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On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties

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Abstract. Let $f : X \dashrightarrow X$ be a dominant rational map of a smooth projective variety defined over a characteristic 0 global field K, let δ_f be the dynamical degree of f, and let $h_X : X(\bar{K}) \to [1, \infty)$ be a Weil height relative to an ample divisor. We prove that for every $\epsilon > 0$ there is a height bound

$$h_X \circ f^n \ll (\delta_f + \epsilon)^n h_X,$$

valid for all points whose f-orbit is well-defined, where the implied constant depends only on X, h_X, f, and ϵ. An immediate corollary is a fundamental inequality $\overline{\alpha}_f(P) \le \delta_f$ for the upper arithmetic degree. If further f is a morphism and D is a divisor satisfying an algebraic equivalence $f^*D \equiv \beta D$ for some $\beta > \sqrt{\delta_f}$, we prove that the canonical height

$$\hat{h}_{f,D} = \lim \beta^{-n} h_D \circ f^n$$

converges and satisfies $\hat{h}_{f,D} \circ f = \beta \hat{h}_{f,D}$ and $\hat{h}_{f,D} = h_D + O(\sqrt{h_X})$. We also prove that the arithmetic degree $\alpha_f(P)$, if it exists, gives the main term in the height counting function for the f-orbit of P. We conjecture that $\overline{\alpha}_f(P) = \delta_f$ whenever the f-orbit of P is Zariski dense and describe some cases for which we can prove our conjecture.

Introduction

Let $X/\mathbb{C}$ be a smooth projective variety, and let $f : X \dashrightarrow X$ be a dominant rational map. The dynamical degree of f is a measure of the geometric complexity of the iterates f^n of f. More precisely, it measures the complexity of the induced maps $(f^n)^*$ of the iterates of f on the Néron–Severi group $\mathrm{NS}(X)_{\mathbb{R}}$ of X, where we note that in general $(f^n)^*$ need not be equal to $(f^*)^n$. (By a divisor we mean a Cartier divisor, and $\mathrm{NS}(X)$ denotes the set of Cartier divisors on X modulo algebraic equivalence. We write $\mathrm{NS}(X)_{\mathbb{R}}$ for $\mathrm{NS}(X) \otimes \mathbb{R}$, and similarly for $\mathrm{NS}(X)_{\mathbb{Q}}$ and $\mathrm{NS}(X)_{\mathbb{C}}$.)

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Definition. Let $X/\mathbb{C}$ be a (smooth) projective variety and let $f : X \dashrightarrow X$ be a dominant rational map as above. The *dynamical degree of f* is

$$\delta_f = \lim_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}})^{1/n},$$

where in general $\rho(A, V)$ denotes the spectral radius of a linear transformation $A : V \to V$ of a real or complex vector space. The limit defining δ_f converges and is a birational invariant, so in particular there is no need to assume that X is smooth; see [20, Proposition 1.2 (iii)], Remark 7, and Corollary 16.

The study of the dynamical degree and its relation to entropy was initiated in [3, 33] and is currently an area of active research; see for example [1, 4–9, 12, 13, 16, 22, 28–30, 32, 37]. In this article we describe how the geometrically defined dynamical degree of a map limits the arithmetic complexity of its orbits, and we prove an inequality relating the dynamical degree to an analogous arithmetic degree defined in [36].

Before stating our main results, we set some notation that will be used throughout this article.

- K: either a number field or a one-dimensional function field of characteristic 0. We let $\bar{K}$ be an algebraic closure of K.
- X, f: either X is a smooth projective variety and $f : X \dashrightarrow X$ is a dominant rational map, all defined over K; or X is a normal projective variety and $f : X \to X$ is a dominant morphism, all defined over K. (See also Remark 8.)
- h_X: an (absolute logarithmic) Weil height $h_X : X(\bar{K}) \to [0, \infty)$ relative to an ample divisor.
- h_X^+: for convenience, we set $h_X^+(P) = \max\{h_X(P), 1\}$.
- $\mathcal{O}_f(P)$: the (*forward*) *f-orbit* of P, i.e., $\mathcal{O}_f(P) = \{f^n(P) : n \geq 0\}$.
- I_f: the *indeterminacy locus* of f, i.e., the set of points at which f is not well-defined.
- $X_f(\bar{K})$: the set of points $P \in X(\bar{K})$ whose forward orbit $\mathcal{O}_f(P)$ is well-defined, i.e., such that $f^n(P) \notin I_f$ for all $n \geq 0$. We note that $X_f(\bar{K})$ always contains many points; see [2].
- $\mathrm{Div}(X)$: the set of Cartier divisors on X.

We refer the reader to [11, 23, 27, 35] for basic definitions and properties of Weil height functions.

Our main theorem gives a uniform upper bound for the growth of points in orbits.

Theorem 1. *Fix $\epsilon > 0$. Then there is a constant $C = C(X, h_X, f, \epsilon)$ so that for all $n \geq 0$ and all $P \in X_f(\bar{K})$,*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P).$$

For rational maps $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ of projective space, Theorem 1 above was essentially proven in [36, Proposition 12]. The same proof works, *mutatis mutandis*, for varieties satisfying $\mathrm{Pic}(X)_{\mathbb{R}} = \mathbb{R}$, and, with a little more work, for varieties satisfying $\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$. But if $\mathrm{NS}(X)_{\mathbb{R}}$ has dimension greater than 1, then the proof of Theorem 1, which we give in Section 5 after several sections of preliminary results, is considerably more intricate.

We next consider the arithmetic degree of a map at a point, as introduced in [36]. We recall the relevant definitions, give an elementary counting result, and then describe an inequality that was a primary motivation for the research that led to this paper.

Definition. Let $P \in X_f(\bar{K})$. The *arithmetic degree of f at P* is the quantity

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{1/n},$$

assuming that the limit exists.

The arithmetic degree of f at P measures the growth rate of the height $h_X(f^n(P))$ as $n \to \infty$. It is thus a measure of the arithmetic complexity of the f-orbit of P.

Conjecture 2. *The limit defining $\alpha_f(P)$ exists for all $P \in X_f(\bar{K})$.*

One reason for studying the arithmetic degree is that it determines the height counting function for points in orbits, as in the following elementary result, which we prove in Section 2.

Proposition 3. *Let $P \in X_f(\bar{K})$ be a wandering point, i.e., a point whose orbit $\#\mathcal{O}_f(P)$ is infinite. Assume further that the arithmetic degree $\alpha_f(P)$ exists. Then*

$$\lim_{B\to\infty} \frac{\#\{Q \in \mathcal{O}_f(P) : h_X(Q) \le B\}}{\log B} = \frac{1}{\log \alpha_f(P)}. \tag{0.1}$$

(If $\alpha_f(P) = 1$, then (0.1) *is to be read as saying that the limit is equal to ∞.)*

Definition. Since for the moment we lack a proof of Conjecture 2, we define *upper and lower arithmetic degrees*,

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n} \quad \text{and} \quad \underline{\alpha}_f(P) = \liminf_{n\to\infty} h_X^+(f^n(P))^{1/n}.$$

As a corollary to Theorem 1, we obtain the following fundamental inequality relating the dynamical degree and the (upper) arithmetic degree. This inequality quantifies the statement that the arithmetic complexity of the f-orbit of an algebraic point P never exceeds the geometrical-dynamical complexity of the map f.

Theorem 4. *Let $P \in X_f(\bar{K})$. Then*

$$\overline{\alpha}_f(P) \le \delta_f.$$

Definition. We will make extensive use of the theory of *Weil heights*. This theory attaches to each divisor $D \in \mathrm{Div}(X)$ a function $h_D : X(\bar{K}) \to \mathbb{R}$, well-defined modulo bounded functions and satisfying various standard properties; see for example [11, 23, 27, 34, 35]. In particular, the linearity relation $h_{D+E} = h_D + h_E + O(1)$ allows us to $\mathbb{R}$-linearly extend the association $D \to h_D$ and define h_D for $D \in \mathrm{Div}(X)_\mathbb{R}$, i.e., we write

$$D = \sum c_i D_i$$

with $D_i \in \mathrm{Div}(X)$ and $c_i \in \mathbb{R}$ and set

$$h_D = \sum c_i h_{D_i}.$$

Classically, a polarized dynamical system is a triple (X, f, D) consisting of a morphism $f : X \to X$ and a divisor D satisfying a *linear equivalence* $f^*D \sim \beta D$ for some $\beta > 1$. (Often the definition also includes the condition that D be ample; cf. [39].) There is a well-known theory of canonical heights associated to polarized dynamical systems; see for example [14]. Using Theorem 1, we are able to partially generalize this theory to cover the case that the relation $f^*D \equiv \beta D$ is only an *algebraic equivalence*.

Theorem 5. *Assume that $f : X \to X$ is a morphism, and let $D \in \mathrm{Div}(X)_{\mathbb{R}}$ be a divisor that satisfies an algebraic equivalence*

$$f^*D \equiv \beta D \quad \textit{for some real number } \beta > \sqrt{\delta_f},$$

where $\equiv$ denotes equivalence in $\mathrm{NS}(X)_{\mathbb{R}}$.

(a) *For all $P \in X(\bar{K})$, the following limit converges:*

$$\hat{h}_{D,f}(P) = \lim_{n\to\infty} \beta^{-n} h_D(f^n(P)).$$

(b) *The canonical height $\hat{h}_{D,f}$ in* (a) *satisfies*

$$\hat{h}_{D,f}(f(P)) = \beta \hat{h}_{D,f}(P) \quad \textit{and} \quad \hat{h}_{D,f}(P) = h_D(P) + O\Big(\sqrt{h_X^+(P)}\Big).$$

(c) *If $\hat{h}_{D,f}(P) \neq 0$, then $\underline{\alpha}_f(P) \geq \beta$.*

(d) *If $\hat{h}_{D,f}(P) \neq 0$ and $\beta = \delta_f$, then $\alpha_f(P) = \delta_f$.*

(e) *Assume that D is ample and that K is a number field. Then*

$$\hat{h}_{D,f}(P) = 0 \iff P \textit{ is preperiodic.}$$

We note that not every morphism $f : X \to X$ admits a polarization (for linear equivalence), but that there always exists at least one nonzero nef divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ satisfying $f^*D \equiv \delta_f D$; see Remark 29. Hence every morphism f of positive algebraic entropy, i.e., with dynamical degree satisfying $\delta_f > 1$, admits a canonical height associated to a nef divisor.

Theorem 4 raises a natural question: Under what conditions is $\alpha_f(P)$ equal to δ_f, i.e., when does the arithmetic complexity of the f-orbit of a point P fully capture the geometrical-dynamical complexity of f? This leads to the following multi-part conjecture, into which we have incorporated Conjecture 2, as well as an integrality conjecture suggested by a classical conjecture [9] on the integrality of δ_f. See also [36, Conjecture 40], in which (b), (c), and (d) were conjectured for $\overline{\alpha}_f(P)$.

Conjecture 6. *Let $P \in X_f(\bar{K})$.*

(a) *The limit defining $\alpha_f(P)$ exists.*

(b) *$\alpha_f(P)$ is an algebraic integer.*

(c) *The collection of arithmetic degrees $\{\alpha_f(Q) : Q \in X_f(\bar{K})\}$ is a finite set.*

(d) *If the forward orbit $\mathcal{O}_f(P)$ is Zariski dense in X, then $\alpha_f(P) = \delta_f$.*

In the final section of this paper we briefly indicate some cases for which we can prove Conjecture 6. These include morphisms f when $\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$, regular affine automorphisms, surface automorphisms, and monomial maps. The proofs of these results, together with other cases for which we can prove the weaker statement that $\alpha_f(P) = \delta_f(X)$ for a Zariski dense set of points $P \in X_f(\bar{K})$ having disjoint orbits, will appear in a companion publication [24]. See also [25] for a proof of Conjecture 6 (a)–(c) when f is a morphism and (d) when f is an endomorphism of an abelian variety.

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1. Some brief remarks

In this section we discuss pull-back maps and make some brief remarks about dynamical degrees, arithmetic degrees, and canonical heights.

Let X be a projective variety and let D be a Cartier divisor. If $f : X \to X$ is a surjective morphism, then the pull-back f^*D is a Cartier divisor. Assume now that X is smooth and that $f : X \dashrightarrow X$ is merely a dominant rational map. In this case, the pull-back f^*D is defined as follows.

We take a smooth projective variety $\widetilde{X}$ and a birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{f} := f \circ \pi : \widetilde{X} \to X$ is a morphism:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{f}"] & \\ X \arrow[r, dashed, "f"] & X. \end{tikzcd}$$

We have the pull-back $\widetilde{f}^*D$, which is a Cartier divisor. We regard $\widetilde{f}^*D$ as a Weil divisor. Then, as Weil divisors, we have the push-forward $\pi_*(\widetilde{f}^*D)$, which we denote by

$$f^*D := \pi_*(\widetilde{f}^*D).$$

Since X is smooth, we regard f^*D as a Cartier divisor. Thus for a Cartier divisor D on X, we have the pull-back Cartier divisor f^*D on X.

We note that f^*D is independent of the choice of $\widetilde{X}$. Indeed, suppose that $\widetilde{X}'$ is a smooth projective variety with a birational morphism $\pi' : \widetilde{X}' \to X$ such that $\widetilde{f}' := f \circ \pi' : \widetilde{X} \to X$ is a morphism. Let $\widetilde{X}''$ be a resolution of the main part of $\widetilde{X} \times_X \widetilde{X}'$, and let $p : \widetilde{X}'' \to \widetilde{X}$ and $p' : \widetilde{X}'' \to \widetilde{X}'$ be the first and the second projections. Since $\widetilde{f}' \circ p'$ and $\widetilde{f} \circ p$ are morphisms from $\widetilde{X}''$ to X that agree on a Zariski open subset of $\widetilde{X}''$, we have

$$\widetilde{f}' \circ p' = \widetilde{f} \circ p.$$

Similarly we obtain

$$\pi \circ p = \pi' \circ p'.$$

Then we have

$$\begin{aligned}\pi_*(\widetilde{f}^* D) &= \pi_*(p_* p^*(\widetilde{f}^* D)) \\ &= (\pi \circ p)_*(\widetilde{f} \circ p)^*(D) \\ &= (\pi' \circ p')_*(\widetilde{f}' \circ p')^*(D) \\ &= \pi'_*(p'_* p'^*(\widetilde{f}'^* D)) \\ &= \pi'_*(\widetilde{f}'^* D).\end{aligned}$$

Thus f^*D is independent of the choice of $\widetilde{X}$.

Remark 7. Let X be a smooth projective variety. Let H be an ample divisor on X, and let $N = \dim(X)$. Then [20, Proposition 1.2 (iii)] says that

$$\lim_{n\to\infty} ((f^n)^* H \cdot H^{N-1})^{1/n} = \limsup_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}}))^{1/n}.$$

(Notice the right-hand side is a lim sup.) We will prove below (Corollary 16) that the limit

$$\lim_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}}))^{1/n}$$

exists, justifying our definition of δ_f in terms of the action of $(f^n)^*$ on $\mathrm{NS}(X)_{\mathbb{R}}$, but we note that the alternative definition of δ_f using intersection theory is more common and often more useful.

Remark 8. We have restricted our variety X to be smooth when f is not a morphism. In our original formulation, we had only assumed that X is normal. We thank Najmuddin Fakhruddin and the referee for pointing out that some conditions are necessary to define the pull-back f^* on $\mathrm{NS}(X)_{\mathbb{R}}$ for a dominant rational map $f : X \dashrightarrow X$. Fakhruddin has indicated that it should suffice to take X to be $\mathbb{Q}$-factorial. We use the Lefschetz Hyperplane Theorem in the proof of Lemma 18, but for a singular variety, one can use a version of the Lefschetz Hyperplane Theorem [18, Theorem, p. 153] for a general member of the linear system of a very ample divisor. Alternatively, if the orbit $\mathcal{O}_f(P)$ of P lies within the smooth locus X^{sm} of X, as is often the case, then one can simply replace X with a smooth model of a projective closure of X^{sm} and reduce to the smooth case.

Remark 9. If $\mathcal{O}_f(P)$ is not Zariski dense, then we can look at the restriction of the map f to the Zariski closure $Y = \overline{\mathcal{O}_f(P)} \subset X$ of the orbit. If Y is nonsingular, then applying Conjectures 6 (a, d) to the dominant rational map $f|_Y : Y \dashrightarrow Y$ and the dense orbit of the point $P \in Y_f(\bar{K})$ gives $\alpha_f(P) = \delta_{f|_Y}$. (Note that $\alpha_f(P)$ is independent of whether we view P as a point of X or a point of Y, since the restriction to Y of an ample height function h_X on X gives an ample height function on Y.)

More generally, suppose that there exist two integers $k \geq 1$ and $m \geq 0$ such that the orbit $\mathcal{O}_{f^m}(f^k(P))$ does not intersect Y^{sing}, the singular locus of Y. Then assuming Conjectures 6 (a, d), an elementary argument gives

$$\alpha_f(P)^k = \alpha_f(f^m(P))^k = \alpha_{f^k}(f^m(P)) = \delta_{f^k|_Y} = \delta_{f|_Y}^k,$$

so again we obtain $\alpha_f(P) = \delta_{f|_Y}$. The existence of such an m and k is tied up with the dynamical Mordell–Lang conjecture, which would imply that the set

$$\{m \geq 0 : f^m(P) \in Y^{\mathrm{sing}}\} \tag{1.1}$$

consists of a finite union of arithmetic progressions. (Note that the set in (1.1) is not all of $\mathbb{N}$, since $\mathcal{O}_f(P)$ is dense in Y and Y^{sing} is a proper closed subset of Y.) We also note that Bellon and Viallet [9] have conjectured that the dynamical degree of a rational map is always an algebraic integer. Thus Conjectures 6 (b, c) more-or-less follow from Conjectures 6 (a, d), the Bellon–Viallet conjecture, and the dynamical Mordell–Lang conjecture.

Remark 10. We use h_X^+ instead of h_X in the definition of arithmetic degree simply to ensure that $\underline{\alpha}_f(P) \geq 1$, even in the rare situation that P is periodic and $h_X(f^n(P)) = 0$ for some n. We also note that the arithmetic degree is independent of the choice of ample height function h_X; see Proposition 12.

Remark 11. Let $f : X \to X$ be a morphism with $\delta_f > 1$, and let $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ be an ample divisor class satisfying the linear equivalence $f^*D \sim \delta_f D$. Then using properties of the classical canonical height $\hat{h}_{D,f}$, as described for example in [14], it is an exercise to show that

$$\hat{h}_f(P) > 0 \implies \alpha_f(P) = \delta_f.$$

In the number field case, it is also an exercise to prove that

$$\hat{h}_f(P) = 0 \implies \#\mathcal{O}_f(P) < \infty,$$

so in particular, Conjecture 6 is true in this case. There are other situations in which one can define a canonical height having sufficiently good properties to prove Conjecture 6; see Section 8 and [24, 36] for examples and further details. But in general, a rational map, or even a morphism, does not have a canonical height with sufficiently good properties to directly imply Conjecture 6 (d). The arithmetic degree $\alpha_f(P)$, although coarser than an ample canonical height, may be viewed as a general nontrivial measure of the arithmetic complexity of the f-orbit of P.

2. Basic properties of the arithmetic degree

In this section we verify that the upper and lower arithmetic degrees are well-defined, independent of the choice of height function h_X on X, and we prove a counting result for points in orbits. We also prove two useful lemmas.

Proposition 12. *The upper and lower arithmetic degrees $\overline{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ are independent of the choice of the height function h_X.*

Proof. If P has finite f-orbit, then it is clear from the definition that the limit $\alpha_f(P)$ exists and is equal to 1, regardless of the choice of h_X. We assume henceforth that P is not preperiodic, which means that we can replace h_X^+ with h_X when taking limits over the orbit of P.

Let h and h' be heights on X relative to ample divisors D and D', and let the corresponding arithmetic degrees be denoted respectively by $\overline{\alpha}_f(P)$, $\underline{\alpha}_f(P)$, $\overline{\alpha}'_f(P)$, and $\underline{\alpha}'_f(P)$. By definition of ampleness [21, Section II.7], there is an integer m such that $mD - D'$ is ample, so standard functorial properties of height functions, as described for example in [27] or [23, Theorem B.3.2], imply that there is a non-negative constant C such that

$$mh(Q) \geq h'(Q) - C \quad \text{for all } Q \in X(\bar{K}). \tag{2.1}$$

We choose a sequence of indices $\mathcal{N} \subset \mathbb{N}$ such that

$$\lim_{n \in \mathcal{N}} h'(f^n(P))^{1/n} = \limsup_{n \to \infty} h'(f^n(P))^{1/n} = \overline{\alpha}'_f(P). \tag{2.2}$$

Then

$$\begin{aligned}
\overline{\alpha}'_f(P) &= \lim_{n \in \mathcal{N}} h'(f^n(P))^{1/n} && \text{from (2.2)} \\
&\leq \lim_{n \in \mathcal{N}} (mh(f^n(P)) + C)^{1/n} && \text{from (2.1)} \\
&\leq \limsup_{n \to \infty} (mh(f^n(P)) + C)^{1/n} \\
&= \limsup_{n \to \infty} h(f^n(P))^{1/n} \\
&= \overline{\alpha}_f(P).
\end{aligned}$$

This gives one inequality for the upper arithmetic degrees, and reversing the roles of h and h' gives the opposite inequality, which proves that $\overline{\alpha}'_f(P) = \overline{\alpha}_f(P)$. We omit the similar proof that $\underline{\alpha}'_f(P) = \underline{\alpha}_f(P)$. □

The following lemma says that both $\overline{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ depend only on the eventual orbit of P.

Lemma 13. *Let $f : X \dashrightarrow X$ be a rational map defined over $\bar{K}$. Then for all $P \in X_f(\bar{K})$ and all $k \geq 0$,*

$$\overline{\alpha}_f(f^k(P)) = \overline{\alpha}_f(P) \quad \textit{and} \quad \underline{\alpha}_f(f^k(P)) = \underline{\alpha}_f(P).$$

Proof. We compute

$$\begin{aligned}
\overline{\alpha}_f(f^k(P)) &= \limsup_{n \to \infty} h_X^+(f^{n+k}(P))^{1/n} \\
&= \limsup_{n \to \infty} \bigl(h_X^+(f^{n+k}(P))^{1/(n+k)}\bigr)^{1+k/n} \\
&= \limsup_{n \to \infty} h_X^+(f^{n+k}(P))^{1/(n+k)} \\
&= \overline{\alpha}_f(P).
\end{aligned}$$

The proof for $\underline{\alpha}_f$ is similar, which completes the proof of Lemma 13. □

We next prove Proposition 3, which we recall says that if the limit defining $\alpha_f(P)$ exists, then the growth of the height counting function of the orbit of P is given by (0.1).

Proof of Proposition 3. Since $\#\mathcal{O}_f(P) = \infty$, it suffices to prove (0.1) with h_X^+ in place of h_X. For every $\epsilon > 0$ there is an $n_0(\epsilon)$ so that

$$(1 - \epsilon)\alpha_f(P) \leq h_X^+(f^n(P))^{1/n} \leq (1 + \epsilon)\alpha_f(P) \quad \text{for all } n \geq n_0(\epsilon).$$

It follows that

$$\{n \ge n_0(\epsilon) : (1+\epsilon)\alpha_f(P) \le B^{1/n}\} \subset \{n \ge n_0(\epsilon) : h_X^+(f^n(P)) \le B\}$$

and

$$\{n \ge n_0(\epsilon) : h_X^+(f^n(P)) \le B\} \subset \{n \ge n_0(\epsilon) : (1-\epsilon)\alpha_f(P) \le B^{1/n}\}.$$

Counting the number of elements in these sets yields

$$\frac{\log B}{\log((1+\epsilon)\alpha_f(P))} - n_0(\epsilon) - 1 \le \#\{n \ge 0 : h_X^+(f^n(P)) \le B\}$$

and

$$\#\{n \ge 0 : h_X^+(f^n(P)) \le B\} \le \frac{\log B}{\log((1-\epsilon)\alpha_f(P))} + n_0(\epsilon) + 1.$$

Dividing by $\log B$ and letting $B \to \infty$ gives

$$\frac{1}{\log((1+\epsilon)\alpha_f(P))} \le \liminf_{B\to\infty} \frac{\#\{Q \in \mathcal{O}_f(P) : h_X^+(Q) \le B\}}{\log B}$$

and

$$\limsup_{B\to\infty} \frac{\#\{Q \in \mathcal{O}_f(P) : h_X^+(Q) \le B\}}{\log B} \le \frac{1}{\log((1-\epsilon)\alpha_f(P))}.$$

Since ϵ is arbitrary, and the lim inf is less than or equal to the lim sup, this completes the proof that

$$\lim_{B\to\infty} \frac{\#\{Q \in \mathcal{O}_f(P) : h_X^+(Q) \le B\}}{\log B} = \frac{1}{\log \alpha_f(P)},$$

including the fact that if $\alpha_f(P) = 1$, then the limit is ∞. □

The following elementary linear algebra result will be used in the proof of Theorem 4.

Lemma 14. *Let $A = (a_{ij}) \in M_r(\mathbb{C})$ be an r-by-r matrix. Let $\|A\| = \max |a_{ij}|$, and as usual let $\rho(A)$ denote the spectral radius of A. Then there are constants c_1 and c_2, depending on A, such that*

$$c_1\rho(A)^n \le \|A^n\| \le c_2 n^r \rho(A)^n \quad \text{for all } n \ge 0. \tag{2.3}$$

In particular, we have $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Proof. For any matrices A and B in $M_r(\mathbb{C})$, the triangle inequality gives the estimate

$$\|AB\| \le r\|A\| \cdot \|B\|.$$

We write $A = P\Lambda P^{-1}$ with Λ in Jordan normal form. Let λ be an eigenvalue of A having largest absolute value such that among such largest eigenvalues, it has the largest Jordan block. Let the dimension of the largest λ-Jordan block be ℓ. Then

$$\|\Lambda^n\| = \max_{0\le i<\ell} \left\{ \binom{n}{i} |\lambda|^{n-i} \right\}.$$

Since $r \le \ell$ and $|\lambda| = \rho(A)$, the trivial estimates $1 \le \binom{n}{i} \le n^r$ give

$$\rho(A)^{n-r} \le \|\Lambda^n\| \le n^r \rho(A)^n. \tag{2.4}$$

We next observe that

$$\|A^n\| = \|P\Lambda^n P^{-1}\| \le r^2\|P\| \cdot \|P^{-1}\| \cdot \|\Lambda^n\|,$$
$$\|\Lambda^n\| = \|P^{-1}A^n P\| \le r^2\|P^{-1}\| \cdot \|P\| \cdot \|A^n\|,$$

so setting $C = C(A) = r^2\|P\| \cdot \|P^{-1}\| > 0$, we have

$$C^{-1}\|\Lambda^n\| \le \|A^n\| \le C\|\Lambda^n\| \quad \text{for all } n \ge 0. \tag{2.5}$$

Combining (2.4) and (2.5) gives (2.3), and then taking n^{th}-roots and letting $n \to \infty$ finally gives $\|A^n\|^{1/n} \to \rho(A)$. □

3. A divisor inequality for rational maps

Let $f : X \dashrightarrow X$ be a rational map. Our goal in this section is to prove the following geometric inequality relating the actions of $(f^*)^n$ and $(f^n)^*$ on the vector space $\mathrm{NS}(X)_{\mathbb{R}}$. This result will provide a crucial estimate in our proof that $h_X \circ f^n \ll (\delta_f + \epsilon)^n h_X$.

Theorem 15. *Let X be a smooth projective variety, and fix a basis $D_1, \dots, D_r$ for the vector space $\mathrm{NS}(X)_{\mathbb{R}}$. A dominant rational map $g : X \dashrightarrow X$ induces a linear map on $\mathrm{NS}(X)_{\mathbb{R}}$, and we write*

$$g^* D_j \equiv \sum_{i=1}^{r} a_{ij}(g) D_i \quad \text{and} \quad A(g) = (a_{ij}(g)) \in M_r(\mathbb{R}).$$

We let $\|\cdot\|$ denote the sup norm on $M_r(\mathbb{R})$. Then there is a constant $C = C(D_1, \dots, D_r) \ge 1$ such that for any dominant rational map $f : X \dashrightarrow X$ we have

$$\|A(f^{m+n})\| \le C\|A(f^m)\| \cdot \|A(f^n)\| \quad \textit{for all } m, n \ge 1, \tag{3.1}$$
$$\|A(f^m)\| \le C\|A(f)^m\| \quad \textit{for all } m \ge 1. \tag{3.2}$$

We remark that an immediate corollary is the convergence of the limit defining the dynamical degree.

Corollary 16. *The limit $\delta_f = \lim_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}})^{1/n}$ converges.*

Proof. With notation as in the statement of Theorem 15, we have

$$\rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}}) = \rho(A(f^n)),$$

so (3.1) gives

$$\log \rho((f^{m+n})^*) \le \log \rho((f^m)^*) + \log \rho((f^n)^*) + O(1).$$

Using this convexity estimate, it is an exercise to show that the sequence $\frac{1}{n}\rho((f^n)^*)$ converges, proving the corollary. □

We start the proof of Theorem 15 with the following preliminary result. This is essentially shown in [20, Proof of Proposition 1.2 (ii)] by an analytic argument; cf. [20, equation (†)]. We give an algebraic proof.

Proposition 17. *Let $X^{(0)}, X^{(1)}, X^{(2)}, \dots, X^{(m-1)}, X^{(m)}$ be smooth projective varieties of the same dimension N, and let $f^{(i)} : X^{(i)} \dashrightarrow X^{(i-1)}$ be dominant rational maps for $1 \le i \le m$. Let D be a nef divisor on $X^{(0)}$. Then for any nef divisor H on $X^{(m)}$, we have*

$$(f^{(1)} \circ f^{(2)} \circ \cdots \circ f^{(m)})^* D \cdot H^{N-1} \le (f^{(m)*}) \cdots (f^{(2)*})(f^{(1)*}) D \cdot H^{N-1}. \tag{3.3}$$

Proof. We consider a sequence of dominant rational maps:

$$X^{(m)} \overset{f^{(m)}}{\dashrightarrow} X^{(m-1)} \dashrightarrow \cdots \dashrightarrow X^{(2)} \overset{f^{(2)}}{\dashrightarrow} X^{(1)} \overset{f^{(1)}}{\dashrightarrow} X^{(0)}.$$

We blow up a closed subscheme with support equal to the indeterminacy locus $I_{f^{(i)}}$ of $f^{(i)}$ in $X^{(i)}$ for $1 \le i \le m$, so that we have

- smooth projective varieties $X_1^{(i)}$,
- birational morphisms $\pi_1^{(i)} : X_1^{(i)} \to X^{(i)}$ and
- morphisms $\widetilde{f}^{(i)} : X_1^{(i)} \to X^{(i-1)}$ such that

$$\widetilde{f}^{(i)} = f^{(i)} \circ \pi_1^{(i)}.$$

Let $f_1^{(i)} : X_1^{(i)} \dashrightarrow X_1^{(i-1)}$ be the induced dominant rational map for $1 \le i \le m$,

$$\begin{tikzcd}
X_1^{(m)} \arrow[r, dashed, "f_1^{(m)}"] \arrow[d, "\pi_1^{(m)}"'] \arrow[dr, "\widetilde{f}^{(m)}"] & X_1^{(m-1)} \arrow[r, dashed] \arrow[d, "\pi_1^{(m-1)}"] & \cdots \arrow[r, dashed] & X_1^{(2)} \arrow[r, dashed, "f_1^{(2)}"] \arrow[d, "\pi_1^{(2)}"'] \arrow[dr, "\widetilde{f}^{(2)}"] & X_1^{(1)} \arrow[d, "\pi_1^{(1)}"'] \arrow[dr, "\widetilde{f}^{(1)}"] & \\
X^{(m)} \arrow[r, dashed, "f^{(m)}"] & X^{(m-1)} \arrow[r, dashed] & \cdots \arrow[r, dashed] & X^{(2)} \arrow[r, dashed, "f^{(2)}"] & X^{(1)} \arrow[r, dashed, "f^{(1)}"] & X^{(0)}.
\end{tikzcd}$$

Next we blow up a closed subscheme with support equal to the indeterminacy locus $I_{f_1^{(i)}}$ of $f_1^{(i)}$ in $X_1^{(i)}$ for $2 \le i \le m$, so that we have

- smooth projective varieties $X_2^{(i)}$,
- birational morphisms $\pi_2^{(i)} : X_2^{(i)} \to X_1^{(i)}$ and
- morphisms $\widetilde{f_1}^{(i)} : X_2^{(i)} \to X_1^{(i-1)}$ such that

$$\widetilde{f_1}^{(i)} = f_1^{(i)} \circ \pi_2^{(i)}.$$

Let $f_2^{(i)} : X_2^{(i)} \dashrightarrow X_2^{(i-1)}$ be the induced dominant rational map for $2 \le i \le m$.

We continue this procedure to obtain the commutative diagram illustrated in Figure 1.

For $1 \le k \le i \le m$, let us define a proper closed subvariety $Z_k^{(i)}$ of $X^{(i)}$ as follows. For $k = 1$, we set

$$Z_1^{(i)} := I_{f^{(i)}} \quad \text{in } X^{(i)} \text{ for } 1 \le i \le m.$$

For $k = 2$, we set

$$Z_2^{(i)} := I_{f^{(i)}} \cup \big(\text{Zariski closure of } (f^{(i)}|_{X^{(i)} \smallsetminus Z_1^{(i)}})^{-1}(Z_1^{(i-1)})\big) \quad \text{in } X^{(i)} \text{ for } 2 \le i \le m.$$

We note that

$$(f^{(i)}|_{X^{(i)} \smallsetminus Z_1^{(i)}})^{-1}(Z_1^{(i-1)}) = (f^{(i)}|_{X^{(i)} \smallsetminus I_{f^{(i)}}})^{-1}(I_{f^{(i-1)}}).$$

$$
\begin{tikzcd}
X_m^{(m)} \arrow[d, "\pi_m^{(m)}"'] \arrow[dr, "\widetilde{f}_{m-1}^{(m)}"] & & & & & \\
X_{m-1}^{(m)} \arrow[r, dashed, "f_{m-1}^{(m)}"] \arrow[d] & X_{m-1}^{(m-1)} \arrow[d] \arrow[dr] & & & & \\
\vdots \arrow[d] & \vdots \arrow[d] & \ddots \arrow[dr] & & & \\
X_2^{(m)} \arrow[r, dashed, "f_1^{(m)}"] \arrow[d, "\pi_2^{(m)}"'] \arrow[dr, "\widetilde{f}_1^{(m)}"] & X_2^{(m-1)} \arrow[r, dashed] \arrow[d, "\pi_2^{(m-1)}"] & \cdots \arrow[r, dashed] & X_2^{(2)} \arrow[d, "\pi_2^{(2)}"'] \arrow[dr, "\widetilde{f}_1^{(2)}"] & & \\
X_1^{(m)} \arrow[r, dashed, "f_1^{(m)}"] \arrow[d, "\pi_1^{(m)}"'] \arrow[dr, "\widetilde{f}^{(m)}"] & X_1^{(m-1)} \arrow[r, dashed] \arrow[d, "\pi_1^{(m-1)}"] & \cdots \arrow[r, dashed] & X_1^{(2)} \arrow[r, dashed, "f_1^{(2)}"] \arrow[d, "\pi_1^{(2)}"'] \arrow[dr, "\widetilde{f}^{(2)}"] & X_1^{(1)} \arrow[d, "\pi_1^{(1)}"'] \arrow[dr, "\widetilde{f}^{(1)}"] & \\
X^{(m)} \arrow[r, dashed, "f^{(m)}"] & X^{(m-1)} \arrow[r, dashed] & \cdots \arrow[r, dashed] & X^{(2)} \arrow[r, dashed, "f^{(2)}"] & X^{(1)} \arrow[r, dashed, "f^{(1)}"] & X^{(0)}
\end{tikzcd}
$$

Figure 1. A commutative diagram.

Inductively, for $k \leq m$, we set

$$Z_k^{(i)} := I_f \cup \big(\text{Zariski closure of } (f^{(i)}|_{X^{(i)} \smallsetminus Z_{k-1}^{(i)}})^{-1}(Z_{k-1}^{(i-1)})\big) \quad \text{in } X^{(i)} \text{ for } k \leq i \leq m.$$

Then we have a sequence of morphisms:

$$
\begin{tikzcd}
X^{(m)} \smallsetminus Z_m^{(m)} \arrow[r, "f^{(m)}|_{X^{(m)} \smallsetminus Z_m^{(m)}}"] & X^{(m-1)} \smallsetminus Z_{m-1}^{(m-1)} \arrow[r, "f^{(m-1)}|_{X^{(m-1)} \smallsetminus Z_{m-1}^{(m-1)}}"] & X^{(m-2)} \smallsetminus Z_{m-2}^{(m-2)} \arrow[d] \\
 & & \vdots \arrow[d] \\
X^{(0)} & X^{(1)} \smallsetminus Z_1^{(1)} \arrow[l, "f^{(1)}|_{X^{(1)} \smallsetminus Z_1^{(1)}}"'] & X^{(2)} \smallsetminus Z_2^{(2)} \arrow[l, "f^{(2)}|_{X^{(2)} \smallsetminus Z_2^{(2)}}"']
\end{tikzcd}
$$

Since nef divisors are limits of ample divisors, we may assume that D is ample. Replacing D by ℓD for sufficiently large ℓ, we may assume that D is very ample and represented by an effective divisor on $X^{(0)}$ with the following property:

(P) The divisor D does not contain the image in $X^{(0)}$ of any effective divisor in $X_m^{(m)}$ whose image in $X^{(m)}$ is contained in $Z_m^{(m)}$.

With these assumptions, we claim that the divisor

$$(f^{(m)*})\cdots(f^{(2)*})(f^{(1)*})D-(f^{(1)}\circ f^{(2)}\circ\cdots\circ f^{(m)})^*D \tag{3.4}$$

is effective, and hence has non-negative intersection with H^{N-1}.

We note that property (P) implies that

$$\begin{aligned}(f^{(1)}\circ f^{(2)}\circ\cdots\circ f^{(m)})^*D \\ &= \text{Zariski closure of } (f^{(1)}\circ f^{(2)}\circ\cdots\circ f^{(m)}|_{X^{(m)}\smallsetminus Z_m^{(m)}})^*D \quad \text{in } X=X^{(m)}.\end{aligned} \tag{3.5}$$

The divisor $(f^{(m)*})\cdots(f^{(2)*})(f^{(1)*})D$ is effective on $X^{(m)}$, and restricted to $X^{(m)}\smallsetminus Z_m^{(m)}$, it contains $(f^{(1)}\circ f^{(2)}\circ\cdots\circ f^{(m)})^*D$. The description (3.5) implies that the divisor

$$(f^{(m)*})\cdots(f^{(2)*})(f^{(1)*})D-(f^{(1)}\circ f^{(2)}\circ\cdots\circ f^{(m)})^*D$$

is effective on $X=X^{(m)}$. Now to complete the proof of Proposition 17, it only remains to note that in general, if E is effective and H is nef on a smooth projective variety X of dimension n, then $E\cdot H^{n-1}\geq 0$. □

We now give the proof of Theorem 15.

Proof of Theorem 15. We set the following notation:

N: the dimension of X, which we assume is at least 2.

$\mathrm{Amp}(X)$: the ample cone in $\mathrm{NS}(X)_{\mathbb{R}}$ of all ample $\mathbb{R}$-divisors.

$\mathrm{Nef}(X)$: the nef cone in $\mathrm{NS}(X)_{\mathbb{R}}$ of all nef $\mathbb{R}$-divisors.

$\mathrm{Eff}(X)$: the effective cone in $\mathrm{NS}(X)_{\mathbb{R}}$ consisting of the classes of all effective $\mathbb{R}$-divisors.

$\overline{\mathrm{Eff}}(X)$: the pseudoeffective cone, i.e., the $\mathbb{R}$-closure of $\mathrm{Eff}(X)$.

As explained in [17, Section 1.4], we have

$$\mathrm{Nef}(X)=\overline{\mathrm{Amp}(X)} \quad\text{and}\quad \mathrm{Amp}(X)=\mathrm{int}(\mathrm{Nef}(X)).$$

In particular, $\mathrm{Nef}(X)$ is a closed convex cone. Also, since $\mathrm{Amp}(X)\subset\mathrm{Eff}(X)$, it follows that $\mathrm{Nef}(X)\subset\overline{\mathrm{Eff}}(X)$.

The next lemma is essentially shown in [31, Proposition II.6.3] over $\mathbb{C}$. (Indeed, in [31, Proposition II.6.3], it is even shown that similar results hold for pseudoeffective cycles of codimension greater than 1 that are not homologically trivial.)

Lemma 18. *With notation as above, let* $D\in\overline{\mathrm{Eff}}(X)\smallsetminus\{0\}$ *and* $H\in\mathrm{Amp}(X)$. *Then*

$$D\cdot H^{N-1}>0.$$

Proof. Since H is ample and D is in the closure of the effective cone, we certainly have $D\cdot H^{N-1}\geq 0$. Our goal is to prove that we have a strict inequality.

We first consider the case $N=2$. Since $D\neq 0$ in $\mathrm{NS}(X)_{\mathbb{R}}$, there is a divisor E such that $D\cdot E\neq 0$. Replacing E by $-E$ if necessary, we may assume that $D\cdot E<0$. Choose $k>0$ sufficiently large so that $kH+E$ is ample. Since D is a limit of effective divisors, it follows that $D\cdot(kH+E)\geq 0$. Hence

$$D\cdot H\geq\frac{-D\cdot E}{k}>0.$$

We now proceed by induction on N. Let $N = \dim X \geq 3$. Replacing H with kH for an appropriate $k \geq 1$, we may assume that H is very ample. Let Y be a (smooth) irreducible variety in the linear system $|H|$. The Lefschetz Hyperplane Theorem [38, Theorem 1.23] says that the restriction map $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ is injective and preserves effective divisors. Our induction hypothesis says that

$$D|_Y \cdot (H|_Y)^{N-2} > 0.$$

Hence $D \cdot Y \cdot H^{N-2} > 0$. But $Y \sim H$ in $\mathrm{Pic}(X)$, so in particular $Y \equiv H$ in $\mathrm{NS}(X)_{\mathbb{R}}$. Hence we have $D \cdot H^{N-1} > 0$, which completes the proof of Lemma 18. □

Lemma 19. *Let $H \in \mathrm{Amp}(X)$, and fix some norm $|\cdot|$ on the $\mathbb{R}$-vector space $\mathrm{NS}(X)_{\mathbb{R}}$. There are constants $C_1, C_2 > 0$ such that*

$$C_1|v| \leq v \cdot H^{N-1} \leq C_2|v| \quad \textit{for all } v \in \overline{\mathrm{Eff}}(X). \tag{3.6}$$

Proof. We consider the map

$$\varphi : \mathrm{NS}(X)_{\mathbb{R}} \to \mathbb{R}, \quad \varphi(w) = w \cdot H^{N-1}.$$

Since φ is continuous, it attains a minimum and (finite) maximum when restricted to the compact set

$$\overline{\mathrm{Eff}}(X) \cap \{w \in \mathrm{NS}(X)_{\mathbb{R}} : |w| = 1\}.$$

Lemma 18 tells us that $\varphi(w) > 0$ for all nonzero $w \in \overline{\mathrm{Eff}}(X)$, so the minimum is strictly positive, say

$$C_1 = \inf\{\varphi(w) : w \in \overline{\mathrm{Eff}}(X) \text{ and } |w| = 1\} > 0.$$

Then for all $v \in \overline{\mathrm{Eff}}(X) \smallsetminus \{0\}$ we have

$$v \cdot H^{n-1} = |v|\varphi\left(\frac{v}{|v|}\right) \geq C_1|v|.$$

Similarly, letting

$$C_2 = \sup\{\varphi(w) : w \in \overline{\mathrm{Eff}}(X) \text{ and } |w| = 1\} < \infty,$$

we have

$$v \cdot H^{n-1} = |v|\varphi\left(\frac{v}{|v|}\right) \leq C_2|v|.$$

This proves the first part of Lemma 19, and the last assertion is then clear, since as noted earlier, we have $\mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}}(X)$. □

We resume the proof of Theorem 15. As in the proof of Lemma 19, we fix a norm $|\cdot|$ on the $\mathbb{R}$-vector space $\mathrm{NS}(X)_{\mathbb{R}}$, and for any linear map $A : \mathrm{NS}(X)_{\mathbb{R}} \to \mathrm{NS}(X)_{\mathbb{R}}$, we set

$$\|A\|' = \sup_{v \in \mathrm{Nef}(X) \smallsetminus 0} \frac{|Av|}{|v|}.$$

We note that for linear maps $A, B \in \mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}})$ and $c \in \mathbb{R}$ we have

$$\|A + B\|' \leq \|A\|' + \|B\|' \quad \text{and} \quad \|cA\|' = |c|\|A\|'.$$

Further, since $\mathrm{Nef}(X)$ generates $\mathrm{NS}(X)_{\mathbb{R}}$ as an $\mathbb{R}$-vector space, we have $\|A\|' = 0$ if and only if $A = 0$. Thus $\|\cdot\|'$ is an $\mathbb{R}$-norm on $\mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}})$.

Similarly, for any linear map $A : \mathrm{NS}(X)_{\mathbb{R}} \to \mathrm{NS}(X)_{\mathbb{R}}$, we set

$$\|A\|'' = \sup_{w \in \overline{\mathrm{Eff}}(X) \smallsetminus 0} \frac{|Aw|}{|w|},$$

then $\|\cdot\|''$ is an $\mathbb{R}$-norm on $\mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}})$.

We note that $\overline{\mathrm{Eff}(X)}$ is preserved by f^* and that $\mathrm{Nef}(X) \subset \overline{\mathrm{Eff}(X)}$. Thus if $v \in \mathrm{Nef}(X)$, then $(f^{m+n})^* v$ and $(f^*)^n v$ belong to $\overline{\mathrm{Eff}(X)}$. This allows us to compute

$$\begin{aligned}
\|(f^{m+n})^*\|' &= \sup_{v \in \mathrm{Nef}(X) \smallsetminus 0} \frac{|(f^{m+n})^* v|}{|v|} \\
&\le C_1^{-1} \sup_{v \in \mathrm{Nef}(X) \smallsetminus 0} \frac{(f^{m+n})^* v \cdot H^{N-1}}{|v|} && \text{from Lemma 19} \\
&\le C_1^{-1} \sup_{v \in \mathrm{Nef}(X) \smallsetminus 0} \frac{(f^m)^*((f^n)^* v) \cdot H^{N-1}}{|v|} && \text{from Proposition 17} \\
&= C_1^{-1} \sup_{\substack{v \in \mathrm{Nef}(X) \smallsetminus 0 \\ (f^n)^* v \neq 0}} \frac{(f^m)^*((f^n)^* v) \cdot H^{N-1}}{|v|} \\
&= C_1^{-1} \sup_{\substack{v \in \mathrm{Nef}(X) \smallsetminus 0 \\ (f^n)^* v \neq 0}} \left(\frac{(f^m)^*((f^n)^* v) \cdot H^{N-1}}{|(f^n)^* v|} \cdot \frac{|(f^n)^* v|}{|v|} \right) \\
&\le C_1^{-1} \left(\sup_{\substack{v \in \mathrm{Nef}(X) \smallsetminus 0 \\ (f^n)^* v \neq 0}} \frac{(f^m)^*((f^n)^* v) \cdot H^{N-1}}{|(f^n)^* v|} \right) \cdot \left(\sup_{v \in \mathrm{Nef}(X) \smallsetminus 0} \frac{|(f^n)^* v|}{|v|} \right) \\
&= C_1^{-1} \left(\sup_{\substack{v \in \mathrm{Nef}(X) \smallsetminus 0 \\ (f^n)^* v \neq 0}} \frac{(f^m)^*((f^n)^* v) \cdot H^{N-1}}{|(f^n)^* v|} \right) \cdot \|(f^n)^*\|' \\
&\le C_1^{-1} \left(\sup_{w \in \overline{\mathrm{Eff}}(X) \smallsetminus 0} \frac{(f^m)^* w \cdot H^{N-1}}{|w|} \right) \cdot \|(f^n)^*\|' && \text{since } (f^m)^* v \in \overline{\mathrm{Eff}}(X) \\
&\le C_1^{-1} C_2 \left(\sup_{w \in \overline{\mathrm{Eff}}(X) \smallsetminus 0} \frac{|(f^m)^* w|}{|w|} \right) \|(f^n)^*\|' && \text{from Lemma 19} \\
&\le C_1^{-1} C_2 \|(f^m)^*\|'' \cdot \|(f^n)^*\|'.
\end{aligned}$$

We recall that we have defined $\|\cdot\|$ to be the sup norm on $M_r(\mathbb{R}) = \mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}})$, where the identification is via the given basis $D_1, \dots, D_r$ of $\mathrm{NS}(X)_{\mathbb{R}}$. We thus have three norms $\|\cdot\|$, $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}})$, so there are positive constants C_3', C_4', C_3'' and C_4'' such that

$$C_3' \|\gamma\| \le \|\gamma\|' \le C_4' \|\gamma\| \quad \text{and} \quad C_3'' \|\gamma\| \le \|\gamma\|'' \le C_4'' \|\gamma\| \quad \text{for all } \gamma \in \mathrm{End}(\mathrm{NS}(X)_{\mathbb{R}}).$$

Hence

$$\begin{aligned}
\|A(f^{n+m})\| = \|(f^{n+m})^*\| &\le C_3'^{-1} \|(f^{n+m})^*\|' \\
&\le C_3'^{-1} C_1^{-1} C_2 \|(f^n)^*\|' \cdot \|(f^m)^*\|'' \\
&\le C_3'^{-1} C_1^{-1} C_2 C_4' C_4'' \|(f^n)^*\| \cdot \|(f^m)^*\| \\
&= C_3'^{-1} C_1^{-1} C_2 C_4' C_4'' \|A(f^n)\| \cdot \|A(f^m)\|.
\end{aligned}$$

This completes the proof of (3.1).

Similarly, if $v \in \mathrm{Nef}(X)$, then $(f^m)^* v$ and $(f^*)^m v$ belong to $\overline{\mathrm{Eff}}(X)$. A similar calculation gives

$$\begin{aligned}
\|(f^m)^*\|' &= \sup_{v\in\mathrm{Nef}(X)\smallsetminus 0} \frac{|(f^m)^* v|}{|v|} \\
&\le C_1^{-1} \sup_{v\in\mathrm{Nef}(X)\smallsetminus 0} \frac{(f^m)^* v \cdot H^{N-1}}{|v|} && \text{from Lemma 19} \\
&\le C_1^{-1} \sup_{v\in\mathrm{Nef}(X)\smallsetminus 0} \frac{(f^*)^m v \cdot H^{N-1}}{|v|} && \text{from Proposition 17} \\
&\le C_1^{-1} C_2 \sup_{v\in\mathrm{Nef}(X)\smallsetminus 0} \frac{|(f^*)^m v|}{|v|} && \text{from Lemma 19} \\
&= C_1^{-1} C_2 \|(f^*)^m\|'.
\end{aligned}$$

Hence

$$\begin{aligned}
\|A(f^m)\| = \|(f^m)^*\| &\le C_3'^{-1} \|(f^m)^*\|' \\
&\le C_3'^{-1} C_1^{-1} C_2 \|(f^*)^m\|' \\
&\le C_3'^{-1} C_1^{-1} C_2 C_4' \|(f^*)^m\| = C_3'^{-1} C_1^{-1} C_2 C_4' \|A(f)^m\|.
\end{aligned}$$

This completes the proof of (3.2), and with it the proof of Theorem 15. □

Remark 20. If we assume that $f : X \to X$ is a morphism, then the conclusions of Theorem 15 are valid for normal varieties X. Indeed, in this situation it suffices to work with $\mathrm{Nef}(X)$; there is no need to introduce $\overline{\mathrm{Eff}}(X)$ into the argument.

4. A height inequality for rational maps

Let $f : X \dashrightarrow X$ be a rational map and let D be a divisor on X. Our goal in this section is to prove an arithmetic inequality relating the height functions $h_D \circ f$ and h_{f^*D}. For rational self-maps $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ of projective space, the desired result follows by an elementary triangle inequality argument [23, Theorem B.2.5 (a)], but the proof for general varieties $f : X \dashrightarrow X$ is more complicated because the pullback of an ample divisor by f need not be ample. With an eye towards future applications, and since the argument is no more difficult, we prove a stronger result in which the domain and range may be different varieties. We again refer the reader to [11, 23, 27, 34, 35] for the theory of height functions and Weil's height machine. In Section 7 we will give an alternative proof of Proposition 21 that avoids blowups.

Proposition 21. *Let $X/\bar{K}$ and $Y/\bar{K}$ be smooth projective varieties, let $f : Y \dashrightarrow X$ be a dominant rational map defined over $\bar{K}$, let $D \in \mathrm{Div}(X)$ be an ample divisor, and fix Weil height functions $h_{X,D}$ and h_{Y,f^*D} associated to D and f^*D. Then*

$$h_{X,D} \circ f(P) \le h_{Y,f^*D}(P) + O(1) \quad \textit{for all } P \in (Y \smallsetminus I_f)(\bar{K}),$$

where the $O(1)$ bound depends on X, Y, f, and the choice of height functions, but is independent of P.

Proof. We blow up a closed subscheme with support equal to the indeterminacy locus I_f of f to get a smooth projective variety Z, a birational morphism $p : Z \to Y$, and a morphism $g : Z \to X$ such that $f = g \circ p^{-1}$. For any effective divisor D on X, the pullback f^*D is defined by

$$f^*D = p_*(g^*D). \tag{4.1}$$

We note that f^*D is independent of the choice of Z.

Lemma 22. *With notation as above, assume that D is nef. Then the divisor*

$$B := p^*p_*(g^*D) - g^*D \tag{4.2}$$

is effective.

Proof. For any curve C on Z such that $p(C)$ is a point, we have

$$-B \cdot C = (g^*D) \cdot C - (p^*p_*(g^*D)) \cdot C = (g^*D) \cdot C \geq 0.$$

Thus $-B$ is p-nef. It follows from the Negativity Lemma (see [26, Lemma 3.39]) that B is effective if and only if p_*B is effective. Since $p_*B = 0$, we conclude that B is effective. □

We now resume the proof of Proposition 21, so in particular we assume that D is ample. For a sufficiently large m, the divisor mD is very ample, so there exists an effective divisor D' that is linearly equivalent to mD. Since f^*D' is linearly equivalent to $f^*(mD)$, we may assume that D is effective.

We let B be the divisor (4.2), so Lemma 22 tells us that B is an effective divisor with the property that $p(\mathrm{Supp}(B)) \subset I_f$. For any $\tilde{P} \in Z(\bar{K}) \smallsetminus \mathrm{Supp}(B)$, we estimate $h_{p^*p_*(g^*D)}(\tilde{P})$ in two ways. First we have

$$\begin{aligned} h_{p^*p_*(g^*D)}(\tilde{P}) &= h_{g^*D+B}(\tilde{P}) \\ &= h_{g^*D}(\tilde{P}) + h_B(\tilde{P}) + O(1) \\ &\geq h_{g^*D}(\tilde{P}) + O(1), \end{aligned} \tag{4.3}$$

where the last inequality follows from the positivity of the height h_B on $Z \smallsetminus \mathrm{Supp}(B)$ for the effective divisor B; see [23, Theorem B.3.2 (e)]. Secondly, using functoriality of height functions for morphisms [23, Theorem B.3.2 (b)], we have

$$\begin{aligned} h_{p^*p_*(g^*D)}(\tilde{P}) &= h_{p_*(g^*D)}(p(\tilde{P})) + O(1) && \text{functoriality} \\ &= h_{f^*D}(p(\tilde{P})) + O(1) && \text{formula (4.1) for } f^*D. \end{aligned} \tag{4.4}$$

Now let $P \in Y(\bar{K}) \smallsetminus I_f$. Then there exists a unique $\tilde{P} \in Z \smallsetminus p^{-1}(I_f)$ with $p(\tilde{P}) = P$. Since $\mathrm{Supp}(B) \subseteq p^{-1}(I_f)$, we have $P \in Z \smallsetminus \mathrm{Supp}(B)$. Hence

$$\begin{aligned} h_{f^*D}(P) &= h_{f^*D}(p(\tilde{P})) && \text{since } P = p(\tilde{P}) \\ &= h_{p^*p_*(g^*D)}(\tilde{P}) + O(1) && \text{from (4.4)} \\ &\geq h_{g^*D}(\tilde{P}) + O(1) && \text{from (4.3)} \\ &= h_D(g(\tilde{P})) + O(1) && \text{since } g \text{ is a morphism} \\ &= h_D(f(P)) + O(1) && \text{since } g(\tilde{P}) = f(P). \end{aligned}$$

This completes the proof of Proposition 21. □

Remark 23. Proposition 21 is true more generally for a nef divisor D such that there exists an $m \geq 1$ such that mD is linearly equivalent to an effective divisor.

5. A bound for the height of an iterate

We now prove the quantitative height upper bound for $h_X^+(f^n(P))$ that constitutes one of the main results of this paper. For the convenience of the reader, the statement includes a reminder of the notation that we set in the introduction.

Theorem 24 (Theorem 1). *Let K be a global field, let $f : X \dashrightarrow X$ be a dominant rational map defined over K, let h_X be a Weil height on $X(\bar{K})$ relative to an ample divisor, let $h_X^+ = \max\{h_X, 1\}$, and let $\epsilon > 0$. Then there is a positive constant $C = C(X, h_X, f, \epsilon)$ such that for all $P \in X_f(\bar{K})$ and all $n \geq 0$,*

$$h_X^+(f^n(P)) \leq C \cdot (\delta_f + \epsilon)^n \cdot h_X^+(P).$$

Before proving Theorem 24, we pause to show how it immediately implies the fundamental inequality $\overline{\alpha}_f(P) \leq \delta_f$ stated in the introduction.

Corollary 25 (Theorem 4). *Let $P \in X_f(\bar{K})$. Then*

$$\overline{\alpha}_f(P) \leq \delta_f. \tag{5.1}$$

Proof. Let $\epsilon > 0$. Then

$$\begin{aligned} \overline{\alpha}_f(P) &= \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n} && \text{definition of } \overline{\alpha}_f(P) \\ &\leq \limsup_{n\to\infty} \bigl(C \cdot (\delta_f + \epsilon)^n \cdot h_X^+(P)\bigr)^{1/n} && \text{from Theorem 24} \\ &= \delta_f + \epsilon. \end{aligned}$$

This holds for all $\epsilon > 0$, which proves that $\overline{\alpha}_f(P) \leq \delta_f$. □

Proof of Theorem 24. If P is preperiodic, then $\overline{\alpha}_f(P) = 1 \leq \delta_f$, so there is nothing to prove. We assume henceforth that $\#\mathcal{O}_f(P) = \infty$. We let m and ℓ be positive integers to be chosen later, and we set

$$g = f^{m\ell}.$$

We note that $X_f(\bar{K}) \subset X_g(\bar{K})$. We choose ample divisors $D_1, \dots, D_r \in \operatorname{Div}(X)$ whose algebraic equivalence classes form a basis for $\operatorname{NS}(X)_{\mathbb{Q}}$, and we fix height functions $h_{D_1}, \dots, h_{D_r}$ associated to the divisors $D_1, \dots, D_r$. We note that any two ample heights are commensurate with one another, i.e., $h_X \asymp h'_X$, so we may take h_X to be

$$h_X(Q) = \max_{1 \leq i \leq r} h_{D_i}(Q).$$

To ease notation, we further assume that h_{D_1} is chosen to satisfy $h_{D_1} \geq 1$, so $h_X^+ = h_X$.

Applying g^* to the divisors in our basis of $\operatorname{NS}(X)_{\mathbb{Q}}$, we have algebraic equivalences

$$g^* D_k \equiv \sum_{i=1}^{r} a_{ik}(g) D_i \quad \text{for some } a_{ik}(g) \in \mathbb{Q}. \tag{5.2}$$

We set the notation

$$A(g) = (a_{ik}(g)) \quad \text{and} \quad \|A(g)\| = \max_{i,k} |a_{ik}(g)|.$$

Algebraic equivalences of divisors as in (5.2) implies a height relation as in the following result.

Lemma 26. *Let $E \in \mathrm{Div}(X)_{\mathbb{R}}$ be a divisor that is algebraically equivalent to* 0*, and fix a height function h_E associated to E. Then there is a constant $C = C(h_X, h_E)$ such that*

$$|h_E(P)| \le C\sqrt{h_X^+(P)} \quad \textit{for all } P \in X(\bar{K}). \tag{5.3}$$

Proof. See for example [23, Theorem B.5.9]. □

Remark 27. A well-known weaker form of Lemma 26 says that

$$\lim_{\substack{P \in X(\bar{K}) \\ h_X(P) \to \infty}} \frac{h_E(P)}{h_X(P)} = 0; \tag{5.4}$$

see for example [23, Theorem B.3.2 (f)] or [27, Chapter 4, Proposition 3.3]. We remark that it is possible to prove that $\overline{\alpha}_f(P) \le \delta_f$ using only the weaker estimate (5.4), but in order to prove the quantitative bound in Theorem 24 and the error estimate in Theorem 5, we need the stronger estimate provided by (5.3).

Applying Lemma 26 to (5.2) and using additivity of height functions, we find a positive constant $C_1 = C_1(\epsilon, g)$ such that

$$\left| h_{g^* D_k}(Q) - \sum_{i=1}^{r} a_{ik}(g) h_{D_i}(Q) \right| \le C_1 \sqrt{h_X(Q)} \quad \text{for all } Q \in X(\bar{K}). \tag{5.5}$$

Here and in what follows, the constants $C_1, C_2, \dots$ are allowed to depend on the divisors $D_1, \dots, D_r$ and their associated height functions, as well as on X, f, ϵ, m, ℓ, and ϵ. However, we will eventually fix m and ℓ, at which point

$$C_i = C_i(X, f, \epsilon, h_{D_1}, \dots, h_{D_r}).$$

We also remind the reader that we have chosen h_X to satisfy $h_X \ge 1$.

Apply Proposition 21 to the rational map g and to each of the ample divisors $D_1, \dots, D_r$. Thus for all points $Q \in X(\bar{K})$, we have

$$\begin{aligned} h_X(g(Q)) &= \max_{1 \le k \le r} h_{D_k}(g(Q)) && \text{definition of } h_X \\ &\le \max_{1 \le k \le r} (h_{g^* D_k}(Q) + C_2) && \text{from Proposition 21} \\ &\le \max_{1 \le k \le r} \left(\sum_{i=1}^{r} a_{ik}(g) h_{D_i}(Q) \right) + C_3 \sqrt{h_X(Q)} && \text{from (5.5)} \\ &\le \Bigl(r \max_{1 \le i,k \le r} |a_{ik}(g)| \Bigr) h_X(Q) + C_3 \sqrt{h_X(Q)} \\ &= r \|A(g)\| h_X(Q) + C_3 \sqrt{h_X(Q)}. \end{aligned} \tag{5.6}$$

We are going to use the following elementary lemma.

Lemma 28. *Let S be a set, let $g : S \to S$ and $h : S \to [0,\infty)$ be maps, let $a \geq 1$ and $b \geq 1$ be constants. Suppose that for all $x \in S$ we have*

$$h(g(x)) \leq ah(x) + c\sqrt{h(x)}. \tag{5.7}$$

Then for all $x \in S$ and all $n \geq 0$,

$$h(g^n(x)) \leq a^n\big(h(x) + (2\sqrt{2}c)^n\sqrt{h(x)}\big). \tag{5.8}$$

Proof. The proof is an elementary induction on n. For the convenience of the reader, we give the details in Appendix A. □

We apply Lemma 28 to (5.6) to obtain

$$\begin{aligned} h_X(g^n(Q)) &\leq (r\|A(g)\|)^n\big(h_X(Q) + C_4^n\sqrt{h_X(Q)}\big) \\ &\leq (C_5 r\|A(g)\|)^n h_X(Q), \end{aligned} \tag{5.9}$$

where we stress that C_4 and C_5 do not depend on Q or n.

We recall that $g = f^{m\ell}$, which lets us estimate

$$\begin{aligned} \|A(g)\| &= \|A((f^\ell)^m)\| \\ &\leq C_6\|A(f^\ell)^m\| && \text{Theorem 15 applied to } f^\ell \\ &\leq C_7 m^r \rho(A(f^\ell))^m && \text{from Lemma 14.} \end{aligned}$$

By definition, the dynamical degree is the limit of $\rho(A(f^\ell))^{1/\ell}$ as $\ell \to \infty$. So we now fix an $\ell = \ell(\epsilon, f)$ such that

$$\rho(A(f^\ell)) \leq (\delta_f + \epsilon)^\ell.$$

For this choice of ℓ, we have

$$\|A(g)\| \leq C_7 m^r (\delta_f + \epsilon)^{\ell m}. \tag{5.10}$$

Substituting (5.10) into (5.9) and using $g = f^{m\ell}$ gives

$$h_X(f^{m\ell n}(Q)) \leq (C_8 r m^r (\delta_f + \epsilon)^{\ell m})^n h_X(Q). \tag{5.11}$$

We now take $P \in X_f(\bar{K})$ as in the statement of the theorem, and we apply (5.11) to each of the points $P, f(P), \dots, f^{m\ell-1}(P)$ to obtain

$$\max_{0 \leq i < m\ell} h_X(f^{m\ell n+i}(P)) \leq (C_8 r m^r (\delta_f + \epsilon)^{\ell m})^n \max_{0 \leq i < m\ell} h_X(f^i(P)). \tag{5.12}$$

For $0 \leq i < m\ell$, we apply Proposition 21 to each of the heights $h_X(f^i(P))$. Using the fact that the ample height h_X dominates any other height h_D, i.e., $h_X \gg h_D$ with a constant depending on D, we obtain

$$\max_{0 \leq i < m\ell} h_X(f^i(P)) \leq C_9 h_X(P). \tag{5.13}$$

Combining (5.12) and (5.13) gives

$$\max_{0 \leq i < m\ell} h_X(f^{m\ell n+i}(P)) \leq C_9(C_8 r m^r (\delta_f + \epsilon)^{\ell m})^n h_X(P). \tag{5.14}$$

Now let $q \geq 1$ be any integer and write

$$q = m\ell n + i \quad \text{with } 0 \leq i < m\ell.$$

Then (5.14) implies that

$$h_X(f^q(P)) \leq C_9 (C_8 r m^r)^{q/m\ell} (\delta_f + \epsilon)^q h_X(P), \tag{5.15}$$

where we have used the trivial estimates $\ell m n \leq q$ and $n \leq q/m\ell$. The key point to note about inequality (5.15) is that the quantity $(C_8 r m^r)^{1/m\ell}$ is independent of q and goes to 1 as $m \to \infty$. So we now fix a value of m such that

$$(C_8 r m^r)^{1/m\ell} \leq (1 + \epsilon).$$

This value of m depends on ϵ, and of course it depends on X and f, but it does not depend on the integer q or the point P. We note that the constant C_9 now also depends on ϵ, but not on q or P. Hence (5.15) becomes

$$h_X(f^q(P)) \leq C_9 (1+\epsilon)^q (\delta_f + \epsilon)^q h_X(P). \tag{5.16}$$

We have proven that (5.16) holds for all $P \in X_f(\bar{K})$ and all $q \geq 0$, where C_9 does not depend on q or P. After adjusting ϵ, inequality (5.16) is the desired result, which completes the proof of Theorem 24. □

6. An application to canonical heights

In this section we use Theorem 24 to prove Theorem 5, which says that the usual canonical height limit converges for certain eigendivisor classes relative to *algebraic* equivalence. We remark that the result is well known (and much easier to prove) for eigendivisor classes relative to *linear* equivalence; cf. [14].

Proof of Theorem 5. To ease notation, we will let $\delta = \delta_f$.

(a) Theorem 24 says that for every $\epsilon > 0$ there is a constant $C_1 = C_1(X, h_X, f, \epsilon)$ such that

$$h_X^+(f^n(P)) \leq C_1 \cdot (\delta + \epsilon)^n \cdot h_X^+(P) \quad \text{for all } n \geq 0. \tag{6.1}$$

We are given that $f^*D \equiv \beta D$. Applying Lemma 26 with $E = f^*D - \beta D$, we find a positive constant $C_2 = C_2(D, A, f)$ such that

$$|h_{f^*D}(Q) - \beta h_D(Q)| \leq C_2 \sqrt{h_X^+(Q)} \quad \text{for all } Q \in X(\bar{K}). \tag{6.2}$$

Since we have assumed that f is a morphism, standard functoriality of the Weil height says that

$$h_{f^*D} = h_D \circ f + O(1),$$

so (6.2) becomes

$$|h_D(f(Q)) - \beta h_D(Q)| \leq C_3 \sqrt{h_X^+(Q)} \quad \text{for all } Q \in X(\bar{K}). \tag{6.3}$$

For $N \geq M \geq 0$ we estimate a telescoping sum,

$$
\begin{aligned}
(6.4)\quad & |\beta^{-N} h_D(f^N(P)) - \beta^{-M} h_D(f^M(P))| \\
&= \left| \sum_{n=M+1}^{N} \beta^{-n} \big(h_D(f^n(P)) - \beta h_D(f^{n-1}(P))\big) \right| \\
&\leq \sum_{n=M+1}^{N} \beta^{-n} |h_D(f^n(P)) - \beta h_D(f^{n-1}(P))| \\
&\leq \sum_{n=M+1}^{N} \beta^{-n} C_3 \sqrt{h_X^+(f^{n-1}(P))} \quad \text{applying (6.3) with } Q = f^{n-1}(P) \\
&\leq \sum_{n=M+1}^{N} \beta^{-n} C_3 \sqrt{C_1(\delta+\epsilon)^{n-1} h_X^+(P)} \quad \text{from (6.1)} \\
&\leq C_4 \sum_{n=M+1}^{\infty} \left(\frac{\delta+\epsilon}{\beta^2}\right)^{n/2} \sqrt{h_X^+(P)}.
\end{aligned}
$$

By assumption we have $\beta > \sqrt{\delta}$, so we can take $\epsilon = \frac{\beta^2-\delta}{2}$, which implies that

$$\gamma := \frac{\delta+\epsilon}{\beta^2} = 1 - \frac{\beta^2-\delta}{2\beta^2} < 1.$$

Hence the series (6.4) converges, and we obtain the estimate

$$(6.5)\qquad |\beta^{-N} h_D(f^N(P)) - \beta^{-M} h_D(f^M(P))| \leq C_5 \gamma^{M/2} \sqrt{h_X^+(P)},$$

where $C_5 = C_5(X, f, D)$ is independent of P, N, and M. Then (6.5) and the fact that $\gamma < 1$ imply that the sequence $\beta^{-n} h_D(f^n(P))$ is Cauchy, which proves (a).

(b) The formula

$$\hat{h}_{D,f}(f(P)) = \beta \hat{h}_{D,f}(P)$$

follows immediately from the limit defining $\hat{h}_{D,f}$ in part (a). Next, letting $N \to \infty$ and setting $M = 0$ in (6.5) gives

$$|\hat{h}_{f,D}(P) - h_D(P)| \leq C_5 \sqrt{h_X^+(P)},$$

which completes the proof of (b).

(c) We are assuming that $\hat{h}_{f,D}(P) \neq 0$. If $\hat{h}_{f,D}(P) < 0$, we change D to $-D$, so we may assume that $\hat{h}_{f,D}(P) > 0$. Let $H \in \mathrm{Div}(X)$ be an ample divisor such that $H + D$ is also ample. (This can always be arranged by replacing H with mH for a sufficiently large m.) Since H is ample, we may assume that the height function h_H is non-negative. We compute

$$
\begin{aligned}
h_{D+H}(f^n(P)) &= h_D(f^n(P)) + h_H(f^n(P)) + O(1) \\
&\geq h_D(f^n(P)) + O(1) && \text{since } h_H \geq 0 \\
&= \hat{h}_{f,D}(f^n(P)) + O\Big(\sqrt{h_X^+(f^n(P))}\Big) && \text{from (b)} \\
&= \beta^n \hat{h}_{f,D}(P) + O\Big(\sqrt{h_X^+(f^n(P))}\Big) && \text{from (b)} \\
&= \beta^n \hat{h}_{f,D}(P) + O\Big(\sqrt{C(\delta+\epsilon)^n h_X^+(P)}\Big) && \text{from Theorem 24.}
\end{aligned}
$$

This estimate is true for every $\epsilon > 0$, where C depends on ϵ. Using the assumption that $\beta > \sqrt{\delta}$, we can choose an $\epsilon > 0$ satisfying $\delta + \epsilon < \beta^2$. This gives

$$h_{D+H}(f^n(P)) \geq \beta^n \hat{h}_{f,D}(P) + o(\beta^n),$$

so taking n^{th}-roots, using the assumption that $\hat{h}_{f,D}(P) > 0$, and letting $n \to \infty$ yields

$$\underline{\alpha}_f(P) = \liminf_{n \to \infty} h_{D+H}(f^n(P))^{1/n} \geq \beta.$$

(Note that Proposition 12 says that we can use h_{D+H} to compute $\underline{\alpha}_f(P)$, since $D + H$ is ample.)

(d) From (c) we get $\underline{\alpha}_f(P) \geq \beta = \delta_f$, while Theorem 4 gives $\overline{\alpha}_f(P) \leq \delta_f$. Hence the limit defining $\alpha_f(P)$ exists and is equal to δ_f.

(e) One direction is trivial. For the other, suppose that $\hat{h}_{D,f}(P) = 0$. Since we are assuming that D is ample, we may take $h_X = h_D$ and $h_D \geq 1$. Then for any $n \geq 0$, we apply part (b) to the point $f^n(P)$ to obtain

$$0 = \beta^n \hat{h}_{D,f}(P) = \hat{h}_{D,f}(f^n(P)) \geq h_D(f^n(P)) - c\sqrt{h_D(f^n(P))}.$$

This gives $h_D(f^n(P)) \leq c^2$, where c does not depend on P or n. This shows that $\mathcal{O}_f(P)$ is a set of bounded height with respect to an ample height. Since $\mathcal{O}_f(P)$ is contained in $X(K(P))$ and since we have assumed that K is a number field, we conclude that $\mathcal{O}_f(P)$ is finite. □

Remark 29. If f is a morphism, then De-Qi Zhang has pointed out that there is always at least one nonzero nef divisor class $D \in \mathrm{NS}(X)_{\mathbb{R}}$ satisfying $f^*D \equiv \delta_f D$. So there is always at least one nontrivial nef divisor class to which Theorem 5 applies, although there need not be any such ample divisor classes. The existence of such a D is an immediate consequence of the following elementary Perron–Frobenius-type result of Birkhoff, applied to the vector space $\mathbb{R}^r = \mathrm{NS}(X)_{\mathbb{R}}$, the linear transformation $T = f^*$, and the cone $C = \mathrm{Nef}(X)$; cf. [15, Lemma 1.12].

Proposition 30 (Birkhoff [10]). *Let $C \subset \mathbb{R}^r$ be a strictly convex closed cone with nonempty interior, and let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an $\mathbb{R}$-linear map with $T(C) \subseteq C$. Then C contains an eigenvector whose eigenvalue is the spectral radius of T.*

Question 31. It would be interesting to know if Theorem 5 is true for algebraically stable rational maps that are not morphisms.

7. An alternative proof of Proposition 21

In this section we give an alternative, more elementary, proof of Proposition 21. The proof uses three lemmas, one geometric, one arithmetic, and the third combining the first two.

Lemma 32. *Let $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_m \in \bar{K}$ with not all of the α_i equal to 0. Then*

$$h([\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_m]) \geq h([\alpha_0, \ldots, \alpha_n]).$$

Proof. Extending K, we may assume that $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m \in K$. Letting M_K be an appropriately normalized set of inequivalent absolute values on K, the definition of the Weil height on $\mathbb{P}^n$ gives

$$\begin{aligned} h([\alpha_0,\dots,\alpha_n]) &= \sum_{v\in M_K} \log\max\{\|\alpha_0\|_v,\dots,\|\alpha_n\|_v\} \\ &\le \sum_{v\in M_K} \log\max\{\|\alpha_0\|_v,\dots,\|\alpha_n\|_v,\|\beta_0\|_v,\dots,\|\beta_m\|_v\} \\ &= h([\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m]), \end{aligned}$$

which completes the proof of Lemma 32. □

Lemma 33. *Let $D \in \operatorname{Div}(X)$ be an effective divisor, let*

$$1 = x_0, x_1, \dots, x_n \in \Gamma(X, \mathcal{O}(D)),$$

and fix a height function h_D on $X(\bar{K})$ associated to the divisor D. Then there is a constant $C = C(X, f, h_D)$ such that for all points $P \in X(\bar{K})$ such that $x_0,\dots,x_n$ are defined at P,

$$h_D(P) \ge h([x_0(P), x_1(P), \dots, x_n(P)]) - C.$$

Proof. Let

$$\tau = [x_0,\dots,x_n] : X \dashrightarrow \mathbb{P}^n$$

be the rational map induced by the functions $x_0,\dots,x_n$.

Let H be an ample divisor on X. It is a consequence of Serre's theorem [21, II.5.17] that there is an integer $m > 0$ such that both mH and $D + mH$ are very ample. We choose a basis $1 = z_0, z_1, \dots, z_\ell$ for $\Gamma(X, \mathcal{O}_X(mH))$. Then the functions $x_i z_j$ satisfy

$$x_i z_j \in \Gamma(X, \mathcal{O}_X(D + mH)) \quad \text{for } 0 \le i \le n \text{ and } 0 \le j \le \ell,$$

so we can find a spanning set $1 = w_0, w_1, \dots, w_k$ for the space $\Gamma(X, \mathcal{O}_X(D + mH))$ whose first $(n+1)(\ell+1)$ elements are the functions $x_i z_j$.

Now for all points in the set

$$\{P \in X(\bar{K}) : x_0,\dots,x_n, w_0,\dots,w_k, z_0,\dots,z_\ell \text{ are regular at } P\}, \tag{7.1}$$

we have

$$\begin{aligned} h_D(P) &= h_{D+mH}(P) - h_{mH}(P) + O(1) \\ &= h([w_0(P),\dots,w_k(P)]) - h([z_0(P),\dots,z_\ell(P)]) + O(1) \\ &\ge h([x_i z_j(P)]_{0\le i\le n,\, 0\le j\le \ell}) - h([z_0(P),\dots,z_\ell(P)]) + O(1) && \text{from Lemma 32} \\ &= h([x_0(P),\dots,x_n(P)]) + O(1) && \text{from [23, Proposition B.2.4 (b)]} \end{aligned}$$

(Segre embedding). This completes the proof of Lemma 33 for all points in the set (7.1). But since $D + mH$ and mH are very ample, we can repeat the argument using a finite number of other bases for $\Gamma(X, \mathcal{O}_X(D + mH))$ and $\Gamma(X, \mathcal{O}_X(mH))$ so as to obtain the desired estimate for all points at which $x_0,\dots,x_n$ are regular. □

Alternative proof of Proposition 21. Replacing D by a multiple, we may assume that D is very ample and effective. We let $1 = x_0, x_1, \dots, x_n$ be a basis for $\Gamma(X, \mathcal{O}_X(D))$.

Our assumption that D is effective implies that f^*D is effective. (To see this, take a projective birational morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ normal so that $\tilde{f} = f \circ \pi$ extends to a morphism. Then $\tilde{f}^*D$ is defined via the pullback of the defining equations of the Cartier divisor D, so $\tilde{f}^*D$ is an effective Cartier divisor, so its associated Weil divisor on $\tilde{X}$ is also effective, and hence the Weil divisor $f^*D := \pi_*(\tilde{f}^*(D))$ is effective.) Further, there is a natural map

$$f^* : \Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Y, \mathcal{O}_X(f^*D)),$$

so in particular,

$$f^*x_0, \ldots, f^*x_n \in \Gamma(Y, \mathcal{O}_X(f^*D)).$$

We consider the set of points

$$\{P \in (X \smallsetminus I_f)(\bar{K}) : x_0, \ldots, x_n \text{ are defined at } f(P)\}. \tag{7.2}$$

For points in this set, we apply Lemma 33 to the divisor f^*D and functions $f^*x_0, \ldots, f^*x_n$. This yields

$$h_{Y,f^*D}(P) \geq h([f^*x_0(P), \ldots, f^*x_n(P)]) - C. \tag{7.3}$$

On the other hand, the functions $x_0, \ldots, x_n$ give an embedding

$$\tau = [x_0, \ldots, x_n] : X \hookrightarrow \mathbb{P}^n \quad \text{satisfying} \quad \tau^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_X(D),$$

so for points $Q \in X(\bar{K})$ at which $x_0, \ldots, x_n$ are regular, we have

$$h_{X,D}(Q) = h(\tau(Q)) = h([x_0(Q), x_1(Q), \ldots, x_n(Q)]) + O(1).$$

Applying this with $Q = f(P)$ and noting that $x_i(f(P)) = f^*x_i(P)$, we find that

$$h_{X,D}(f(P)) = h([f^*x_0(P), \ldots, f^*x_n(P)]) + O(1). \tag{7.4}$$

Combining (7.3) and (7.4) gives

$$h_{Y,f^*D}(P) \geq h_{X,D}(f(P)) + O(1),$$

which gives the desired result for points in the set (7.2). By taking a finite number of different effective divisors in the very ample divisor class of D, we obtain analogous inequalities that cover all points P at which f is defined. □

8. Some instances of Conjecture 6

Let $P \in X_f(\bar{K})$. We recall that Conjecture 6 asserts:

- The limit defining $\alpha_f(P)$ exists and is an algebraic integer.
- The set $\{\alpha_f(P) : P \in X_f(\bar{K})\}$ is finite.
- If $\mathcal{O}_f(P)$ is Zariski dense in X, then $\alpha_f(P) = \delta_f$.

The following theorem describes some cases for which we can prove Conjecture 6.

Theorem 34. *Conjecture* 6 *is true in the following situations:*

(a) f *is a morphism and* $\mathrm{NS}(X)_{\mathbb{R}} = \mathbb{R}$.

(b) $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ *extends a regular affine automorphism* $\mathbb{A}^N \to \mathbb{A}^N$.

(c) X *is a smooth projective surface and* f *is an automorphism.*

(d) $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ *is a monomial map and* $P \in \mathbb{G}_m^N(\bar{K})$.

(e) X *is an abelian variety and* $f : X \to X$ *is an endomorphism.*

Proof. See [24] for (a)–(c), see [36] for (d), and see [25] for (e). □

Remark 35. The maps in Theorem 34 (a)–(c) are algebraically stable. (This is automatic for morphisms, and it is also true for regular affine automorphisms.) We note that if f is algebraically stable, then

$$\delta_f = \lim_{n\to\infty} \rho((f^n)^*)^{1/n} = \lim_{n\to\infty} \rho((f^*)^n)^{1/n} = \rho(f^*),$$

so δ_f is automatically an algebraic integer. Monomial maps are not, in general, algebraically stable, but their dynamical degrees are known to be algebraic integers [22].

We also mention the following result from [24] which shows in certain cases that $\alpha_f(P)$ is equal to δ_f for a "large" collection of points. The proof uses p-adic methods, weak lower canonical heights, and Guedj's classification of degree 2 planar maps [19].

Theorem 36. *Let* $f : \mathbb{A}^2 \to \mathbb{A}^2$ *be an affine morphism defined over* $\bar{K}$ *whose extension to* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *is dominant. Assume that either of the following is true:*

(a) *The map* f *is algebraically stable.*

(b) $\deg(f) = 2$.

Then $\{P \in \mathbb{A}^2(\bar{K}) : \alpha_f(P) = \delta_f\}$ *contains a Zariski dense set of points having disjoint orbits.*

Proof. See [24]. □

A. Proof of Lemma 28

In this section we prove Lemma 28, which we restate for the convenience of the reader:

Lemma (Lemma 28). *Let* S *be a set, let* $g : S \to S$ *and* $h : S \to [0, \infty)$ *be maps, and let* $a \geq 1$ *and* $c \geq 1$ *be constants. Suppose that for all* $x \in S$ *we have*

$$h(g(x)) \leq a h(x) + c\sqrt{h(x)}. \tag{A.1}$$

Then for all $x \in S$ *and all* $n \geq 0$,

$$h(g^n(x)) \leq a^n \big(h(x) + (2\sqrt{2}c)^n \sqrt{h(x)}\big). \tag{A.2}$$

Proof of Lemma 28. To ease notation, we let $\gamma = 2\sqrt{2}$. The proof is by induction on n. Inequality (A.2) is trivially true for $n = 0$, and for $n = 1$, the desired inequality (A.2) is weaker than the assumed estimate (A.1). Suppose now that (A.2) is true for n. Then

$$\begin{aligned}
h(g^{n+1}(x)) &= h(g^n(g(x))) \\
&\le a^n\bigl(h(g(x)) + (\gamma c)^n\sqrt{h(g(x))}\bigr) && \text{from the induction hypothesis} \\
&\le a^n\Bigl(ah(x) + c\sqrt{h(x)} + (\gamma c)^n\sqrt{ah(x) + c\sqrt{h(x)}}\Bigr) && \text{from (A.1)} \\
&\le a^n\bigl(ah(x) + c\sqrt{h(x)} + (\gamma c)^n\sqrt{2ach(x)}\bigr) && \text{since } a, c, h(x) \ge 1 \\
&= a^{n+1}h(x) + \bigl(a^n c + (\gamma ac)^n\sqrt{2ac}\bigr)\sqrt{h(x)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&a^{n+1}\bigl(h(x) + (\gamma c)^{n+1}\sqrt{h(x)}\bigr) - h(g^{n+1}(x)) \\
&\quad\ge \bigl(a^{n+1}h(x) + (\gamma ac)^{n+1}\sqrt{h(x)}\bigr) - \bigl(a^{n+1}h(x) + \bigl(a^n c + (\gamma ac)^n\sqrt{2ac}\bigr)\sqrt{h(x)}\bigr) \\
&\quad= \sqrt{h(x)}a^n c\bigl(\gamma^{n+1}ac^n - 1 - \gamma^n a^{1/2}c^{n-1/2}\sqrt{2}\bigr) \\
&\quad\ge \sqrt{h(x)}a^n c\bigl(\gamma^{n+1}ac^n - 1 - \gamma^n ac^n\sqrt{2}\bigr) \\
&\quad= \sqrt{h(x)}a^n c\bigl(\gamma^n ac^n(\gamma - \sqrt{2}) - 1\bigr) \\
&\quad= \sqrt{h(x)}a^n c\bigl(\gamma^n ac^n\sqrt{2} - 1\bigr) \quad \text{since } \gamma = 2\sqrt{2} \\
&\quad> 0 \quad \text{since } a, c \ge 1.
\end{aligned}$$

This shows that (A.2) is true for $n + 1$, which completes the proof of the lemma. □

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