RECENT RESULTS AND SOME OPEN QUESTIONS ON SIEGEL'S LINEARIZATION THEOREM OF GERMS OF COMPLEX ANALYTIC DIFFEOMORPHISMS OF C^n NEAR A FIXED POINT

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INTRODUCTION.

We propose to survey some of the classical and recent results on the linearization of germs of analytic diffeomorphisms.

The main point will be the analytic difficulties due to small divisors, and we will concentrate on the case where all eigenvalues have modulus 1. The results are illustrated by many open questions and numerous examples which, for the sake of simplicity, will not be studied in the most general setting. Some new results are stated without complete proofs ; the details will appear elsewhere.

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TABLE OF CONTENTS.

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Chapter I

THE PROBLEM AND THE STATEMENT OF THE RESULTS ; SIEGEL'S AND BRJUNO'S THEOREM

1. Let $f \in (\mathbf{C}[[z_1, \cdots, z_n]])^n$ a germ of formal diffeomorphism of $(\mathbf{C}^n, 0)$. With $z = (z_1, \cdots, z_n)$, we write :

$$f(z) = Az + O(z^2) \ , \tag{1}$$

with $A \in GL(n, \mathbf{C})$, and denote by $\lambda_1, \cdots, \lambda_n$ the eigenvalues of A.

For $k = (k_1, \cdots, k_n) \in \mathbf{N}^n$, we write λ^k for $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$ and $|k|$ for $\sum k_i$. We say that the matrix A satisfies condition (*) if we have :

$$\lambda^k - \lambda_j \neq 0 \tag{$*$}$$

for any $1 \leq j \leq n$, any $k \in \mathbf{N}^n$ with $|k| \geq 2$.

The following proposition is elementary.

PROPOSITION 1. *If A satisfies condition (*), there exists a unique germ of formal diffeomorphism h of $(\mathbf{C}^n, 0)$ of the form :*

$$h(z) = z + O(z^2) \tag{2}$$

which satisfies (formally) :

$$f \circ h(z) = h(Az) \ . \tag{3}$$

2. Example : For $n = 1$, $Az = \lambda z$, with $\lambda \in \mathbf{C}^*$; then condition (*) means that λ is not a root of unity.

3. Let $g : \mathbf{C} \to \mathbf{C}$ be a holomorphic function (entire function) of the form $g(z) = z + a_2 z^2 + \cdots$ such that :

$$g(z) \not\equiv z \,. \tag{4}$$

For $\lambda \in \mathbf{C}$, let $f_\lambda = \lambda g$. If $\lambda^q = 1$, one has :

$$f_\lambda^q(z) \not\equiv z, \tag{5}$$

so one cannot find h satisfying (3) if (*) is violated.

When one has :

$$\lambda = \lambda_\alpha = \exp(2\pi i \alpha), \quad \alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z}) \,, \quad \mathbf{T}^1 = \mathbf{R}/\mathbf{Z} \,, \tag{6}$$

Proposition 1 applies and there exists $h_\alpha(z) = z + b_2 z^2 + \cdots$ such that (formally) we have $f_{\lambda_\alpha} \circ h_\alpha(\lambda_\alpha z) = h_\alpha(\lambda_\alpha z)$.

The following proposition is easy ([H 7], [L 4], see also [C 7]), [C 10], [C 11]).

PROPOSITION 2. : *There is a G_δ-dense set of $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ for which the radius of convergence of h_α is* 0.

In [H 4] we constructed, for a G_δ-dense set of $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$, rational functions f_α of degree $d \geq 2$ such that $f_\alpha(z) = \lambda_\alpha z + O(z^2)$ at 0 and f_α has a dense orbit on the Riemann sphere. I.N. Baker and P.J. Rippon ([B 1]) showed that, with $g(z) = e^z - 1$, f_{λ_α} has a dense orbit in $\mathbf{C}$ for a G_δ-dense subset of $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$.

4. STATEMENT OF THE PROBLEM OF CONVERGENCE.

4.1 We suppose in the following that f is a germ of $\mathbf{C}$-analytic diffeomorphism of $\mathbf{C}^n$ at 0, with $f(0) = 0$ and

$$A = Df(0) \quad \text{satisfies } (*) \quad , \tag{7}$$

where $Df(z)$ denotes the complex derivative or tangent map of f at the point $z \in \mathbf{C}^n$.

In the following, analytic will always mean $\mathbf{C}$-analytic.

4.2 One asks whether f can be linearized in a neighbourhood of 0, that is whether the formal conjugacy h defined by Proposition 1 defines a germ of analytic diffeomorphism at 0. When $n = 1$, this problem is also called the Schröder equation ([S 0]).

4.3 It is an easy result due to H. Poincaré ([P 5, t. I, p. XXXVI-CXXIX], see also [K 2], [F 3], [P 4] for references before 1912, and for other references [D 1]), that the problem has a positive answer if (7) holds and :

$$\sup_j |\lambda_j| < 1 \quad \text{or} \quad \sup_j |\lambda_j^{-1}| < 1 \,. \tag{8}$$

A matrix A satisfying (8) is said to be in the Poincaré domain. The proof is elementary, using, for instance, majorant series. It does not require that A is diagonizable. Moreover, if (8) holds, (7) is violated but nevertheless f is formally linearizable, then f is analytically linearizable. When (8) is violated, the matrix A is said to be in the Siegel set, which have non empty interior for $n \geq 2$. For $n = 1$, the Siegel set is $\{\lambda, |\lambda| = 1\}$.

4.4 When (7) holds and :

$$|\lambda_j| \neq 1 \,, 1 \leq j \leq n \,, \tag{7'}$$

then, by Sternberg's theorem ([S 8], [S 9]) and its generalization by Chaperon ([C 4]), f is C^∞-linearizable.

When (7') holds (but not necessarily (7)), f is topologically linearizable by Hartman-Grobman's theorem ([C 4], [P 10]). Note that when (7) and (7') hold but (8) is violated, the germ of conjugacy (as a C^∞-diffeomorphism) is not unique.

When $n = 1$ and $Az = \lambda z$, with $|\lambda| = 1$, then f is analytically linearizable if it is topologically linearizable (see §13 and §23).

4.5 When (7) holds but (8) is violated, the question of analytic linearization is much more delicate. The problem was certainly known by H. Poincaré ([P 5, t. IV, p. 36-59 and especially p. 43]), and probably, when $n = 1$, by many other mathematicians at the end of the last century ([P 4]) ; the question was explicitly asked by Kasner ([K 1]) in 1912. See [W 1], and [C 7] to [C 11] for other historical references.

Proposition 3 shows that arithmetical conditions on the λ_j's are certainly necessary when A is in the Siegel set.

4.6 We assume that A satisfies (*). For $m \in \mathbb{N}$, $m \geq 2$, we define :

$$\Omega(m) = \inf_{\substack{2 \leq |k| \leq m \\ 1 \leq j \leq n}} |\lambda^k - \lambda_j| \,.$$

Condition (*) means that $\Omega(m)$ is non zero for $m \geq 2$. If A is in the Siegel set, one has :

$$\lim_{m \to +\infty} \Omega(m) = 0 \,. \tag{9}$$

This relation, which gives birth to the so called "small divisors", is what makes it difficult to prove the convergence of h when A is in the Siegel set.

4.7 When $n = 1$ and (7) is satisfied, the first published example of a germ f with a non convergent h was given by Pfeiffer in 1917 ([P 3]).

The known cases where, given A, one can find a germ f with a non convergent h are summarized in the following proposition. It follows from a simple and elegant result of Il'yachenko ([I 1]). The divergence of h under the hypothesis (10) was first obtained by H. Cremer ([C 11]) when $n = 1$, and then generalized by Brjuno ([B 6]). The divergence under the hypothesis (11) follows from a result of J.C. Yoccoz ([Y 5]) and requires no other hypothesis on λ_1 than $|\lambda_1| = 1$.

PROPOSITION 3. *Suppose that A satisfies* (*) *and one of the conditions* (10), *or* (11) *:*

$$\limsup_{m\to+\infty}\left(-\frac{1}{m}\operatorname{Log}\Omega(m)\right) = +\infty\ ; \tag{10}$$

$$A \quad \text{has a Jordan block} \quad \begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix} \quad \text{with} \quad |\lambda_1| = 1\ ; \tag{11}$$

then there exists an analytic germ f with $f(0) = 0$, $Df(0) = A$, such that the formal conjugacy h defined by Proposition 1 has radius of convergence equal to 0.

Using [I 1], it is sufficient, to prove Proposition 3, to check that the linear operator :

$$\eta \to A\eta - \eta\circ A \tag{12}$$

on the space of holomorphic germs η from $(\mathbf{C}^n, 0)$ to itself which satisfy $\eta(0) = D\eta(0) = 0$, is not surjective. Observe that (12) is the linearization of (3) at $f = A$, $h = \mathrm{id}$.

The non surjectivity of (12) is immediate when (10) holds. The matrices A satisfying (10) form a G_δ-dense subset of the Siegel set.

5. ARITHMETIC CONDITIONS.

(For diophantine approximation, the reader can consult [S]).

5.1 Let $A \in GL(n, \mathbf{C})$ satisfying condition (*).

DEFINITION. *The matrix A satisfies the Brjuno condition if :*

$$\sum_{k=0}^{+\infty} 2^{-k}\operatorname{Log}(\Omega^{-1}(2^{k+1})) < +\infty\ . \tag{13}$$

We refer to [B 6] and [R 4] for numerous equivalent formulations of (13).

If there exists $\beta \geq 0$, $\gamma > 0$ such that :

$$\Omega^{-1}(m) \leq \gamma^{-1} m^{\beta}\ ,\ \forall\, m \geq 2 \tag{14}$$

we say that A satisfies a *diophantine condition (of exponent β).*

When $\beta > n$, almost every n-tuple $(\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ satisfies a diophantine condition of exponent β.

5.2 Let $n=1$; we write $\lambda = \exp(2\pi i\alpha)$, with $\alpha \in \mathbf{R} - \mathbf{Q}$.

Let $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$ the continued fraction of α, and $\left(\frac{p_n}{q_n}\right)_{n\geq -2}$ the convergence of α : $p_{-2}=0$, $q_{-2}=1$, $p_{-1}=1$, $q_{-1}=0$, and $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 0$; one alway has

$$q_n \geq 2^{\frac{n-1}{2}} \ , \ n \geq 1 \ . \tag{15}$$

The continued fraction of α determines its approximation by rationals (cf. [H 3, ch. V]) ; one has :

$$(a_{n+1}+2)^{-1} q_n^{-2} \leq \left|\alpha - \frac{p_n}{q_n}\right| \leq a_{n+1}^{-1} q_n^{-2} \ , \ n \geq 0 \ . \tag{16}$$

The Brjuno condition on λ is equivalent to :

$$\sum_{n=0}^{+\infty} q_n^{-1} \operatorname{Log} q_{n+1} < +\infty \ , \tag{17}$$

and we then say that α is a Brjuno number, writing B for the set of Brjuno numbers. Condition (10) is equivalent to $\limsup_{n\to+\infty} q_n^{-1} \operatorname{Log} q_{n+1} = +\infty$. A diophantine condition of exponent β for λ is equivalent to $q_{n+1} = O(q_n^\beta)$ (so $\beta \geq 1$) ; we write DC for the set of $\alpha \in \mathbf{R}$ such that λ satisfies a diophantine condition.

We say that α is of constant type, and write $\alpha \in CT$, if λ satisfies a diophantine condition of exponent 1.

The set $L = \mathbf{R} - (DC \cup \mathbf{Q})$ is by definition the set of *Liouville numbers* ; it is a G_δ-dense subset of $\mathbf{R}$ and has Haussdorff dimension 0 (hence a fortiori Lebesgue measure 0).

5.3 When $n \geq 2$, nothing as simple as continued fractions exists. The construction of matrices A which satisfy (13) but not (14), with all eigenvalues of modulus 1, is left to the reader.

When $n=1$, one has $\alpha \in DC$ (resp. $\alpha \in B$) if and only if $-\alpha \in DC$ (resp. $-\alpha \in B$). For $n \geq 2$, however, we only take into account for (13) or (14) the quantities $|\lambda^k - \lambda_j|$, with $k \in \mathbf{N}^n$ ($|k| \geq 2$) ; this is different from considering all quantities $|\lambda^k - \lambda_j|$, with $k \in \mathbf{Z}^n$, and $|k| = \sum |k_j| \geq 2$.

6.

THEOREM 1. *Let f be a germ of analytic diffeomorphisms of $(\mathbf{C}^n,0)$, of the form :*

$$f(z) = Az + O(z^2)\ ,\ A \in GL(n,\mathbf{C}).$$

We assume that A is diagonalizable, and satisfies (*) and Brjuno condition (13). Then the formal conjugacy h of Proposition 1 defines a germ of analytic diffeomorphism in a neighbourhood of 0.

The above theorem was first proved by C.L. Siegel in 1942 ([S 4]), when $n = 1$ and A satisfies a diophantine condition (i.e. (14)). Still under the diophantine condition (14), it was generalized by Siegel ([S 5, Band III, p. 178-187]) to vector fields in $\mathbf{C}^n$ near singular a point, and by S. Sternberg ([S 7]) and Gray ([G 4]) to germs of diffeomorphisms of $(\mathbf{C}^n,0)$. Under Brjuno condition (13), it was first proved by Brjuno ([B 5], [B 6]), and afterwards, when $n = 1$, by H. Rüssmann ([R 2]).

Both Siegel's and Brjuno's proofs use majorant series ; this is the most natural method, and gives the best results : the estimates of the radius of convergence are very reasonable, with the right weights for the contributions of the diophantine approximations. The key point, and probably the whole problem, is of arithmetical nature.

These are many other proofs using rapid iteration methods and the so-called KAM techniques. For Newton's method (i.e. the existence of an inverse up to a quadratic term) see H. Rüssmann ([R 3]) and E. Zehnder ([Z 1]) ; for rapid iteration techniques with an infinite number of change of coordinates, see Brjuno ([B 6]), Rüssmann ([R 2]), Siegel and Moser ([S 6]) and Arnold ([A 1]).

We refer to Brjuno ([B 6]), Rüssmann ([R 5]) and J. Pöschel ([P 9]) for various generalizations of Theorem 1.

Theorem 1 was generalized to non-archimedean complete valued fields of characteristic 0 (instead of $\mathbf{C}$) by Sibuya-Sperber ([S 1]) and Herman-Yoccoz ([H 12]) ; we refer to [S 2], [S 3] for applications.

When A is (conjugate to) a unitary matrix, satisfies condition (*) and a suitable diophantine condition, one can deduce Siegel's theorem from results on normal forms of analytic diffeomorphisms of $\mathbf{T}^n$, i.e. Arnold-Moser's theorem ([A 2], [M 7],and also [H 3, Annexe] and [Z 2]) : see [H 4, VIII] and [H 6] for the precise statements. This requires KAM techniques and is very useful to study the boundaries of Siegel singular disk and the global properties of radii of convergence ; see § 19.

Historical comments. To the best knowledge of the author, the first to use, for solving non linear equations in the complex analytic category, the technique of diminishing domains together with Cauchy estimates (which amounts essentially to the same thing as smoothing operators), was M. Gevrey in 1914 ([G 1]). This method of Gevrey is very standard in complex analysis ; it was for instance used in

1940 by H. Cartan ([C 2]) in combination with Newton's method (without being explicitly stated). A.N. Kolmogorov in 1954 ([K 4]) was the first to state[1] that Gevrey's method in conjunction with Newton's method could be used for questions related to small divisors ; various proofs, due to V.I. Arnold and J. Moser, were given in the sixties ; see [B 4] for a recent survey. The technique of introducing an infinite number of change of coordinates was used, in a different context, by Newlander and Nirenberg in 1957 ([N 3]).

In 1919, G. Julia claimed, in an incorrect paper ([J 1]), to disprove Siegel's theorem ; it was rapidly known that this was in fact an open problem (see H. Cremer's paper) until C.L. Siegel settled it in 1942. Most of the difficult questions of convergence involving small divisors where known to H. Poincaré ; he frequently made the incorrect "conjectures" (for example, about the convergence of perturbation series with fixed frequencies, i.e. the existence of KAM invariant torii), but to the best knowledge of the author, never *claimed to disprove* the convergence. I take the opportunity to quote H. Poincaré about the existence of invariant torii ([P 6, t. II, § 149, p. 104-105]) :

"Ne peut-il pas arriver que les séries (2) convergent, quand on donne aux x_i^0 certaines valeurs convenablement choisies ?

Supposons, pour simplifier, qu'il y ait deux degrés de liberté ; les séries ne pourraient-elles pas, par exemple, converger quand x_1^0 et x_2^0 ont été choisies de sorte que le rapport $\frac{n_1}{n_2}$ soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport $\frac{n_1}{n_2}$ est assujetti à une autre condition analogue à celle que je viens d'énoncer un peu au hasard) ?

Les raisonnements de ce chapitre ne me permettent pas d'affirmer que ce fait ne se présente pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable."

See also [P 5, t. XI, p. 69-78].

[1] A.N. Kolmogorov never published a complete proof of [K 4].

7. Some conjectures.

Conjecture 1. *The Brjuno condition is necessary for Theorem 1 to hold ; in other words, assuming A diagonalizable and satisfying condition (*), the Proposition 3 holds when (10) is replaced by :*

$$(10') \qquad A \quad \text{does not satisfy Brjuno condition.}$$

Observe that for the linearized equation (12), the condition (10) is the best possible (i.e. surjectivity of (12) is equivalent to the negation of (10)).

For very reasonable support of this conjecture, the reader should look at (11.3). Part of the conjecture was claimed (in a stronger form) by T. Cherry ([C 5]) ; but no proof has appeared, and the claim of [C 5] might well be incorrect.

Conjecture 2. *If one replaces* $\mathbb{C}$ *by a locally compact non-trivial complete valued field of* strictly positive *characteristic, Siegel's theorem is usually false, even for polynomials of one variable.*

What goes wrong in such a field is that there are no Brjuno numbers (i.e. satisfying (13)), cf. [H 12].

For lack of space, we refer the reader to the partial survey of V.I. Arnold ([A 1]) about Il'yachenko's, Pyartli's and his own work on the geometric materializations of resonances when A has no eigenvalues of modulus 1. Other references are [A 3], [I 2], [I 3], [M 1], [M 4], [M 5].

We also will not discuss here the infinite dimensional versions of Theorem 1 ; we only give references to N.V. Nikolenko ([N 4], [N 5], [N 6]) and E. Zehnder ([Z 3]).

Chapter II

THE IDEA OF THE PROOF, BY MAJORANT SERIES, OF SIEGEL'S AND BRJUNO'S THEOREMS ; VARIOUS REMARKS

8. IDEA OF THE PROOF OF THEOREM 1.

8.1 We describe here the main ideas of the proof, due to Siegel ([S 4]). We only consider the case $n = 1$; the case $n \geq 2$ is essentially similar : see [B 6], [P 9].

Let $f(z) = \sum_{j \geq j} c_j z^j$, with $c_1 = \exp(2\pi i\alpha) = \lambda$, the germ of holomorphic diffeomorphism we are considering ; replacing if necessary $f(z)$ by $\frac{1}{t} f(tz)$ with small $t \neq 0$, we can assume that :

$$|c_j| \leq 1 \ , \ \forall j \geq 1 \ ; \tag{8.1}$$

in particular, f is holomorphic on $\{|z| < 1\}$.

The (formal) conjugacy $h(z) = \sum\limits_{j \geq 1} h_j z^j, c_1 = 1$, satisfies :

$$h(\lambda z) = f(h(z)) = \lambda z + \sum_{j \geq 2} c_j (h(z))^j \ .$$

For $j \geq 2$, let $s_j = |\lambda^j - \lambda|$, and ω_j a strictly positive number such that $0 < \omega_j \leq s_j$. Consider then the formal germ $g(z) = \sum_{j \geq 1} b_j z^j$, with $b_1 = 1$, which satisfies :

$$\sum_{j \geq 2} \omega_j b_j z^j = \sum_{j \geq 2} (g(z))^j \ . \tag{8.2}$$

This gives for $j \geq 2$:

$$b_j = \omega_j^{-1} \sum b_{k_1} \cdots b_{k_l} \ , \tag{8.2'}$$

the sum being taken for $l \geq 2$, $k_p \geq 1$ and $\sum\limits_p k_p = j$.

By induction, one sees that, for $j \geq 1$:

$$|h_j| \leq b_j \ . \tag{8.3}$$

8.2 By Dirichlet principle, we know that :

$$\liminf_{n\to+\infty} ns_n < +\infty \ .$$

However, if we take $\omega_j = j^{-\beta}$, for some $\beta > 0$, then g diverges ; this illustrates the difficulty, due to the small divisors s_j (or ω_j), in proving the convergence of h (or g).

By (8.6) and (8.8) a necessary (but not sufficient, see (11.3)) condition for the convergence of g is :

$$\limsup_{j\to+\infty} (\omega_2 \cdots \omega_j)^{-1/j} < +\infty \ . \tag{8.4}$$

Tambs Lyche ([T 1]) and Hardy and Littlewood ([H 2]) have shown that, for any $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$:

$$\limsup_{j\to+\infty}(s_2 \cdots s_j)^{-1/j} = \limsup_{j\to+\infty} q_{j+1}^{-1/q_j} \ . \tag{8.4'}$$

When α is a Brjuno number, both *lim sup* are limits and equal to 1 (but this is not equivalent to the Brjuno condition).

The result (8.4') of Tambs Lyche and Hardy and Littlewood is *not* a consequence of Birkhoff's ergodic theorem for a.e. α. It is related to the following question asked by A.Y. Khintchine :

QUESTION 1. *For which $\varphi \in L^2(\mathbf{T}^n, d\theta)$ does one have, for almost all $\alpha \in \mathbf{T}^n$:*

$$\lim_{j\to+\infty} \frac{1}{j}\sum_{l=1}^{j} \varphi(j\alpha) = \int_0^1 \varphi(\theta)d\theta \ ?$$

By Marstrand ([M 2]), this may be false, even when $n = 1$ and φ is bounded ; it is true, by [K 3], when $n = 1$ and $\varphi(\theta) = \operatorname{Log}\|\theta\|$ or $\operatorname{Log}|\sin 2\pi\theta|$; see also [H 2].

8.3 We take $\omega_n = s_n$, $\delta_1 = 1$, and define δ_j for $j \geq 2$:

$$\delta_j = \omega_j^{-1} \sup \delta_{k_1} \cdots \delta_{k_l} \ , \tag{8.5}$$

the supremum being taken for $l \geq 2$, $k_p \geq 1$, $\sum_p k_p = j$. We certainly have :

$$\delta_j \geq (\omega_2 \cdots \omega_j)^{-1} \ , \ j \geq 2 \ . \tag{8.6}$$

Define $l(z) = \sum_{j \geq 1} l_j z^j$ by $l_1 = 1$ and

$$l(z) = z + \frac{(l(z))^2}{1 - l(z)}$$

which gives :

$$l_j = \sum l_{k_l} \cdots l_{k_l} \ ,$$

the sum being taken as in (8.2'). Induction shows that :

$$b_n \leq \delta_n l_n \ ; \tag{8.7}$$

$$\delta_n \leq b_n \ . \tag{8.8}$$

We conclude that g is convergent if and only if :

$$\sup_j j^{-1} \operatorname{Log} \delta_j < +\infty \ . \tag{8.9}$$

8.4 To obtain (8.9), Brjuno uses the following property of the sequence (ω_j) : there exists a sequence $(v_j)_{j \geq 1}$ in $]0, 1/2]$, with the v_j all distinct and having 0 as accumulation point, and constants $\theta \in]0, 1/2]$, $c > 0$ such that :

$$v_j \leq c\omega_{j+1} \ , \ j \geq 1 \ ; \tag{8.10}$$

$$\sum_{k=1}^{\infty} -t_k^{-1} \operatorname{Log}(\theta v_{t_k}) = d < +\infty \ . \tag{8.11}$$

$$\text{If} \quad v_j < \theta v_{t_k} \ , \ 1 \leq l < t_{k+1}, \quad \text{then} \quad v_{j-l} \geq \theta v_{t_k} \ , \tag{8.12}$$

where the numbers t_k are defined inductively by $t_1 = 1$ and :

$$t_{k+1} = \inf(l \mid v_l < v_{t_k}) \ , \ k \geq 1 \ .$$

Indeed, from (8.10), (8.11), (8.12), one gets, using a counting lemma and clever (but elementary) manipulations of majorants series, that :

$$\sup_{j \geq 1} j^{-1} \operatorname{Log} \delta_j \leq C_1 d + C_2 \ , \tag{8.13}$$

with positive universal constants C_1, C_2.

Remark : Instead of g, Brjuno works with $\eta(z) = z^{-1} g(z) - 1$ which gives a slightly different inductions (but also (8.13)).

8.5 In our case, we take, for $j \geq 1$, $v_j = \|j\alpha\|$, where $\|x\| = \inf_{p\in\mathbf{Z}} |x+p|$ for $x \in \mathbf{R}$. We then have :

$$s_{j+1} = \omega_{j+1} = |\lambda^{j+1} - \lambda| \geq 4v_j \ , \ j \geq 1 \ . \tag{8.14}$$

On the other hand, one has :

$$\|j\alpha\| + \|(j-l)\alpha\| \geq \|l\alpha\| \ .$$

So (8.10), (8.12) hold with $\theta = 1/2$, $c = 1/4$. One has $t_k = q_{k-1}$ if $\alpha \in (0, 1/2)$, and $t_k = q_k$ if $\alpha \in (1/2, 1)$; hence (8.11) follows from (17), using (16).

Let $R(g)$, $R(h)$ the radii of convergence of g, h ; we finally conclude from (8.3), (8.13) that :

$$\mathrm{Log}[R(h)^{-1}] \leq \mathrm{Log}[R(g)^{-1}] \leq C_3 \left(\sum_{k=0}^{\infty} q_k^{-1} \, \mathrm{Log}(\|q_k\alpha\|^{-1}) \right) + C_4 \ , \tag{8.15}$$

where C_3, C_4 are universal constants, under the assumption (8.1) on f.

9. In the special case :

$$f_\alpha(z) = e^{2\pi i\alpha}(z + z^2) = \lambda_\alpha(z + z^2)$$

a slight modification of Brjuno's proof ([B 6]) gives :

$$\mathrm{Log}[R(h_\alpha)^{-1}] \leq \mathrm{Log}[R(g_\alpha)^{-1}] \leq \mathrm{Log}\, \frac{1}{4} + 2 \sum q_k^{-1} \, \mathrm{Log}[(2 \sin \frac{\pi}{2} \|q_k\alpha\|)^{-1}] \ , \tag{9.1}$$

where the sum starts with $k = 0$ for 0, α, $1/2$ and $k = 1$ if $1/2 < \alpha < 1$.

Idea of the proof : The term Log 1/4 comes from the special type of induction for h_α ; in order to get better constants, one uses the counting argument of [B 6], but with the fuction $\mathrm{Log}[(2|\sin \pi\theta|)^{-1}]$. ∎

When $\alpha = \frac{\sqrt{5}-1}{2}$, (9.1) gives, up to a factor less than 50, the correct value for $R(h_\alpha)$; this is very reasonable.

To get estimates *from above* for $R = R(h_\alpha)$, one observes that $\tilde{h}(z) = \frac{1}{R} h(Rz)$ is univalent on $\{|z| < 1\}$ (see §13). The critical value $-\lambda_\alpha/4$ of f_α cannot belong to the image of this disk, so we obtain an elementary bound for R by Koebe's 1/4-theorem :

$$R \leq 1 \ . \tag{9.2}$$

A much better estimate is obtained by considering :

$$\tilde{h}(z) \left(1 + \frac{4R}{\lambda_\alpha} \tilde{h}(z) \right)^{-1} = z + b_2 z^2 + O(z^3) \ ,$$

which is univalent on $\{|z| \leq 1\}$, and applying Bieberbach's inequality ($|b_2| \leq 2$) to get :

$$R \leq \frac{2|\lambda_\alpha - 1|}{|4 - 3\lambda_\alpha|} \leq \frac{4}{7} \ . \tag{9.3}$$

10. RELATION BETWEEN THE RADIUS OF CONVERGENCE AND THE DIOPHANTINE PROPERTIES OF α.

If there exists $\beta \geq 0$, $\gamma > 0$, the estimate (8.15) shows that :

$$R(g)^{-1} \leq C\gamma^{-1} , \tag{10.1}$$

for some universal constant $C > 0$.

Actually, one can get from (8.15) a much better result, giving up to constants the right contribution of the various rational approximations of α.

Suppose that the continuous fraction $1/(a_1 + 1/(a_2 + \cdots))$ of α is such that all a_i are 1, except one, say a_{n+1}, which is large.

Then we get from (8.15) :

$$\operatorname{Log}[R(g)^{-1}] \leq C_1 + C_2 q_n^{-1} \operatorname{Log} q_{n+1} , \tag{10.2}$$

for some universal constants C_1, C_2.

10.3 The following example shows that this type of estimate is optimal. Consider $f(z) = \lambda z + z^{q+1}$, where $\lambda = \exp(2\pi i\alpha)$, α is above, and $q = q_n$. The conjugacy h has the form :

$$h(z) = z + h_{q+1}z^{q+1} + h_{q+2}z^{q+2} + \cdots$$

with $h_{q+1} = (\lambda^{q+1} - \lambda)^{-1}$.

The image by h of its disk of convergence $D(h)$ is contained in $\{|z| \leq 2^{1/q}\}$, because $\lim_{n\to+\infty} |f^n(z)| = +\infty$ when $|z| > 2^{1/q}$. As h is injective on $D(h)$ (see §13), we obtain :

$$\text{Area } (h(D(h))) = \pi \sum_{j\geq 1} |h_j|^2 j [R(h)]^{2j} \leq \pi 2^{2/q} ,$$

and therefore :

$$\begin{aligned} \operatorname{Log}[R(h)^{-1}] &\geq \frac{1}{2q} \operatorname{Log}(q2^{-2/q}|\lambda^q - 1|^{-2}) \\ &\geq C_3 + C_4 q_n^{-1} \operatorname{Log} q_{n+1} \end{aligned}$$

for universal constants C_3, C_4.

11. For $\alpha \in \mathbf{T} - (\mathbf{Q}/\mathbf{Z}), \lambda = \exp(2\pi i\alpha)$, let $l(\alpha)$ be the radius of convergence of the function g_α defined by (8.2), with $\omega_n = |\lambda^n - \lambda|$.

It is not difficult to see that :

$$l(\alpha) = 0 \quad \text{on a dense } G_\delta \text{ subset of } \quad \mathbf{T}^1 \ ; \tag{11.1}$$

the measurable function $:\alpha \to [l(\alpha)]^{-1}$is in the weak L^1-space of $(\mathbf{T}^1, d\theta)$. (11.2)

Indeed, with $|E|$ denoting the Lebesgue measure of $E \in \mathbf{T}^1$, we have, for $\beta > 0$, $\gamma > 0$:

$$\left|\left\{\alpha, |\alpha - (p/q)| \leq \frac{\gamma}{q^{2+\beta}} \quad \text{for some} \quad p/q\right\}\right| \leq C(\beta)\gamma \ , \ C(\beta) > 0 \ ,$$

and from this and (10.1), we deduce that :

$$|\{\alpha \in \mathbf{T}^1, l(\alpha) \leq \gamma\}| \leq C\gamma \ .$$

J.C. Yoccoz has shown ([Y 4]) that :

$$\text{if} \quad l(\alpha) \neq 0, \quad \text{then } \alpha \text{ is a Brjuno number .} \tag{11.3}$$

This strongly supports Conjecture 1 (at least if one does not believe in wild cancellations due to the fact that $\lambda^n - \lambda$ are complex).

In counterpart, determining *exactly* $l(\alpha)$ seems to be, in view of (11.1), (11.2), an untractable and unreasonable problem. It probably requires the exact knowledge of the continued fraction of α and of all possible cancellations !

We let the reader try to calculate $R(h_\alpha)$ or even give a reasonable lower bound for it, when $\alpha = \pi$ or $\alpha = 2^{k+(1/k)}$, $k \geq 3$, $k \in \mathbf{N}$.

QUESTION 2. *Is it possible to find an algorithm to decide, given a small $\epsilon > 0$, if $R(g_\alpha) \geq \epsilon$ from the base 2 expansion of α (as computers suggest) ?*

For numbers α of constant type, we found [H 8] a very simple general method which applies to almost all small divisors problems ; it gives in particular a very simple proof of Siegel's theorem ([H 4]), and yields, for more difficult problems, very reasonable constants ([H 8, vol. 2, ch. VII]). The constants depend only on the calculation of the logarithms and sines of a couple of numbers, and a pocket calculator is more than enough !

For a remarkably simple minoration of $R(h)$, due to J.C. Yoccoz, in the special case $f(z) = e^{2\pi i\alpha}(z + z^2)$, we refer to 18.4 ; see also [L 1] when α is the golden mean.

Using the simple idea that one can calculate the Taylor coefficients of h and then conjugate f by appropriate truncations of h, various authors ([L 2], [L 3]) have claimed much better estimates for $R(h)$ when α is the golden mean. Unfortunately, this approach requires large computers and a huge amount of numerical work, so the claims are not easily checked. These authors have also used the fact that one can follow Newton's method on computers, a point of view which was first adopted by O. Hald and Braess and Zehnder ([B 7]).

12. Generalizations and remarks on the proof.

The conditions (8.11), (8.12) we have required in 8.4 on the sequence $(v_j)_{j\geq 1}$ may be replaced by the following less restrictive conditions, due to Brjuno ([B 6], see also [P 9]) : there exists $\theta \leq 1/2$ and integers $1 = p_0 < p_1 < p_2 < \cdots$ such that we have, with $\check{\Omega}(m) = \inf_{1\leq k\leq m} v_k$:

$$\sum_{k=0}^{+\infty} -p_k^{-1} \operatorname{Log}[\theta\check{\Omega}(p_{k+1})] = d_1 < +\infty \; ; \tag{8.11'}$$

$$\text{If} \quad v_n < \theta\check{\Omega}(p_k) \quad \text{and} \quad l < p_{k+1}, \quad \text{then} \quad v_{n-l} \geq \theta\check{\Omega}(p_k) \; . \tag{8.12'}$$

These conditions together with (8.10) imply :

$$\operatorname{Log}[R(g)^{-1}] \leq C_3 d_1 + C_4 \; , \tag{8.13'}$$

for universal constants $C_3, C_4 > 0$. When $v_n = \|n\alpha\|$, (8.11') is equivalent to the Brjuno condition, see [B 6]. One uses the sequence (v_j) instead of (ω_j) because (8.12) might not hold for (ω_j).

C. L. Siegel, in [S 4], makes the following assumptions : there exist $\gamma \in]0,1[$ and $\nu > 0$ such that :

$$0 < \omega_n^{-1} < \gamma^{-1}(n-1)^{\nu}, \quad \text{for} \quad n \geq 2 \; ; \tag{12.1}$$

$$\min(\omega_p^{-1}, \omega_q^{-1}) < \gamma^{-1}(q-p)^{\nu} \quad \text{for} \quad 1 < p < q \; . \tag{12.2}$$

He then shows that :

$$\sup_{n\geq 1} n^{-1} \operatorname{Log} \delta_n \leq \gamma^{-1} L, \tag{12.3}$$

where δ_n is defined by (8.5) and L is a universal constant.

The conditions (12.1) or (8.11') alone are *not* sufficient to obtain (12.3) ; the conditions (8.12') or (12.2) imply that the ω_n^{-1} are not frequently large, and this is the crucial point.

The conditions (8.11') and (8.12') (or (12.1) and (12.2)) appear in many other problems of small divisors.

12.4 Both Siegel and Brjuno prove that g converges and therefore (by (8.3)), so does h. The other existing proofs of the convergence of h (for instance, using rapid iteration methods, [S 6]) do not show that g converges !! See § 32 and § 34.

Chapter III
THE BOUNDARIES OF SIEGEL SINGULAR DISKS

13. Study of the boundaries of Siegel singular domains.

13.1 For this study to make sense, we will require global assumptions on f.

D. Sullivan asked the following question :

Assume that U is a simply connected domain with compact closure $\bar{U}$, containing 0, and that f is an analytic diffeomorphism of U, extending continuously to $\bar{U}$, such that $f(0) = 0$ and $f'(0) = e^{2\pi i\alpha}$, with $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$; does this imply that ∂U is a Jordan curve ?

Couterexamples were given by Moeckel ([M 6], see also [P 7]) and independently in [H 5]. In the example of [H 5] (which is adapted from one of M. Handel [H 1]), f extends to a C^∞-diffeomorphism of $\mathbf{C}$ and ∂U can be taken as the "pseudo-circle".

13.2 For the sake of simplicity, we assume that $f = f_\lambda = \lambda g$, with g as in § 3 and :

$$\lambda = e^{2\pi i\alpha} \ , \ \alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z}) \ . \tag{13.3}$$

We assume that f is linearizable at 0 ; this is the case if α is a Brjuno number.

Let U ($\neq \emptyset$) be the maximal connected open set containing 0 on which the family $(f^n)_{n\geq 0}$ is normal. We have $f(U) = U$, and, by a result of P. Fatou (see [F 2]), $U \neq \mathbf{C}$. By the maximum principle, U is simply connected.

Let $h_1 : \mathbf{D} = \{|z| < 1\} \to U$ the conformal representation of U which satisfies $h_1(0) = 0$, $h_1'(0) = t > 0$. By Schwarz's lemma, we have :

$$h_1^{-1} \circ f \circ h_1(z) = \lambda z \ ,$$

so $h(z) = h_1\left(\frac{z}{t}\right)$ is univalent and satisfies the same equation on $\{|z| < t\}$; moreover $h'(0) = 1$ and t is the radius of convergence of h.

We call U a *singular domain (or disk)*. (We add "singular" because "Siegel domains" classically refer to the symmetric spaces $Sp(\mathbf{R}^{2n})/U(n)$.)

Question 3.

a) *If U has compact closure, does there exist a critical point of f on ∂U ?*

b) *If f has no critical points, is U unbounded ?*

Clearly a positive answer to a) implies a positive answer to b).

QUESTION 4. *If U has compact closure, is it true that :*
a) ∂U is a Jordan curve (i.e. a simple closed curve)?
b) ∂U is a quasicircle?

(A quasicircle is the image of the standard circle $\mathbf{S}^1 = \{z \in \mathbf{C}, |z| = 1\}$ by a quasi-conformal homeomorphism of $\mathbf{C}$, hence is a Jordan curve.)

When f is polynomial, U has always compact closure. These questions were first asked by A. Douady (1980) and D. Sullivan (1981) for rational functions ([S 10]).

One of the reasons for asking Question 3.a) is the following result of P. Fatou ([F 1], [H 7]) : let SV be the set of singular values of f, i.e. the critical or asymptotic values of f ; then the ω-limit set of SV by f (i.e. $\cap_{n\geq 0}\overline{\cup_{k\geq n} f^k(SV)}$) contains the boundaries of all singular domains of f. When f is a polynomial, SV is just the set of critical values of f.

14. We assume in this section that :

$$\alpha \in DC \quad \text{(see (5.2))} . \tag{14.1}$$

14.2 Under hypothesis (14.1), E. Ghys ([G 2]) has shown that a positive answer to Question 4.a) implies a positive answer to question 3.a). This was generalized by the following theorem ([H 7]) :

14.3

THEOREM 2. *If U has compact closure, $f_{|\partial U}$ is injective and (14.1) holds, then there is a critical point of f in ∂U.*

Hence, to answer positively to Question 3.a), under the hypothesis (14.1), supposing that there is no critical point of f on ∂U, one has to check that $f_{|\partial U}$ is injective ; curiously enough, it is not easy at all, and we have only been able to do that in special examples : see [H 7] where the following theorem is proved.

THEOREM 3. *If f is a polynomial with only one critical point c, and (14.1) holds, then $c \in \partial U$.*

One an take for instance, $f(z) = \frac{\lambda}{n}((z+1)^n - 1)$, $n \geq 2$; the theorem also applies for periodic elliptic points.

We also obtained in [H 7] many examples with a critical value of f on ∂U.

14.4 Consider the case where $f(z) = \lambda(e^z - 1)$. It was shown in [H 7] that, when (14.1) holds, U is unbounded ; this implies that ∂U is not, in the Riemann sphere, a Jordan curve and is rather complicated : $\{\infty\}$ is contained in the impression of every prime end of U.

This example is the reason why we asked in Question 4 for U to have compact closure.

In the following questions, $f(z) = \lambda(e^z - 1)$.

QUESTION 4.

c) *When U is unbounded, does the omitted value $-\lambda$ of f belong to ∂U ?*

d) *When f is still linearizable at 0, but (14.1) does not hold, is U always unbounded ?*

14.5 After [H 7] was obtained, L. Carleson and P. Jones gave a simpler proof of Theorem 3, showing that $c \in \partial U$ for a.e. $\lambda \in \mathbf{S}^1$; they use the ingredients of J.C. Yoccoz's proof of Siegel's theorem for this particular class of polynomials (see § 18).

15. In both Ghys' partial result ([G 2]) and in the proof of Theorem 2, the main ingredient is the following result.

THEOREM 4. *Let f be a $\mathbf{R}$-analytical diffeomorphism of the circle, with rotation number $\alpha \in DC$; then f is $\mathbf{R}$-analytically conjugated to the rotation $R_\alpha : \theta \to \theta + \alpha$.*

This theorem was first proved by the author in 1975 for $\alpha \in CT$, in 1976 for a.e. α ([H 3]), and was generalized to $\alpha \in DC$ by J.C. Yoccoz in 1982 ([Y 1], see also [Y 2]). By Denjoy's theorem, f is topologically conjugated to R_α. One first shows that the conjugacy h is C^1, then that it is C^∞ ; the proof is more direct and natural than the usual techniques in KAM theory (which, anyway, give only perturbative or local results). Finally, one shows that if h is C^∞ and $\alpha \in DC$ then h is $\mathbf{R}$-analytic : for this we used in [H 3] an improvement of a theorem of Arnold and Moser ([H 3, annexe]), but this can be avoided for a.e. α by adapting [H 6].

The author does not know of any other *global* result than Theorem 4 in small divisors theory ; in fact, it could well be the only simple one, cf. [H 9] and [H 3, XIII].

QUESTION 5. *Does Theorem 4 hold when α is a Brjuno number ?*

A positive answer would imply that Theorems 2 and 3 are still valid when $\alpha \in B$. The *local* conjugacy theorem, for $\alpha \in B$, is true and is proved by adapting [R 3] and [H 3, annexe].

16. An example of application of Theorem 3.

Let $f(z) = \lambda(z+z^2)$, with $\lambda = \exp(2\pi i\alpha)$; we assume that $\alpha \in DC$ Let $c = -\frac{1}{2}$ the critical point of f_λ ; a theorem of Fatou ([F 1]) says that ∂U is contained in the closure of the orbit of c under f.

Using Theorem 3 we conclude that :

$$(16.1) \qquad (f_\lambda^n(c))_{n\geq 0} \quad \text{is dense in} \quad \partial U .$$

This, as the following example shows, can force ∂U to be geometrically complicated.

16.2 *Example :* Let $n \geq 1$, $p \in \mathbf{N}^\star \cup \{\infty\}$; suppose that the continued fraction of α satisfies $a_i = 1$ if $i \neq n$ and $a_n = p$ (when $p = \infty$, this means that the continued fraction stops at stage $(n-1)$). We suppose that n and p are very large, and write f_p to indicate the dependance on p of $f(z) = e^{2\pi i\alpha}(z+z^2)$.

When $p = 1$, α is the golden mean. Given $\epsilon > 0$ and $h \in \mathbf{N}^\star$, if n is very large (independently of p), the distance between $f_1^l(c)$ and $f_p^l(c)$ will be less than ϵ for $l \leq k$.

On the other hand, α is rational when $p = \infty$, so by a result of G. Julia and P. Fatou ([F 1]) we have $\lim_{l\to+\infty} f_\infty^l(c) = 0$. Given $k_1 \in \mathbf{N}^\star$ and $\epsilon > 0$, the orbit $(f_\infty^l(c))_{l\leq k_1}$ is ϵ-close to $(f_p^l(c))_{l\leq k_1}$ is p is large enough (note that here, "large" depends on n). As p is still finite, α is of constant type and (16.1) applies, showing wild oscillations for ∂U.

This example shows that one cannot conclude anything from the numerical computations of $(f^n(c))_{n\geq 0}$ if one does not control the error terms ; on the other hand, to keep track of these terms seems difficult, as c is on the boundary of the basin of ∞.

17. Arithmetic conditions are necessary.

17.1 Recall that in the result of E. Ghys (14.2) as well as in Theorems 2 and 3, we assume that $\alpha \in DC$.

The following theorem ([H 10]), which is obtained using Ghys's construction ([G 2]), shows that an arithmetic hypothesis is necessary.

THEOREM 5. *There exists $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ such that $f(z) = e^{2\pi i\alpha}(z+z^2)$ is linearizable at 0, and its singular Siegel disk U satisfies :*

i) ∂U is a quasicircle ;

ii) no point $f^n(c)$, $n \geq 0$ lies on ∂U.

17.2 Fatou's theorem says that $\partial U \subset \overline{\{f^n(c), n \geq 0\}}$. By Theorem 3, the number α in Theorem 4 cannot belong to DC. In [H 10], it is also shown that Question 3.b) has a negative answer. Theorem 5 shows that Ghys's result 14.2 is false for some Liouville number α, and that most results of [H 7] are false without arithmetic assumptions.

17.3 Taking into account § 15, at least one of the following statements is true :

a) Question 5 has a negative answer ;

b) The number α in Theorem 5 is not a Brjuno number.

If b) holds, Cherry's claim in [C 5] is incorrect.

QUESTION 6. *Which of these statements is true ?*

QUESTION 7. *In Theorem 5, can one find α such that one can replace i) by one of the following statements :*

a) ∂U is a C^k-submanifold for some $1 \leq k \leq \infty$?

b) ∂U is a Jordan cruve, but not a quasi-circle?

17.4 Ghys's result 14.2 shows that ∂U cannot be a C^1-submanifold when $\alpha \in DC$

For any α such that f is linearizable at 0, and any $z_0 \in \partial U$, the intersection of ∂U with a neighbourhood of z_0 cannot be an analytic arc. Otherwise, using Schwarz's reflection principle, the conjugacy equation, and the minimality of $z \to \lambda z$ on $|z| = \text{constant}$, one would be able to extend the conjugacy to a bigger disk, in contradiction with the maximality of U.

18. YOCCOZ'S PROOF OF SIEGEL'S THEOREM FOR $f(z) = \lambda(z+z^2)$ ([Y 3]).

For $|\lambda| \leq 1$, let $f_\lambda(z) = \lambda(z+z^2)$. Let $h_\lambda(z) = z + O(z^2)$ the formal conjugacy :

$$f_\lambda \circ h_\lambda(z) = h_\lambda(\lambda z) \ . \tag{18.1}$$

Poincaré's result shows that for $|\lambda| < 1$ (including $\lambda = 0$), the radius of convergence $R(\lambda)$ of h_λ is strictly positive ; the image L_λ by h_λ of its circle of convergence is a Jordan curve, analytic except at $c = -1/2$ where it has a right angle. One has, for $|\lambda| < 1$, the following elementary facts :

$$h_\lambda(\{|z| < R(\lambda)\}) \subset \{|z| \leq 2\} \ . \tag{18.2}$$

$$u(\lambda) = \lim_{n \to +\infty} \lambda^{-n} f_\lambda^n(c) \quad \text{exists ;} \tag{18.3}$$

$$u \quad \text{is analytic in} \quad \{|\lambda| < 1\} \ ; \tag{18.4}$$

$$|u(\lambda)| \leq 2 \ ; \tag{18.5}$$

$$h_\lambda(u(\lambda)) = c \ ; \tag{18.6}$$

$$|u(\lambda)| = R(\lambda) \ ; \tag{18.7}$$

$$u(0) = -\frac{1}{4} \ . \tag{18.8}$$

(To show (18.5) one uses the maximum principle and that $(f_\lambda^n(c))_{n \geq 0}$ is bounded (by P. Fatou's and G. Julia's result), which imply $|f_\lambda^n(c)| \leq 2$ when $|\lambda| = 1$.)

18.9

LEMMA 1. *If $(\lambda_i)_{i\geq 0}$ is a sequence such that :*

i)
$$|\lambda_i| < 1, \lim_{i\to+\infty} \lambda_i = 1 \;;$$

ii)
$$R(\lambda_i) \geq \delta > 0, \quad \textit{for some } \delta \;;$$

then a subsequence of $(h_{\lambda_i})_{i\geq 0}$ converges to a function $H_\lambda(z) = z + O(z^2)$ analytic on $\{|z| < \delta\}$ which satisfies on this disk $f_\lambda \circ H_\lambda(z) = H_\lambda(\lambda z)$.

Using (18.2) it is a straightforward application of Montel's theorem. Unicity shows that $H_\lambda = h_\lambda$ and $R(\lambda) \geq \delta$.

By P. Fatou's theorem, there is a function $U \in L^\infty(\mathbf{T}^1, d\theta)$ such that the radial limits :

$$\lim_{\substack{t\to 1\\ 0<t<1}} u(te^{2\pi i\alpha}) = U(\alpha), \quad \text{for a.e. } \alpha \;. \tag{18.10}$$

By a theorem of F. Riesz, the function Log $|U|$ belongs to $L^1(\mathbf{T}^1, d\theta)$, so we have :

$$U(\alpha) \neq 0, \quad \text{for a.e. } \alpha \;. \tag{18.11}$$

Siegel's theorem, for a.e. α, now follows from (18.7), (18.10), (18.11) and Lemma 1. We actually conclude from Lemma 1 that :

$$R(\lambda) \geq \limsup_{\substack{\lambda_i\to\lambda\\ |\lambda_i|<1}} R(\lambda_i). \tag{18.12}$$

J.C. Yoccoz proves more ([Y 3]) : for every λ with $|\lambda| = 1$ one has that radial limit :

$$\lim_{\substack{t\to 1\\ 0>t>1}} |u(t\lambda)| \quad \text{exists and is equal to} \quad R(\lambda) \;. \tag{18.13}$$

The function u has curious properties, see [Y 3]. By (18.5), it belongs to $H^\infty(\mathbf{D})$; by (11.2) it is an outer function.

18.14 From (18.8), the remarkably simple following observation follows : *there exists a subset of $\mathbf{S}^1$ of positive Lebesgue measure for which $R(\lambda) \geq 1/4$.*

On the other hand, by Proposition 2 and (8.12), $|U(\alpha)| = R(\lambda)$ vanishes on a dense G_δ subset of $\mathbf{S}^1$.

19. Properties of the radius of convergence.

19.1 Let $\lambda = e^{2\pi i\alpha}$, with $\alpha \in DC$; we denote O_λ the space of entire functions f which satisfy $f(0) = 0$, $f'(0) = \lambda$ equipped with the compact open topology ; it is a complex codimension 1 affine subspace of the space of all entire functions.

For $f \in O_\lambda$, let h_f the analytic function defined near 0 by :

$$f \circ h_f(z) = h_f(\lambda z),\ h_f(0) = 0,\ h'_f(0) = 1\ ; \qquad (19.2)$$

let $R(h_f)$ the radius of convergence of h_f, and $g(f) = \mathrm{Log}[R(h_f)^{-1}]$.

We have $g(f) = -\infty$ if and only if $f(z) = \lambda z$.

Theorem 6. *The function g on O_λ is continuous and plurisubharmonic (i.e. subharmonic on any complex line in O_λ).*

Idea of the proof :
Lower semi-continuity : this is proved as in Lemma 1, using that the functions h_f, univalent on a disk $\{|z| < r\}$ and satisfying $h_f(0) = 0$, $h'_f(0) = 1$, form a compact set for the compact open topology.
Upper semi-continuity : this is more delicate and requires the appropriate generalization of the theorem of Arnold and Moser on diffeomorphisms of the circle : see [H 4], [H 6].
Plurisubharmonicity : the Taylor coefficients $h_n(f)$ of h_f depend analytically on f and one has :

$$g(f) = \limsup_{n\to+\infty} \frac{1}{n} \mathrm{Log}\,|h_n(f)|\ . \blacksquare$$

19.3 Define :

$$U_\epsilon = \{f \in O_\lambda \mid \sup_{|z|\le 4} \left|f(z) - \lambda(z+z^2)\right| < \epsilon\}.$$

If ϵ is small enough, g is pluriharmonic on U_ϵ ; to see this, one uses Douady-Hubbard's theory of polynomial-like mappings and adapts, using [H 4, VIII], Yoccoz's proof (§ 18).

19.4 An example :

Let $f_b = \lambda(z + bz^2) + z^3$, for $b \in \mathbf{C}$, and denote $g(f_b)$ simply by $g(b)$. One has $g(b) > -\infty$ for any $b \in \mathbf{C}$.

PROPOSITION. *The function g is not harmonic.*

Proof : The conjugacy h_b has the form :

$$h_b = z + h_2(b)z^2 + O(z^3) ,$$

with $h_2(b) = b(\lambda-1)^{-1}$ satisfying $|h_2(b)| \leq 2e^{g(b)}$ by Bieberbach's inequality. This shows that $\mathrm{Log}\,|b| - g(b)$ is bounded from above ; if g were harmonic, one should have $g(b) = \mathrm{Log}\,|b| +$ constant, in contradiction with $g(0) > -\infty$. ∎

From 19.3, considering $bf_b(b^{-1}z)$, one sees that g is harmonic for large b and that :

$$\operatorname*{Lim}_{b\to\infty} (g(b) - \mathrm{Log}\,|b|) = g(f_\lambda) ,$$

with $f_\lambda(z) = \lambda(z + z^2)$.

By F. Riesz's theorem ([T 2]), one can write :

$$g(b) = l(b) + \int_{\mathbf{C}} \mathrm{Log}\,|b - u|\, d\mu(u) ,$$

where μ is a Radon measure with compact support and l is harmonic ; l is actually constant, because $l(b) \leq c_1 \mathrm{Log}\,|b|$ where $|b|$ is large (then, an entire function k with modulus e^l is a polynomial and does not vanish). The measure μ has no atoms, and its support K has strictly positive capacity.

CONJECTURE 3. *K is not a countable union of C^1-embedded arcs.*

Possibly K is not even locally connected. The structure of K is related to the set of parameters b where the orbits of the two critical points of f_b interact with each other.

20. A FORMULA FOR THE RADIUS OF CONVERGENCE.

Let $f \in O_\lambda$; we assume that the Siegel singular domain of f is bounded. From (19.1) we have :

$$\frac{1}{n}\sum_{j=0}^{n-1} \operatorname{Log}\left|f^j(u)\right| = \frac{1}{n}\sum_{j=0}^{n-1} k(\lambda^j z) \ ,$$

where $u \in U-\{0\}$, $u = h_f(z)$ and $k(z) = \operatorname{Log}|h(z)|$ is harmonic and bounded from above on $\{0 < |z| < R(h_f)\}$. As $z \to \lambda z$ is uniquely ergodic on $\{|z| = \text{constant}\}$, the right-hand side converges to $\int_0^1 k(e^{2\pi i\theta}|z|)d\theta = \operatorname{Log}|z|$ when $n \to +\infty$. Taking radial limits, and applying the ergodic theorem to $\lim_{t\to 1} k(tz)$, $|z| = R(h_f)$, we conclude :

$$\operatorname*{Lim}_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \operatorname{Log}\left|f^j(u)\right| \le \operatorname{Log} R(h_f)$$

for every $u \in U$, and :

$$\operatorname*{Lim}_{n\to+\infty} \frac{1}{n}\sum_{j=o}^{n-1} \operatorname{Log}\left|f^j(u)\right| = \operatorname{Log} R(h_f) \ , \tag{20.1}$$

for almost every $u \in \partial U$, with respect to harmonic measure.

QUESTION 8. *When does (20.1) hold for every $u \in \partial U$?*

If Question 4.a) has a positive answer, the unique ergodicity of $z \to \lambda z$ on $\{|z| = R(h_f)\}$ implies that question 8 has a positive answer. If moreover Question 4.b) has a positive answer (with estimates on the quasiconformal constants), one can use (20.1), [H 3, VI 3.2] and the classical estimates on univalent functions ([P 8]) to get numerical values for $R(h_f)$ (see remarks at the end of § 16).

21. SOME QUESTIONS ON NON LINEARIZABLE FIXED POINTS.

21.1 Consider the special case $f(z) = f_\alpha(z) = e^{2\pi i\alpha}(z+z^2)$, for $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$: there exists a dense G_δ-subset of $\mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ such that f_α is not linearizable at 0.

A. Douady ([D 3]) has shown that one can find α such that f_α is non linearizable at 0 and the Julia set $J(f_\alpha)$ (which here consists of the points with bounded orbits) is not locally connected. In his example, the critical point $c = -1/2$ is not accessible in $\mathbf{C} - J(f_\alpha)$. He asked :

QUESTION 9. *Can $J(f_\alpha)$ have positive Lebesgue measure when f_α is not linearizable at 0 ?*

One can also ask Question 9 when f_α is linearizable. The following question seems to be important in order to understand the Siegel singular domain of f_α.

QUESTION 10. *Can one find $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ such that the orbit of the critical point is dense in the Julia set ?*

PROBLEM. *Calculate, or at least find, reasonable estimates of*
$\sup_\alpha \sup_{n\geq 0} |f_\alpha^n(c)|$.

21.2 We mention that, in the general problem of classification of the germs of the form $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$, Naĭshul' has shown ([N 1]) that $\pm\alpha$ is an invariant of topological conjugacy.

22. A NON-ARCHIMEDEAN EXAMPLE

Let $(\mathbf{Q}_2, |\ |_2)$ be the 2-adic field with its standard absolute value, defined by $|2|_2 = \frac{1}{2}$ and $|p|_2 = 1$ for every odd prime.

For $\lambda \in \mathbf{Q}_2$, with $|\lambda| = 1$, consider $f_\lambda(z) = \lambda(z + z^2)$; the critical point $c = -1/2$ satisifes :

$$\lim_{n\to+\infty} |f_\lambda^n(c)|_2 = +\infty \ .$$

By [H 12], f_λ is linearizable at 0 if λ is not a root of unity. This shows that the answers to Questions 3 and 4 certainly depend on the base field.

Chapter IV

SOME EXAMPLES ON $\mathbf{C}^n$, $n \geq 2$.

23. We consider an entire mapping $f : \mathbf{C}^n \to \mathbf{C}^n$; we assume that $f(0) = 0$ and $Df(0) = A$ is unitary.

If f is linearizable at 0 (in particular, if A satisfies Brjuno condition (13)), one can define as in § 13.2 the *Siegel singular domain* U of f at 0. Then one defines for $z \in U$:

$$h(z) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} A^{-j} f^j(z) \ , \tag{23.1}$$

(a standard formula due to Bochner and Martin ([B 3], see also [P 1])), and one has :

$$h(0) = 0 \ , \ Dh(0) = \mathrm{id}$$

$$h(f(z)) = Ah(z) \ , \ z \in U \ .$$

Adapting Fatou's arguments ([F 1], see also [D 0]), one concludes that $f_{|U}$ is a diffeomorphism of U. The closure of $(f^n_{|U})_{n \geq 0}$ for the compact open topology is a compact group, isomorphic to $\mathbf{T}^n$ when A satisfies condition (*).

QUESTION 11. *Is $h : U \to h(U)$ a diffeomorphism ?*

I have only been able to prove this when the jacobian of f is constant and A satisfies (*) ; then the jacobian of h is constant and equal to 1.

23.2 When A is diagonal and satisfies condition (*), the $(A^n)_{n \in \mathbf{Z}}$ form a dense subgroup of the standard action of $\mathbf{T}^n$ on $\mathbf{C}^n$, defined by :

$$(\theta_1, \cdots, \theta_n) \cdot (z_1, \cdots, z_n) = (e^{2\pi i \theta_1} z_1, \cdots, e^{2\pi i \theta_n} z_n);$$

hence $h(U)$ is a Reinhardt domain.

QUESTION 12. *Describe which Reinhardt domains one can obtain in this way, up to biholomorphic diffeomorphisms.*

23.3 A theorem of Cartan and Thullen, solving a conjecture of G. Julia, says that U is polynomially convex (i.e. a Runge domain) : see [V 1, § 24.8, p. 207].

When A is diagonal, and Question 11 has a positive answer, $h(U)$ is the domain of normal convergence of h^{-1} ; then $h(U)$ is a pseudoconvex complete Reinhardt domain ; complete means that $(\lambda_1 z_1, \cdots, \lambda_n z_n) \in h(U)$ if $(z_1, \cdots, z_n) \in h(U)$ and $|\lambda_j| \leq 1$).

24. One can have $U = h(U) = \mathbf{C}^n$ even when f is not linear. Indeed, $\mathbf{C}^n$ has a large group of biholomorphic diffeomorphisms by which one can conjugate A to define f. Examples of such diffeomorphisms are :

$$h(z_1,\cdots,z_n) = (\exp(\varphi_1(z_2,\cdots,z_n))z_1 + \varphi_2(z_2,\cdots,z_n), z_2,\cdots,z_n)$$

where φ_1, φ_2 are entire functions.

25. **Example 1 :** Let $\lambda_1,\ \lambda_2 \in \mathbf{C}$, $|\lambda_1| = |\lambda_2| = 1$, and $E \in (-2,2)$. Define a biholomorphic diffeomorphism of $\mathbf{C}^3$ by :

$$f(z_1,z_2,z_3) = \left(\lambda_1 B(z_3)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \lambda_2 z_3\right),$$

where $B(z_3) = \begin{pmatrix} E + z_3 & -1 \\ 1 & 0 \end{pmatrix}$. We have :

$$Df(0) = A = \begin{pmatrix} \lambda_1 E & -\lambda_1 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

For a.e. $(\lambda_1,\lambda_2,E) \in \mathbf{S}^1 \times \mathbf{S}^1 \times (-2,2)$, A satisfies Brjuno condition and we showed in [H 9] that $U = \{(z_1,z_2,z_3) \in \mathbf{C}^3 \mid |z_3| < 1\}$; in this case ∂U is an analytic manifold, whose Levi form vanishes (we recall that this is not possible in one variable, cf. 17.4).

Example 2 : Let $n \geq 2$, $\varphi : \mathbf{C}^{n-1} \to \mathbf{C}$ an entire function satisfying $\varphi(0) = 0$, $D\varphi(0) = 0$, and $\lambda_1,\cdots,\lambda_n$ complex numbers of modulus 1. Define a biholomorphic diffeomorphism f of $\mathbf{C}^n$ by :

$$f(z_1,\cdots,z_n) = (\lambda_1 z_1 + \varphi(z_2,\cdots,z_n), \lambda_2 z_2,\cdots,\lambda_n z_n) .$$

We assume that condition (*) is satisfied by the λ_j's. The unique formal conjugacy of Proposition 1 has the form :

$$h(z_1,z_2,\cdots,z_n) = (z_1 + \eta(z_2,\cdots,z_n), z_2,\cdots,z_n)$$

and η is solution of :

$$\eta(\lambda_2 z_2,\cdots,\lambda_n z_n) - \lambda_1 \eta(z_2,\cdots,z_n) = \varphi(z_2,\cdots,z_n) .$$

Using Baire category arguments (cf. [H 12]) one can consruct the (λ_j) and a sequence $(k_j)_{j\geq 0}$ in $\{0\} \times \mathbf{N}^{n-1}$ with $|k_j| \to +\infty$ such that :

$$|\lambda_1 - \lambda^{k_j}| \leq |k_j|^{-|k_j|},\ j \geq 0 .$$

If we then choose :

$$\varphi(z) = \sum_j (\lambda^{k_j} - \lambda_1) z^{k_j}$$

(with $z = (z_2, \cdots, z_n)$) we will have :

$$\eta(z) = \sum_j z^{k_j} .$$

The Siegel singular domain of f at 0 in this case has the form $\mathbf{C} \times V$, where V is a Reinhardt domain different from $\mathbf{C}^{n-1}$.

26. **Example 3 :** For $\lambda_1, \lambda_2 \in \mathbf{C}$, with $|\lambda_1| = |\lambda_2| = 1$, we consider the biholomorphic diffeomorphism of $\mathbf{C}^2$:

$$f(z_1, z_2) = (\lambda_1 z_1 + z_2, \lambda_2 z_2 + (\lambda_1 z_1 + z_2)^2).$$

We assume that $\lambda_1 \neq \lambda_2$ and that (λ_1, λ_2) satisfy Brjuno condition. Then f has a Siegel singular domain U at 0.

PROPOSITION. *The closure of U in $\mathbf{C}^2$ is compact.*

Idea of the proof : As f has constant jacobian, we know from 23.2, 23.3 that there exists a biholomorphic diffeomorphism g from a pseudoconvex complete Reinhardt domain R onto U which satisfies :
- $f(g(z)) = g(\lambda_1 z_1, \lambda_2 z_2)$ for $z = (z_1, z_2) \in R$;
- The Taylor series of g at 0 is normally convergent. With $g = (g_1, g_2)$, we obtain :

$$g_2(z) + \lambda_1 g_1(z) = g_1(\lambda_1 z_1, \lambda_2 z_2) \; ; \tag{26.1}$$

$$(g_1(\lambda_1 z_1, \lambda_2 z_2))^2 = g_1(\lambda_1^2 z_1, \lambda_2^2 z_2) - (\lambda_1 + \lambda_2) g_1(\lambda_1 z_1, \lambda_2 z_2) + \lambda_1 \lambda_2 g_1(z_1, z_2) \, . \tag{26.2}$$

If $\tilde{R} = \{(z_1, z_2) \mid |z_1| \leq r_1, |z_2| \leq r_2\}$ is included in R, we get from (26.2) that $|g_1| \leq 4$ on $\tilde{R}$ and then from (26.1) that $|g_2| \leq 8$ on $\tilde{R}$; we conclude that $U \subset \{(z_1, z_2) \mid |z_1| \leq 4, |z_2| \leq 8\}$. ∎

Keeping the same notations, R is logarithmically convex ([V 1]) and g_1, g_2 are bounded and analytic on R ; for each complex line L passing through 0, at least one of g_1, g_2 has non constant restriction to $L \cap R$, so this intersection has the form $\{|z| < r\}$, with $r < +\infty$ and z a coordinate on L (vanishing at 0). We conclude that R is bounded and $\partial R \cap \{(z_1, z_2), z_1 z_2 \neq 0\}$ is a topological manifold.

QUESTION 13. *Is ∂U a C^∞-submanifold of $\mathbf{C}^2$?*

QUESTION 14. *Is ∂U a topological submanifold of $\mathbf{C}^2$? When it is, is it a locally flat submanifold ?*

CONJECTURE 4. *For almost every (λ_1, λ_2) (with $|\lambda_1| = |\lambda_2| = 1$), ∂U is not a C^∞-submanifold of $\mathbf{C}^2$.*

Observe that f is a diffeomorphism, so has no critical point. J. H. Hubbard ([H 13]) has a more geometrical approach to examples where the proposition holds.

The fact that ∂U is compact is probably related to the fact that normal forms for diffeomorphisms of $\mathbf{T}^2$ are only local (cf. [H 9]).

27. Remarks on Fatou-Julia theory on $\mathbf{C}^n$, $n \geq 2$.

(See also [N 7] ; this paper claims, p. 367, using [J 1], that Siegel's theorem is incorrect !)

27.1 For an entire mapping $f : \mathbf{C}^n \to \mathbf{C}^n$, one defined the *Julia set* $J(f)$ as follows : $x \in \mathbf{C}^n - J(f)$ if and only if there is a neighbourhood U of x such that $(f^n_{|U})_{n\geq 0}$ is normal with values in $\mathbf{C}^n \cup \{\infty\}$ (i.e. we allow $\{\infty\}$ as a limit function). It follows from the definition that $J(f)$ is closed and that one has :

$$f(J(f)) \subset J(f) ;$$
$$f^{-1}(J(f)) = J(f) .$$

27.2 The reader should consult [H 7] for the properties of $J(f)$ when $n = 1$; for $n \geq 2$, the properties of $J(f)$ are quite different :

- $J(f)$ can be empty even if f is not linear : see § 24.
- Periodic points are not always dense in $J(f)$: see Examples 1 and 2 above.
- One can have $\text{int}(J(f)) \neq \emptyset$ but $J(f) \neq \mathbf{C}^n$, and $f_{|J(f)}$ does not always have a dense orbit : Examples 1, 2.
- $J(f)$ can have isolated points : for instance, take for f a Cremona diffeomorphism of $\mathbf{C}^2$, with a repulsive linearizable fixed point at 0, whose basin B of attraction of 0 by f^{-1} is a Poincaré-Fatou- Bierberbach domain (i.e. $\mathbf{C}^2 - B$ has non-empty interior) ; see [D 1], [P 5, t. IV, p. 537-582] ; observe that for any small neighbourhood V of 0, $\cup_{n\geq 0} f^n(V)$ is not dense in $\mathbf{C}^2$.

27.3 If f is a biholomorphic diffeomorphism of $\mathbf{C}^2$, and x_0 is a fixed point of f not in $J(f)$ or $J(f^{-1})$, then f is linearizable at x_0 and $Df(x_0)$ is conjugate to a unitary matrix : this is implied by Bochner-Martin's formula (23.1) and the fact that $((Df(x_0))^n)_{n\in\mathbf{Z}}$ is bounded in $\mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$.

27.4 One can construct ([H 11]) a biholomorphic diffeomorphism f of $\mathbf{C}^2$ with the following properties :

- $f(0) = 0$;
- $Df(0)$ *is a diagonal, unitary and satisfies condition (*) :*
- 0 *is the unique periodic point of* f ;
- f *has a dense orbit in* $\mathbf{C}^2$.

Question 15. *Does there exist a polynomial Cremona diffeomorphism of* $\mathbf{C}^2$ *which preserves Lebesgue measure and has an infinite number of periodic points outside its Julia set ?*

Using S. Newhouse's techniques ([N 2]) one can probably construct polynomial Cremona diffeomorphisms f of $\mathbf{C}^2$ (preserving $\mathbf{R}^2$) with an infinite number of periodic sinks.

Chapter V

CENTER MANIFOLDS AND THEIR RELATION TO INVARIANT CIRCLES OF TWIST-MAPS. EXAMPLES

28. Center manifolds.

Let $f(z) = Az + O(z^2)$ a germ of analytic diffeomorphism of $\mathbf{C}^n$. When the matrix A does not satisfy condition (*), usually f is not formally linearizable ; when, for instance, $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $|\lambda| = 1$, λ is not a root of unity, f is formally linearizable but usually not analytically linearizable.

Nevertheless, under suitable arithmetical hypotheses on the eigenvalues of A, one can frequently find, even when condition (*) is not fulfilled, germs of analytic submanifolds through 0 which are invariant under f and on which f acts like a diagonal matrix. We refer to J. Pöschel ([P 9]) for some general results and other references.

We want here to give some examples (slightly different from Pöschel's) and show the strong relation between these invariant submanifolds and the invariant torii of symplectic diffeomorphisms.

29. Poincaré-Lindstedt perturbation series[1].

We only look at simple examples and refer to H. Poincaré ([P 6]) and J. Moser ([M 9]) for a more general (but elementary) approach.

Let φ a $\mathbf{Z}$-periodic real valued C^∞ function, satisfying $\int_0^1 \varphi(\theta)d\theta = 0$. We consider, fot $a \in \mathbf{R}$, the diffeomorphism f_a of $\mathbf{T}^1 \times \mathbf{R}$ defined by :

$$f_a(\theta, r) = (\theta + r, r + a\varphi(\theta + r)) \ . \tag{29.1}$$

Let $\alpha \in DC$, and C_a a C^∞ simple closed curve, homotopic to $\{r = 0\}$, such that f_a leaves C_a invariant and $f_{a|C_a}$ has rotation number α. One can then (as $f_{a|C_a}$ is smoothly conjugated to a rotation) choose a parametrization of C_a of the form :

$$\theta \to (\theta + \eta_a(\theta), \alpha + l_a(\theta))$$

where η_a, $l_a \in C^\infty(\mathbf{T}^1)$ satisfy :

$$\eta_a(\theta + \alpha) = \eta_a(\theta) + l_a(\theta) \ ; \tag{29.2}$$

[1] H. Poincaré attributes to Lindstedt the discovery of these series ; but most of Poincarés attributions are not quite correct, since he frequently forgets his own contributions.

$$L_\alpha \eta_a(\theta) = a\varphi(\theta + \eta_a(\theta)) \ , \tag{29.3}$$

the linear map L_α being defined by :

$$L_\alpha \eta_a(\theta) = 2\eta_a(\theta) - \eta_a(\theta + \alpha) - \eta_a(\theta - \alpha) \ .$$

One can expand η_a as a formal power series :

$$\eta_a(\theta) = \sum_{n \leq 1} b_n(\theta) a^n \ ,$$

with the b_n in $C^\infty(\mathbf{T}^1)$, $\int_0^1 b_n(\theta) d\theta = 0$: these are the Poincaré -Lindstedt perturbation series (with fixed frequency).

We restrict ourselves to the *standard map*, obtained by taking $\varphi(\theta) = \frac{1}{2\pi}\sin(2\pi\theta)$. One shows inductively that b_n is an odd trigonometric polynomial of degree n. D. Goroff showed me, and many authors have noticed, ([G 3], [G 5], [R 1]) how to calculate inductively the b_n. We define :

$$\exp(2\pi i(\theta + \eta_a(\theta))) = \sum_{n \geq 0} c_n(\theta) a^n \ ; \tag{29.4}$$

then we have :

$$c_0(\theta) = e^{2\pi i \theta} \ ; \tag{29.5}$$

$$c_n = 2\pi i n^{-1} \sum_{k=1}^{n} k b_k c_{n-k}, \quad \text{for} \quad n \geq 1 \ ; \tag{29.6}$$

$$b_n = \frac{1}{4\pi i} L_\alpha^{-1}(c_{n-1} - \bar{c}_{n-1}), \quad \text{for} \quad n \geq 1 \ . \tag{29.7}$$

Relation (29.6) is obtained from (29.4) differentiating with respect to a ; in (29.7), L_α is invertible because α is irrational and $c_{n-1} - \bar{c}_{n-1}$ is a trigonometric polynomial without constant term.

When $\alpha \in DC$ and φ is $\mathbf{R}$-analytic, the series for η_a converge ; in fact, for $\epsilon > 0$ sufficiently small, the function : $(\theta, a) \to \eta_a(\theta)$ extends to an analytic map on $\{|\text{Im}\theta| < \epsilon\} \times \{|a| < \epsilon\}$ (where $\theta \in \mathbf{C}/\mathbf{Z}, a \in \mathbf{C}$). This is shown adapting a proof of E. Zehnder ([Z 2], see also [M 8]).

The most natural proof (for the standard map) of the convergence of the Poincaré-Lindstedt perturbation series (with fixed frequency) would be to deduce it directly from the relations (29.4)-(29.7).

Unfortunately, this is a delicate problem, as explained now.

The relations (29.6), (29.7) allow to express the coefficients of the trigonometric polynomial c_n as polynomials in the coefficients of the c_j with $j < n$. These polynomials have real coefficients. If one replaces these real coefficients by their absolute value (as one does in Siegel's theorem), the majorant series obtained in this way for η_a are divergent ! The reason is the following : for $l(\theta) = \sum_{k \neq 0} l_k e^{2\pi i k \theta}$, one has :

$$L_\alpha^{-1} l(\theta) = \sum l_k v_k^{-1} e^{2\pi i k \theta} ,$$

with $v_k = \sin^2 k\pi\alpha$, and in particular :

$$v_{q_k}^{-1} \geq (4\pi^2)^{-1} q_k^2 ,$$

if q_k is the denominator of a convergent of α. In the expression for the coefficient of c_n, with $q_k < n < q_{k+1}$, $v_{q_k}^{-1}$ appear in some terms with an exponent which is larger than approximately $\frac{1}{2}(n - q_k)$, and this is too much for convergence.

This means that to prove the convergence of the Poincaré-Lindstedt perturbation series (with fixed frequency), one has to take into account the cancellations due to the signs of the different terms in the induction. Eliasson claims ([E 1], [E 2]) to overcome in part this difficulty. His arguments are (and have to be) very delicate ; the author of these lines is far from understanding the whole story, but is convinced that eventually this type of approach will give a remarkably simple, natural and beautiful new proof of the results of KAM-theory in the $\mathbf{R}$-analytic case.

30. We indicate how *some* of the coefficients of the b_n (for the standard map) are closely related to the Siegel's theorem and the induction (8.2).

Let u_n be the coefficient of $e^{2\pi i n \theta}$ in $2\pi i b_n$ (this is the higher degree term). Then one has :

$$u_{n+1} = \frac{1}{2} v_{n+1}^{-1} t_n, \quad \text{for} \quad n \geq 0 , \tag{30.1}$$

$$\text{with} \quad t_0 = 1 , t_n = \frac{1}{n} \sum_{k=1}^{n} k u_k t_{n-k}, \ n \geq 1 .$$

This gives :

$$u_n = \frac{1}{(n-1) v_n} \sum_{k=1}^{n-1} k u_k u_{n-k} v_{n-k} . \tag{30.2}$$

This implies :

$$t_n > 0, \ u_n > 0 \quad \text{for} \quad n \geq 1 ; \tag{30.3}$$

$$u_n \geq (2v_n)^{-1} u_{n-1} \quad \text{for} \quad n \geq 2 ; \tag{30.4}$$

$$u_{2n} \geq \frac{1}{2} v_n v_{2n}^{-1} u_n^2 \quad \text{for} \quad n \geq 1 . \tag{30.5}$$

31. THE STANDARD MAP (cf. [G 5]).

We consider the biholomorphic diffeomorphism G of $(\mathbf{C}/\mathbf{Z}) \times \mathbf{C}$ defined by :

$$G(\theta, r) = (\theta + r, r - (2\pi i)^{-1} e^{2\pi i(\theta + r)}) .$$

Let $\alpha \in \mathbf{R} - \mathbf{Q}$; we define the u_n, for $n \geq 1$, by (30.1), (30.2) and put :

$$\begin{aligned} q_\alpha(z) &= (2\pi i)^{-1} \sum_{n \geq 1} u_n z^n, \\ \tilde{\eta}_\alpha(\theta) &= q_\alpha(e^{2\pi i \theta}) . \end{aligned} \tag{31.1}$$

Then, we obtain :

$$L_\alpha \tilde{\eta}_\alpha = (4\pi i)^{-1} e^{2\pi i(\theta + \tilde{\eta}(\theta))} ,$$

which we rewrite, with $\lambda = e^{2\pi i \alpha}$, as :

$$2q_\alpha(z) - q_\alpha(\lambda z) - q_\alpha(\lambda^{-1} z) = \frac{1}{2} z e^{q_\alpha(z)} . \tag{31.2}$$

When α is a Brjuno number (see [B 6], the remark after (8.13), and (12.3) when $\alpha \in DC$), the series (31.1) for q_α define an analytic function. Let $R(\alpha)$ be its radius of convergence, $\delta(\alpha) = -\operatorname{Log} R(\alpha)$ and $D_\alpha^\star = \{\theta \in \mathbf{C}/\mathbf{Z}, \operatorname{Im}(\theta) > \delta(\alpha)\}$: then $\tilde{\eta}_\alpha$ is defined on $D_\alpha^\star$,which is a biholomorphic image of $\mathbf{D}^\star = \{0 < |z| < 1\}$. The mapping S_α defined by :

$$S_\alpha(\theta) = (\theta + \tilde{\eta}_\alpha(\theta), \alpha + \tilde{\eta}_\alpha(\theta + \alpha) - \tilde{\eta}_\alpha(\theta))$$

from $D_\alpha^\star$ to $(\mathbf{C}/\mathbf{Z}) \times \mathbf{C}$, satisfies :

$$G(S_\alpha(\theta)) = S_\alpha(\theta + \alpha) . \tag{31.3}$$

As α is irrational, S_α is injective on $D_\alpha^\star$ and $\tilde{\eta}_\alpha$ cannot be extended beyond $D_\alpha^\star$.

31.4 Relation with center manifolds.

Consider the biholomorphic diffeomorphism F of $\mathbf{C}^2$ defined by :

$$F_1(z_1,z_2) = \left(z_1 e^{2\pi i z_2}, z_2 - \frac{1}{4\pi i} z_1 e^{2\pi i z_2}\right).$$

Then, with $h(\theta,r) = (e^{2\pi i\theta}, r)$, the diagram :

$$\begin{array}{ccc} (\mathbf{C}/\mathbf{Z}) \times \mathbf{C} & \xrightarrow{G} & (\mathbf{C}/\mathbf{Z}) \times \mathbf{C} \\ h\big\downarrow & & \big\downarrow h \\ \mathbf{C}^2 & \xrightarrow[F]{} & \mathbf{C}^2 \end{array}$$

is commutative. Each point $(0,z_2)$ is fixed by F ; the image $S_\alpha(D_\alpha^\star)$ is associated to the *center manifolds* passing through the point $(0,\alpha)$, and tangent at this point to $\{z_2 = \alpha\}$.

The analogy between $S_\alpha(D_\alpha^\star)$ and a $\mathbf{R}$-analytic invariant curve R of rotation number α, for a $\mathbf{R}$-analytic twist diffeomorphism f of $\mathbf{T}^1 \times \mathbf{R}$, appears when one complexifies R and f to some domaim $\{(z_1,z_2) \in (\mathbf{C}/\mathbf{Z}) \times \mathbf{C}, \max_j |\mathrm{Im}(z_j)| < \delta\}$; this is possible for small $\delta > 0$, giving f_1, R_1 and a ring R_2 invariant under f_1, satisfying $R \subset R_2 \subset R_1$, on which f_1 is conjugate to the translation by α.Observe that the restriction to R_2 of Df_1 is parabolic : R_2 is a center manifold of f_1 just as $S_\alpha(D_\alpha^\star)$ is for F.

PROBLEM. *To describe globally in the complex domain the (complexified) invariant curves of twist maps (29.1), when φ extends to an entire function.*

There are delicate questions of global analytic continuation.

32. We list here some of the properties of the center manifolds $S_\alpha(D_\alpha^\star)$, which may be considered as a simple model for the invariant circles of twist maps (with power series involved instead of trigonometric series).

32.1 For every Brjuno number α, the series defining q_α converge and the center manifold exists.

32.2 For a dense G_δ-subset of $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$, the series defining q_α has radius of convergence 0 ; this follows from (8.4') since we have from (30.4) :

$$u_n^{1/n} \geq \frac{1}{2}(v_1 \cdots v_n)^{-1/n} . \qquad (32.3)$$

32.4 An a priori estimate.

Let α be such that $R = R(\alpha) > 0$; for $0 < r < R$, let M_r be the maximum modulus of q_α on $\{|z| \leq r\}$. By (30.3), we have $M_r = q_\alpha(r)$; from (31.2), we obtain :

$$4\ M_r \geq \frac{1}{2} r e^{M_r} \ , \tag{32.5}$$

from which we deduce :

$$r \leq 8 M_r e^{-M_r} \leq 8 \max_{x \geq 0} x e^{-x} = 8e^{-1} \ ,$$

$$R(\alpha) \leq 8e^{-1}$$

$$R(\alpha) \leq 8 \sup_{r<R} M_r e^{-M_r} \ .$$

Moreover M_r increases with r, so we get from the above relations :

$$M_R = \sup_{r<R} M_r < +\infty \ ; \tag{32.6}$$

$$R \leq 8 M_R e^{-M_R} \ . \tag{32.7}$$

We conclude from (30.3), (32.6) and Abel's elementary Tauberian theorem that the series defining q_α converge absolutely uniformly on $\{|z| \leq R\}$; using (31.3) we see that $S_\alpha(\{z \in \mathbf{C}/\mathbf{Z}, \operatorname{Im} z = \delta(\alpha)\})$ is an *embedded Jordan curve.*

32.7 For any $\epsilon > 0$, the set $\{\alpha, R(\alpha) > \epsilon\}$ is *nowhere dense.* Indeed, if $R(\alpha) > \epsilon$ we get from (32.7) :

$$M_{R(\alpha)} \leq c(\epsilon),$$

$$u_n \leq c'(\epsilon)\epsilon^{-n}$$

where $c(\epsilon)$, $c'(\epsilon)$ only depend on ϵ. But the last inequality is violated on an open and dense set of α (see (32.3)).

This property is analogous to the fact that generic twist maps have invariant curves only for a nowhere dense set of rotation numbers : see [H 8, chap. I].

32.8 It is not difficult to show that $R(\alpha) \leq C\,|\alpha|^2$, using (30.4) and (30.5). For the similar property property for twisp maps, see [H 8, ch. II].

32.9 One can adapt J.C. Yoccoz's proof ([Y 4]) of (11.3) to show that if $R(\alpha) > 0$ *then α is a Brjuno number.*

From this, we conclude that if the perturbation series for the standard map define an $\mathbf{R}$-analytic function :

$$a \to \eta_a \in L^2(\mathbf{T}^1, d\theta)$$

for small $|a|$, then α is a Brjuno number.

(For twist maps an Aubry-Mather sets of fixed rotation number α, η_a is of bounded variation so is in L^∞).

The reader should *not* conclude that this implies that the standard map admits only Brjuno numbers as rotation numbers of its invariant circles (J.C. Yoccoz actually remarked that this is false for the generic C^∞ twist map) ; the only thing one can say is that these invariant circles whose rotation numbers are not Brjuno numbers cannot be obtained by analytic perturbation techniques.

33. Remark : Most of what we have said for the standard map is still true when in (29.1) the function φ is $\mathbf{R}$-analytic, extends to an entire function and its Fourier series has the form :

$$\varphi(\theta) = \sum_{n\geq 1} a_n \sin(2\pi n\theta), \; a_n \geq 0 .$$

34. AN EXAMPLE WITH A PARABOLIC (for related examples, see [P 2]).

For $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ and $\lambda = \exp(2\pi i\alpha)$, we consider the biholomorphic diffeomorphism f_α of $\mathbf{C}^2$ defined by :

$$f_\alpha(z_1, z_2) = (\lambda(z_1 + z_2), \lambda z_2 - \lambda^2(z_1 + z_2)^2) . \tag{34.1}$$

We look for an invariant (complex) curve through 0, tangent to $\{z_2 = 0\}$ at 0, invariant under f and such that the restriction of f to it is linearizable. This means that we look at a germ $S_\alpha = (\eta_\alpha, l_\alpha)$ of analytic map from $(\mathbf{C}^2, 0)$ with $\eta_\alpha(z) = z + O(z^2)$, $l_\alpha(z) = O(z^2)$ and satisfying :

$$\eta_\alpha(\lambda z) = \lambda(\eta_\alpha(z) + l_\alpha(z)) ; \tag{34.2}$$

$$(\eta_\alpha(z))^2 = 2\eta_\alpha(z) - \bar\lambda\eta_\alpha(\lambda z) - \lambda\eta_\alpha(\bar\lambda z) . \tag{34.2'}$$

With $\eta_\alpha(z) = \sum_{n\geq 1} b_n z^n$, $b_1 = 1$, we obtain the induction relation :

$$b_n = (v_{n-1})^{-1} \sum_{k=1}^{n-1} b_k b_{n-k}, \; n \geq 2 , \tag{34.3}$$

where v_n is as in 29.

We see that :

$$b_n > 0 \quad \text{for} \quad n \geq 1 . \tag{34.4}$$

34.5 It follows from (8.9) and (11.3) that the radius of convergence of the series defining η_α is strictly positive if and only if α is a Brjuno number. Let $R(\alpha)$ be this radius of convergence, and for $r \in [0, R(\alpha))$, define :

$$M_r = \eta_\alpha(r) = \max_{|z| \leq r} |\eta_\alpha(z)| \quad \text{(see (34.4))} .$$

From (34.2'), we have $M_r^2 \leq 4M_r$, hence :

$$M_r \leq 4 . \tag{34.6}$$

34.7 A consequence of (34.6) is that $R(\alpha)$ is finite and the series defining η_α are absolutely uniformly convergent on the closed disk $\{|z| \leq R(\alpha)\}$. Furthermore, from (34.2) and (34.5), S_α is an embedding of this disk into $\mathbf{C}^2$.

34.8 The set of $\alpha \in \mathbf{T}^1 - (\mathbf{Q}/\mathbf{Z})$ for which $R(\alpha) > \epsilon$ is nowhere dense in $\mathbf{T}^1$, for any $\epsilon > 0$. This follows from (34.6), Cauchy's inequality and the estimate :

$$b_n \geq (v_1 \cdots v_n)^{-1} .$$

Caution : If one replaces in (34.1) λ by $t\lambda$, with $0 < t < 1$, we obtain an infinite radius of convergence for the series defining η ; compare with §18.

QUESTION 16. *Is η_α univalent on $\{|z| < R(\alpha)\}$? In other terms, is the image of S_α a graph over $\{z_2 = 0\}$?*

A similar question occurs for the semi-standard map.

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