, \tag{1.3}$$

for all $n \in \mathbb{N}$, as linear maps on $Pic(Z)$. It follows (see for example [11]) that $\delta(F)$ is the spectral radius $sp(F_Z^*)$ of F_Z^*.

For the case of general matrices, we constructed in [5] a space Z for which $sp(K_Z^*) = \delta(K|\mathcal{C}_q)$. This immediately implies $\delta(K) = sp(K_Z^*) = \delta(K|\mathcal{C}_q)$. (Remark: The same argument as that of the proof of Lemma 1 below shows that in fact the map K_Z in [5] satisfies condition (1.3), thus gives another proof to the cited result in [5].)

For the proof of Theorem 1 in this paper, we will construct a space Z via a construction which is similar to, but more complicated than, the one in [5]. Although we do not prove (1.3), we show that $\delta(K|\mathcal{S}_q) = \delta(K) = \delta(K|\mathcal{C}_q)$ are all equal to the spectral radius of K_Z^*. The results that allow us to circumvent (1.3) in this case are Proposition 7 and Theorem 2.

This paper is organized as follows: In Sect. 2, we give some basic properties of the map $K|\mathcal{S}_q$. In Sect. 3 we construct a space Z by a series of blowups starting from $\mathcal{S}_q$. In Sect. 4 we explore the behavior of the iterates of the map K_Z on the exceptional hypersurfaces, and obtain a lower bound for $\delta(K|\mathcal{S}_q)$. In Sect. 5 we show that the lower bound is equal to the largest modulus of the roots of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$, thus complete the proof of Theorem 1.

2 Basic properties of the map K

By [5], we know that $1 \le \delta(K|\mathcal{S}_q) \le \delta(K) \le 1$ for $q = 2, 3, 4$, so in the sequel we will assume that $q \ge 5$. For convenience we will use the simple notation K for $K|\mathcal{S}_q$.

First, we introduce some notation that will be helpful in the course of the proof of Theorem 1. Most of the notation used here have a counterpart for the case of general matrices, which was used in [5].

For $1 \leq j \leq q-1$, define R_j to be the set of matrices in $\mathcal{S}_q$ of rank less than or equal to j. Elements of R_1, the symmetric matrices of rank 1, may be represented as $\nu \otimes \nu = (\nu_i \nu_j)_{1 \leq i,j \leq q}$ for $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{C}^q$. In particular, R_1 is a smooth subvariety of $\mathcal{S}_q$.

For $i, j = 1, \ldots, q$ denote:

$$\Sigma_{i,j} = \{x = (x_{k,l}) \in S_q : x_{i,j} = 0\},$$

and define

$$A_{i,j} = \bigcap_{k=i \text{ or } l=j} \Sigma_{k,l}.$$

Thus $\Sigma_{i,j}$ is the set of symmetric matrices whose (i, j)th entry is zero, and $A_{i,j}$ is the set of symmetric matrices whose ith and jth rows and columns are zero. In particular, $A_{i,j} = A_{i,i} \cap A_{j,j}$ for all $1 \leq i, j \leq q$. This leads to a difficulty that does not arise in the non-symmetric case.

We summarize some properties of the map K in the following proposition

Proposition 1 (a) *The exceptional hypersurfaces of K are JR_{q-1} and $\Sigma_{i,j}$'s.*
(b) *The indeterminacy locus K is contained in the set*

$$JR_{q-2} \cup \bigcup_{(i,j)\neq(k,l)} (\Sigma_{i,j} \cap \Sigma_{k,l}).$$

(c) $deg(K) = q^2 - q + 1$.

Proof The proofs of (a) and (b) are similar to those of Propositions 2.1 and 3.1 in [5] (see also the results in Sect. 3 of this paper).

We now proceed to proving (c). Regarding $\mathcal{S}_q$ as the projective space $\mathbb{P}^{(q^2+q-2)/2}$, then a point $y \in \mathcal{S}_q$ can be represented by the homogeneous coordinates $(y_{i,j},\ 1 \leq i \leq j \leq q)$. Then the corresponding matrix in $\mathcal{M}_q$ is the symmetric matrix $\widehat{y}$ whose entries are $\widehat{y}_{i,j} = y_{i,j}$ for $1 \leq i \leq j \leq q$.

It suffices to show that the homogeneous representation $\widehat{K}$ of K is:

$$\widehat{K}_{i,j}(y) = C_{i,j}(1/\widehat{y}) \prod(\widehat{y}),$$

for $1 \leq i \leq j \leq q$, where $\prod(\widehat{y}) := \prod_{1 \leq i,j \leq q} \widehat{y}_{k,l}$ and $C_{i,j}(1/\widehat{y})$ is the (i, j)-cofactor of the matrix $1/\widehat{y}$. That is, to show that the GCD of all polynomials $\widehat{K}_{i,j}(y)$ (for $1 \leq i \leq j \leq q$) is 1. To this end, it suffices to show that the GCD of all polynomials $\widehat{K}_{i,i}(y)$ (where $1 \leq i \leq q$) is 1.

Note that the rational function $C_{i,i}(1/\widehat{y})$ does not depend on the variables $\widehat{y}_{i,k}$ and $\widehat{y}_{k,i}$ for $1 \leq k \leq q$. Moreover, since $C_{i,i}(1/\widehat{y})$ is the determinant of the $(q-1) \times (q-1)$ symmetric matrix obtained by deleting the ith row and ith column from the matrix $1/\widehat{y}$, it is easy to see that

$$D_i(y) := C_{i,i}(1/\widehat{y}) \prod_{(k-i)(l-i)\neq 0} \widehat{y}_{k,l}$$

is a polynomial independent of variables $\widehat{y}_{i,k}$ and $\widehat{y}_{k,i}$ for $1 \leq k \leq q$, and is not divisible by any of the variables $\widehat{y}_{k,l}$ where $1 \leq k, l \leq q$. Then we have

$$\widehat{K}_{i,i}(y) = D_i(y) E_i(y)$$

where $E_i(y) = \prod_{(k-i)(l-i)=0} \widehat{y}_{k,l}$. Observe that

(1) For any i and j, $GCD(D_i, E_j) = 1$. This is because as noted above, D_i is not divisible by any of the variables $\widehat{y}_{k,l}$, while E_j is a monomial in these variables.
(2) $GCD(E_1, E_2, \ldots, E_q) = 1$. In fact, E_i depends only on the variables in $S_i = \{\widehat{y}_{i,1}, \widehat{y}_{i,2}, \ldots, \widehat{y}_{i,q}\}$. Hence if ϕ is a divisor of E_i, ϕ depends only on the variables in S_i. Since $\bigcap_{i=1,\ldots,q} S_i = \emptyset$, it follows that the $GCD(E_1, \ldots, E_q)$ must be a constant.
(3) $GCD(D_1, \ldots, D_q) = 1$. The argument is similar to that of (2).

From (1), (2) and (3), it follows that $GCD(\widehat{K}_{1,1}, \widehat{K}_{2,2}, \ldots, \widehat{K}_{q,q}) = 1$. □

3 Construction of the space Z

Let us describe the sequence of blowups used to construct Z.

(A) First we let $\pi_1 : Z_1 \to \mathcal{S}_q$ be the blowing up with center R_1 and exceptional divisor $\mathcal{R}^1 = \pi_1^{-1}(R_1)$. To give a local coordinate system we fix $2 \le i_0, j_0 \le q$, $1 \le k_0 \le q$. Let $s \in \mathbb{C}$; $v = (v_{i,j})_{2\le i,j\le q} \in \mathcal{S}_{q-1}$ and $v_{i_0,j_0} = 1$; $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{C}^q$ and $\nu_{k_0} = 1$, and $\nu \otimes \nu \in \mathcal{M}_q$ whose (i, j)th entry is $\nu_i \nu_j$. Without loss of generality, we may assume that $k_0 = 1$, i.e. $\nu_1 = 1$. Then, in the local coordinate (s, v, ν) the projection $\pi_1 = \pi_{\mathcal{R}^1}$ is given by

$$\pi_{\mathcal{R}^1}(s, v, \nu) = \nu \otimes \nu + s \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}. \tag{3.1}$$

In this local coordinate system, $\mathcal{R}^1 = \{s = 0\}$.
(B) Next we let $\pi_2 : Z_2 \to Z_1$ be the blow up of Z_1 along the strict transforms of $A_{i,j}$ for all $1 \le i < j \le q$. The space Z_2 depends on the order in which these blowups are performed. But it does not matter for our purpose, the Picard group $Pic(Z_2)$ of Z_2 is generated by $Pic(Z_1)$ and the exceptional divisors $\mathcal{A}^{i,j} = \pi_2^{-1}(A_{i,j})$. The object we will use is $Pic(Z_2)$, which is essentially independent of the order of blowups. We describe a local coordinate system of π_2 near the exceptional divisor $\mathcal{A}^{1,2}$. We fix $3 \le i_0, j_0 \le q$, $1 \le \min\{k_0, l_0\} \le 2$. Let $s \in \mathbb{C}$; $v = (v_{i,j})_{3\le i,j\le q} \in \mathcal{S}_{q-2}$ and $v_{i_0,j_0} = 1$;

$$\begin{pmatrix} \zeta_{1,1} & \zeta_{1,2} & \cdots & \zeta_{1,q} \\ \zeta_{2,1} & \zeta_{2,2} & \cdots & \zeta_{2,q} \\ \vdots & \vdots & 0_{q-2} & \\ \zeta_{q,1} & \zeta_{q,2} & & \end{pmatrix} =: \begin{pmatrix} \zeta & \zeta & \zeta \\ \zeta & \zeta & \zeta \\ \zeta & \zeta & 0_{q-2} \end{pmatrix} \in \mathcal{S}_q,$$

where 0_{q-2} is the $(q-2) \times (q-2)$ zero matrix; $\zeta = (\zeta_{k,l})_{1\le\min\{k,l\}\le 2}$, and $\zeta_{k_0,l_0} = 1$. In the local coordinate (s, ζ, v), the projection $\pi_2 = \pi_{\mathcal{A}^{1,2}}$ is given by

$$\pi_{\mathcal{A}^{1,2}}(s, \zeta, v) = \begin{pmatrix} s\zeta & s\zeta & s\zeta \\ s\zeta & s\zeta & s\zeta \\ s\zeta & s\zeta & v \end{pmatrix}. \tag{3.2}$$

In this local coordinate system, $\mathcal{A}^{1,2} = \{s = 0\}$. Local coordinates near other $\mathcal{A}^{i,j}$'s $(i \ne j)$ are similarly defined.
(C) Next we let $\pi_3 : Z_3 \to Z_2$ be the blow up of Z_2 along the strict transforms of $A_{i,i}$ for all $1 \le i \le q$, with exceptional divisors $\mathcal{A}^{i,i} = \pi_3^{-1}(A_{i,i})$. We describe a local

coordinate system of π_2 near the exceptional divisor $\mathcal{A}^{1,1}$. We fix $2 \leq i_0, j_0 \leq q$, $1 \leq k_0 \leq q$. Let $s \in \mathbb{C}$; $v = (v_{i,j})_{2\leq i,j\leq q} \in \mathcal{S}_{q-1}$ and $v_{i_0,j_0} = 1$; $\zeta = (\zeta_{k,l})_{\min\{k,l\}=1}$ and $\zeta_{1,k_0} = 1$. In the local coordinate (s, ζ, v), the projection $\pi_3 = \pi_{\mathcal{A}^{1,1}}$ is given by

$$\pi_{\mathcal{A}^{1,1}}(s, \zeta, v) = \begin{pmatrix} s\zeta & s\zeta \\ s\zeta & v \end{pmatrix}. \tag{3.3}$$

In this local coordinate system, $\mathcal{A}^{1,1} = \{s = 0\}$.

Let $K_{Z_3} = \pi_{Z_3}^{-1} \circ K \circ \pi_{Z_3}$ be the induced map of K in Z_3.

Proposition 2 (i) $K_{Z_3}(\mathcal{R}^1) = R_{q-1}$.
(ii) $K_{Z_3}(JR_{q-1}) = \mathcal{R}^1$.
(iii) *For all* $1 \leq i \leq q$, $K_{Z_3}(\Sigma_{i,i}) = \mathcal{A}^{i,i}$.
(iv) *For all* $1 \leq i < j \leq q$, $K_{Z_3}(\Sigma_{i,j}) = \mathcal{A}^{i,j} \cap \Sigma_{i,i} \cap \Sigma_{j,j}$.

Proof (i) It suffices to show that: for $\nu = (1, \nu_2, \dots, \nu_q)$, $z = \pi_{\mathcal{R}^1}(0, v, \nu) \in \mathcal{R}^1$ then

$$K_{Z_3}(z) = A^t \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1}(v') \end{pmatrix} A,$$

where I_{q-1} is the matrix inverse on $\mathcal{M}_{q-1}$,

$$v' = \left(-\frac{v_{j,k}}{\nu_j^2 \nu_k^2}\right)_{2\leq j,k\leq q}, \quad A = \begin{pmatrix} 1 & & 0 \dots 0 \\ -\frac{1}{\nu_2} & & 1 & \\ \vdots & & & \ddots \\ frac1\nu_q & & & 1 \end{pmatrix},$$

and A^t is the transpose of A. Here the entries of A outside the main diagonal and the first column are zero.

Without loss of generality, we work at v and ν such that v' in the above is invertible. We have

$$J(\pi_{\mathcal{R}^1}(s, v, \nu)) = \frac{1}{\nu \otimes \nu} + sv' + O(s^2) = \pi_{\mathcal{R}^1}\left(s + O(s^2), v' + O(s), \frac{1}{\nu}\right).$$

Let $e_1 = (1, 0, \dots, 0)$ be the first standard basis vector in $\mathbb{C}^q$. Then

$$A\left(\frac{1}{\nu \otimes \nu}\right)A^t = A\left(\frac{1}{\nu} \otimes \frac{1}{\nu}\right)A^t = \left(A\frac{1}{\nu}\right) \otimes \left(A\frac{1}{\nu}\right) = e_1 \otimes e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $A_{[1,1]}$ (respectively $A^t_{[1,1]}$), the matrix in $\mathcal{M}_{q-1}$ obtained by deleting the first row and column of A (correspondingly of A^t), is the identity matrix in $\mathcal{M}_{q-1}$, we obtain:

$$sAv'A^t = \begin{pmatrix} 0 & 0 \\ 0 & sA_{[1,1]}v'A^t_{[1,1]} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & sv' \end{pmatrix}.$$

Hence

$$\begin{aligned} K_{Z_3}(z) &= \pi_{Z_3}^{-1} \circ I \circ J \circ \pi_{Z_3}(z) \\ &= \pi_{Z_3}^{-1} \circ I\left(\frac{1}{\nu \otimes \nu} + sv' + O(s^2)\right) \\ &= \pi_{Z_3}^{-1}\left(A^t I\left[A\left(\frac{1}{\nu \otimes \nu} + sv' + O(s^2)\right)A^t\right]A\right). \end{aligned}$$

The principal part (first terms of Taylor expansion) of the latter is equal to

$$\pi_{Z_3}^{-1}\left(A^t I \begin{pmatrix} 1 & 0 \\ 0 & sv' \end{pmatrix} A\right) = \pi_{Z_3}^{-1}\left(A^t \begin{pmatrix} s & 0 \\ 0 & I_{q-1}(v') \end{pmatrix} A\right),$$

and (i) follows by letting $s \to 0$.

Proofs of (ii), (iii), and (iv) are similar (cf. [5], Sects. 2, 3). □

Remark 1 Proposition 2 (iv) shows that $\Sigma_{i,j}$ $(i < j)$ is still exceptional for the map K_{Z_3}, which differs from the corresponding situation in [5] for general matrices. This motivates us to perform blowups in subsection (E) below.

(D) Next we let $\pi_4 : Z_4 \to Z_3$ be the blow up of Z_3 along the strict transforms of $B_{i,i} = \mathcal{A}^{i,i} \cap \Sigma_{i,i}$ (where $1 \leq i \leq q$), with exceptional divisors $\mathcal{B}^{i,i} = \pi_4^{-1}(B_{i,i})$. We describe two local coordinate systems of π_4 near the exceptional divisor $\mathcal{B}^{1,1}$.
For the first local coordinate system, we fix $2 \leq i_0, j_0 \leq q$, $1 \leq k_0 \leq q$. Let $t, \xi \in \mathbb{C}$; $v = (v_{i,j})_{2\leq i,j\leq q} \in \mathcal{S}_{q-1}$ and $v_{i_0,j_0} = 1$; $\zeta = (\zeta_{k,l})_{\min\{k,l\}=1,\ k\neq l}$ and $\zeta_{1,k_0} = 1$. In the local coordinate (t, ξ, ζ, v), the projection $\pi_4 = \pi^1_{\mathcal{B}^{1,1}}$ is given by

$$\pi^1_{\mathcal{B}^{1,1}}(t, \xi, \zeta, v) = \begin{pmatrix} t^2\xi & t\zeta \\ t\zeta & v \end{pmatrix}. \tag{3.4}$$

In this local coordinate system, $\mathcal{B}^{1,1} = \{t = 0\}$.

To cover the points corresponding to $\xi = \infty$ in the first projection $\pi^1_{\mathcal{B}^{1,1}}$, we let $t, \xi \in \mathbb{C}$; $v = (v_{i,j})_{2\leq i,j\leq q} \in \mathcal{S}_{q-1}$ and $v_{i_0,j_0} = 1$; $\zeta = (\zeta_{k,l})_{\min\{k,l\}=1,\ k\neq l}$ and $\zeta_{1,k_0} = 1$. In the local coordinate (t, ξ, ζ, v), the projection $\pi_4 = \pi^2_{\mathcal{B}^{1,1}}$ is given by

$$\pi^2_{\mathcal{B}^{1,1}}(t, \xi, \zeta, v) = \begin{pmatrix} t^2\xi & t\xi\zeta \\ t\xi\zeta & v \end{pmatrix}. \tag{3.5}$$

In this local coordinate system, $\mathcal{B}^{1,1} = \{t = 0\}$. The set $\{t = 0, \xi = \infty\}$ in the first projection $\pi^1_{\mathcal{B}^{1,1}}$ corresponds to the set $\{t = 0, \xi = 0\}$ in this second projection $\pi^2_{\mathcal{B}^{1,1}}$.

Let $K_{Z_4} = \pi_{Z_4}^{-1} \circ K \circ \pi_{Z_4}$ be the induced map of K in Z_4.

Proposition 3 *For $1 \leq i \leq q$:*

(i) $K_{Z_4}(\mathcal{A}^{i,i}) = \mathcal{B}^{i,i} \cap I(\Sigma_{i,i})$. *In fact, if $(s = 0, \zeta, v) \in \mathcal{A}^{1,1}$ as in (3.3) then*

$$K_{Z_4}(s = 0, \zeta, v) = (t = 0, \xi', \zeta', v') \in \mathcal{B}^{1,1}, \tag{3.6}$$

where

$$\begin{pmatrix} \xi' & \zeta' \\ \zeta' & v' \end{pmatrix} = I \begin{pmatrix} 0/\zeta_{1,1} & 1/\zeta \\ 1/\zeta & 1/v \end{pmatrix}.$$

(ii) $K_{Z_4}(\mathcal{B}^{i,i}) = \mathcal{B}^{i,i}$.
Moreover, the restriction of K_{Z_4} to each of the spaces $\mathcal{B}^{i,i}$ is the same as K, in the sense that

$$K_{Z_4}(t = 0, \xi, \zeta, v) = (t = 0, \xi', \zeta', v'),$$

at generic points $(t = 0, \xi, \zeta, v)$ of $\mathcal{B}^{1,1}$, where

$$\begin{pmatrix} \xi' & \zeta' \\ \zeta' & v' \end{pmatrix} = K \begin{pmatrix} \xi & \zeta \\ \zeta & v \end{pmatrix}.$$

Similar results hold for the other $\mathcal{B}^{i,i}$'s $(1 \leq i \leq q)$.

Proof (i) We make use of the following property (see formula (4.4) in [5]): If

$$K\begin{pmatrix}\xi & \zeta\\ \zeta & v\end{pmatrix}=\begin{pmatrix}\xi' & \zeta'\\ \zeta' & v'\end{pmatrix}$$

then

$$K\begin{pmatrix}t^2\xi & t\zeta\\ t\zeta & v\end{pmatrix}=\begin{pmatrix}t^2\xi' & t\zeta'\\ t\zeta' & v'\end{pmatrix}. \tag{3.7}$$

Using the projection (3.3), to determine $K_{Z_4}(\mathcal{A}^{1,1})$ it suffices to compute the limit when $s\to 0$ of $K(x)$ where

$$x=\begin{pmatrix}s\zeta & s\zeta\\ s\zeta & v\end{pmatrix}.$$

Rewriting x as

$$x=\begin{pmatrix}s^2\zeta_{1,1}/s & s\zeta\\ s\zeta & v\end{pmatrix},$$

using the formula (3.7), we have

$$K(x)=\begin{pmatrix}s^2\xi' & s\zeta'\\ s\zeta' & v'\end{pmatrix},$$

where

$$\begin{pmatrix}\xi' & \zeta'\\ \zeta' & v'\end{pmatrix}=K\begin{pmatrix}\zeta_{1,1}/s & \zeta\\ \zeta & v\end{pmatrix}=I\begin{pmatrix}s/\zeta_{1,1} & 1/\zeta\\ 1/\zeta & 1/v\end{pmatrix}.$$

The last formula shows that when $s\to 0$, the limit of $K(x)$ is in $\mathcal{B}^{1,1}\cap I(\Sigma_{1,1})$, and we obtain (3.6). Hence $K_{Z_4}(\mathcal{A}^{1,1})=\mathcal{B}^{1,1}\cap I(\Sigma_{1,1})$.

The proof of (ii) is similar. □

Let us consider a matrix

$$x=\begin{pmatrix}\xi & \zeta\\ \zeta & v\end{pmatrix},$$

written as in (3.4). That is, ξ and the $\zeta's$ fill out the first row and column, where $\xi\in\mathbb{C}$. We will consider algebraic subvarieties $W\subset\mathcal{S}_q$ with the property that whenever $x\in W$, then

$$\begin{pmatrix}t^2\xi & t\zeta\\ t\zeta & v\end{pmatrix}\in W, \tag{3.8}$$

for all $\mathbb{C}\ni t\neq 0$. If W has this property, and if no component of W is contained in the indeterminacy loci of I, J, and K, then so do $I(W)$, $J(W)$, and $K(W)$.

We say that an irreducible hypersurface $W\subset\mathcal{S}_q$ is compatible with $\mathcal{B}^{1,1}$ if condition (3.8) is satisfied and if moreover

$$W\not\subset JR_{q-1}\cup\bigcup_{(k,l)\neq(1,1)}\Sigma_{k,l}.$$

When W is compatible, then W is not contained in any of the centers of blowups in the construction of Z_4, thus we can take its strict transform inside Z_4 and define $\mathcal{B}^{1,1}\cap W\subset Z_4$. Using coordinate projections analogous to (3.4), we may also define what it means for W to be compatible with $\mathcal{B}^{i,i}$ for $2 \leq i \leq q$. Note that both hypersurfaces $\Sigma_{1,1}$ and $I(\Sigma_{1,1})$ are compatible with $\mathcal{B}^{1,1}$.

Proposition 4 *For* $1 \leq i \leq q$*:*

If W *is compatible with* $\mathcal{B}^{i,i}$ *and* $W \not\subseteq \Sigma_{i,i}$*, then* $K_{Z_4}(\mathcal{B}^{i,i} \cap W) = \mathcal{B}^{i,i} \cap K(W)$*.*
If $W = \Sigma_{i,i}$*, then* $K_{Z_4}(K_{Z_4}(\mathcal{B}^{i,i} \cap \Sigma_{i,i})) = \mathcal{B}^{i,i} \cap I(\Sigma_{i,i})$*.*

Moreover, $K_{Z_4}(\mathcal{B}^{i,i} \cap \Sigma_{i,i})$ *can be written explicitly. For example, if* $i = 1$ *then in the local coordinate system* (3.5) *we have:* $K_{Z_4}(\mathcal{B}^{1,1} \cap \Sigma_{1,1}) = \{t = \xi = 0\}$*.*

Proof The first claim follows from the discussion in last paragraph and Proposition 3.

The proof of the third claim is similar to that of Proposition 2 (iii).

The second claim follows from the third claim and an argument similar to that of the proof of Proposition 3 (i). □

(E) Next we let $\pi_5 : Z_5 \to Z_4$ be the blow up of Z_4 along the strict transforms of $C_{i,j} = \mathcal{A}^{i,j} \cap \Sigma_{i,i} \cap \Sigma_{j,j}$ (where $1 \leq i < j \leq q$), with exceptional divisors $\mathcal{C}^{i,j}$. We describe a local coordinate system of π_5 near the exceptional divisor $\mathcal{C}^{1,2}$. We fix $3 \leq i_0, j_0 \leq q$, $1 \leq \min\{k_0, l_0\} \leq 2$, $k_0 \neq l_0$. Let $t \in \mathbb{C}$; $v = (v_{i,j})_{3\leq i,j\leq q} \in \mathcal{S}_{q-2}$ and $v_{i_0,j_0} = 1$; $\xi = (\xi_{1,1},\ \xi_{2,2}) \in \mathbb{C}^2$; $\zeta = (\zeta_{k,l})_{1\leq\min\{k,l\}\leq 2,\ k\neq l}$, and $\zeta_{k_0,l_0} = 1$. In the local coordinate (t, ξ, ζ, v), the projection $\pi_5 = \pi_{\mathcal{C}^{1,2}}$ is given by

$$\pi_{\mathcal{C}^{1,2}}(t, \xi, \zeta, v) = \begin{pmatrix} t^2\xi_{1,1} & t\zeta & t\zeta \\ t\zeta & t^2\xi_{2,2} & t\zeta \\ t\zeta & t\zeta & v \end{pmatrix}. \tag{3.9}$$

In this local coordinate system, $\mathcal{C}^{1,2} = \{t = 0\}$.

(F) Finally, we let $\pi_6 : Z_6 \to Z_5$ be the blow up of Z_5 along the strict transforms of $D_{i,j} = \mathcal{C}^{i,j} \cap \Sigma_{i,j}$ (where $1 \leq i < j \leq q$), with exceptional divisors $\mathcal{D}^{i,j} = \pi_6^{-1}(D_{i,j})$. We describe two local coordinate systems of π_6 near the exceptional divisor $\mathcal{D}^{1,2}$. For the first local coordinate system, we fix $3 \leq i_0, j_0 \leq q$, $1 \leq \min\{k_0, l_0\} \leq 2 < \max\{k_0, l_0\}$. Let $t \in \mathbb{C}$; $v = (v_{i,j})_{3\leq i,j\leq q} \in \mathcal{S}_{q-2}$ and $v_{i_0,j_0} = 1$; $\xi = (\xi_{1,1},\ \xi_{1,2},\ \xi_{2,2}) \in \mathbb{C}^3$; $\zeta = (\zeta_{k,l})_{1\leq\min\{k,l\}\leq 2<\max\{k,l\}}$, and $\zeta_{k_0,l_0} = 1$. In the local coordinate (t, ξ, ζ, v), the projection $\pi_6 = \pi^1_{\mathcal{D}^{1,2}}$ is given by

$$\pi^1_{\mathcal{D}^{1,2}}(t, \xi, \zeta, v) = \begin{pmatrix} t^2\xi_{1,1} & t^2\xi_{1,2} & t\zeta \\ t^2\xi_{1,2} & t^2\xi_{2,2} & t\zeta \\ t\zeta & t\zeta & v \end{pmatrix}. \tag{3.10}$$

In this local coordinate system, $\mathcal{D}^{1,2} = \{t = 0\}$.

To cover the points corresponding to $\xi_{1,2} = \infty$ in the first projection $\pi^1_{\mathcal{D}^{1,2}}$, we let $t \in \mathbb{C}$; $v = (v_{i,j})_{3\leq i,j\leq q} \in \mathcal{S}_{q-2}$ and $v_{i_0,j_0} = 1$; $\lambda \in \mathbb{C}$; $\xi = (\xi_{1,1},\ \xi_{1,2},\ \xi_{2,2}) \in \mathbb{C}^3$ and one of its coordinates is 1; $\zeta = (\zeta_{k,l})_{1\leq\min\{k,l\}\leq 2<\max\{k,l\}}$, and $\zeta_{k_0,l_0} = 1$. In the local coordinate (t, ξ, ζ, v), the projection $\pi_6 = \pi^2_{\mathcal{D}^{1,2}}$ is given by

$$\pi^2_{\mathcal{D}^{1,2}}(t, \lambda, \xi, \zeta, v) = \begin{pmatrix} t^2\lambda^2\xi_{1,1} & t^2\lambda\xi_{1,2} & t\lambda\zeta \\ t^2\lambda\xi_{1,2} & t^2\lambda^2\xi_{2,2} & t\lambda\zeta \\ t\lambda\zeta & t\lambda\zeta & v \end{pmatrix}. \tag{3.11}$$

In this local coordinate system, $\mathcal{D}^{1,2} = \{t = 0\}$. The set $\{t = 0, \xi_{1,2} = \infty\}$ in the first projection $\pi^1_{\mathcal{D}^{1,2}}$ corresponds to the set $\{t = 0, \lambda = 0\}$ in this second projection $\pi^2_{\mathcal{D}^{1,2}}$.

(F) We define $Z = Z_6$. Let $K_Z = \pi_Z^{-1} \circ K \circ \pi_Z : Z \to Z$ be the induced map of K on Z.

Proposition 5 *For* $1 \le i < j \le q$*:*

(i) $K_Z(\Sigma_{i,j}) = \mathcal{C}^{i,j}$.
(ii) $K_Z(\mathcal{A}^{i,j}) = \mathcal{D}^{i,j} \cap I(\Sigma_{i,i} \cap \Sigma_{j,j} \cap \Sigma_{i,j})$.
(iii) $K_Z(\mathcal{C}^{i,j}) = \mathcal{D}^{i,j} \cap I(\Sigma_{i,j})$.
(iv) $K_Z(\mathcal{D}^{i,j}) = \mathcal{D}^{i,j}$.

Moreover, the restriction of K_Z *to each of the spaces* $\mathcal{D}^{i,j}$ *is the same as* K*, in the sense that*

$$K_Z(t = 0, \xi, \zeta, v) = (t = 0, \xi', \zeta', v'),$$

at generic points $(t = 0, \xi, \zeta, v)$ *of* $\mathcal{D}^{1,2}$*, where*

$$\begin{pmatrix} \xi' & \xi' & \zeta' \\ \xi' & \xi' & \zeta' \\ \zeta' & \zeta' & v' \end{pmatrix} = K \begin{pmatrix} \xi & \xi & \zeta \\ \xi & \xi & \zeta \\ \zeta & \zeta & v \end{pmatrix}.$$

Similar results hold for other $\mathcal{D}^{i,j}$*'s* $(1 \le i < j \le q)$*.*

Proof The proofs of all these claims are similar to the proof of Proposition 3, but instead of using formula (3.7), we use a similar formula:

If

$$K \begin{pmatrix} \xi & \xi & \zeta \\ \xi & \xi & \zeta \\ \zeta & \zeta & v \end{pmatrix} = \begin{pmatrix} \xi' & \xi' & \zeta' \\ \xi' & \xi' & \zeta' \\ \zeta' & \zeta' & v' \end{pmatrix}$$

then

$$K \begin{pmatrix} t^2\xi & t^2\xi & t\zeta \\ t^2\xi & t^2\xi & t\zeta \\ t\zeta & t\zeta & v \end{pmatrix} = \begin{pmatrix} t^2\xi' & t^2\xi' & t\zeta' \\ t^2\xi' & t^2\xi' & t\zeta' \\ t\zeta' & t\zeta' & v' \end{pmatrix}.$$

□

Corollary 1 *The exceptional hypersurfaces of* K_Z *are* $\mathcal{A}^{i,i}$ *(for* $1 \le i \le q$*),* $\mathcal{A}^{i,j}$ *(for* $1 \le i < j \le q$*), and* $\mathcal{C}^{i,j}$ *(for* $1 \le i < j \le q$*).*

Let us consider a matrix

$$x = \begin{pmatrix} \xi_{1,1} & \xi_{1,2} & \zeta \\ \xi_{1,2} & \xi_{2,2} & \zeta \\ \zeta & \zeta & v \end{pmatrix},$$

written as in (3.10). That is, the ξ's and $\zeta's$ fill out first two rows and first two columns. We will consider algebraic subvarieties $W \subset \mathcal{S}_q$ with the property that whenever $x \in W$, then

$$\begin{pmatrix} t^2\xi_{1,1} & t^2\xi_{1,2} & t\zeta \\ t^2\xi_{1,2} & t^2\xi_{2,2} & t\zeta \\ t\zeta & t\zeta & v \end{pmatrix} \in W, \tag{3.12}$$

for all $\mathbb{C} \ni t \neq 0$. If W has this property, and if no component of W is contained in the indeterminacy loci of I, J, and K, then so do $I(W)$, $J(W)$, and $K(W)$.

We say that an irreducible hypersurface W is compatible with $\mathcal{D}^{1,2}$ if condition (3.12) is satisfied and if moreover

$$W \not\subseteq JR_{q-1} \cup \bigcup_{(k,l) \neq (1,1),(1,2),(2,2)} \Sigma_{k,l}.$$

When W is compatible, then W is not contained in any of the centers of blowups in the construction of Z, thus we can take its strict transform inside Z and define $\mathcal{D}^{1,2} \cap W \subset Z$. Using coordinate projections analogous to (3.10), we may also define what it means for W to be compatible with $\mathcal{D}^{k,l}$ for $1 \leq k < l \leq q$. Note that both hypersurfaces $\Sigma_{1,2}$ and $I(\Sigma_{1,2})$ are compatible to $\mathcal{D}^{1,2}$.

Similarly to Proposition 4, we obtain

Proposition 6 *For $1 \leq i < j \leq q$:*

If W is compatible with $\mathcal{D}^{i,j}$ and $W \not\subseteq \Sigma_{i,i} \cup \Sigma_{i,j} \cup \Sigma_{j,j}$, then $K_Z(\mathcal{D}^{i,j} \cap W) = \mathcal{D}^{i,j} \cap K(W)$.
If $W = \Sigma_{i,j}$, then $K_Z(K_Z(\mathcal{D}^{i,j} \cap \Sigma_{i,j})) = \mathcal{D}^{i,j} \cap I(\Sigma_{i,j})$.

Moreover, $K_Z(\mathcal{D}^{i,j} \cap \Sigma_{i,j})$ can be explicitly written. For example, if $i = 1$, $j = 2$, then in the local coordinate system (3.11) we have: $K_Z(\mathcal{D}^{1,2} \cap \Sigma_{1,2}) = \{t = \lambda = 0\}$.

4 A lower bound for $\delta(K)$

We will use the notation:

$$S = \bigcup_{i \neq j} \mathcal{A}^{i,j}, \ U = Z \backslash S.$$

In this section we will show that instead of establishing the property (1.3) for K_Z, we can work with the restriction of K_Z to the Zariski dense open subset U of Z.

We denote by $\mathcal{I}(K_Z)$ the indeterminacy locus of K_Z.

Lemma 1 *For any $n \geq 1$, and for any $1 \leq i < j \leq q$:*

$K_Z^n(\mathcal{A}^{i,i})$ is a subvariety of codimension 1 of $\mathcal{B}^{i,i}$, and is not contained in $\mathcal{I}(K_Z) \cup S$.
$K_Z^n(\mathcal{C}^{i,j})$ is a subvariety of codimension 1 of $\mathcal{D}^{i,j}$, and is not contained in $\mathcal{I}(K_Z) \cup S$.

Proof In the following, as noted before, we assume that $q \geq 5$. We present the proof only for $\mathcal{A}^{1,1}$, since the proofs for other $\mathcal{A}^{i,i}$'s and for $\mathcal{C}^{i,j}$'s are similar.

By Proposition 3, we know that $K_Z(\mathcal{A}^{1,1}) = \mathcal{B}^{1,1} \cap I(\Sigma_{1,1})$. Hence from Proposition 4, as long as $K^m(I(\Sigma_{1,1})) \not\subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}$ for all $m = 0, \ldots, n$ then $K_Z^{m+1}(\mathcal{A}^{1,1}) = \mathcal{B}^{1,1} \cap K^m(I(\Sigma_{1,1}))$, for all $m = 0, \ldots, n$. Each of these varieties is a subvariety of codimension 1 of $\mathcal{B}^{1,1}$, and is not contained in the indeterminacy locus of K_Z. Moreover, $K^m(I(\Sigma_{1,1}))$ is then compatible to $\mathcal{B}^{1,1}$, hence $\mathcal{B}^{1,1} \cap K^m(I(\Sigma_{1,1}))$ is defined in the local coordinate (3.4) by $\{t = 0, P(\xi, \zeta, \upsilon) = 0)\}$ where $P(x_{i,j}) = 0$ is the equation in $\mathcal{S}_q$ of $K^m(I(\Sigma_{1,1}))$. From this, it is easy to see that $\mathcal{B}^{1,1} \cap K^m(I(\Sigma_{1,1}))$ is not contained in $\bigcup_{k \neq l} \mathcal{A}^{k,l}$.

Hence it remains to explore what happens in case $K^n(I(\Sigma_{1,1})) \subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}$ for some n. We choose $n = n_0$ to be the smallest integer satisfying $K^n(I(\Sigma_{1,1})) \subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}$. It is not difficult to see that $I(\Sigma_{1,1}) \not\subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}$, hence $n_0 > 0$, and then by definition of n_0:

$$K^m(I(\Sigma_{1,1})) \not\subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}, \tag{4.1}$$

for all $m = 0, \ldots, n_0 - 1$, and

$$K^m(I(\Sigma_{1,1})) \subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}. \tag{4.2}$$

Since $I(\Sigma_{1,1})$ is an irreducible hypersurface, K is a birational map, and since JR_{q-1} and $\Sigma_{k,l}$'s are the only exceptional hypersurfaces of K, (4.1) and (4.2) imply that for all $m = 0, \ldots, n_0$: $K^m(I(\Sigma_{1,1}))$ is an irreducible hypersurface in $\mathcal{S}_q$. Moreover, either

$$K^{n_0}(I(\Sigma_{1,1})) = JR_{q-1}, \tag{4.3}$$

or

$$K^{n_0}(I(\Sigma_{1,1})) = \Sigma_{i,j}, \tag{4.4}$$

for some $1 \le i, j \le q$.

Now we show that in fact

$$K^{n_0}(I(\Sigma_{1,1})) = \Sigma_{1,1}. \tag{4.5}$$

To this end, we will use the operations $\rho_{l,m}$ defined as follows: For $1 \le l, m \le q$, let $\rho_{l,m} : \mathcal{S}_q \to \mathcal{S}_q$ denote the matrix operation which interchanges the lth and mth rows, and then interchanges the lth and mth columns of a matrix $x \in \mathcal{S}_q$. Observe that on the space $\mathcal{S}_q : \rho_{l,m}(I(x)) = I(\rho_{l,m}(x))$, $\rho_{l,m}(J(x)) = J(\rho_{l,m}(x))$, and $\rho_{l,m}(K(x)) = K(\rho_{l,m}(x))$. In particular, $\rho_{l,m} JR_{q-1} = JR_{q-1}$.

First we rule out the possibility (4.3). Assume in order to reach a contradiction that $K^{n_0}(I(\Sigma_{1,1})) = JR_{q-1}$. Then for all i we have

$$K^{n_0}(I(\Sigma_{i,i})) = K^{n_0}(I(\rho_{i,1}\Sigma_{1,1})) = \rho_{i,1}K^{n_0}(I(\Sigma_{1,1})) = \rho_{i,1}JR_{q-1} = JR_{q-1}.$$

Hence q different irreducible hypersurfaces $I(\Sigma_{1,1}), \ldots, I(\Sigma_{q,q})$ are mapped under K^{n_0} to the same irreducible hypersurfaces JR_{q-1}. But this would be a contradiction to the fact that K^{n_0} is birational. Thus we showed that (4.3) does not occur. Hence (4.4) must occur.

We next show that $K^{n_0}(I(\Sigma_{1,1})) = \Sigma_{1,1}$. We know that $K^{n_0}(I(\Sigma_{1,1})) = \Sigma_{i,j}$, for some $1 \le i, j \le q$. We need to show that $i = j = 1$. Assume in order to reach a contradiction that $i \ne 1$ or $j \ne 1$. We have two cases:

Case 1: Both $i, j \ne 1$. Choose $k \ne i, j, 1$, we have then:

$$K^{n_0}(I(\Sigma_{k,k})) = K^{n_0}(I(\rho_{k,1}\Sigma_{1,1})) = \rho_{k,1}K^{n_0}(I(\Sigma_{1,1})) = \rho_{k,1}\Sigma_{i,j} = \Sigma_{i,j}.$$

Hence two different irreducible hypersurfaces $I(\Sigma_{1,1})$ and $I(\Sigma_{k,k})$ have the same image $\Sigma_{i,j}$ under the birational mapping K^{n_0}, which is a contradiction.

Case 2: One of i, j is 1, but the other is not. Without loss of generality, we may assume that $i = 1$ and $j \ne 1$. Then

$$K^{n_0}(I(\Sigma_{j,j})) = K^{n_0}(I(\rho_{1,j}\Sigma_{1,1})) = \rho_{1,j}K^{n_0}(I(\Sigma_{1,1})) = \rho_{1,j}\Sigma_{1,j} = \Sigma_{1,j}.$$

Hence two different irreducible hypersurfaces $I(\Sigma_{1,1})$ and $I(\Sigma_{j,j})$ have the same image $\Sigma_{1,j}$ under the birational map K^{n_0}, which is again a contradiction.

Hence we showed that if $n_0 > 0$ is the smallest integer such that $K^{n_0}(I(\Sigma_{1,1})) \subset JR_{q-1} \cup \bigcup_{k,l} \Sigma_{k,l}$, then for all $m = 0, \ldots, n_0$, $K^m(I(\Sigma_{1,1}))$ is an irreducible hypersurface of $\mathcal{S}_q$, and $K^{n_0}(I(\Sigma_{1,1})) = \Sigma_{1,1}$. Hence by Proposition 4, for all $m = 0, \ldots, n_0$: $K_Z^m(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1})) = \mathcal{B}^{1,1} \cap K^m(I(\Sigma_{1,1}))$ is a subvariety of codimension 1 of $\mathcal{B}^{1,1}$, and such that (by Proposition 3) $K_Z^{n_0+1}(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1})) = K_Z(\mathcal{B}^{1,1} \cap \Sigma_{1,1})$ is a subvariety of codimension 1 of $\mathcal{B}^{1,1}$. Moreover

$$K_Z^{n_0+2}(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1})) = K_Z(K_Z(\mathcal{B}^{1,1} \cap \Sigma_{1,1})) = \mathcal{B}^{1,1} \cap I(\Sigma_{1,1}) = K_Z(\mathcal{A}^{1,1}).$$

Hence if (4.2) happens, then the orbit of $K_Z(\mathcal{A}^{1,1})$ under K_Z is periodic. Thus the orbit of $K_Z(\mathcal{A}^{1,1})$ never lands in $\mathcal{I}(K_Z)$.

To complete the proof, we need to show that the orbit never lands in $S = \bigcup_{i \neq j} \mathcal{A}^{i,j}$. That $K_Z^{n_0}(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1}))$, which equals $\mathcal{B}^{1,1} \cap \Sigma_{1,1}$, is not contained in S can be checked directly. For values m when $K_Z^m(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1})) \neq \mathcal{B}^{1,1} \cap \Sigma_{1,1}$, we can use the argument at the end of the second paragraph of this proof to show that $K_Z^m(\mathcal{B}^{1,1} \cap I(\Sigma_{1,1}))$ (which is then equal to $\mathcal{B}^{1,1} \cap K_Z^m(I(\Sigma_{1,1})))$ is not contained in S as well. □

By Lemma 1, we obtain the following result

Corollary 2 *If V is an irreducible hypersurface which is not contained in S then for any $n \geq 1$: $K_Z^n(V)$ is not contained in $\mathcal{I}(K_Z) \cup S$.*

Let V be a hypersurface (or divisor) of Z. We let $V|_U$ denote the restriction to U. Let $R_U(V)$ denote the "extension by zero" of $V|_U$ to Z. We let $(K_Z^n)^*(V)$ denote the pull-back of V by the map K_Z^n.

Proposition 7 *If V is a hypersurface on Z, then for all $n \geq 1$:*

$$R_U((K_Z^n)^*V) = R_U((K_Z^n)^*R_U(V)) = R_U((K_Z^*)^nV) = R_U((K_Z^*)^nR_U(V)), \tag{4.6}$$

as divisors on Z. In particular, if $R_U(V) = 0$ then for all $n \geq 1$: $R_U((K_Z^n)^*V) = 0$.

Proof Before applying R_U on the left, the difference between any two of the divisors in Eq. (4.6) is a hypersurface supported in $K_Z^{-j}(\mathcal{I}(K_Z) \cup S)$. However, by Corollary 2, this last set is disjoint from U, hence the difference vanishes on applying R_U. □

Define $\Lambda := Pic(Z)/ker(R_U)$, and let $pr_\Lambda : Pic(Z) \to \Lambda$ be the canonical projection. By Proposition 7, the maps $pr_\Lambda \circ (K_Z^n)^* : Pic(Z) \to \Lambda$ induce well-defined maps $L_n : \Lambda \to \Lambda$ which satisfy the identities: $L_n = (L_1)^n$ for all $n \geq 1$.

Theorem 2 *$\delta(K) \geq sp(L_1)$, where $sp(L_1)$ is the spectral radius of L_1.*

Proof The dynamical degree $\delta(K_Z) = \lim_{n\to\infty} ||(K_Z^n)^*||^{1/n}$ is independent of the choice of norm $||.||_{Pic(Z)}$ on $Pic(Z)$. Further, since π_Z is a birational map, we have that $\delta(K_Z) = \delta(K)$ (see for example [10], and see [9] for more general results). Finally, if we use the induced norm on Λ, we have

$$\lim_{n\to\infty} ||(K_Z^n)^*||_{Pic(Z)}^{1/n} \geq \lim_{n\to\infty} ||L_n||_\Lambda^{1/n} = \lim_{n\to\infty} ||(L_1)^n||_\Lambda^{1/n} = sp(L_1).$$

□

5 The spectral radius of L_1

A basis for the Picard group $Pic(Z)$ is given by H (the class of a generic hyperplane in $\mathcal{S}_q$), and the classes of the strict transforms of $\mathcal{R}^1$, $\mathcal{A}^{i,i}$'s $(1 \leq i \leq q)$, $\mathcal{B}^{i,i}$'s $(1 \leq i \leq q)$, $\mathcal{A}^{i,j}$'s $(1 \leq i < j \leq q)$, $\mathcal{C}^{i,j}$'s $(1 \leq i < j \leq q)$, and $\mathcal{D}^{i,j}$'s $(1 \leq i < j \leq q)$. The images under pr_Λ of classes of H and of the strict transforms of $\mathcal{R}^1$, $\mathcal{A}^{i,i}$ $(1 \leq i \leq q)$, $\mathcal{B}^{i,i}$ $(1 \leq i \leq q)$, $\mathcal{C}^{i,j}$ $(1 \leq i < j \leq q)$, and $\mathcal{D}^{i,j}$ $(1 \leq i < j \leq q)$ form a basis for Λ. For convenience, we will use the same letters to denote the images of these classes in Λ. Further, we define

$$\mathcal{A} = \sum_i \mathcal{A}^{i,i},\ \mathcal{B} = \sum_i \mathcal{B}^{i,i},\ \mathcal{C} = 2\sum_{i<j} \mathcal{C}^{i,j},\ \mathcal{D} = 2\sum_{i<j} \mathcal{D}^{i,j}. \tag{5.1}$$

Let Λ_0 be the subspace of Λ generated by the ordered basis $H,\ \mathcal{R}^1,\ \mathcal{A},\ \mathcal{B},\ \mathcal{C}$ and $\mathcal{D}$.

Lemma 2 *The map L_1 restricted to Λ_0 is given by*

$$
\begin{aligned}
L_1(H) &= (q^2-q+1)H-(q-2)\mathcal{R}^1-(2q-3)\mathcal{A}-(2q-2)\mathcal{B}-(2q-3)\mathcal{C}-(2q-2)\mathcal{D},\\
L_1(\mathcal{R}^1) &= (q^2-q)H-(q-1)\mathcal{R}^1-(2q-3)\mathcal{A}-(2q-2)\mathcal{B}-(2q-3)\mathcal{C}-(2q-2)\mathcal{D},\\
L_1(\mathcal{A}) &= qH-\mathcal{A}-2\mathcal{B}-2\mathcal{C}-2\mathcal{D},\\
L_1(\mathcal{B}) &= \mathcal{A}+\mathcal{B},\\
L_1(\mathcal{C}) &= (q^2-q)H-(2q-2)\mathcal{A}-(2q-2)\mathcal{B}-(2q-3)\mathcal{C}-(2q-2)\mathcal{D},\\
L_1(\mathcal{D}) &= \mathcal{C}+\mathcal{D}.
\end{aligned}
$$

In particular, Λ_0 is invariant under L_1, and the spectral radius of $L_1|\Lambda_0$ is the largest root of the polynomial $\lambda^2-(q^2-4q+2)\lambda+1$.

Proof The proof is similar to the proof of Proposition 6.1 in [5]. For example, we determine $L_1(H)$. There are integers $a,\ b,\ \alpha_{i,i},\ \beta_{i,i},\ \gamma_{i,j}$ and $\lambda_{i,j}$ such that

$$
\begin{aligned}
L_1(H) = aH - b\mathcal{R}^1 - \sum_{1\le i\le q} \alpha_{i,i}\mathcal{A}^{i,i}\\
- \sum_{1\le i\le q} \beta_{i,i}\mathcal{B}^{i,i} - \sum_{1\le i<j\le q} \gamma_{i,j}\mathcal{C}^{i,j} - \sum_{1\le i<j\le q} \lambda_{i,j}\mathcal{D}^{i,j}.
\end{aligned}
$$

By symmetry, there are constants $\alpha,\ \beta,\ \gamma$ and λ such that $\alpha_{i,i}=\alpha,\ \beta_{i,i}=\beta,\ \gamma_{i,j}=\gamma$ and $\lambda_{i,j}=\lambda$ for all $1\le i<j\le q$. Thus

$$L_1(H) = aH - b\mathcal{R}^1 - \alpha\mathcal{A} - \beta\mathcal{B} - \frac{1}{2}\gamma\mathcal{C} - \frac{1}{2}\lambda\mathcal{D}.$$

Recall from Proposition 1 that the homogeneous form of K is

$$\widehat{K}_{i,j}(x) = C_{i,j}(1/x)\prod(x),$$

where $x=(x_{k,l})_{1\le k,l\le q}\in\mathcal{S}_q$.

The coefficient a is the degree of K, so by Proposition 1, we have $a=q^2-q+1$. To find the other coefficients, we let $H=\{l=0\}$ where $l=\sum c_{i,j}x_{i,j}$, and we determine the order of vanishing of $\widehat{K}\circ l$ at the various divisors.

The constant b is the order of vanishing of $\widehat{K}\pi_{\mathcal{R}^1}(s,v,\nu)$ in s, where $\pi_{\mathcal{R}^1}$ is given in (3.1). For $\nu=(\nu_1,\dots,\nu_q)$ with $\nu_1\dots\nu_q\ne 0$, $\prod(\pi_{\mathcal{R}^1}(s,v,\nu))\ne 0$ when $s=0$. Further

$$\frac{1}{\pi_{\mathcal{R}^1}(s,v,\nu)} = \frac{1}{\nu}\otimes\frac{1}{\nu}+O(s).$$

Since $\frac{1}{\nu}\otimes\frac{1}{\nu}$ has rank 1, $C_{i,j}(1/\pi_{\mathcal{R}^1}(s,v,\nu))=O(s^{q-2})$. Thus $b=q-2$.

The constant α is the order of vanishing of $\widehat{K}\pi_{\mathcal{A}^{1,1}}(s,\zeta,v)$ in s, where $\pi_{\mathcal{A}^{1,1}}$ is given in (3.3). The order of vanishing of $\prod(\pi_{\mathcal{A}^{1,1}}(s,\zeta,v))$ in s is $2q-1$, since only the entries on the first row and first column of the matrix $\pi_{\mathcal{A}^{1,1}}(s,\zeta,v)$ vanish when $s=0$, and moreover all of these entries vanishes to order 1 in s. The minimal order of vanishing of $C_{i,j}(1/(\pi_{\mathcal{A}^{1,1}}(s,\zeta,v)))$ $(1\le i,j\le q)$ in s is -2, since $C_{i,j}(1/(\pi_{\mathcal{A}^{1,1}}(s,\zeta,v)))$ is a sum whose summands are of the form $\pm\sigma_1\sigma_2\dots\sigma_{q-1}$, where σ_i are entries of $1/\pi_{\mathcal{A}^{1,1}}(s,\zeta,v))$ and not any two of them are from a same row or column. Thus $\alpha=2q-3$.

The constants $\beta=2q-2$, $\gamma=4q-6$, and $\lambda=4q-4$ are similarly determined. Hence $L_1(H)$ is as in the statement of the lemma. $\square$

Proof of Theorem 1 By Theorem 2 and Lemma 2, we have $\delta(K) \geq sp(L_1) \geq sp(L_1|\Lambda_0) =$ the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$. Because the degree complexity of the matrix inversion restricted to $\mathcal{S}_q$ is not larger than that of the general matrices, and since the value of the later is equal to the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$ (see [5]), we conclude that $\delta(K) =$ the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$.

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