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COMPLEX HÉNON MAPPINGS IN $\mathbb{C}^2$ AND FATOU-BIEBERBACH DOMAINS

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Introduction. For $(a, c) \in \mathbb{C}^* \times \mathbb{C}$ the formula

$$g(z, w) = (z^2 + c + aw, z)$$

defines a biholomorphism in $\mathbb{C}^2$ whose Jacobian is $-a$. These are the complex continuations of the maps studied by Hénon when $(z, w) \in \mathbb{R}^2$ and $(a, c) \in \mathbb{R}^* \times \mathbb{R}$.

In [H] Hubbard introduced the following terminology. Let

$$K^{\pm} = \{p; p \in \mathbb{C}^2, g^{\pm n}(p) \text{ is a bounded sequence}\}.$$

Also let $J^{\pm} = \partial K^{\pm}$ and $K = K^+ \cap K^-$. As in [Br] and [H], we define the functions

$$G^+(z, w) = \lim_{n \to \infty} \frac{1}{2^n} \log^+ \|g^n(z, w)\|$$

and

$$G^-(z, w) = \lim_{n \to \infty} \frac{1}{2^n} \log^+ \|g^{-n}(z, w)\|.$$

It was shown in [H], [BS1], that $G^{\pm}$ are continuous functions in $\mathbb{C}^2$ plurisubharmonic in $U^+ = \mathbb{C}^2 \backslash K^+$ and $U^- = \mathbb{C}^2 \backslash K^-$, respectively. It follows that K^+ and K^- are nonpluripolar closed sets.

Define $\mu^{\pm} = dd^c G^{\pm}$. The positive, closed (1, 1) currents $\mu^{\pm}$ satisfy the functional equations

$$g^* \mu^{\pm} = 2^{\pm 1} \mu^{\pm}.$$

It was shown by Bedford and Smillie [BS2] that, if V is an algebraic curve in $\mathbb{C}^2$, then the currents $(1/2^n)[g^{-n}(V)]$ converge to a constant multiple of μ^+. Assuming that g is hyperbolic on $J = J^+ \cap J^-$, Bedford and Smillie showed that the interior of K^+ consists of the basins of finitely many sink orbits and that J^+ has a foliation $\mathscr{F}^+$ whose leaves are complex manifolds biholomorphically equivalent to $\mathbb{C}$.

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Hubbard and Oberste-Vorth [HO] have studied the foliations of $U^{\pm}$ connected to the functions $G^{\pm}$, when $|c|$ is large or when the polynomial $z^2 + c$ has an attractive fixed point and $|a|$ is small enough. Finally, Benedicks-Carleson [BC] have shown the existence of strange attractors in the real case for some values of parameters. See also Mora-Viana [MV].

In this paper we continue the study of dynamical properties of polynomial automorphisms in $\mathbb{C}^2$.

In the first section we show that the functions $G^{\pm}$ are Hölder continuous, which implies estimates for the Hausdorff dimension of $K^{\pm}$ when the interior of $K^{\pm}$ is empty. We then show that, if T is a closed positive (1, 1) current in $\mathbb{P}^2$ such that the point $p_+ = [1:0:0]$ is not in the support of T, then the currents $(1/2^n)g^{*n}(T)$ converge to $c\mu^+$ for some positive constant c. This gives a stronger version of the result by Bedford-Smillie. The results generalize to the case where g is a finite composition of Hénon mappings as considered in [FM] and [BS].

Let f be a polynomial automorphism of $\mathbb{C}^2$. Suppose $f(0) = 0$ and that the eigenvalues λ_1, λ_2 of $f'(0)$ satisfy $|\lambda_1| < 1, |\lambda_2| < 1$. Let $\Omega = \{q; q \in \mathbb{C}^2, \lim_{n\to\infty} f^n(q) = 0\}$. It is well known that Ω is biholomorphic to $\mathbb{C}^2$; Ω is called a Fatou-Bieberbach domain. If the matrix $f'(0)$ has no resonances, then f can be linearized in Ω.

In the second section we consider the case when the eigenvalues λ_1, λ_2 of $f'(0)$ satisfy the condition that $\lambda_1 = e^{2i\pi\theta}, |\lambda_2| < 1$, and θ satisfies a diophantine condition. Then the domain of linearization Ω of f is either $\mathbb{C}^2$ or biholomorphic to $\Delta \times \mathbb{C}$ where Δ is the unit disc in $\mathbb{C}$. In this later case we call Ω a Siegel cylinder.

In the third section we turn to the family

$$g(z, w) = (z^2 + c + aw, az).$$

Such g is just conjugate to $(z, w) \to (z^2 + c + a^2 w, z)$. Let $P_c(z) = z^2 + c$. We first assume that $P_c^n(0)$ is unbounded. In that case, if a is small enough, then $K^{\pm}$ have empty interior and are foliated by complex manifolds, and moreover each leaf is dense. It then follows that g is hyperbolic on K. We also give estimates for the Hausdorff dimension of $K^{\pm}$.

If $P_c(z)$ has in $\mathbb{C}$ an attractive orbit of order k, then if a is small enough, g has an attractive cycle of order k: $\{p_1, \dots, p_k\}$, the interior of K^+ consists of k connected components, and J^+ is their common boundary. As a consequence, if $k \geqslant 2$, the boundaries of these Fatou-Bieberbach domains are not topological manifolds. We also show, when $k = 1$, that it is generically not a $\mathscr{C}^1$ manifold. We show that J^+ has a foliation with dense leaves, and also $K^- \setminus \{p_1, \dots, p_k\}$ has a foliation with dense leaves. The foliation of K^- cannot be extended to $\{p_1, \dots, p_k\}$. In this case also g is hyperbolic on $J = J^+ \cap J^-$.

When $k = 1$, then J^+ is a topological manifold, and for $|w_0|$ small enough $J^+ \cap \{w = w_0\}$ is a quasicircle. Moreover, J^+ has Lebesgue measure 0. This property can also be deduced from the general fact, due to Bowen, that a hyperbolic set with empty interior has Lebesgue measure zero. We thank the referee for this observation. We also obtain estimates for the Hausdorff dimension of K^-.

1. Green function and currents. Let

$$g(z, w) = (z^2 + c + aw, z)$$

$$g^{-1}(z, w) = \left(w, \frac{-w^2 + c + z}{a}\right).$$

Let $\mathbb{P}^2$ be the projective space and let $p_\pm$ be the points in $\mathbb{P}^2$ defined in homogeneous coordinates by

$$p_+ = [1:0:0], \qquad p_- = [0:1:0].$$

We consider the extension $\tilde{g}$ of g from $\mathbb{P}^2\backslash\{p_-\}$ to $\mathbb{P}^2\backslash\{p_-\}$ defined by the formula

$$\tilde{g}[z:w:t] = [z^2 + ct^2 + awt : zt : t^2].$$

Similarly, $\widetilde{g^{-1}}$ is defined from $\mathbb{P}^2\backslash\{p_+\}$ to $\mathbb{P}^2\backslash\{p_+\}$ by

$$\widetilde{g^{-1}}[z:w:t] = \left[wt : \frac{-w^2 - ct^2 + zt}{a} : t^2\right].$$

We observe the important fact that p_+ is a fixed point for $\tilde{g}$ and $(D\tilde{g}^2)(p_+) = 0$. Indeed, in the (w, t) coordinates

$$D\tilde{g}(p_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

therefore,

$$(D\tilde{g}^2)(p_+) = 0.$$

We recall a few elementary facts concerning the dynamics of g; see [FM] and [BS1] for a proof.

There exists $R > 0$ such that $|z| > R$ implies that either $|z^2 + c + aw| > |z|$ or $|w| > |z|$ or both. For such R let

$$V^- = \{(z, w); |z| > R, |z| > |w|\}$$

$$V^+ = \{(z, w); |w| > R, |w| > |z|\}$$

$$V = \{(z, w); |z| \leqslant R, |w| \leqslant R\}.$$

Proposition 1.1 ([FM], [BS1]). *For R as above the following holds.*
(i) *$K^+ \cap \overline{V^-} = \varnothing$ and $K^- \cap \overline{V^+} = \varnothing$.*

(ii) $g(\overline{V^-}) \subset V^-$, $g(V \cup \overline{V^-}) \subset \overline{V^-} \cup V$; $g^{-1}(\overline{V^+}) \subset V^+$, $g^{-1}(\overline{V^+} \cup V) \subset \overline{V^+} \cup V$. $U^+ = \mathbb{C}^2 \backslash K^+ = \bigcup_{n \geq 0} g^{-n} V^-$, *and* $U^- = \mathbb{C}^2 \backslash K^- = \bigcup_{n \geq 0} g^n V^+$.

(iii) *If* $(z, w) \in K^\pm$, *then* $\lim_{n\to\infty} \operatorname{dist}(g^{\pm n}(z, w), K) = 0$.

(iv) *Given* $\varepsilon > 0$, *suppose* $(z_n, w_n) = g^n(z, w)$. *If* $(z, w) \in U^+$, *then for n large enough* $|w_n| < \varepsilon |z_n|$.

Let $\widetilde{U^+} = U^+ \cup (H \backslash p_-)$, where H is the hyperplane at infinity of $\mathbb{P}^2$. It is easy to deduce from Proposition 1.1 that $\widetilde{U^+}$ is the domain of attraction of p_+ for $\tilde{g}$. Similarly, $\widetilde{U^-} = U^- \cup (H \backslash p_+)$ is the domain of attraction of p_- for $\widetilde{g^{-1}}$.

Let $\| \;\|$ denote a norm in $\mathbb{C}^2$. The function

$$G^+(z, w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|g^n(z, w)\|$$

describes the rate of escape at infinity of $g^n(z, w)$. It follows from Proposition 1.1 that

$$G^+(z, w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |z_n|.$$

Therefore, G^+ is pluriharmonic outside K^+ and clearly

$$G^+(g) = 2G^+, \qquad G^-(g^{-1}) = 2G^-.$$

In what follows we will give the results only for G^+. The adaptation for G^- is easy.

As mentioned in the introduction, G^+ is plurisubharmonic and continuous on $\mathbb{C}^2$ and $K^+ = \{(z, w); G^+(z, w) = 0\}$. Moreover, for $|w| \leq R$

$$G^+(z, w) = \log^+ |z| + 0(1);$$

see [BS1].

THEOREM 1.2. *There exists* $\tau > 0$ *such that for every compact* $X \subset \mathbb{C}^2$ *there is a constant* $C > 0$ *such that for* (z, w) *and* $(z', w') \in X$

$$|G^+(z, w) - G^+(z', w')| \leq C \|(z, w) - (z', w')\|^\tau.$$

Proof. Let V be as in Proposition 1.1. Property (iii) of Proposition 1.1 implies that we can assume $X = V$. Let $\delta > 0$ such that $G^+(z, w) \geq 3\delta$ on $g(V) \backslash V \subset V^-$. Define

$$\Omega_\delta = \{(z, w) \in V, \delta < G^+(z, w) \leq 2\delta\}$$

and let

$$C = \sup\left\{\left|\frac{\partial G^+}{\partial z}\right|, \left|\frac{\partial G^+}{\partial w}\right| \text{ for } (z, w) \in \Omega_\delta\right\}.$$

For $(z, w) \in V \backslash K^+$ let $n \in \mathbb{Z}$ be such that

$$\delta < 2^n G^+(z, w) \leqslant 2\delta. \tag{1}$$

Clearly, there exists $n_0 \geqslant 0$ such that $n \geqslant -n_0$. Assume first that $n \geqslant 0$. Since $G^+(g^n(z, w)) = 2^n G^+(z, w)$, we have

$$2^n \frac{\partial G^+}{\partial \bar{z}}(z, w) = \frac{\partial G^+}{\partial \bar{z}}(g^n) \overline{\frac{\partial (g^n)_1}{\partial z}} + \frac{\partial G^+}{\partial \bar{w}} \overline{\frac{\partial (g^n)_2}{\partial z}}. \tag{2}$$

We know that $g(V \cup V^-) \subset V \cup V^-$ and $g(V^-) \subset V^-$. Therefore, $g^k(z, w) \in V$ for $k \leqslant n$. Let M be a majorant of the derivatives of g on V. Inductively, we have $|\partial g_j^n/\partial z| \leqslant 2^{n-1} M^n$ and $|\partial g_j^n/\partial w| \leqslant 2^{n-1} M^n$ for $j = 1, 2$. From (2) we deduce

$$\left| 2^n \frac{\partial G^+}{\partial \bar{z}}(z, w) \right| \leqslant C(2M)^n. \tag{3}$$

This estimate holds also for $-n_0 \leqslant n < 0$, increasing C if necessary. Define $\gamma = \log 2M/\log 2$. Relation (3) is equivalent to

$$\left| \frac{\partial G^+}{\partial \bar{z}} \right| \leqslant C 2^{n(\gamma - 1)} \leqslant C \left(\frac{2\delta}{G^+} \right)^{\gamma - 1}$$

or

$$\left| \frac{\partial}{\partial \bar{z}} (G^+)^\gamma \right| \leqslant C\gamma (2\delta)^{\gamma - 1}.$$

With a similar argument for $\partial/\partial \bar{w} (G^+)^\gamma$ we get

$$|\mathrm{grad}(G^+)^\gamma| \leqslant C_1.$$

Hence, $(G^+)^\gamma$ extends to a Lipschitz function on V, and therefore G^+ is Hölder with exponent $\tau = 1/\gamma = \log 2/\log 2M$. □

Remarks. (i) It follows that a minorant of τ is computable as soon as one has localized K. It is easy to show that $v(a, c) = \log 2/\log(2[\sup_{K_{a,c}} |g'|])$ is a lower semicontinuous function of (a, c) and that, if $\tau > v(a, c)$, then G^+ is Hölder continuous of exponent τ.

(ii) Friedland and Milnor have introduced in [FM] the semigroup $\mathcal{G}$ of generalized Hénon mappings

$$\mathcal{G} = \{g = g_m \circ \cdots \circ g_1, g_j(z, w) = (p_j(z) - a_j w, z)\}$$

where p_j is a monic polynomial of degree $d_j \geqslant 2$. If g is of degree $d := d_1 d_2 \ldots d_m$, one defines similarly

$$G^+(z, w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|g^n(z, w)\|$$

where $d = d_1 \dots d_m$.

The same proof, with minor changes, shows that G^+ is Hölder continuous.

We develop a consequence of the previous theorem for the Hausdorff dimension of $K^{\pm}$. We first mention the following result from [FM].

PROPOSITION 1.3. *Let σ denote the Lebesgue measure.*
(i) *If $|a| = 1$, then $\sigma(K) = \sigma(K^+) = \sigma(K^-)$ are finite and equal.*
(ii) *If $|a| < 1$, then $\sigma(K^-) = 0$ and $\sigma(K^+) = 0$ or ∞.*
(iii) *If $|a| > 1$, then $\sigma(K^+) = 0$ and $\sigma(K^-) = 0$ or ∞.*

The proof follows easily from the fact that the Jacobian of g is equal to $-a$.

COROLLARY 1.4. *Let τ be as in Theorem 1.2. Then K^+ is of Hausdorff dimension $\geqslant 2 + \tau$ at every point of K^+.*

Proof. We recall the following result from [Ca]. Let O be an open set in $\mathbb{C}$ and Y a closed set in O. Let φ be a Hölder continuous function of order τ in O. Assume φ is harmonic in $O \backslash Y$ with no harmonic extension to O. Then $\Lambda^\tau(Y) > 0$, where Λ^τ denotes the Hausdorff measure of dimension τ.

Let $p \in J^+$ and let X be a germ of analytic manifold at p; assume X is not contained in K^+. The restriction of G^+ to X is harmonic in $(G^+ > 0) \cup \mathrm{Int}_X(G^+)^{-1}(0))$ and has no harmonic extension to X. The previous result implies that $\Lambda^\tau(X \cap K^+) > 0$. Suppose $p = (z_0, w_0)$ and fix $r > 0$. For $|\alpha| \leqslant 1$ let

$$L_{\alpha,\delta} = \{(z, w); \delta \leqslant |z - z_0| < r, w - w_0 = \alpha(z - z_0)\}.$$

We have

$$\Lambda^\tau(L_{\alpha,0} \cap K^+) > 0.$$

Hence,

$$0 < \int_{|\alpha| \leqslant 1} \Lambda^\tau(L_{\alpha,0} \cap K^+)\, d\lambda(\alpha).$$

So for $0 < \delta < r$ small enough

$$0 < \int_{|\alpha| \leqslant 1} \Lambda^\tau(L_{\alpha,\delta} \cap K^+)\, d\lambda(\alpha) \leqslant C\Lambda^{2+\tau}(K^+ \cap [|z - z_i| \leqslant r, |w - w_0| \leqslant r])$$

by a standard geometric inequality, where λ denotes the Lebesgue measure in $\mathbb{C}$.

The result is of interest only when K^+ (resp. K^-) are of empty interior in a neighborhood of p.

Before proving a convergence result of $(1/2^n)(g^n)^*(T)$ to $\mu^+ = dd^c G^+$ as mentioned in the introduction, we recall a few basic facts. See [Le2] for background.

Let $d = \bar{\partial} + \partial$ and $d^c = i(\bar{\partial} - \partial)$. Define $\beta = dd^c(|z|^2 + |w|^2)$ and $\omega = dd^c \log(1 + |z|^2 + |w|^2)$. Let

$$\mathscr{L} = \{v;\, v \text{ plurisubharmonic in } \mathbb{C}^2,\, v(z, w) \leqslant \log^+ \|(z, w)\| + O(1) \text{ at infinity}\}.$$

In [Le1] (see also [Sk1]) Lelong proved the following result.

THEOREM 1.5 ([Le1]). *If $v \in \mathscr{L}$, then the positive* (1.1) *current $dd^c v$ satisfies*

$$\int_{\mathbb{C}^2} dd^c v \wedge \omega < \infty.$$

Conversely, if T is a closed positive (1, 1) *current in $\mathbb{C}^2$ such that*

$$\int_{\mathbb{C}^2} T \wedge \omega < \infty,$$

then there exists a constant $c \geqslant 0$ and $v \in \mathscr{L}$ such that $T = c\, dd^c v$.

We will call *minimal* a positive closed (1, 1) current T in $\mathbb{C}^2$ such that $\int_{\mathbb{C}^2} T \wedge \omega < \infty$. Let $\mathbb{P}^2$ be the projective space of dimension 2 and let H be the hyperplane at infinity. The current T in $\mathbb{P}^2 \backslash H$ has bounded mass near H. It follows that T has an extension $\tilde{T}$ to $\mathbb{P}^2$; $\tilde{T}$ is a positive closed (1, 1) current on $\mathbb{P}^2$ with zero mass on H. See [Si] or [Sk] for a discussion of this extension problem. We will identify T and $\tilde{T}$.

The currents $\mu^+ = dd^c G^+$ and $\mu^- = dd^c G^-$ are examples of minimal currents. Since G^+ is pluriharmonic on V^-, it follows that $p_+ \notin \operatorname{supp} \mu^+ = J^+$; we also have that $p_- \notin \operatorname{supp} \mu^- = J^-$.

THEOREM 1.6. *Let T be a positive closed* (1, 1) *current on $\mathbb{P}^2$. Suppose $p_+ \notin \operatorname{supp} T$. Then the sequence of currents $T_n = (1/2^n) g^{n*}(T)$ converges to $c\mu^+$ in $\mathbb{C}^2$ for some constant $c > 0$.*

LEMMA 1.7. *Let $\varphi \in \mathscr{L}$. Fix $\varepsilon > 0$ and let*

$$C_\varepsilon = \left\{(z, w);\, |z| \geqslant \frac{1}{\varepsilon}|w|\right\}.$$

Assume that at infinity in the cone C_ε we have

$$\varphi(z, w) = \log|z| + O(1).$$

Then $\varphi_n = (1/2^n)\varphi(g^n)$ converges in $L^1_{\mathrm{loc}}(\mathbb{C}^2)$ to G^+.

Proof. It is clear that the sequence φ_n has a uniform upper bound on any compact set. It follows from Proposition 1.1 that φ_n converges uniformly on compact sets of U^+ to the function G^+.

We recall the following result; see [Hö, p. 95]. Let v_j be a sequence of subharmonic functions on a domain W of $\mathbb{R}^n$, which have a uniform upper bound on compact sets. Then the following holds.

(a) If v_j does not converge uniformly on compact sets of w to $-\infty$, then there is a subsequence v_{j_k} which is convergent in $L^1_{\mathrm{loc}}(W)$ to a subharmonic function v.

(b) If v_j converge in $L^1_{\mathrm{loc}}(W)$ to a subharmonic function v, then

$$\overline{\lim_{j\to\infty}}\, v_j(x) \leqslant v(x) \qquad x \in W,$$

and for every compact $X \subset W$

$$\overline{\lim_{j}}\, \sup_X v_j \leqslant \sup_X v.$$

Assume that there is a ball B in $\mathbb{C}^2$ and a subsequence n_j such that $\int_B |\varphi_{n_j} - G^+| \geqslant \varepsilon$ for every j. We apply the previous result to the sequence $v_j = \varphi_{n_j}$. Let v be the plurisubharmonic function such that v_{j_k} converges to v in $L^1_{\mathrm{loc}}(\mathbb{C}^2)$. We know that $v = G^+$ on U^+, and we have to show that $v = 0$ on K^+; this will contradict the assumption on the sequence φ_{n_j}. Property (iv) of Proposition 1.1 implies that $v \leqslant 0$. By upper semicontinuity it is impossible to have $v < 0$ at a point of J^+; so $v = G^+$ on $U^+ \cup J^+$. Suppose there is $\alpha > 0$ and $W \subset\subset \operatorname{Int} K^+$ such that $v \leqslant -\alpha$ on W. Property (b) implies that for j large enough

$$\varphi_{n_j}(z, w) \leqslant -\frac{\alpha}{2} \text{ on } W.$$

Let

$$E^n = \{(z, w) \in V;\ \varphi(z, w) \leqslant -\tfrac{1}{2}\alpha 2^n\}$$

and

$$E^n_{w_0} = \{z;\ \varphi(z, w_0) \leqslant -\tfrac{1}{2}\alpha 2^n\}'$$

We recall the following result. Let Y be a nonpolar compact set in $\mathbb{C}$. Then $G_Y(\zeta)$, the Green function on Y with pole at infinity, is equal to the supremum of the subharmonic functions v in $\mathbb{C}$, such that $v \leqslant 0$ on Y and $v(\zeta) = \log|\zeta| + O(1)$ at infinity. Moreover, $G_Y(\zeta) = \log|\zeta| + \gamma + O(1)$ with $\gamma \in \mathbb{R}$. The logarithmic capacity of Y is by definition $\operatorname{cap}(Y) = \exp(-\gamma)$; see [Ts] for reference.

Let X be a compact set in E^{n_j} and let $X_{w_0} = X \cap \{w = w_0\}$. The hypothesis on φ implies that $\varphi(z, w_0) + \frac{1}{2}\alpha 2^{n_j}$ is $\leqslant 0$ on X_{w_0} and grows like $\log|z| + O(1)$ at infinity.

Therefore, there is a constant $C > 0$ such that for every compact $X \subset E^{n_j}$

$$\mathrm{Cap}(X_{w_0}) \leqslant Ce^{-(1/2)\alpha 2^{n_j}}.$$

Using a standard inequality between area and capacity (see [Ts]), we get that for some constant $C_1 > 0$ and for $|w_0| < R$

$$\mathrm{Area}(X_{w_0}) \leqslant C_1 e^{-(1/2)\alpha 2^{n_j}}.$$

Fubini's theorem and the regularity of the Lebesgue measure imply that for a constant $C_2 > 0$

$$\mathrm{vol}(E^{n_j}) \leqslant C_2 e^{-(\alpha/2)2^{n_j}}.$$

But for j large, $g^{n_j}(W) \subset V \cap E^{n_j}$; therefore,

$$|a|^{2n_j}\,\mathrm{vol}(W) = \mathrm{vol}(g^{n_j}(W)) \leqslant C_2 e^{-(\alpha/2)2^{n_j}}$$

which is a contradiction. Hence, $v = G^+$ and φ_n converge to G^+ in $L^1_{\mathrm{loc}}(\mathbb{C}^2)$.

Proof of Theorem 1.6. A theorem of Siu [Siu] implies that the restriction of T to the hyperplane at infinity H is a multiple of the current of integration on H. Since $p_+ \notin \mathrm{Supp}\, T$, it follows that T has no mass on H, hence the restriction of T to $\mathbb{C}^2$ is nonzero. Since $p_+ \notin \mathrm{Supp}\, T$, then there exists $R_1 > 0$ and $\varepsilon > 0$ such that, if $C_\varepsilon = \{(z, w); |z| \geqslant (1/\varepsilon)|w|, |z| \geqslant R_1\}$, then $\mathrm{Supp}\, T \cap C_\varepsilon \neq \varnothing$. By Theorem 1.5 on $\mathbb{C}^2$, there exists a constant $c > 0$ and $\varphi \in \mathscr{L}$ such that $dd^c \varphi = cT$ on $\mathbb{C}^2$. Since φ is pluriharmonic on C_ε and has logarithmic growth at infinity, we have that for every w fixed $\varphi(z, w) = \alpha(w) \log |z| + O(1)$ at infinity. Pluriharmonicity of φ on C_ε implies that α is harmonic on $\mathbb{C}$, the growth of φ, and the fact that $T \not\equiv 0$ imply that α is a nonzero constant. Changing c eventually, we can assume that for w fixed $\varphi(z, w) = \log |z| + O(1)$. Lemma 1.7 implies that $\varphi_n = \varphi(g^n)/2^n$ converge in L^1_{loc} to G^+. Hence, $(1/2^n)\, dd^c \varphi_n = c(1/2^n)(g^n)^*T$ converges in the sense of currents to μ^+.

Remarks. 1. Let T be a positive closed (1, 1) current in $\mathbb{P}^2$ *without mass on H*. For every $k \in \mathbb{N}$ the current $(g^k)^*T$ is well defined as a current on $\mathbb{C}^2$; it is positive closed and has minimal growth; hence, it has a positive closed extension as a current in $\mathbb{P}^2$ that we still denote $(g^k)^*T$. If, for some k, $p_+ \notin \mathrm{Supp}\, g^{k*}(T)$, then we also have that $(1/2^n)g^{n*}(T)$ converges to $c\mu^+$. When T is the current of integration on an algebraic curve S, the condition on the support of $g^{k*}[S]$ is always verified for some k; see [BS2]. So Theorem 1.6 implies the Bedford-Smillie result, that for an algebraic variety S the currents $(1/2^n)g^{n*}[S]$ converge to $c\mu^+$.

2. A condition on the support of T is needed for Theorem 1.5 to hold. Suppose $T = \mu^-$; then $g^*\mu^- = \frac{1}{2}\mu^-$. Therefore, $(1/2^n)(g^n)^*\mu^-$ converges to 0 in the topology of currents on $\mathbb{C}^2$.

3. Theorem 1.6 is easily generalized if we only assume that $g \in \mathscr{G}$, the semigroup of finite compositions of Hénon mappings.

COROLLARY 1.8. *If T is a positive closed* $(1, 1)$ *current in* $\mathbb{C}^2$ *such that* Supp $T \subset K^+$ *and* $g^*(T) = 2T$, *then* $T = c\mu^+$, *where c is a positive constant.*

Proof. Since $K^+ \cap V^- = \varnothing$, the current T is defined by 0 in a neighborhood of p_+ in $\mathbb{P}^2$. So T can be considered as a $(1, 1)$ positive closed current in $\mathbb{P}^2 \backslash \Delta$, where Δ is a compact disc in the hyperplane at infinity H. It follows from [Si, Corollary 1.5] that the trivial extension of T to $\mathbb{P}^2$ is closed. Therefore, T is a minimal current and $T = (1/2^n)g^{n*}(T)$; hence, by Theorem 1.5, $T = c\mu^+$. □

COROLLARY 1.9. *Let* $T \not\equiv 0$ *be a positive closed* $(1, 1)$ *current in* $\mathbb{C}^2$. *If* Supp $T \subset K^-$, *then T has a closed extension* $\tilde{T}$ *to* $\mathbb{P}^2$ *and* $p_+ \in$ Supp $\tilde{T}$. *If* Supp $T \subset K^+$, *then T has a closed extension* $\tilde{T}$ *to* $\mathbb{P}^2$ *and* $p_- \in$ Supp $\tilde{T}$.

Proof. We have already seen that under the above assumptions T has a closed extension to $\mathbb{P}^2$. Suppose for example that $p_+ \notin$ Supp $\tilde{T}$ and Supp $T \subset K^-$. Then $T_n = (1/2^n)(g^n)^*T$ converges to $c\mu^+$. On the other hand, Supp $T_n \subset K^-$ since $g^{-1}(K^-) = K^-$. Therefore, the support of μ^+ should be contained in $K = K^+ \cap K^-$ which is impossible since every point of J^+, which is not compact, is an essential singularity for G^+; i.e., G^+ has no pluriharmonic extension in a neighborhood of a point of K^+. The proof when Supp $T \subset K^+$ is similar. □

COROLLARY 1.10. *Let Ω be an open set in* $\mathbb{C}^2$ *such that* $g^{-1}(\Omega) \subset \Omega$. *Assume* $|\det g'| < 1$ *and that Ω contains a fixed point p of g. Let* $T \not\equiv 0$ *be a positive closed* $(1, 1)$ *current in* $\mathbb{P}^2$ *with* $p_- \notin$ Supp T. *Then* Supp $T \cap \Omega$ *is nonempty.*

Proof. Since $p_- \notin$ Supp T, then the sequence $T_n = (1/2^n)(g^{-n})^*T$ converges to $c\mu_-$. If Supp $T \cap \Omega = \varnothing$, then since Ω is invariant under g^{-1}, we get that Supp $T_n \cap \Omega = \varnothing$ for every n, and hence Supp $\mu^- \cap \Omega = \varnothing$. But, by Proposition 1.3, $J^- = K^-$ if $|\det g'| < 1$, and we know that Supp of $\mu^- = J^-$. Since $p \in J^-$, we have a contradiction. Hence, Supp $T \cap \Omega$ is nonempty.

Remarks. 1. It is enough to assume that $K^- \cap \Omega$ is nonempty without assuming the existence of a fixed point.

2. It follows that Supp T intersects any Fatou-Bieberbach domains and the Siegel cylinders we will construct in Section 2.

The case where T is the current associated to an algebraic variety and where Ω is a Fatou-Bieberbach domain, has been proved by Bedford and Smillie [BS2].

3. With obvious modifications, Corollaries 1.8, 1.9, and 1.10 are valid for $g \in \mathscr{G}$, the semigroup of finite compositions of Hénon mappings.

2. Siegel cylinders. In this section we study a linearization problem around a fixed point for polynomial automorphisms of $\mathbb{C}^2$.

Following [FM], we will say that a polynomial automorphism e is elementary if it can be written as

$$e(z, w) = (\alpha z + p(w), \beta w + \gamma)$$

where α, β, $\gamma \in \mathbb{C}$, $\alpha\beta \neq 0$, and p is a polynomial. We call E the group of such automorphisms. A Hénon mapping is an automorphism which has the form

$$g(z, w) = (p(z) - aw, z)$$

with $\deg p \geqslant 2$, $a \neq 0$.

It is proved in [FM] that a polynomial automorphism of $\mathbb{C}^2$ is either conjugate to an elementary automorphism or to a composition of a finite number of Hénon mappings. The dynamics of elementary mappings is quite simple, and we just recall the following result from [FM].

THEOREM 2.1. *Every element of E is E-conjugate to one of the following types of automorphisms.*

(i) $(z, w) \to (\alpha z, \beta w)$.

(ii) $(z, w) \to (\alpha z, w + 1)$ *or* $(z, w) \to (z + 1, \beta w)$.

(iii) $(z, w) \to (\beta^d(z + w^d), \beta w)$, *where* $d \in \mathbb{N}$, $d \geqslant 1$.

(iv) $(z, w) \to (\beta^\mu(z + w^\mu q(w^r)), \beta w)$, *where β is a primitive rth root of unity, q a nonconstant polynomial, and* $\mu \geqslant 0$.

Let f be a germ of an automorphism of $\mathbb{C}^2$ near the origin, such that $f(0) = 0$. Let $A = f'(0)$ and let λ_1, λ_2 be the eigenvalues of A. Assume $|\lambda_1| < 1$, $|\lambda_2| < 1$ and for all $(k_1, k_2) \in \mathbb{N}^2$, $|k| \geqslant 2$,

$$\lambda_1^{k_1}\lambda_2^{k_2} - \lambda_j \neq 0.$$

Then f can be linearized near 0. More precisely, there exists a unique germ h of a biholomorphism of the form

$$h = \mathrm{Id} + O(\|(z, w)\|^2)$$

such that in a neighborhood of 0 the identity

$$f \circ h = h \circ A$$

holds. If f is a polynomial automorphism of $\mathbb{C}^2$, then h extends to a biholomorphic map from $\mathbb{C}^2$ to Ω, the basin of attraction of 0. When one of the eigenvalues of A is of modulus 1, the situation is more delicate, and we refer to the recent survey by Herman [He] for a discussion and references. We just recall the following result.

Let A be a (2, 2) diagonalizable matrix with eigenvalues (λ_1, λ_2). Define

$$\Omega(m) = \inf_{2 \leqslant |k| \leqslant m} |\lambda^k - \lambda_j|$$

where as usual $\lambda^k = \lambda_1^{k_1}\lambda_2^{k_2}$ if $k = (k_1, k_2)$, k_1, k_2 are nonnegative integers. The matrix

A satisfies the Brjuno condition if and only if

$$\text{(B)} \qquad \sum_m 2^{-m} \log \frac{1}{\Omega(2^m)} < \infty .$$

Observe that, if $|\lambda_1| = 1$ and $|\lambda_2| < 1$, then A satisfies the Brjuno condition if and only if

$$\text{(B}'\text{)} \qquad \sum_m 2^{-m} \log \frac{1}{\tilde{\Omega}(2^m)} < \infty$$

where

$$\tilde{\Omega}(m) = \inf_{2 \leqslant k \leqslant m} |\lambda_1^k - \lambda_1| .$$

If $\lambda_1 = e^{2i\pi\theta}$ and there are positive constants α, β such that $|\theta - (p/q)| \geqslant \alpha/|q|^\beta$ for all $p, q \in \mathbb{Z}$, then λ_1 satisfies condition (B$'$).

THEOREM 2.2. *Let $f = A + O(\|(z, w)\|^2)$ be a germ of a biholomorphism of $\mathbb{C}^2$. Assume A is diagonal with eigenvalues $\lambda_1 = e^{2i\pi\theta}$, $|\lambda_2| < 1$, which satisfy condition (B). Then there exists a germ of a biholomorphism in a neighborhood of $0 \in \mathbb{C}^2$ such that*

$$f \circ h = h \circ A .$$

The above theorem was proved under weaker conditions by Siegel and in general by Brjuno. See, however, [He] for a more precise discussion and for references.

We want to prove a global result when f is a polynomial automorphism.

THEOREM 2.3. *Let f be a polynomial automorphism of $\mathbb{C}^2$. Suppose $f(0) = 0$ and that $A = f'(0)$ has two eigenvalues $\lambda_1 = e^{2i\pi\theta}$, $|\lambda_2| < 1$, where λ_1 satisfies the Brjuno condition* (B$'$). *Then either f is conjugate to A in the group E or there exists a biholomorphic map $h: \Delta \times \mathbb{C} \to \Omega$, where Ω is the connected component of* $\operatorname{int}(K^+)$ *containing* 0 *and Δ is the unit disc in $\mathbb{C}$ such that*

$$f \circ h = h \circ A .$$

Proof. Suppose f is conjugate to an elementary mapping. We use the Friedland Milnor classification given in Theorem 2.1. Case (i) is clear. In case (ii) there is no fixed point. In cases (iii) and (iv) the eigenvalues at the fixed points do not satisfy the hypothesis of Theorem 2.3.

We now assume that f is conjugate to a composition of a finite number of Hénon mappings. Conjugating with a translation, we can still assume that $f(0) = 0$ and that Proposition 1.1 holds for f. Brjuno's theorem implies the existence of a germ of a biholomorphism h in a neighborhood of 0 such that

$$f \circ h = h \circ A.$$

We assume that A is in diagonal form and $A(z, w) = (\lambda_1 z, \lambda_2 w)$.

Let M be a germ of a complex manifold of dimension 1 around 0 such that, on M, f is conjugate to a rotation. More precisely, for some $r > 0$, $M = \{h(\zeta, 0), |\zeta| < r\}$. Since f^n is a normal family on Ω, we can choose a subsequence n_k with the property that $f^{n_k}|_M$ converges to $\mathrm{Id}|_M$. We also assume that f^{n_k} converges on compact sets of Ω.

LEMMA 2.4. *Let $F = \lim f^{n_k}$. Define*

$$\mathscr{M} = \{q \in \Omega; F(q) = q\}.$$

Let $\tilde{M}$ be the connected component of $\mathscr{M}$ containing 0. *Then $\tilde{M}$ is a closed complex manifold biholomorphic to a disc.*

Proof. In a neighborhood of 0 we have

$$f^n h(z, w) = h(e^{2i\pi n\theta} z, \lambda_2^n w).$$

Therefore, we also have that

$$Fh(z, w) = h(z, 0), \tag{1}$$

and hence $F \circ F = F$ in a neighborhood of 0.

From (1) it follows that F' has at most rank one in Ω; hence, $I - F'(z, w)$ has at least rank one. Therefore, $\mathscr{M}$ is a complex manifold of dimension 1 or 0. So $\tilde{M}$ is a complex manifold of dimension 1 since $M \subset \tilde{M}$. Since $f(M) = M$, by analytic continuation for f and f^{-1} we have that f is an automorphism of $\tilde{M}$ which is conjugate to an irrational rotation. Hence, the group generated by $f_{|\tilde{M}}$ is infinite and has a fixed point; so $\tilde{M}$ is biholomorphic to the unit disc Δ since $\tilde{M} \subset K$. $\square$

Let $\pi: \Delta \to \tilde{M}$ be a biholomorphic map from Δ onto $\tilde{M}$ with $\pi(0) = 0$. For $0 < r < 1$ let $\tilde{M}_r = \pi(\Delta(0, r))$.

LEMMA 2.5. *Fix $r < 1$. The open set $\Omega_r = F^{-1}(\tilde{M}_r)$ can be exhausted by biholomorphic images of the bidisc.*

Proof of Lemma 2.5. Let $\pi = (\pi_1, \pi_2)$. Choose holomorphic functions h_1, h_2 on Δ such that

$$\frac{\partial \pi_1}{\partial t_1}(t_1) h_2(t_1) - \frac{\partial \pi_2}{\partial t_1}(t_1) h_1(t_1) = 1.$$

This is possible since $\partial \pi_1/\partial t_1, \partial \pi_2/\partial t_1$ do not vanish simultaneously on Δ. Consider the map H defined on $\Delta \times \mathbb{C}$ as

$$H(t_1, t_2) = (\pi_1(t_1) + t_2 h_1(t_1), \pi_2(t_1) + t_2 h_2(t_1)).$$

For every $r' < 1$ there exists $\delta > 0$ such that H is a biholomorphism from $\Delta_{r'} \times \Delta_\delta$ onto its image; hence, $\tilde{M}_{r'}$ has a basis of neighborhoods biholomorphic to a bidisc. As a consequence, there exists a holomorphic function φ defined on a neighborhood of $\overline{\tilde{M}_r}$ and vanishing to first order exactly on M.

For ε small enough let $D^\varepsilon = \Omega_r \cap \{|\varphi| < \varepsilon\}$. For ε small enough define $\Phi^\varepsilon(z, w) = (\pi^{-1} \circ F(z, w), (1/\varepsilon)\varphi(z, w))$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, Φ^ε is a biholomorphism from D^ε to $\Delta_r \times \Delta$.

On $\tilde{M}$, f is conjugate to a rotation; hence, $f(\Omega_r) = \Omega_r$. Since f^{n_k} converges uniformly on compact sets to F, it follows that for k large enough $f^{n_k}(D^\varepsilon) \subset D^\varepsilon$; so there exists a subsequence n_{k_j} such that $f^{-n_{k_j}}(D^\varepsilon) \nearrow \Omega_r$.

LEMMA 2.6. *The connected component Ω of* Int K^+ *containing* 0 *is equal to* $\Omega_1 = F^{-1}(\tilde{M})$.

Proof of Lemma 2.6. It is not a priori clear that $F(\Omega) \subset \Omega$ although $F(\Omega) \subset \bar{\Omega}$. However, the local images of F on Ω are one-dimensional varieties. Hence, we can abstractly think of $F(\Omega)$ as a connected normal complex variety $\tilde{M}^*$ over $\mathbb{C}^2$ containing $\tilde{M}$. We know that $f(\tilde{M}) = \tilde{M}$; so f extends to $\tilde{M}^*$, $f(\tilde{M}^*) = \tilde{M}^*$. As in Lemma 2.5, we prove that $\tilde{M}^*$ is biholomorphic to a disc and that f is conjugate to a rotation on $\tilde{M}^*$. Suppose $q \in \tilde{M}^*$ is on the boundary of $\tilde{M}$ in $\tilde{M}^*$. We can construct a disc M_1 containing $0 \in \tilde{M}$ and $q \in \tilde{M}^*$ such that M_1 has a neighborhood U_1 biholomorphic to a bidisc; this latter fact is proved as in Lemma 2.6. On U_1, f is conjugate to

$$\tilde{f}(z, w) = (e^{2i\pi\theta} z + 0(w), 0(w)).$$

At every point of $(w = 0)$ the matrix $(f^n)'$ has an eigenvalue equal to $e^{2i\pi n\theta}$. Since $|\mathrm{Det}\, f'| = \alpha < 1$, the other eigenvalue decreases geometrically to 0. Therefore, there is a neighborhood U_2 of M_1 such that the sequence f^n is normal on U_2, and hence $q \in \Omega$. □

End of proof of Theorem 2.3. From Lemma 2.6 it follows that $\Omega = \Omega_1$ can be exhausted by biholomorphic images of polydiscs. Since F is a nonconstant bounded holomorphic map on Ω, it follows that the infinitesimal Kobayashi metric of Ω is not identically 0. A theorem of [FS] implies that in this situation Ω is biholomorphic to $\Delta \times \mathbb{C}$ or to Δ^2. But $f \in \mathrm{Aut}\, \Omega$ has a fixed point, and f^{n_k} converges to a degenerate mapping; therefore, Ω is not biholomorphic to Δ^2.

Since $F(\Omega) = \tilde{M}$, F is a retraction on $\tilde{M}$ and $F \circ F = \mathrm{Id}$. The biholomorphism $\Phi: \Omega \to \Delta \times \mathbb{C}$ constructed in [FS] is such that $\Phi(\tilde{M}) = \Delta \times \{0\}$; i.e., the first component Φ_1 satisfies $\Phi_1 = \pi^{-1} \circ F$.

If $\psi = \Phi \circ f \circ \Phi^{-1}$, then ψ is a biholomorphic map on $\Delta \times \mathbb{C}$, $\psi(z, 0) = (\lambda_1 z, 0)$, and therefore $\psi(z, w) = (\lambda_1 z, \lambda_2 w e^{u(z)})$, where u is a holomorphic function on Δ with $u(0) = 0$.

For a holomorphic function v on Δ with $v(0) = 0$, define

$$T_v(z, w) = (z, e^{v(z)}w).$$

We have to determine v such that

$$(*) \qquad T_v^{-1} \circ \psi \circ T_v(z, w) = (\lambda_1 z, \lambda_2 w).$$

Equation (*) is equivalent to

$$u(z) + v(z) - v(\lambda_1 z) = 0.$$

Let $u(z) = \sum_{n\geqslant 1} a_n z^n$ and $v(z) = \sum_{n\geqslant 1} b_n z^n$. Equation (*) is satisfied if and only if for every $n \geqslant 1$

$$a_n + b_n(1 - \lambda_1^n) = 0;$$

i.e.,

$$b_n = -\frac{a_n}{(1 - \lambda_1^n)}.$$

Since λ_1 satisfies the Brjuno condition, then given $\varepsilon > 0$, for n large enough,

$$\frac{1}{|\lambda_1^n - 1|} \leqslant e^{\varepsilon n}.$$

Therefore, the series $\sum_{n\geqslant 1} b_n z^n$ has a radius of convergence at least equal to 1. □

It should be observed that the component Ω is a Runge domain. This is indeed a general fact.

PROPOSITION 2.7. *Let $f \in \mathcal{G}$ be a polynomial automorphism of $\mathbb{C}^2$. If ω is a component of* Int K^+, *then ω is a Runge domain.*

Proof. Recall that

$$K^+ = \{q; f^n(q) \text{ is bounded}\}.$$

Recall that a domain of holomorphy is Runge if and only if every holomorphic function in U is uniformly approximable on compact sets by polynomials. If X is a compact set in $\mathbb{C}^k$, let $\hat{X}$ denote the polynomial hull of X; i.e.,

$$\hat{X} = \left\{\zeta \in \mathbb{C}^k, |p(\zeta)| \leqslant \sup_{z\in X} |p(z)| \text{ for every polynomial } p\right\}.$$

Equivalently, U is Runge if and only if, for every compact $X \subset U$, $\hat{X} \subset U$.

If X is a compact in $\mathbb{C}^k$, let

$$X_r = \{\zeta \in \mathbb{C}^k;\ \mathrm{dist}(\zeta, X) \leqslant r\}.$$

Observe that $(\hat{X})_r \subset \widehat{X_r}$. Indeed, let p be a polynomial and $\eta \in \mathbb{C}^k$, $\|\eta\| \leqslant r$. If $\zeta \in \hat{X}$, then

$$|p(\zeta + \eta)| \leqslant \sup_{z \in X} |p(z + \eta)| \leqslant \sup_{X_r} |p|$$

which implies that $\zeta + \eta \in \hat{X}_r$.

Suppose $X \subset\subset \operatorname{Int} K^+$. Let $r > 0$ such that $X_r \subset \operatorname{Int} K^+$. It follows from Proposition 1.1 that $\hat{X}_r \subset K^+$. Since $(\hat{X})_r \subset \hat{X}_r$, we have $\hat{X} \subset \operatorname{Int} K^+$. It is well known that connected components of a Runge open set are Runge. □

Problem. Let f be a polynomial automorphism of $\mathbb{C}^k$ such that $f(0) = 0$ and the eigenvalues (λ_i) of the matrix $A = f'(0)$ satisfy $|\lambda_i| \leqslant 1$, $i = 1, \ldots, k$. Let Ω be the maximal connected open set containing 0 on which the sequence f^n is normal. Assume f is linearizable around 0. Is Ω biholomorphic to $\omega \times \mathbb{C}^{k-\ell}$, where ω is a bounded Reinhardt domain in $\mathbb{C}^\ell$ for some ℓ, $0 \leqslant \ell \leqslant k$? When all λ_j satisfy $|\lambda_j| = 1$ and f is linearizable around 0, Herman [He] asks for a description of the Reinhardt domains that one can obtain in this way.

3. Structure of K^+ and K^- for small a. Let P be a monic polynomial in $\mathbb{C}$ of degree at least two. Recall that a point z belongs to the Julia set $J(P)$ if and only if the sequence f^n (considered as maps to $\mathbb{P}^1$) is not a normal family in a neighborhood of z. The polynomial P is hyperbolic if and only if all the critical points of P are attracted by attractive cycles (including ∞). Equivalently, P is hyperbolic if and only if there are two constants $C > 0$, $\gamma > 1$, such that for every $z \in J(P)$

$$|P^{n\prime}(z)| \geqslant C\gamma^n.$$

It is standard to find a Riemannian smooth metric on $\mathbb{C}$ with the property that in the new metric there is a $\gamma > 1$ such that for $z \in J(P)$

$$|P'(z)| > \gamma.$$

In what follows, we will assume that such a metric has been chosen.

In this paragraph we study the dynamics of maps

$$g_{a,c}(z, w) = (z^2 + c + aw, az)$$

assuming that the polynomial $P_c(z) = z^2 + c$ is hyperbolic and that a is small enough. The idea is as in Benedicks-Carleson [BC] to consider g as a perturbation of the degenerate map

$$g_{0,c}(z, w) = (z^2 + c, 0).$$

Let $J(c)$ denote the Julia set of the polynomial P_c. Let $\mathscr{M}$ be the Mandelbrot set

$$\mathscr{M} = \{c; J(c) \text{ is connected}\}.$$

It is well known that, for $c \in \mathbb{C}\backslash\mathscr{M}$, $J(c)$ is a Cantor set and that the polynomial P_c is hyperbolic; see [De], [Do]. This happens in particular if $|c| > 2$.

It is however not known whether for $c \in \operatorname{Int}(\mathscr{M})$ the polynomial P_c is hyperbolic. We will study separately the case where $c \in \mathbb{C}\backslash\mathscr{M}$ and the case where c belongs to a hyperbolic component of Int $\mathscr{M}$.

We recall the following definition of hyperbolicity; see [Sh]. Let f be a diffeomorphism of a smooth manifold N. Let Λ be a closed invariant set under f. Then f is hyperbolic on Λ if there is a continuous invariant splitting for f'; i.e., there are invariant continuous subbundles E_s and E_u such that $TN_{|\Lambda} = E_s \oplus E_u$, and there is a metric on N, and constants $c > 0$, $\lambda > 1$, such that in this metric

$$|(f^n)'_{|E_s}| < \frac{c}{\lambda^n}, \qquad |(f^n)'_{|E_u}| > c\lambda^n.$$

THEOREM 3.1. *Let $c \in \mathbb{C}\backslash\mathscr{M}$. Then there exists $a_0(c) > 0$ such that, if $|a| < a_0(c)$ and $g(z, w) = (z^2 + c + aw, az)$, then the sets K^+ and K^- associated to g have the following structure.*

(i) *K^+ and K^- are foliated by complex manifolds which intersect transversally. Every leaf in the foliation $\mathscr{F}^+$ of K^+ (resp. in the foliation $\mathscr{F}^-$ of K^-) is dense in K^+ (resp. in K^-) and is biholomorphic to $\mathbb{C}$.*

(ii) *If $K = K^+ \cap K^-$, then g is hyperbolic on K and K is a Cantor set.*

(iii) *The Hausdorff dimension $h_{a,c}$ of $K^\pm$ has the following properties: $h_{a,c}(K^+) > 2$ and for any fixed a $\lim_{c\to\infty} h_{a,c}(K^+) = 2$. We also have the inequalities*

$$2 < h_{a,c}(K^-) \leqslant 2 + \frac{\log 2}{\log 1/|a|}$$

for the Hausdorff dimension of K^-.

In the proof of Theorem 3.1 we basically deal with the horseshoe construction.

We recall the following facts from the dynamics of $P_c(z) = z^2 + c$. See [De] or [Do].

LEMMA 3.2. *Suppose $c \notin \mathscr{M}$. There exist 3 real analytic simple closed curves γ_1, γ_2, γ_3 bounding D_1, D_2, D_3 with the following properties.*

(i) $\bar{D}_1 \cap \bar{D}_2 = \varnothing$;

(ii) $\bar{D}_1 \subset D_3$, $\bar{D}_2 \subset D_3$;

(iii) $0 \in D_3 \backslash \bar{D}_1 \cup \bar{D}_2$, $c \notin \bar{D}_3$;
(iv) *P_c maps each D_i biholomorphically to D_3, $i = 1, 2$;*
(v) *If $z \notin D_3$, then $P_c(z) \notin \bar{D}_3$ and $P_c^n(z) \to \infty$ as $n \to \infty$.*

For any sequence $\lambda = (i_1, \ldots, i_n, \ldots)$ where $i_j \in \{1, 2\}$, let $\lambda_n = (i_1, \ldots, i_n)$. The domains D_{λ_n} and the curves $\gamma_{\lambda_n} = \partial D_{\lambda_n}$ are defined inductively using the following relations: $\gamma_{i_1 \ldots i_{n+1}} \subset D_{i_1 \ldots i_n}$ and $P_c^n(\gamma_{i_1 \ldots i_{n+1}}) = \gamma_{i_{n+1}}$. Let $U_n = \bigcup_{|\lambda| = n} D_\lambda$. Then $U_{n+1} \subset\subset U_n$, $\cap U_n$ has empty interior, and $J(c)$ is given by

$$J(c) = \bigcap_n U_n.$$

Since P_c is hyperbolic, we know as above that there is Riemannian metric on $\mathbb{C}$ and a constant $\gamma > 1$ such that

$$|P_c'(z)| > \gamma \quad \text{on } J(c).$$

By continuity there exists an integer $M > 1$ so that

$$|P_c'(z)| > \gamma \quad \text{on } \bar{U}_M.$$

We fix $R > 1$ and $R > |c|$ such that $\bar{D}_3 \subset \Delta(0, R)$. Let

$$D = D_3 \times \{|w| < R\}.$$

We will make restrictions on a of the type $|a| < a(c)$; the conjunction of all restrictions will give the constant $a_0(c) > 0$ of the theorem.

LEMMA 3.3. *There exists $a(c) > 0$ such that, if $0 < |a| < a(c)$, the following holds.*
(i) *If $(z, w) \in \bar{D}$ and for some $n \geq 1$, $g^n(z, w) \notin D$, then $g^{n+k}(z, w) \notin \bar{D}$ for every $k \geq 1$ and $g_1^k(z, w) \to \infty$ as $k \to \infty$.*
(ii) *If $(z, w) \in K^+$, then there exists n_0 such that $g^n(z, w) \in D$ for every $n \geq n_0$.*

Proof. (i) If $|w| < R$ and $z \notin D_3$, then $z^2 + c + aw$ is just a perturbation of $z^2 + c$; therefore, $g_1(z, w) \notin D_3$ and $g_1^k(z, w) \to \infty$ as $k \to \infty$.
(ii) Suppose $(z, w) \in \mathbb{C}^2$, $|w| \geq R$, and $|z| \geq |w|$, if a is small enough, $|z^2 + c + aw| \gg |z|$ and clearly $(z, w) \notin K^+$.

Let $(z_0, w_0) \in K^+$ and $(z_n, w_n) = g^n(z_0, w_0)$ be the forward orbit. It follows from the previous observation that, if $|w_n| \geq R$, we must have $|z_n| < |w_n|$. Hence,

$$|w_{n+1}| = |az_n| \leq |a||w_n|.$$

Therefore, for some n_0, $|w_{n_0}| < R$, and since $(z_{n_0}, w_{n_0}) \in K^+$, $z_{n_0} \in D_3$. Using (i), we see that $g^n(z, w) \in D$ for all $n \geq n_0$. □

The main ingredient in the proof of Theorem 3.1 is the following estimate on the derivative of g^n.

PROPOSITION 3.4. *Fix* $\tilde{R} \gg 1$. *There are constants* C_1, $C_2 > 0$, $\lambda > 1$, *and* $a(c, \tilde{R}) > 0$ *such that the following holds.*

Suppose (z_j, w_j), $0 \leqslant j \leqslant n$, *is an orbit in* $\bar{D}$ *and let*

$$(g^n)'(z_0, w_0) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Then

$$|a_n| \geqslant C_1 \lambda^n, \qquad |b_n| \leqslant \frac{1}{\tilde{R}} |a_n|, \qquad |c_n| \leqslant C_2 |a| |a_n|, \qquad |d_n| \leqslant 3C_2 |a| |a_n|.$$

We prove at first a lemma.

LEMMA 3.5. *Given* $\tilde{R} \gg 1$, *let* $(1, \alpha) \in \mathbb{C}^2$ *with* $|\alpha| \leqslant \tilde{R}$. *There are constants* $C(\tilde{R}) > 0$, $\lambda > 1$, *such that the following holds. Assume* $(z_i, w_i) \in \bar{D}$ *for* $i = 0, \ldots, k$, *where* $(z_j, w_j) = g(z_{j-1}, w_{j-1})$. *Let* $k = m + r$ *with* $r \leqslant M + 2$ *and suppose* $z_0, \ldots, z_m \in U_M$. *Let*

$$(g^k)'(z_0, w_0) = (g^r)'(z_m, w_m) g'(z_{m-1}, w_{m-1}) \ldots g'(z_0, w_0) = A_{m+1} A_m \ldots A_1 .$$

Denote

$$(x_1, y_1) = A_1(1, \alpha), \qquad (x_2, y_2) = A_2(x_1, y_1), \ldots, (x_{m+1}, y_{m+1}) = A_{m+1}(x_m, y_m).$$

Then

(i) *for* $1 \leqslant j \leqslant m - 1$, $|x_{j+1}| > \lambda |x_j|$ *and* $|y_j| \leqslant 2C(\tilde{R})|a||x_j| \leqslant |x_j|$; *and*

(ii) *there exists* $\delta > 0$ *such that*

$$|x_{m+1}| > \tfrac{1}{2}(2\delta)^r |x_m|$$

and

$$|y_{m+1}| \leqslant \frac{4C|a|}{(2\delta)^r} |x_{m+1}| \leqslant C(\tilde{R})|a||x_{m+1}| \leqslant |x_{m+1}|.$$

Proof. Suppose $m = 0$; then $k = r$

$$(g^r)'(z_0, w_0) = \begin{bmatrix} 2z_0 \ldots 2z_{r-1} + f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$$

where $|f_j| \leqslant C|a|$. Since $(z_i, w_i) \in D$, there is a constant $\delta > 0$ such that $|z_i| \geqslant \delta$ if $i \leqslant r - 1$; hence,

$$\left| \prod_{j=0}^{r-1} 2z_j \right| \geqslant (2\delta)^r.$$

We then have

$$(g^r)'(z_0, w_0)\begin{pmatrix}1\\ \alpha\end{pmatrix} = ((2z_0 \ldots 2z_{r-1} + f_1) + f_2\alpha, f_3 + f_4\alpha) = (x_1, y_1).$$

If $|a| \leqslant a(C, \tilde{R})$ is small enough, we have

$$|x_1| \geqslant (2\delta)^r - C|a| - C|a|\tilde{R} \geqslant (2\delta)^r - 2C|a|\tilde{R} \geqslant \tfrac{1}{2}(2\delta)^r$$

and

$$|y_1| \leqslant C|a| + C|a|\tilde{R} \leqslant 2C|a|R \leqslant R \leqslant \frac{4C|a|R}{(2\delta)^r}|x_1| \leqslant |x_1|.$$

Suppose $m \geqslant 1$. Then

$$(x_1, y_1) = (2z_0 + a\alpha, a).$$

On $\mathbb{C}$ we use the metric with respect to which $|P_c'(z)| > \gamma$ on U_M. The first part can be reinterpreted with this metric as well. We have

$$|x_1| \geqslant 2|z_0|\left(1 - \frac{|a||\alpha|}{2|z_0|}\right) > \frac{\gamma + 1}{2} = \lambda \text{ if } a \text{ is small enough},$$

$$|y_1| \leqslant C|a| \leqslant C'|a||x_1|.$$

For $1 \leqslant j \leqslant m$ we have by induction since $z_{j-1} \in U_M$ and

$$(x_j, y_j) = (2z_{j-1}x_{j-1} + ay_{j-1}, ax_{j-1})$$

$$|x_j| \geqslant 2|z_{j-1}||x_{j-1}|\left(1 - \left|\frac{ay_{j-1}}{2z_{j-1}x_{j-1}}\right|\right) > \lambda|x_{j-1}|$$

$$|y_j| \leqslant C|a||x_{j-1}| \leqslant C|a||x|.$$

For (x_{m+1}, y_{m+1}) the estimate is just as in the case $m = 0$, i.e., $(x_{m+1}, y_{m+1}) = ((2z_m \ldots 2z_{m+r-1} + f_1)x_m + f_2y_m, f_3x_m + f_4y_m)$, where $|f_j| \leqslant C|a|$ for some constant C. Recall that $r \leqslant M + 2$; hence,

$$|x_{m+1}| \geqslant (2\delta)^r|x_m| - 2C|a||x_m| \geqslant \frac{1}{2}(2\delta)^r|x_m|$$

$$|y_{m+1}| \leqslant 2C|a||x_m| \leqslant \frac{4C|a|}{(2\delta)^r}|x_{m+1}|. \quad \square$$

Proof of Proposition 3.4. We assume that $a(c)$ is small enough such that, if $z_i \in U_M$ for $i \leqslant m$ and $z_{m+1}, \ldots, z_n \in D_3$, by the choice of $a(c)$ we have that $n - m \leqslant M + 2$. We have

$$(g^n)'(z_0, w_0)\begin{pmatrix}1\\ \alpha\end{pmatrix} = \begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}\begin{pmatrix}1\\ \alpha\end{pmatrix} = \begin{pmatrix}a_n & b_n\alpha\\ c_n & d_n\alpha\end{pmatrix} = \begin{pmatrix}x_n\\ y_n\end{pmatrix}.$$

We apply Lemma 3.5 to the vector (1, 0). It follows that there is a constant C_1 such that

$$|x_n| \geqslant \lambda^m \tfrac{1}{2}(2\delta)^r \geqslant C_1 \lambda^n.$$

Since for $|\alpha| \leqslant \tilde{R}$, $a_n + b_n\alpha \neq 0$,

$$|b_n| \leqslant \frac{1}{\tilde{R}}|a_n|.$$

Next, $y_n = c_n + d_n\alpha$. If $\alpha = 0$, then $|c_n| = |y_n| \leqslant C_{\tilde{R}}|a||x_n| \leqslant C_{\tilde{R}}|a||a_n|$. If $\alpha = -1$, then

$$\begin{aligned}|d_n| &\leqslant |c_n - d_n| + |c_n| = |y_n| + |c_n|\\ &\leqslant |y_n| + C_{\tilde{R}}|a||a_n| \leqslant C_{\tilde{R}}|a||x_n| + C_{\tilde{R}}|a||a_n|\\ &\leqslant C_{\tilde{R}}|a|[2|a_n| + |b_n|] \leqslant 3C_{\tilde{R}}|a||a_n|. \quad \square\end{aligned}$$

We now introduce some special neighborhoods of $K^+ \cap D$. For each n let

$$S_n = \{(z, w) \in D;\, g^n(z, w) \in D\}.$$

Then clearly, $S_n \supset S_{n+1}$ and $\bigcap_n S_n = K^+ \cap D$. The following lemma gives a description of S_n.

LEMMA 3.6. (i) *For each* n

$$S_n = \{(z, w), |w| < R, g_1^n(z, w) \in D_3\}.$$

(ii) *For each fixed* $|w| < R$, $S_{n,w} = \{z; (z, w) \in S_n\}$ *consists of* 2^n *smoothly bounded simply connected regions. Moreover, for every* $(z_0, w_0) \in S_n$ *there exists a unique holomorphic function* $z = \varphi(w)$, $|w| < R$, *with* $z_0 = \varphi(w_0)$ *and* $g_1^n(\varphi(w), w) = g_1^n(z_0, w_0)$. *There is a constant* C_3 *such that the diameter of each component of* $S_{n,w}$ *is* $\leqslant C_3/\lambda^n$.

(iii) *Each* S_n *has* 2^n *connected components,* S_{λ_n}, $\lambda_n = (i_1, \ldots, i_n)$, $i \in \{1, 2\}$, *and* $D \cap \bar{S}_{i_1 \ldots i_{n+1}} \subset S_{i_1, \ldots, i_n}$; *moreover,* $\bar{S}_{\lambda_n} \cap \bar{S}_{\lambda'_n} = \varnothing$ *if* $\lambda_n \neq \lambda'_n$.

Proof. We assume that a is sufficiently small in order that all the previous results hold. Assertion (i) follows easily from Lemma 3.3.

(ii) Let r be a defining function for D_3, i.e.,

$$D_3 = \{z; z \in \mathbb{C}, r(z) < 0\}$$

and $\nabla r \neq 0$ on ∂D_3. We have

$$\partial S_n \cap D = \{(z, w) \in D; r(g_1^n(z, w)) = 0\}.$$

Let $(a_n, b_n) = ((\partial g_1^n/\partial z), (\partial g_1^n/\partial w))$. We know that $|b_n| \leqslant (1/\tilde{R})|a_n|$; therefore,

$$\left|\frac{\partial}{\partial w} r \circ g_1^n\right| \leqslant \frac{1}{\tilde{R}}\left|\frac{\partial}{\partial z} r \circ g_1^n\right|.$$

Also, $\nabla(r \circ g_1^n) \neq 0$ on $\partial S_n \cap D$ since g^n is a biholomorphism. It follows that S_n has a real analytic boundary and is almost vertical if $\tilde{R}$ is large enough (which we can assume if a is small enough). Moreover, each S_n is foliated by complex manifolds given by $g_1^n(z, w) = c^{te}$. Each such level set is a graph over the w axis (i.e. $z = \varphi(w)$) since $|b_n| \leqslant (1/\tilde{R})|a_n|$. It follows also that $S_{n,w}$ consists of finitely many components and that the number of components is independent of w; indeed, $\{w = w_0\}$ is transverse to ∂S_n.

Let $|\beta| < R$ and $\Delta^\beta = \{(z, \beta), z \in D_3\}$. Let $\Delta_1^\beta \dots \Delta_{\ell(n)}^\beta$ be the components of $\Delta^\beta \cap S_n$. We will prove inductively that $\ell(n) = 2^n$, that $g_1^n | \Delta_i^\beta$ is biholomorphic onto D_3, and that S_n has 2^n components.

For $n = 1$ we know that S_1 has 2 components and that $\ell(1) = 2$ and $g_{1|\Delta_i^\beta}$ are biholomorphisms onto D_3. Suppose the assertion holds for n. Then $g^n(\Delta_i^\beta)$, $i \leqslant 2^n$, is a graph over D_3 given by

$$w = \varphi_i^\beta(z) = g_2^n((g_1^n)^{-1}(z, \beta), \beta).$$

Proposition 3.4 implies the existence of a constant C such that

$$\left|\frac{\partial \varphi_i^\beta}{\partial z}(z)\right| \leqslant C|a|.$$

Therefore, if a is small enough, the graph $w = \varphi_i^\beta(z)$ intersects S_1 transversally (i.e., $S_1 \cap \{w = \varphi_i^\beta(z)\}$ has two components); so the assertion holds for $n + 1$.

Since $|\partial g_1^n/\partial z| \geqslant C_1 \lambda^n$ on Δ_i^β, it follows that the diameter of Δ_i^β is smaller than C_3/λ^n, where C_3 is independent of n and of $|\beta| < R$. This proves (ii) and (iii) is immediate. □

We now prove some of the assertions of Theorem 3.1 concerning K^+. Since $K^+ \cap D = \bigcap S_n$ and each S_n is foliated by "vertical" complex manifolds and since the diameter of each component of $S_{n,w}$ goes to zero, it follows that $K^+ \cap D$ is foliated by complex manifolds and that the leaves are of the form $z = \varphi(w)$ with $|\varphi'(w)| \leqslant 1/\tilde{R}$. On $K^+ \cap D$ we have symbolic dynamics. To a leaf L in D we associate the sequence $\{j_i\}_{i=1}^{\infty}$ if $L = \bigcap_n S_{j_1 \ldots j_n}$, or equivalently, $g^n(L) \subset S_{j_{n+1}}$. Two leaves L, L' in D are in the same global leaf if and only if $j_i \neq j_i'$ for only finitely many i's. The action of g on the space of leaves is conjugate to the shift operator on the sequences $\{j_i\}$. It follows that each leaf is dense in K^+.

We know that, for $|\beta| < R$, S_n^β has 2^n components $S_{\lambda_n}^\beta$, $|\lambda_n| = n$, and that $\operatorname{diam}(S_{\lambda_n}^\beta) \leqslant C/\lambda^n$. Therefore, $K_\beta^+ = K^+ \cap \{w = \beta\}$ has a Hausdorff dimension less than $\log 2/\log \lambda$. Since λ is arbitrarily large when c is large, it follows that, for a fixed, $\lim_{c\to\infty} h_{a,c}(K_+) = 0$. The fact that $h_{a,c}(K^+) > 2$ is the content of Corollary 1.4.

We now study K^-.

LEMMA 3.7. *There is $a(c) > 0$ such that, if $|a| < a(c)$, the following holds. If $(z, w) \in K^-$, there exists n_0 such that, for $n \geqslant n_0$, $g^{-n}(z, w) \in D$.*

Proof. Recall that $g^{-1}(z, w) = ((w/a), (z/a) - (c/a) - (w^2/a^3))$. Suppose first that $|w| \geqslant R$ and $|z| \leqslant |w|$. Let $(z', w') = g^{-1}(z, w)$. We have

$$|w'| \geqslant \frac{1}{|a|}\left(\left|\frac{w}{a}\right|^2 - |c| - |z|\right) \geqslant \frac{1}{|a|}\left(\left|\frac{w}{a}\right|^2 - R - |w|\right) \geqslant \frac{|w|^2}{2|a|^3} \geqslant 2|w|$$

if a is small enough, and

$$|z'| = \left|\frac{w}{a}\right| \leqslant \frac{|w|^2}{2|a|^3} \leqslant |w'|.$$

So inductively, $g^{-n}(z, w) \to \infty$; hence,

$$K^- \cap \{(z, w); |w| \geqslant R \text{ and } |z| \leqslant |w|\} = \varnothing.$$

Let $\Lambda = \{(z, w); |z| \geqslant R \text{ and } |w| \leqslant |z|\} \cup \Delta^2(0, R)\backslash D$. If $(z, w) \in \Lambda$, we have seen in Lemma 3.3 that $g^n(z, w) \to \infty$ uniformly on Λ; therefore, if $(z, w) \in K^-$, necessarily there is an n_0 such that $g^{-n_0}(z, w) \in D$. So we can assume $(z, w) \in D$. If $(z', w') = g^{-1}(z, w)$, then necessarily $z' \in D_3$, and if $|w'| \geqslant R$, then $|z'| < |w'|$ and $g^{-n}(z', w') \to \infty$. So $(z', w') \in D$. $\square$

LEMMA 3.8. *Let $T_n = g^n(S_n)$ and $T_\lambda = g^n(S_\lambda)$ for $|\lambda| = n$. Then $T_{j_1 \ldots j_{n+1}} \subset T_{j_2 \ldots j_{n+1}}$ and $\partial T_{n+1} \cap \partial T_n \subset \partial D_3 \times \{|w| < R\}$. We also have $K^- \cap D = \bigcap_n T_n$.*

Proof. We have

$$(g^n)^{-1}(T_{j_1 \ldots j_{n+1}}) = g(S_{j_1 \ldots j_{n+1}}) \subset S_{j_2 \ldots j_{n+1}} = (g^n)^{-1}(T_{j_2 \ldots j_{n+1}}).$$

Therefore, $T_{j_1 \ldots j_{n+1}} \subset T_{j_2 \ldots j_{n+1}}$, and consequently $T_{n+1} \subset T_n$. Assume $(z, w) \in \partial T_{n+1} \cap$

$\{(z, w); z \in D_3\}$. Then $(z', w') = g^{-(n+1)}(z, w)$ satisfies $|w'| = R$; hence, $(z'', w'') = g(z', w') = g^{-n}(z, w)$ satisfies $|w''| < R$, and then $(z, w) \in T_n$.

Suppose $(z, w) \in K^- \cap D$. Lemma 3.7 implies that, for $n \geqslant 1, g^{-n}(z, w) \in D$. Hence, $g^{-n}(z, w) \in S_n$, which means that for all $n \geqslant 1, (z, w) \in T_n$. Assume that $(z, w) \in \bigcap T_n$. Then, for all n, $g^{-n}(z, w) \in S_n \subset D$; so $(g^{-n}(z, w))$ is bounded. Hence, $(z, w) \in K^- \cap D$. $\square$

For each λ, S_λ is foliated by the horizontal sections $S_\lambda \cap (w = w_0)$. The image under g^n of each such leaf is a graph of the form $w = \varphi(z)$. This was shown in the proof of Lemma 3.6. To prove that the limit of these foliations is a foliation of $K^- \cap D$, we only need to show that for fixed α the diameter of $T_\lambda \cap \{z = \alpha\}$ approaches zero when $|\lambda| \to \infty$.

Consider the foliation of $S_\lambda \cap D$ and let $L = \{(z, w), z = \varphi(w), |w| < R\}$ be a leaf. The tangent to L is almost vertical if $\tilde{R}$ is large enough. We also know that $g^n(L)$ is vertical and that $g^k(L)$, $1 \leqslant k \leqslant n$, is almost vertical (Lemma 3.5). But

$$g'(z, w).(\varphi'(w), 1) = \begin{bmatrix} 2z & a \\ a & 0 \end{bmatrix} \begin{pmatrix} \varphi'(w) \\ 1 \end{pmatrix} = \begin{pmatrix} 2z\varphi'(w) + a \\ a\varphi'(w) \end{pmatrix}.$$

In particular, $|2z\varphi'(w) + a| \leqslant |a\varphi'(w)|$ since the vector is close to vertical and the second component shrinks by the factor $|a\varphi'(w)|$. Similarly, the vertical vector $g^{n'}(\varphi(w), w)(\varphi'(w), 1)$ has length smaller than $C|a|^n$, where C is a fixed constant. So the diameter of $T_\lambda \cap (z = \alpha)$ is at most $C|a|^n$. It follows that the Hausdorff dimension of $(D \cap K^-) \cap (z = \beta) = K_\beta^-$ is less than $\log 2/\log(1/|a|)$, and we know it is positive because the function G^- is Hölder continuous and has a nonremovable singularity on K_β^-.

Using the same symbolic dynamics argument as for K^+, we can show that each leaf is dense in K^-.

We prove that each global leaf is biholomorphic to $\mathbb{C}$. Suppose L is a leaf of K^+. If γ is a closed curve in L, $g^n(\gamma)$ can be contracted in $g^n(L) \cap D$ for large n. Hence, L is biholomorphic to the unit disc or $\mathbb{C}$. Assume there is a nonconstant holomorphic function on L, $|h| < 1$, $h(p) = 0$, $h(q) = \frac{1}{2}$, for some $p, q \in L$. Pulling the function forward to $g^n(L) \cap D$ and observing that $\|g^n(p) - g^n(q)\| \to 0$ and $|g_2^n(p)| \leqslant |a| R$, we obtain a contradiction to the Schwarz lemma. So L is biholomorphic to $\mathbb{C}$. The argument for K^- is quite similar.

To complete the proof of Theorem 3.1 we only need to prove that g is hyperbolic on $K = K^+ \cap K^-$.

For $q \in K$ let E_u be the tangent plane to the leaf in K^- through q and let E_s be the tangent plane of the leaf in K^+ through q. Since $K \subset D$, then E_u and E_s meet transversally; so

$$T\mathbb{C}^2 = E_s \oplus E_u,$$

and $g'(q)E_s^q = E_s^{g(q)}$, $g'(q)E_u^q = E_u^{g(q)}$, with obvious notations. The leaves of $K^+ \cap D$

are almost vertical, and we have seen that

$$|(g^n)'|_{E_s}| \leqslant C|a|^n, \qquad n \geqslant 1.$$

For E_u we use the estimate of Proposition 3.4, namely,

$$(g^n)'(z, w) = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}.$$

Now a_n dominates the other terms and $|a_n| \geqslant C\lambda^n$ with $\lambda > 1$. Therefore,

$$|(g^n)'|_{E_u}| \geqslant C'\lambda^n$$

for some positive constant C'.

It is also clear that the subbundles E_u and E_s vary continuously since the foliations on K^+ and K^- are uniform limits of smooth foliations on S_n and T_n, respectively.

Remarks. 1. Suppose P is a polynomial in $\mathbb{C}$ such that all critical points of P are in the basin of attraction of ∞. Then P is a hyperbolic polynomial, and $J(P)$ is a Cantor set. We can show that there is a constant $a_0(P) > 0$ such that for $|a| \leqslant a_0(P)$ the Hénon map

$$g(z, w) = (P(z) + aw, az)$$

is hyperbolic on $K = K^+ \cap K^-$ and such that K^+ and K^- are foliated by dense leaves which are biholomorphic to $\mathbb{C}$.

2. In [BS1] Bedford and Smillie show that, when g is hyperbolic on K, then J^+ and J^- have foliations by dense leaves biholomorphic to $\mathbb{C}$.

We consider now the case where the Hénon mapping has an attractive cycle in $\mathbb{C}^2$.

Let ω be an open set in $\mathbb{C}$. We will assume in what follows, that for every $c \in \omega$ the polynomial $P_c(z) = z^2 + c$ has an attractive cycle of order k: for example, if $|c| < \frac{1}{4}$, $P_c(z)$ has an attractive fixed point, i.e., $k = 1$. Also, if $|c - 1| < \frac{1}{4}$, $P_c(z)$ has an attractive cycle of order 2.

It is a result due to Fatou (see [Do]) that in this case P_c has no other attractive or indifferent cycle. Since the critical point is in the immediate basin of attraction of the cycle, it follows that P_c is hyperbolic on $J(c)$. Choose a Riemannian metric on $\mathbb{C}$ such that there is a constant $\gamma > 1$ with the property that

$$|P_c'(z)| > \gamma$$

if z belongs to a neighborhood U of the Julia set $J(c)$ of P_c. Let $U_1, \dots, U_k$ be the immediate basins of attraction of $z_1, \dots, z_k$. Without loss of generality we can assume that $0 \in U_1$. Let K_c be the filled-in Julia set of P_c; i.e., $z \in K_c$ if and only if $P_c^n(z)$ is a bounded sequence. It is a result due to Sullivan [Su] that any other component V

of $K_c \setminus J(c)$ is preperiodic to $\{U_1, \ldots, U_k\}$; i.e., for some $m \geqslant 1$ and j, $1 \leqslant j \leqslant k$

$$P_c^m\colon V \to U_j$$

is a biholomorphic map.

We first fix some notations for the dynamics of P_c. Choose simple closed real analytic curves $\gamma_0, \ldots, \gamma_k, \gamma_k'$ in U with the following properties.

(i) γ_0 bounds a domain $D_0 \supset K_c$, and we can assume $P_c^{-1}(D_0) \subset\subset D_0$.

(ii) γ_j bounds a domain $D_j \subset\subset U_j$ for $1 \leqslant j \leqslant k$ and for $1 \leqslant j < k$, $\gamma_j \cap P_c^{-1}(U_{j+1} \setminus D_{j+1}) = \varnothing$, $0 \in D_1$.

(iii) γ_k' bounds a domain D_k' such that $D_k \subset\subset D_k' \subset U_k$ and $\gamma_k' \cap P_c^{-1}(U_1 \setminus D_1) = \varnothing$.

The following theorem gives a description of K^+ and K^- for the Hénon maps associated to the polynomial P_c.

THEOREM 3.9. *Suppose $c \in \omega$ and let*

$$g(z, w) = (z^2 + c + aw, az).$$

Fix $R \gg 1$. There exists a positive constant $a_0(c, R)$ such that for $0 < |a| < a_0(c, R)$ the following properties hold.

(i) *g has an attractive cycle of order k, $\{p_1, \ldots, p_k\}$, and the interior of K^+ consists of k connected components, each of which is the immediate basin of attraction of one of $p_1, \ldots, p_k$.*

(ii) *J^+ is foliated by complex manifolds, biholomorphic to $\mathbb{C}$, which are dense in J^+. The k basins have the same boundary J^+.*

(iii) *If $|\beta| \leqslant R$, then $K_\beta^+ = K^+ \cap \{w = \beta\}$ is a connected compact set. If $k = 1$, then $\partial(K_\beta^+)$ is of Lebesgue measure zero, and consequently, J^+ is of measure zero in $\mathbb{C}^2$.*

(iv) *$K^- \setminus \{p_1, \ldots, p_k\}$ is also foliated by complex manifolds, biholomorphic to $\mathbb{C}$, and leaves are dense in K^-. All leaves cluster at each p_j, but no leaf has an extension as a complex variety through any p_j.*

(v) *The Hausdorff dimension of K^- satisfies the inequalities*

$$2 < h_{a,c}(K^-) \leqslant 2 + \frac{\log 2}{\log 1/|a|}.$$

We will decompose the proof in a series of lemmas. We fix $R > 1$ such that $K_c \subset\subset D(0, R)$. We will have various restrictions on a of the type $|a| < a_0(c, R)$, and we will always consider that the restrictions on a introduced in previous lemmas are satisfied.

LEMMA 3.10. *There exists $a_0 > 0$ such that for $0 < |a| < a_0$ the map*

$$g(z, w) = (z^2 + c + aw, az)$$

has an attractive cycle of order k, $\{p_1, \dots, p_k\}$, and $D_j \times \{|w| < R\}$ is contained in the immediate basin of p_j.

Proof. We have

$$g^k(z, w) = (P_c^k(z) + P(z, w), Q(z, w)) = G(z, w, a, c)$$

where all the coefficients of P and Q contain positive powers of a. The rank of the derivative of $G(z, w, a, c) - (z, w)$ at $z = z_1$, $w = 0$, $a = 0$ is two. So by the implicit function theorem there are two holomorphic functions $Z(a)$, $W(a)$ defined for $|a| < a_0$ such that

$$G(Z(a), W(a), a, c) = (Z(a), W(a)).$$

We denote, for $1 \leqslant j \leqslant k$, $p_j = g^{j-1}(Z(a), W(a))$. We can assume that, for $|a| < a_0$, $|G'(z, w, a, c)| < 1$ on a polydisc $\Delta^2((Z(0), W(0)); \delta)$. If $(z, w) \in \bar{D} \times \{|w| < R\}$ and if a is small enough, then $G^\ell(z, w, a, c) \in \Delta^2(Z(0), W(0), \delta)$ for a fixed ℓ large enough, and therefore $G^n(z, w, a, c)$ converges to $(Z(a), W(a))$. The same applies on $D_j \times \{|w| < R\}, j = 2, \dots, k$.

LEMMA 3.11. *Let $(z, w) \in K^+$. Then there exists n_0 such that, for $n \geqslant n_0$, $g^n((z, w)) \in \bar{D}_0 \times \{|w| \leqslant R\} =: \bar{D}$. If $(z, w) \in K^+ \cap \bar{D}$, then $g(z, w) \in D$. If $(z, w) \in K^-$, there exists n_0 such that, for $n \geqslant n_0$, $g^{-n}(z, w) \in D$.*

The proof is very similar to Lemma 3.3 and Lemma 3.7, and we omit it.

Let $\mathscr{D} := D_0 \backslash \bigcup_{i=1}^k \bar{D}_i$. We want to estimate the derivative $(g^n)'(z_0, w_0)$ if the orbit $(z_i, w_i) = g(z_{i-1}, w_{i-1})$, $1 \leqslant i \leqslant n$, stays in

$$\mathscr{U} := \mathscr{D} \times \{|w| < R\}.$$

LEMMA 3.12. *Assume $(z_i, w_i) = g(z_{i-1}, w_{i-1})$, $0 \leqslant 1 \leqslant n$, is in $\mathscr{U}$. Given $\tilde{R} \gg 1$, let $(1, \alpha) \in \mathbb{C}^2$ with $|\alpha| \leqslant \tilde{R}$. Denote $(x_j, y_j) = (g^j)'(z_0, w_0)(1, \alpha)$, $1 \leqslant j \leqslant n$. There exists $a_0(c, \tilde{R}) > 0$ and a constant $C_{\tilde{R}} > 0$, both independent of n, such that, if $|a| < a_0(c, \tilde{R})$, then*

(i) $|y_j| < 2C_{\tilde{R}}|a||x_j| < |x_j|$, *and*
(ii) $|x_j| > (1/C_{\tilde{R}})\lambda^j$, $\lambda = (\gamma + 1)/2$.

The proof is like the proof of Lemma 3.5, and we omit it.

As a consequence, we have the following estimate for $(g^n)'$.

PROPOSITION 3.13. *Fix $\tilde{R} \gg 1$. There exists $a_0(c, \tilde{R}) > 0$ and positive constants $C_1, C_2, \lambda > 1$ such that, if (z_j, w_j), $0 \leqslant j \leqslant n$ is an orbit in $\mathscr{U}$ and*

$$(g^n)'(z_0, w_0) = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix},$$

then $|a_n| \geqslant C_1\lambda^n$, $|b_n| \leqslant (1/\tilde{R})|a_n|$, $|c_n| \leqslant C_2|a||a_n|$, $|d_n| \leqslant C_2|a||a_n|$.

Now we define

$$\mathscr{U}_n = \{(z, w) \in \mathscr{U};\, g^n(z, w) \in \mathscr{U}\}.$$

The following lemma can be deduced from the estimate on $(g^n)'$ in the previous proposition.

LEMMA 3.14. *Each $\mathscr{U}_n$ has a real analytic boundary in $\{|w| < R\}$, $\partial\mathscr{U}_n$ is almost vertical, and $\mathscr{U}_n$ is foliated by almost vertical complex discs of the form $z = \varphi(w)$, $|w| < R$, such that $g_1^n(\varphi(w), w)$ is constant for $|w| < R$.*

We remark that, if we choose $\tilde{R}$ very large, then $a(c, \tilde{R})$ is very small, and then the leaves $z = \varphi(w)$ are close to vertical; i.e., $|\varphi'| \ll 1$.

LEMMA 3.15. *Fix $|\beta| < R$. Fix*

$$\mathscr{U}_{n,\beta} = \{z;\, (z, \beta) \in \mathscr{U}_n\}.$$

Then $\mathscr{U}_{n,\beta}$ is a connected domain with smooth real analytic boundary. The number of holes in $\mathscr{U}_{n,\beta}$ is independent of β and is equal to $k + (2^n - 1)(k - 1)$. They are hierarchically ordered according to their bounding curves

$$\{\gamma_{0,1}^n, \ldots, \gamma_{0,k}^n\}, \{\gamma_{1,1}^n, \ldots, \gamma_{1,k-1}^n\}, \{\gamma_{2,1}^n, \ldots, \gamma_{2,2(k-1)}^n\}, \ldots, \{\gamma_{n,1}^n, \ldots, \gamma_{n,2^{n-1}(k-1)}^n\}$$

and with outer boundary γ^n. Furthermore, $\gamma_{i,j}^{n+1}$ always surrounds $\gamma_{i,j}^n$ for $i \leqslant n$ while γ^n surrounds γ^{n+1}.

Proof. The proof is by induction on n. Observe first that $\mathscr{U}_0 = \mathscr{U}$ has vertical boundary in $(\mathbb{C} \times |w| < R)$ and that $\mathscr{U}_{0,\beta}$ has k holes for every $|\beta| < R$. The lemma is clear if we let $\beta = 0$ and $a = 0$. Recall that we have assumed that 0, the critical point of $P_c(z)$, belongs to $D_1 \subset U_1$. The choice of $D_1, \ldots, D_k$ was such that $P_c^{-1}(\mathscr{D})$ contains holes slightly larger than $D_1, \ldots, D_k$. Also, $P_c^{-1}(\mathscr{D})$ has $(k - 1)$ new holes of $K_c \backslash J(c)$ since P_c is 2 to 1. Hence, if a is small enough, $\mathscr{U}_1 \cap \{w = \beta\}$ is also a simply connected domain minus holes slightly larger than $D_1, \ldots, D_k$ and $(k - 1)$ new holes.

Assume the lemma is true for n. Let $|\beta| < R$. The map g^n maps $\mathscr{U}_{n,\beta}$ properly to $\mathscr{D} \times \{|w| < R\}$ and $g_1^n \partial\mathscr{U}_{n,\beta} \subset \partial\mathscr{D}$. So $g_1^n\colon \mathscr{U}_{n,\beta} \to \mathscr{D}$ is a proper holomorphic map which is an unbranched covering of degree 2^n. The estimate of the derivative of g^n in Proposition 3.13 implies that $g^n(U_{n,\beta})$ cuts $\partial\mathscr{U}_1$ transversally; hence, for $U_{n+1,\beta}$ we obtain $2^n(k - 1)$ extra holes from the $(k - 1)$ extra holes of $\mathscr{U}_1$. We list the boundary curves of these new holes as $\{\gamma_{n+1,1}^{n+1}, \ldots, \gamma_{n+1,2^n(k-1)}^{n+1}\}$. $\square$

LEMMA 3.16. *For $|\beta| < R$, $\bigcap \mathscr{U}_{n,\beta} = \bigcap_{n \geqslant 1} \overline{\bigcup_{m \geqslant n} \gamma^m} = J_\beta^+$ is connected and has empty interior.*

Proof. Recall that there exists $\lambda > 1$ such that $|(g_1^n)'| > C\lambda^n$ on $\mathscr{U}_n$ and that g_1^n: $\mathscr{U}_{n,\beta} \to \mathscr{D}$ is an unbranched cover of degree 2^n. Let $(z_0, \beta) \in \mathscr{U}_{n,\beta}$. Let σ be a curve of length ℓ from $g_1^n(z_0, \beta)$ to the outer boundary of $\mathscr{D}$. Let $\tilde{\sigma}$ be the pullback of σ

from z_0 to the exterior boundary γ^n of $\mathscr{U}_{n,\beta}$. The length of $\tilde{\sigma}$ in the z-direction is smaller than $\ell/C\lambda^n$. Therefore, $\bigcap \mathscr{U}_{n,\beta}$ has empty interior and is connected. The other assertions are clear. □

LEMMA 3.17. *Through every point in $J^+ \cap \{|w| < R\} = \bigcap_n \mathscr{U}_n$, there is a unique leaf contained in J^+ of the form $z = \varphi(w)$, $|w| < R$, with φ holomorphic. Moreover, $|\varphi'(w)|$ is arbitrarily small provided a is small enough.*

Proof. Since $\bigcap \mathscr{U}_{n,\beta}$ has empty interior, it follows from Hurwitz's lemma that the limit of the foliations of $\partial\mathscr{U}_n$ is a foliation of $\bigcap_n \mathscr{U}_n$. The other assertions are clear from the previous discussion. □

We now prove that the interior of K^+ consists of k connected components, each of which is the immediate basin of attraction of one of the points $\{p_1, \ldots, p_k\}$ in the attractive cycle. Let $(z, w) \in K^+$. By Lemma 3.11 there exists n_0 such that, for $n \geqslant n_0$, $g^n(z, w) \in D$. If $g^n(z, w)$ is never in $\bigcup_{j=1}^k D_j \times \{|w| < R\}$, then $g^{n_0}(z, w) \in \mathscr{U}_n$ and consequently $g^{n_0}(z, w) \in J^+ \cap \{|w| < R\}$ and $(z, w) \in J^+$. In particular, we have shown that the only attractive cycle for g is $\{p_1, \ldots, p_k\}$. This proves part (i) of Theorem 3.9.

Lemma 3.17 implies that J^+ if foliated by complex manifolds. To prove density of leaves in J^+ we need the following lemma.

LEMMA 3.18. *Let $w = \psi(z)$ be a germ of a complex manifold at $(z_0, w_0) \in J^+$. Suppose $|\psi(z)| < R$ and that $|\psi'(z)| < 1$. Then there exists an n such that $g^n[z, \psi(z)] \cap (\mathscr{D} \times \{|w| < R\})$ is a locally horizontal manifold intersecting all leaves of $J^+ \cap (|w| < R)$.*

Proof. That the manifolds $g^n[z, \psi(z)]$ are locally horizontal follows from the fact that $|(\partial/\partial z)g_1^n|$ dominates the other derivatives as shown in Proposition 3.13.

Let Γ be the family of graphs $w = \psi(z)$ defined for $|z - z_0| < \varepsilon$ and satisfying the requirements in the lemma. Since $|(\partial/\partial z)g_1^n| \geqslant c\lambda^n$, it follows that there exists $r > 0$ independent of ε such that, for every graph in $\Gamma(z, \psi(z))$, $|z - z_0| < \varepsilon$, there is an N such that $g^N[z, \psi(z)] \supset [z, \tilde{\psi}(z)]$, where $\tilde{\psi}$ is defined in $|z - z_0'| < r$, $(z_0', \tilde{\psi}(z_0')) \in J^+$. For the polynomial $P_c(z) = z^2 + c$ there exists n_0 such that, if $z_0 \in J(c)$ and $\Delta = \{|z - z_0| \leqslant r/2\}$, then $P_c^{n_0}(\Delta)$ contains a fixed open neighborhood of $J(c)$. Consequently, if a is small enough, $g^{n_0}[g^N[z, \psi(z)]]$ meets all leaves in $J^+ \cap \{|w| < R\}$. □

To show that the global leaves of J^+ are dense in J^+, it suffices to prove density in $J^+ \cap (D_0 \times \{|w| < R\})$. Let L be a leaf in J^+ and let W be a neighborhood of $p = (z_0, w_0) \in J^+ \cap (D_0 \times \{|w| < R\})$. Let $\varepsilon > 0$ such that the disc $\Delta_1 =: \{|z - z_0| < \varepsilon, w = w_0\}$ is contained in W. Suppose $z = \varphi(w)$, $|w| < R$, is contained in $L \cap (D_0 \times \{|w| < R\})$. By the above lemma there exists an n such that $g^n(\Delta_1)$ cuts all leaves in $D_0 \times \{|w| \times R\}$. In particular, it intersects the leaf containing $g^n(\varphi(w), w)$, $|w| < R$. Hence, one of the leaves intersecting Δ_1 lies globally in the same leaf as L, and therefore L clusters at p. The proof that L is biholomorphic to $\mathbb{C}$ is the same as in the case considered in Theorem 3.1.

Let $(z_0, w_0) \in J^+$ and choose $j = 1, \dots, k$. We will show that (z_0, w_0) is a boundary point of the basin of attraction of p_j. For n_0 large, $g^{n_0 k}(z_0, w_0) = (z_1, \beta) \in \mathscr{U}$. As in Lemma 3.16, we can find a curve $\tilde{\sigma}$ of length $O(1/\lambda^{nk})$ in $\mathscr{U}_{nk,\beta}$ from (z_1, β) to the basin of attraction of p_j. Here, $\lambda > 1$ is the constant from Proposition 3.13. Hence, $g^{-n_0 k}(\tilde{\sigma})$ is a curve of length $O(1/\lambda^{nk})$ from (z_0, w_0) to the basin of attraction of p_j.

This completes the proof of part (ii) of Theorem 3.9.

We suppose now that $k = 1$, and we prove that the Lebesgue measure of J_+ is zero.

LEMMA 3.19. *Let $P_c(z) = z^2 + c$. For all $|c|$ small enough there is a constant $\gamma > \sqrt{2}$ and an open set $U \supset J(c)$ such that $|P_c'(z)| > \gamma$ on U, where $|\ |$ denotes the Euclidean norm.*

The proof is left to the reader.

For $|\beta| < R$ we have seen that $z \to g_1^n(z, \beta)$ is a covering of $\mathscr{U}_{n,\beta}$ onto $\mathscr{D}$ of degree 2^n. To emphasize the dependence on a and c, we denote by $g(z, w, a, c)$ the Hénon mapping we consider. We have

$$\sigma(\mathscr{U}_{n,\beta}(a, c)) = \int_{\mathscr{D}} \sum_{i=1}^{2^n} \frac{1}{\left|\dfrac{\partial}{\partial z} g_1^n(z_i(z), \beta, a, c)\right|^2} \, d\sigma(z)$$

where σ is the Lebesgue measure on $\mathbb{C}$ and $g_1^n(z_i(z), \beta, a, c) = z$ for $1 \leqslant i \leqslant 2^n$. The domain $\mathscr{D}$ is independent of (a, c) for (a, c) in a neighborhood of (a_0, c_0).

Observe that $\log \sigma(\mathscr{U}_{n,\beta}(a, c))$ is a plurisubharmonic function of (β, a, c). Consequently, $\log \sigma(J_\beta^+(a, c)) = \lim \searrow \log \sigma(\mathscr{U}_{n,\beta}(a, c))$ is also a plurisubharmonic function.

On the other hand, for $|c| \ll 1$, $|a| \ll 1$ and $|\beta| \leqslant R$ we have

$$\sigma(\mathscr{U}_{n,\beta}(a, c)) \leqslant 2^n \frac{1}{C\lambda^{2n}}$$

where λ is a constant such that $|(\partial/\partial z) g_1^n(z, w, a, c)| > C\lambda^n$ in Proposition 3.13. But if a is small enough, λ can be chosen arbitrarily close to γ; so we can assume $\lambda > \sqrt{2}$. Consequently, $\sigma(J_\beta^+(a, c)) = 0$ for $|a|$, $|c|$ small enough and $|\beta| < R$. But since $\log \sigma(J_\beta(a, c))$ is plurisubharmonic for $|a| < a_0(c)$ provided, the polynomial $z^2 + c$ has one attractive fixed point; we get that $\sigma(J_\beta^+(a, c))$ is identically zero.

Together with Lemma 3.16, this completes the proof of part (iii) of Theorem 3.9.

We prove assertion (iv). We first show that $K^- \cap \mathscr{U}$ is foliated by complex manifolds.

LEMMA 3.20. *Let $(z, w) \in K^-$. There exists n_0 such that, for $n \geqslant n_0$, $g^{-n}(z, w) \in D_0 \times \{|w| < R\} = D$.*

Proof. Recall that $g^{-1}(z, w) = (w/a, z/a - c/a - w^2/a^3)$. Suppose first that $|w| \geqslant R$ and $|z| \leqslant |w|$. Then if $(z', w') = g^{-1}(z, w)$, we have $|w'| \geqslant 2|w|$ and $|z'| \leqslant |w'|$.

(Recall that a is small.) So $g^{-n}(z, w)$ converges to infinity; i.e., K^- does not intersect $|w| \geqslant R$, $|z| \leqslant |w|$.

Let $\Lambda = (\{(z, w); |z| \leqslant R, |w| < R\} \backslash D) \cup \{|z| \geqslant R, |w| \leqslant |z|\}$. We have seen that, on Λ, g^n converges uniformly to ∞. Suppose $(z, w) \in K^-$ and infinitely many of the $g^{-n}(z, w)$ belong to Λ. This contradicts convergence to infinity of g^n on Λ.

LEMMA 3.21. *If* $(z, w) \in K^- \backslash \{p_1, \ldots, p_k\}$, *then* $g^{-n}(z, w)$ *clusters on* $J^+ \cap \{|w| \leqslant |a| R\}$. *Consequently, for n large enough*, $g^{-n}(z, w) \in \mathscr{U} = \mathscr{D} \times \{|w| < R\}$.

Proof. Suppose at first that $(z, w) \notin K^+$. By Lemma 3.20, $g^{-n}(z, w) \in D$ for n large enough. But then it follows even that, for each m, $g^{-n}(z, w) \in \mathscr{U}_m$ for n large enough. Hence, all the cluster points of the sequence are in $J^+ \cap \{|w| \leqslant R\}$. Since the set of cluster points is invariant under g, it follows that they are in $J^+ \cap \{|w| \leqslant |a| R\}$.

If $(z, w) \in J^+$, there is nothing to prove; so we can assume $(z, w) \in \operatorname{Int} K^+ \backslash \{p_1, \ldots, p_k\}$. Since $\operatorname{int} K^+$ is the basin of attraction of the cycle, then $g^{-n}(z, w)$ clusters on J^+. In all cases, for n large, $g^{-n}(z, w) \in \mathscr{U}$.

We will prove that $K^- \cap \mathscr{U}$ is foliated. This will imply that $K^- \backslash \{p_1, \ldots, p_k\}$ is foliated by complex manifolds. Observe however that there is no analytic disc through p_j *in* K^-. Suppose $\Phi\colon \Delta \to K^-$ is a nonconstant analytic map from the unit disc with values in K^- and such that $\Phi(0) = p_j$. Then $g^{-n} \circ \Phi$ is a normal family on Δ, a limit function h will satisfy $h(0) = p_j$, and, except on $\Phi^{-1}(p_j)$, h should have values on J^+, which is impossible.

Recall that $\mathscr{U}_n = \{(z, w) \in \mathscr{U}; g^n(z, w) \in \mathscr{U}\}$. Define $\mathscr{V}_n = g^n(\mathscr{U}_n)$.

LEMMA 3.22. *We have* $\bar{\mathscr{V}}_{n+1} \cap \mathscr{U} \subset \mathscr{V}_n$ *and* $\bigcap_n \mathscr{V}_n = K^- \cap \mathscr{U}$.

Proof. If $(z, w) \in \mathscr{V}_{n+1} \cap \mathscr{U}$, then $g^{-(n+1)}(z, w) \in \mathscr{U}_{n+1}$ and $g^{-n}(z, w) \in g(\mathscr{U}_{n+1}) \subset \mathscr{U}_n$; hence, $(z, w) \in \mathscr{V}_n$.

For $(z, w) \in \partial \mathscr{V}_{n+1} \cap (\mathscr{D} \times \{|w| < R\})$ let $(z'', w'') = g^{-(n+1)}(z, w)$. We have $(z'', w'') \in \partial \mathscr{U}_{n+1}$. Let $(z', w') = g(z'', w'')$. Clearly, $|w'| < R$ and $g^n(z', w') = (z, w) \in \mathscr{U}$. Hence, $(z', w') \in \mathscr{U}_n$ and $(z, w) \in \mathscr{V}_n$.

If $(z, w) \in \bigcap \mathscr{V}_n$, then, for every n, $g^{-n}(z, w) \in \mathscr{U}$; so $(z, w) \in K^-$. Suppose next that $(z, w) \in K^- \cap \mathscr{U}$. We need to show that for every n, $(z, w) \in \mathscr{V}_n$. Lemma 3.21 implies that $g^{-n}(z, w) \in \mathscr{U}$ for all large enough n. Therefore, $g^{-n}(z, w) \in \mathscr{U}_n$ and $(z, w) \in \mathscr{V}_n$ for large enough n. Since $\mathscr{V}_{n+1} \subset \mathscr{V}_n$, we have $(z, w) \in \bigcap \mathscr{V}_n$. □

The description of the dynamics of g^{-1} on $K^- \cap \mathscr{U}$ is contained in the following lemma.

LEMMA 3.23. *For each n*, $\mathscr{V}_n$ *has a real analytic boundary relative to* $\mathscr{U}$, *whose normal is almost vertical everywhere. Moreover, for each* $z \in \mathscr{D}$ *the z-section* $\mathscr{V}_{n,z}$ *of* $\mathscr{V}_n$ *consists of* 2^n *components, each of which is simply connected with a real analytic boundary. Furthermore, each component of* $\mathscr{V}_{n,z}$ *contains exactly two components of* $\mathscr{V}_{n+1,z}$. *Moreover, we have* $\operatorname{diam}(\mathscr{V}_{n,z}) \leqslant C|a|^n$, *where C is a constant.*

Proof. Using the horizontal foliation of $\mathscr{U}_n$ by level sets $\omega = \beta$ and taking the image under g^n, we obtain a foliation of $\mathscr{V}_n$. We have shown in the proof of Lemma

3.15 that the image of $\mathscr{U}_{n,\beta}$ under g^n is an unbranched covering over $\mathscr{D}$ of degree 2^n. The estimate on the derivative of g^n implies that locally each such leaf is of the form $w = \varphi(z)$. We have that $\partial\mathscr{V}_n \cap \mathscr{U}$ is foliated by $g^n(\mathscr{U}_{n,\beta})$ with $|\beta| = R$. These manifolds are almost horizontal. Therefore, the normal is almost vertical.

Fix $z^0 \in \mathscr{D}$ and $|\beta| < R$. Let $z_1, \ldots, z_{2^n}$ be the 2^n points in $\mathscr{U}_{n,\beta}$ such that $g_1^n(z_i, \beta) = z^0$, $i = 1, \ldots, 2^n$. Let $z = \varphi_i(w)$ be the almost vertical leaf of $\mathscr{U}_n$ such that $\varphi_i(\beta) = z_i$. By construction of the foliation of $\mathscr{U}_n$ we have $g_1^n(\varphi_i(w), w) = z^0$ for $i = 1, \ldots, 2^n$. These graphs $(\varphi_i(w), w)$ are the preimage of $\mathscr{V}_{n,z^0}$ since g_1^n is of degree 2^n and since the graphs are disjoint. Since g is biholomorphic, $g_2^n(\varphi_i(w), w)$, $|w| < R$, are the 2^n components of $\mathscr{V}_{n,z_0}$. Observe that, as after Lemma 3.8, g contracts vertical vectors by a factor almost $|a|$; hence, the diameter of a component of $\mathscr{V}_{n,z}$ is of order of magnitude almost $|a|^n$.

It remains only to show that each component of $\mathscr{V}_{n,z}$ contains exactly two components of $\mathscr{V}_{n+1,z}$. For fixed $z^0 \in \mathscr{D}$ let $z = \varphi_{n,i}(w)$, $i = 1, \ldots, 2^n$ be the preimages of $\mathscr{V}_{n,z^0}$. Since $g_1^n[g[\varphi_{n+1,i}(w), w)]] = z^0$, it follows that $g(\varphi_{n+1,i}(w), w)$ is contained in one of the graphs $(\varphi_{n,j}(w), w)$. We have to prove that for every (n, j) there are at most two $(n+1, i)$ for which the inclusion holds.

We study $g^{-1}(\varphi_{n,j}(w), w)$. We have

$$g^{-1}(\varphi_{n,j}(w), w) = \left(\frac{w}{a}, \frac{\varphi_{n,j}(w)}{a} - \frac{c}{a} - \frac{w^2}{a^3}\right).$$

We want to show that there are at most two values of w, $|w| < R$, such that the second coordinate is equal to w_0 with $|w_0| < R$. But this is an immediate consequence of Rouché's theorem. □

We prove that $K^- \cap \mathscr{U}$ is foliated by complex manifolds. Let $Z_n^\beta = g^n(\{z, \beta\})$ with $(z, \beta) \in \mathscr{U}_n$ and $|\beta| = R$. As n varies, the Z_n^β are disjoint and connected. Moreover, locally in $\mathscr{D}$, Z_n^β is a graph of the form $w = \psi_n(z)$. We can therefore apply Hurwitz's theorem to prove that the analytic manifolds Z_n^β converge to complex manifolds that foliate $K^- \cap \mathscr{U}$. (Recall that the diameter of the components of $\mathscr{V}_{n,z_0}$ has limit zero.) Since each of the Z_n^β goes through all components of $\mathscr{V}_{n,z_0}$, it follows that each leaf of K^- is dense in $K^- \cap \mathscr{U}$, and hence in K^-.

Note that leaves of $K^- \cap \mathscr{U}$ enter the basins of attraction of $\{p_1, \ldots, p_j\}$. Hence, by the invariance under g, each p_j is a cluster point of $K^- - \{p_j\}$.

To show that each leaf is biholomorphic to $\mathbb{C}$, consider a closed curve γ in a leaf L. Then $g^{-n}(\gamma) \subset K^- \cap \mathscr{U}$ for all n large enough. Also, from the estimate of the derivative of g it follows that the length of $g^{-n}(\gamma) \to 0$ as $n \to \infty$. Hence, for large n, $g^{-n}(\gamma)$ is contractible in $g^{-n}(L)$. Hence, L is simply connected. We can show that L has no nonconstant bounded holomorphic functions in the same way as for leaves of K^+ in Theorem 3.1.

This completes the proof of part (iv) of Theorem 3.9. □

It only remains to prove part (v). The estimate on the Hausdorff dimension of K^- is quite simple. We know from the Hölder continuity of Green's function and the theorem of Carleson already mentioned that the Hausdorff dimension of $K^- \cap \{z = z_0\}$ is strictly positive. As in Corollary 1.4, the Hausdorff dimension of K^- is then strictly larger than 2. The fact that the 2^n components of $\mathscr{V}_{n,z_0}$ have a diameter decreasing like $|a|^n$ gives the other estimate. This completes the proof of Theorem 3.9.

THEOREM 3.24. *Suppose (a, c) is as in Theorem 3.9. If $P_c(z) = z^2 + c$ has an attractive cycle of order $k \geqslant 2$, then J^+ is nowhere a topological manifold. If $P_c(z)$ has an attractive fixed point and g satisfies the following generic condition, condition* (C), *then J^+* is nowhere a *$\mathscr{C}^1$ manifold.*

Condition (C). *The map g satisfies condition* (C) *if there is a $k \in \mathbb{N}$ such that g^k has a fixed hyperbolic point P with eigenvalues $|\lambda| < 1$, $|\mu| > 1$, and $\mu = |\mu| e^{2i\pi\theta}$ with θ irrational.*

Proof. If $P_c(z) = z^2 + c$ has an attractive cycle of order $k \geqslant 2$, then at every point $p \in J^+$, J^+ is on the boundary of at least 3 components. Therefore, J^+ is not a topological manifold.

Suppose now that P_c has an attractive fixed point. We will use the following lemma.

LEMMA 3.25. *Let f be a Hénon automorphism of $\mathbb{C}^2$. Let p be a hyperbolic fixed point of f^k. Let v, w be two nonzero vectors in $\mathbb{C}^2$ such that $(f^k)'(p)v = \lambda v$ and $(f^k)'(p)w = \mu w$ with $|\lambda| < 1$ and $|\mu| > 1$. Suppose $\mu = |\mu| e^{2i\pi\theta}$. If the tangent cone to J^+ is not equal to $\mathbb{C}v$, then for some $\beta \neq 0$ it contains any limit point of $\{e^{2i\pi n\theta}\beta w\}$.*

Proof. Recall that the tangent cone $C(p, J^+)$ at p is the set of positive multiples of limits of $(p_n - p)/\|p_n - p\|$ for $p_n \in J^+ | \{p\}$ and $\lim_n p_n = p$. Since J^+ is stable under f, it follows that $C(p, J^+)$ is stable under $(f^k)'(p)$. Suppose $\xi = \alpha v + \beta w \in C(p, J^+)$, $\beta \neq 0$·. The limits of $((f^n)'(p)\xi)/\|(f^n)'(p)\xi\|$ give all the limit points of $\{e^{2i\pi n\theta}\beta w\}$.

End of the proof of Theorem 3.24. Suppose p is a hyperbolic fixed point for some g^k and that the expansive value associated to it satisfies $\mu = |\mu| e^{2i\pi\theta}$, θ irrational. The stable manifold W_s of $f = g^k$ at p is contained in J^+. Hence, $\mathbb{C}v \subset C(p, J^+)$. Since in our situation $C(p, J^+) \neq \mathbb{C}v$, by Lemma 3.25 we have in $C(p, J^+)$ all the limits of $\{e^{2i\pi n\theta}\beta w\}$ for some $\beta \neq 0$. Therefore, $C(p, J^+)$ is not contained in a real hyperplane.

We want to show that J^+ is not $\mathscr{C}^1$ at any point of W_s. Since W_s is dense in J^+, this will prove the theorem.

Without loss of generality assume that J^+ is $\mathscr{C}^1$ at $q \in \mathscr{U} \cap W_s$. Let Δ_q be a small horizontal disc centered at q. Since W_s is almost vertical at q, Δ_q is transverse to J^+ at q, and the intersection of Δ_q with J_+ is a $\mathscr{C}^1$ curve in Δ_q. By the stable manifold theorem (see [Sh]), $g^{nk}(\Delta_q)$ converge to the germ W_u^0 of the unstable manifold through p. We therefore get that $W_u \cap J^+$ contains a $\mathscr{C}^1$ curve γ at p. Let $h\colon \mathbb{C} \to W_u$ be a holomorphic parametrization of the global unstable manifold W_u. Then h satisfies the equation $g^k(h(\zeta)) = h(\mu\zeta)$. We may assume that $h^{-1}(\gamma)$ can be parametrized as $(s, \sigma(s))$, $\sigma'(0) = 0$. Using the functional equation satisfied by h, we see that we may assume $\sigma \equiv 0$. Since g^{nk} is uniformly bounded on γ, we get, using the functional equation, that h is bounded uniformly on all the rays $e^{2i\pi n\theta}$. Since θ is irrational, we deduce that h is bounded, which is impossible. □

Remark. Condition (C) can be weakened to the following. If $\theta = (r/\ell)\varepsilon\mathbb{Q}$, we have to assume that $\log|\mu|$ is large enough for the Phragmen-Lindelöf theorem to apply. Indeed, h is an entire map of order $\alpha \leqslant \log 2/\log|\mu|$. If we know that h is bounded on some ray in any open sector of angular width π/α, then h is bounded.

The following result gives a parametrization of $\Omega_R = (|w| < R)\backslash K^+$.

THEOREM 3.26. *For (a, c) as in Theorem 3.9, the open set $\Omega_R = (|w| < R)\backslash K^+$ is biholomorphic to the domain $(|\zeta| > 1, |w| < R)$.*

Proof. For $|w| < R$ fixed, $G^+(z, w) = \log|z| + O(1)$ at infinity and G^+ vanishes on the compact K_w^+. Hence, $z \to G^+(z, w)$ is the Green's function of K_w^+ with pole at infinity. It is easy to check that $G^+(z, w) = \log|z| + O(1)$ at infinity, i.e., that $\operatorname{cap}(K_w^+) = 1$ for $|w| < R$.

We have proved in Theorem 3.9 that K_w^+ is connected for $|w| < R$. Therefore, the function $z \to G^+(z, w)$ has no critical point in $\mathbb{C}|K_w^+$. Let $z \to H(z, w, a, c)$ be the conjugate of $G^+(z, w, a, c)$ which is defined only modulo $2k\pi$, $k \in \mathbb{Z}$. The function $F(z, w, a, c) = \exp(G^+(z, w, a, c) + iH(z, w, a, c))$ is well defined, and $\Phi\colon \Omega \to (|\zeta| > 1, |w| < R)$ defined by $\Phi(z, w, a, c) = (F(z, w, a, c), w)$ is a biholomorphism depending holomorphically on the parameters.

Remark. Let $\Phi^{-1}(\zeta, w, a, c) = (\psi(\zeta, w, a, c), w)$. For each fixed ζ we get a leaf of a foliation of Ω_R. Using the λ-lemma ([MSS]), we see that this foliation extends to give a foliation of $\bar{\Omega}$. Hurwitz's lemma implies that this foliation on $J^+ \cap \{|w| < R\}$ coincides with the previous one.

In the following result we study the quasi-conformal geometry of slices $J_{w_0}^+$ with $|w_0| < R$. We emphasize the dependence on the parameters (a, c).

Recall that a homeomorphism in $\mathbb{C}$, f, is quasi-conformal if and only if f has derivatives in $L^2_{\mathrm{loc}}(\mathbb{C})$ and $\partial f/\partial\bar{z} = \mu(\partial f/\partial z)$, where $\mu \in L^\infty(\mathbb{C})$ and $\|\mu\|_\infty < 1$. For the properties of quasiconformal mappings we refer to Lehto [Le].

A quasi circle is the image of a circle under a quasi-conformal homeomorphism of the plane.

We will also use the following notion from [ST]. Let X be a subset of $\mathbb{C}$. A holomorphic motion of X in $\mathbb{C}$ is a map

$$f\colon T \times X \to \mathbb{C}$$

defined on an open disc $T \subset \mathbb{C}$ containing 0 such that

(a) for any fixed $x \in X$, $f_t(x) = f(t, x)$ is a holomorphic map;

(b) for any fixed t, f_t is injective; and

(c) f_0 is the identity on X.

The following result is proved in Sullivan ad Thurston [ST]; see also [MSS] and [Sl].

THEOREM 3.27. *A holomorphic motion of a set $X \subset \mathbb{C}$ can be extended to a holomorphic motion of $\mathbb{C}$ defined on $T \times \mathbb{C}$, and each map f_t is a quasi-conformal homeomorphism of $\mathbb{C}$ onto $\mathbb{C}$. Moreover, the map $(t, x) \to f(t, x)$ is continuous.*

THEOREM 3.28. *Assume that, for $c \in \bar{\omega}$, $P_c(z) = z^2 + c$ has an attractive cycle of order $k \geqslant 1$. Suppose $|a| \leqslant a_0(c)$ as in Theorem 3.9. Let $J^+_{w_0}(a, c)$ be the slice of $J^+(a, c)$ by the plane $w = w_0$. Then, for $|w| < R$, $c \in w$, $|a| \leqslant a_0(c)$ all the $J_{w_0}(a, c)$ are quasi-conformally equivalent. If $k = 1$, $J_{w_0}(a, c)$ is a quasi circle.*

Proof. We first prove that for (a, c) fixed, $|a| < a_0(c)$, $|w| < R$, the slice $J_w(a, c)$ is quasi-conformally equivalent to $J_0(a, c)$. For $x \in J_0(a, c)$, there is a leaf in $(J^+ \cap |w| < R)$ through x; the leaf is a graph $z = \varphi_x(w)$. If $T = \{w \in \mathbb{C}, |w| < R\}$ and $X = J_0(a, c)$, the map $(x, w) \to \varphi_x(w)$ is a holomorphic motion of X in $\mathbb{C}$. This is a consequence of the fact that each graph $(\varphi_x(w), w)$ is a leaf of a foliation and of the fact that $\varphi_x(0) = x$. Theorem 3.27 implies that $(w, x) \to \varphi_x(w)$ is continuous and for each w, $x \to \varphi_x(w)$ extends to a quasi-conformal homeomorphism. Hence, $x \to \varphi_x(w)$ is a homeomorphism between $J_0(a, c)$ and $J_w(a, c)$.

Each $\partial\mathscr{U}_n(a, c)$ is foliated by graphs $z = \varphi_n(w, a, c, x)$, where x varies in $\partial\mathscr{U}_n(a, c) \cap (w = 0)$. Each φ_n depends holomorphically on (w, a, c) as follows from Lemma 3.14. These foliations converge to a foliation of $J^+(a, c)$, and the leaves are graphs $z = \varphi(w, a, c, x)$, where x varies in $J_0(0, c)$. Moreover, each function $(w, a, c) \to \varphi(w, a, c, x)$ is holomorphic for x fixed.

Let $\alpha = \varphi(0, 0, c, x)$. Fix $c \in \omega$ and let $T = \{a : |a| < a_0(c)\}$. Define $h(a, \alpha) = \varphi(0, a, c, x)$. The map h is a holomorphic motion of $J_0(0, c)$ since, if $\alpha \neq \alpha'$, then $x \neq x'$ and the corresponding graphs are disjoint.

We apply Theorem 3.27 to get that $J_0(0, c)$ is quasi-conformally homeomorphic to $J_0(a, c)$ for $|a| < a_0(c)$. Hence, all the $J_w(a, c)$ are quasi-conformally homeomorphic.

If $P_c(z) = z^2 + c$ has an attractive fixed point, it was proved in [MSS] that $J_0(0, c)$ is a quasi circle. Consequently, $J_w(a, c)$ is also a quasi circle. □

COROLLARY 3.29. *Let $g(z, w) = (z^2 + c + aw, az)$. Suppose a, c are as in Theorem 3.28. Then J^+ is of Lebesgue measure 0.*

Proof. We have seen in Theorem 3.27 that, for $|w_0| \leqslant R$, $J^+_{w_0}(a, c)$ is quasi-conformally equivalent to $J^+_0(0, c)$ which is the Julia set for the polynomial $P_c(z) = z^2 + c$. We have assumed that the polynomial P_c has an attractive fixed point of order $k \geqslant 1$, and hence P_c is a hyperbolic polynomial. In that case it is a result due to Sullivan [Su2] that the corresponding Julia set is of Lebesgue measure 0 in $\mathbb{C}$. Since the image of a set of Lebesgue measure 0 under a quasi-conformal homeomorphism is of measure zero (see [LV, p. 150]), it follows that $J^+_{w_0}(a, c)$ is of Lebesgue measure zero; hence, J^+ has zero volume in $\mathbb{C}^2$. □

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