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Structure Properties of Laminar Currents on $\mathbb{P}^2$

By *Romain Dujardin*

ABSTRACT. *We study the structure of a class of laminar closed positive currents on* $\mathbb{CP}^2$, *naturally appearing in birational dynamics. We prove such a current admits natural non intersecting leaves, that are closed under analytic continuation. As a consequence it can be seen as a foliation cycle a weak lamination.*

1. Introduction

Positive closed currents play an important role in higher dimensional holomorphic dynamics. Since the very beginnings of the theory, they have served as an elementary bridge between the ambient complex geometry and the dynamics. Let us focus on the case of polynomial automorphisms of $\mathbb{C}^2$ (see N. Sibony [21] for a more thorough study and bibliographical data). In case f is hyperbolic on its non wandering set, the laminar structure of the Julia sets J^+ and J^- is predicted by general Stable Manifold theory, whereas D. Ruelle and D. Sullivan proved in [19] the existence of foliation cycles (uniformly laminar currents) subordinate to those laminations. E. Bedford and J. Smillie proved in [2] that the Ruelle-Sullivan currents coincide with the invariant "Green" currents obtained by equidistributing preimages of generic subvarieties.

Laminar currents were introduced by E. Bedford et al. [1] as analogues of the Ruelle-Sullivan foliation cycles in the general (nonuniformly hyperbolic) setting. They proved the invariant currents of polynomial automorphisms of $\mathbb{C}^2$ are laminar and derived some dynamical consequences, among them the local product structure of the maximal entropy measure, as well as equidistribution of saddle periodic orbits. Also, the laminar structure of the Julia sets is required in the notions of unstable critical points and external rays (see [3, 4]).

Our purpose here is to continue the development of the general theory of laminar currents with a view to new dynamical applications. We proved in [11] that laminar currents are abundant in rational dynamics on the complex projective plane, by exhibiting a general criterion ensuring laminarity of the limit of a sequence of $\mathbb{Q}$-divisors on $\mathbb{P}^2$. We proved in [12] that these currents are well behaved with respect to taking wedge products.

In this article we give a structure theorem for this class of laminar currents, which is also

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new in the case of plane polynomial automorphisms. In the article [14], we apply these results to birational surface dynamics.

Let us be more specific. Recall a laminar current in an open $\Omega \subset \mathbb{C}^2$ is a current "filled" in the sense of measure, by compatible holomorphic disks (see below Section 2 for more details). The point is that the disks are not assumed to be properly embedded in Ω. On the other hand, a current is said to be uniformly laminar, if it is locally made up of currents of integration over disjoint complex submanifolds.

There are examples showing that closed laminar currents may have somehow strange structure; this prevents us from studying general laminar currents. Nevertheless, the currents arising in some dynamical situations (e.g., those constructed in [3, 11]) have an additional property: They are limits of sequences of $\mathbb{Q}$-divisors $\frac{1}{d_n}[C_n]$ on $\mathbb{P}^2$ with controlled geometry. We call such currents strongly approximable. An important class of examples is provided by invariant currents of polynomial automorphisms, and more generally, birational maps. Our main result is the following.

Theorem 1.1. *Let T be a diffuse strongly approximable laminar current on $\mathbb{P}^2$. Then*

1. *If $\mathcal{L} \subset \Omega \subset \mathbb{P}^2$ is an embedded lamination by Riemann surfaces, then $T|_{\mathcal{L}}$ is uniformly laminar (analytic continuation statement).*
2. *Two disks subordinate to T are compatible, i.e., their intersection is either empty, or open in the disk topology (non self-intersection).*

Here, the notion of disk subordinate to T is stronger than just appearing in the decomposition of T as integral over a measured family of disks. It appears that the good notion to be considered is that of uniformly laminar current subordinate to T (see Definition 2.3 below). Note, that in the case of non diffuse currents, that is currents giving mass to algebraic curves, if $\mathcal{L}$ is a curve, the first item is a consequence of the Skoda-El Mir Extension Theorem (see Demailly [7]). A main issue in this theorem is that there is no regularity hypothesis on the potentials, which seems important in view of wide dynamical applications. On the other hand, assuming the wedge product $T \wedge T$ (is well defined and) vanishes, then the second item in the theorem is automatic—this was shown to hold in case T admits local continuous potentials in [12].

We use this result to show, using a construction of Meiyu Su [23], that disks subordinate to such T form a lamination, in a weak sense, and T induces an invariant transverse measure on this lamination. We believe this construction provides a useful language for treating problems related to strongly approximable currents. In particular, this clarifies the question of differing representations of a laminar current by measurable families of disks.

As an application we prove in Section 5 that, if such a T is extremal—which is common in dynamical situations—then the transverse measure is ergodic. We also apply Item 2 of the theorem to prove (Theorem 6.7) that the potential of a strongly approximable current is either harmonic, or identically $-\infty$ on almost every leaf.

The precise outline of the article is as follows. In Section 2 we recall some basic notions related to laminar currents. In Sections 3 and 4 we prove our main theorem, whereas the interpretation in terms of weak lamination structure is given in Section 5. We also relate invariant transverse measures on the weak laminations and closed laminar currents dominated by T. In Section 6 we give some applications of our study, with some pluripotential theoretic flavor: We study the potential of strongly approximable laminar currents along the leaves (Theorem 6.7) and prove such currents decompose as sums of two closed laminar currents, one not charging pluripolar sets, and the other with full mass on a pluripolar set (Theorem 6.8).

2. Preliminaries on laminar currents

We begin by recalling some definitions and preparatory results on laminar currents that will be useful to us in the sequel. Additional references are [11]–[13], [1]. A good reference on positive closed currents is Demailly's survey article [7].

The first definitions are local so we consider an open subset $\Omega \subset \mathbb{C}^2$, and T a positive $(1, 1)$ current in Ω. We let $\mathrm{Supp}(T)$ denote the (closed) support of T, $\|T\|$ the trace measure and $\mathbf{M}(T)$ the mass norm; $[V]$ denotes the integration current over the subvariety V, possibly with boundary. Also $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Definition 2.1. T is uniformly laminar, if for every $x \in \mathrm{Supp}(T)$ there exists open sets $V \supset U \ni x$, with V biholomorphic to the unit bidisk $\mathbb{D}^2$ so that in this coordinate chart $T|_U$ is the direct integral of integration currents over a measured family of disjoint graphs in $\mathbb{D}^2$, i.e., there exists a measure λ on $\{0\} \times \mathbb{D}$, and a family (f_a) of holomorphic functions $f_a : \mathbb{D} \to \mathbb{D}$ such that $f_a(0) = a$, the graphs Γ_{f_a} of two different f_a's are disjoint, and

$$T|_U = \int_{\{0\}\times\mathbb{D}} [\Gamma_{f_a} \cap U] \, d\lambda(a) \, . \tag{2.1}$$

A family of disjoint horizontal graphs in $\mathbb{D}^2$ form a lamination. More precisely, the holonomy map is Hölder continuous in this case: It is a corollary of the celebrated Λ-lemma on holomorphic motions [18], since laminations by graphs in $\mathbb{D}^2$ and holomorphic motions in the unit disk are two sides of the same object. Another useful consequence of the theory of holomorphic motions is that a lamination by graphs of some vertically compact $X \subset \mathbb{D}^2$ has an extension to a neighborhood of X—this is a special case of a deep theorem of Slodkowski's, see [22, 10]. A uniformly laminar current induces an invariant transverse measure on its underlying lamination.

Definition 2.2. T is laminar in Ω, if there exists a sequence of open subsets $\Omega^i \subset \Omega$, such that $\|T\| (\partial\Omega^i) = 0$, together with an increasing sequence of currents $(T^i)_{i\geq 0}$, T^i uniformly laminar in Ω^i, converging to T.

Equivalently (see [1]), T is laminar in Ω, if there exists a measured family $(\mathcal{A}, \mu)$ of holomorphic disks $D_a \subset \Omega$, such that for every pair (a, b), $D_a \cap D_b$ is either empty, or open in the disk topology (compatibility condition), and

$$T = \int_{\mathcal{A}} [D_a] \, d\mu(a) \, . \tag{2.2}$$

Notice that both representations of a laminar current as increasing limits or integral of disks are far from being unique, since they may be modified on sets of zero $\|T\|$ measure. One aspect of the results in this article (in particular the construction in Section 5) is to provide a natural representation of T as a foliated cycle on a measured lamination, which is the "maximum" of all possible representations. Notice also that our laminar currents are called "weakly laminar" in [1].

There is a useful alternate representation of T as an integral of disks. Following [1, Equation (6.5)], one may reparametrize the laminar representation (2.2) to obtain an alternate representation of T as an integral over a family of disjoint disks. Indeed, there exists a measured family $(\tilde{\mathcal{A}}, \tilde{\mu})$ of disjoint disks $(D_a)_{a\in\tilde{\mathcal{A}}}$, and for almost every $a \in \tilde{\mathcal{A}}$ a function p_a, nonnegative and a.e. equal to a lower semicontinuous function such that

$$T = \int_{\tilde{\mathcal{A}}} p_a [D_a] \, d\tilde{\mu}(a) \, . \tag{2.3}$$

A basic feature here is that there is some choice to be made in choosing the collection of disks that appear in this representation. We now define a non ambiguous notion of disk subordinate to T. It has the advantage of being independent of the representation of T as a laminar current.

Definition 2.3. A holomorphic disk D is subordinate to T, if there exists in some $\Omega' \subset \Omega$ a uniformly laminar current $S \leq T$ with positive mass, such that $D \subset \mathrm{Supp}(S)$ lies inside a leaf of the lamination induced by S.

This definition is motivated by the fact that the usual ordering on positive currents is compatible with the laminar structure. This is the content of the next proposition, which is implicit in [1, Section 6] (see [13], or [5] for a precise proof).

Remark that, unlike the disks of representation (2.2), disks subordinate to T may intersect. Such examples are provided by sums of uniformly laminar currents with transversals of zero area, see e.g., [12, Example 2.2].

Proposition 2.4. *Let $T_1 \leq T_2$ be laminar currents in Ω. Assume T_2 has the representation $T_2 = \int_{\mathcal{A}} p_a[D_a]\,d\mu(a)$. Then T_1 and T_2 have compatible representations in the sense that T_1 may be written as $T_1 = \int_{\mathcal{A}} q_a[D_a]\,d\mu(a)$, with $q_a \leq p_a$ almost everywhere.*

The currents we will be interested in this article have a crucial additional property: There is an explicit bound on the "residual mass" $\mathbf{M}(T - T^i)$. We call such currents strongly approximable. We need first introduce a few concepts.

Let us consider a sequence of (one-dimensional) analytic sets $[C_n]$ defined in some neighborhood of $\overline{\Omega}$, with area d_n, and such that $d_n^{-1}[C_n] \rightharpoonup T$. The disks of T are to be obtained as cluster values of sequences of graphs for some linear projection π. Let L be a complex line, transverse to the direction of the projection π, and a subdivision $\mathcal{Q}$ of L into squares of size r. For $Q \in \mathcal{Q}$, we say that a connected component Γ of $\pi^{-1}(Q) \cap C_n$ is *good*, if $\pi : \Gamma \to Q$ is a homeomorphism, and the area of Γ is bounded by some universal constant, *bad,* if not. The assumption on area(Γ) ensures that families of good components form normal families.

Definition 2.5. T is a strongly approximable laminar current, if there exist a sequence (C_n) of analytic subsets of some neighborhood $\mathcal{N}$ of $\overline{\Omega}$, with $d_n^{-1}[C_n] \rightharpoonup T$, at least two distinct linear projections π_j, and a constant C such that, if $\mathcal{Q}$ is any subdivision of the projection basis L into squares of size r, and $T_{\mathcal{Q},n}$ denotes the current made up of good components of $d_n^{-1}[C_n]$ in $\mathcal{N}$, one has the following estimate in Ω

$$\left\langle d_n^{-1}[C_n] - T_{\mathcal{Q},n}, \pi_j^*(\omega|_L)\right\rangle \leq Cr^2\,, \tag{2.4}$$

where $\omega|_L$ is the restriction of the ambient Kähler form to the complex line L.

We say T is strongly approximable in $\mathbb{P}^2$, if the C_n are plane algebraic curves and assumption (2.4) on good components holds for a generic linear projection $\mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$.

Here "generic" means "for p outside a countable union of Zariski closed subsets." An important consequence of the definition is that such currents are closed. The definition of strongly approximable currents (locally, or on $\mathbb{P}^2$), though seeming rather inelegant, is designed to fit with the constructions in [11] and [3], where it is of course satisfied.

Remark 2.6. We did deliberately state a local definition, for we believe this is the good setting for future applications. However, item 1 (analytic continuation) of the main Theorem 1.1, even though it is a local result, requires global hypotheses. More precisely, the important fact is the following: We need the approximating curves C_n to have a controlled number of intersection points with the fibers $\pi^{-1}(x)$, which is only known to hold in global situations. It would be interesting to extend it to the purely local case, and more generally, using only the mass estimate (2.5) below. De Thelin [8] has a local approach to approximation of laminar currents, nevertheless we do not know, if the crucial estimate (2.5) is true in his case.

In the remainder of the article, we will have the occasion to deal with results that require the global hypothesis, and those that do not, we will then, respectively speak of currents strongly approximable in $\mathbb{P}^2$, or in some Ω.

Equation (2.4) can be turned into a real mass estimate when combining both projections. For a proof of the next proposition, see [12, Proposition 4.4].

Proposition 2.7. *Let T be a strongly approximable laminar current in Ω. Fix $\Omega' \subset\subset \Omega$, and π_1, π_2 projections satisfying Definition 2.5. Then for any subdivisions $\mathcal{S}_1, \mathcal{S}_2$ of the respective projection bases into squares of size r, if*

$$\mathcal{Q} = \left\{\pi_1^{-1}(s_1) \cap \pi_2^{-1}(s_2), (s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2\right\}$$

denotes the associated subdivision of Ω into affine cubes of size r, there exists a current $T_{\mathcal{Q}} \leq T$ in a neighborhood of $\overline{\Omega}$, uniformly laminar in each $Q \in \mathcal{Q}$, and satisfying the estimate

$$\mathbf{M}(T - T_{\mathcal{Q}}) \leq Cr^2 , \tag{2.5}$$

in Ω', with C independent of r.

3. The defect function and analytic continuation

The aim of this section is to prove the first part of Theorem 1.1. We introduce a notion of defect of a laminar current with respect to a projection, analogous to the Ahlfors-Nevanlinna defect for entire functions: The defect measures the amount of good components in the slice mass of the current in some fiber. We already stressed in Remark 2.6 that this section uses global arguments; for ease of reading we consider the case of plane algebraic curves, nevertheless the cases of horizontal-like curves in the bidisk (see [13]), or curves on an algebraic surface are similar—what is needed is the approximating curves to have a controlled number of intersection points with the fibers of the projection.

The way to the proof is quite simple, but precise formulation requires some care, and we apologize by advance for possible stylistic heaviness.

We begin by recalling the statement of the analytic continuation theorem. If T is a laminar current in Ω there is a representation (2.2) of T as an integral of compatible disks

$$T = \int_{\mathcal{A}} [D_a]\, d\mu(a) .$$

If $\mathcal{L} \subset \Omega' \subset \Omega$ is an embedded lamination, we define the restriction $T|_{\mathcal{L}}$ as

$$T|_{\mathcal{L}} = \int_{\{a \in \mathcal{A},\ D_a \subset L \in \mathcal{L}\}} [D_a]\, d\mu(a) , \tag{3.1}$$

where the notation $L \in \mathcal{L}$ means "L is a leaf of $\mathcal{L}$." The current $T|_{\mathcal{L}}$ is laminar in Ω'.

Theorem 3.1. *Let T be a strongly approximable laminar current on $\mathbb{P}^2$. Assume $\mathcal{L}$ is an embedded lamination in $\Omega \subset \mathbb{P}^2$, then $T|_{\mathcal{L}}$ is uniformly laminar in Ω.*

It may not seem so clear why this is an analytic continuation result. Recall the alternate representation (2.3)

$$T = \int_{\tilde{\mathcal{A}}} p_a [D_a]\, d\tilde{\mu}(a)\,.$$

The functions p_a need not be locally constant, even, if T is closed; See Demailly's example [11, Example 2.3]. The assertion of the theorem is that in case T is strongly approximable, the functions p_a are globally constant along disks subordinate to T (in the sense of Definition 2.3).

The scheme of the proof is quite natural. The approximating curves form branched coverings (global assumption) of growing degree and branching over a line in $\mathbb{P}^2$. Slicing Theory provides a way to "count" the number of points in fibers as *measures* on the fibers and we try to construct as many local sections as possible, matching over overlapping disks in the basis—the sections of the covering being uniformly laminar currents subordinate to T.

For this section let us fix a sequence of curves $C_n \subset \mathbb{P}^2$ of degree d_n satisfying Definition 2.5, $T = \lim d_n^{-1}[C_n]$, and fix a linear projection $\pi_p : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$, such that the Lelong number $\nu(T, p)$ vanishes. We also assume $p \notin C_n$ for every n. For almost every line through p, one may define the slice $T|_L$ which is a probability measure on L, moreover, $T \wedge [L]$ is well defined and $T \wedge [L] = T|_L$. In this case, Siu's stability property of Lelong numbers by slicing (see Demailly [7]) shows that for almost every L, $T|_L$ gives no mass to $\{p\}$. We also choose p such that the set of vertical disks for the projection π_p in the laminar decomposition of T has measure zero.

We are given a lamination in $\Omega \subset \mathbb{P}^2$, and we want to prove the restriction $T|_{\mathcal{L}}$ is uniformly laminar. Since the problem is local (on $\mathcal{L}$), we may assume $\mathcal{L}$ is made up of graphs over the unit square $Q_0 \subset \mathbb{C}$ for some projection π_p satisfying the requirements above; moreover we may assume $\mathcal{L}$ is vertically compact. So for now we denote π_p by π and restrict the problem to $\pi^{-1}(Q_0) \simeq Q_0 \times \mathbb{C}$. We consider the following three sequences of overlapping subdivisions of $\mathbb{C}$, where $\mathcal{Q}$ is the standard subdivision (tesselation) of $\mathbb{C}$ into translates of Q_0 and $r_k \to 0$—say $r_k = 2^{-k}$—

$$\mathcal{Q}_k^0 = r_k \mathcal{Q},\ \mathcal{Q}_k^1 = r_k \mathcal{Q} + \left(\frac{r_k}{3} + \frac{2ir_k}{3}\right),\ \mathcal{Q}_k^2 = r_k \mathcal{Q} + \left(\frac{2r_k}{3} + \frac{ir_k}{3}\right);$$

these subdivisions induce subdivisions of Q_0 that form a neighborhood basis of Q_0. For each $Q \in \mathcal{Q}_k^j$, we let $\mathcal{G}(Q, n)$ be the family of good components of C_n over Q, and

$$T_{Q,n} = \frac{1}{d_n} \sum_{\Gamma \in \mathcal{G}(Q,n)} [\Gamma]\,;$$

if $\mathcal{Q}$ is one of the subdivisions $\mathcal{Q}_k^j$ we also use the notation $T_{\mathcal{Q},n} = \sum_{Q \in \mathcal{Q}} T_{Q,n}$. Also, for $Q \in \mathcal{Q}_k^j$, there is a subsequence, still denoted by n, such that $T_{Q,n} \rightharpoonup T_Q$, where T_Q is a uniformly laminar current in $Q \times \mathbb{C}$; see e.g., [11, Proposition 3.4] and recall that by definition, good components form normal families. We perform a diagonal extraction so that for every j, k and every $Q \in \mathcal{Q}_k^j$, one has the convergence $T_{Q,n} \to T_Q$. Let $T_{\mathcal{Q}_k^j} = \sum_{Q \in \mathcal{Q}_k^j} T_Q$; from (2.4) one deduces the estimate

$$\left\langle T - T_{\mathcal{Q}_k^j}, \pi^*\left(idz \wedge d\bar{z}\right)\right\rangle \leq C r_k^2\,. \tag{3.2}$$

The currents $T_{Q,n}$, T_Q have an important property of *invariance of the slice mass,* that we now describe. For a vertical fiber $F_x = \pi^{-1}(x)$ one has

$$T_{Q,n}|_{F_x} = T_{Q,n} \wedge [F_x] = \sum_{y \in F_x \cap \mathcal{G}(Q,n)} \delta_y$$

(there are no multiplicities because the current is made of good components). The mass of the slice measures is constant and denoted by m.s.$(T_{Q,n})$. The uniformly laminar currents T_Q do have the same property, and $T_Q \wedge [F_x]$ is the image on F_x of the underlying transverse measure of T_Q.

Now we claim that m.s.$(T_{Q,n}) \to$ m.s.(T_Q). Convergence on compact subsets implies $\liminf$ m.s.$(T_{Q,n}) \geq$ m.s.(T_Q) and we need to check the other inequality. We know that $d_n^{-1}[C_n] \rightharpoonup T$ in $Q \times \mathbb{C}$, so for almost every fiber F_x, $d_n^{-1}[C_n] \wedge [F_x] \rightharpoonup T \wedge [F_x]$, and, as noted before, the hypothesis $\nu(T, p) = 0$ implies $T \wedge [F_x]$ is a probability measure on F_x. Now write $d_n^{-1}[C_n] = T_{Q,n} + R_{Q,n}$, and assume $R_{Q,n_\ell} \rightharpoonup R_Q$ so that $T = T_Q + R_Q$. As T is a current on $\mathbb{P}^2$, m.s.$(T) = 1$ is well defined, and so is the case for R_Q. We conclude using the inequality $\liminf$ m.s.$(R_{Q,n_\ell}) \geq$ m.s.(R_Q).

Definition 3.2. For $Q \in \mathcal{Q}_k^j$, $k \geq 0$, $0 \leq j \leq 2$, one defines the defect of Q by

$$\mathrm{dft}(Q) = 1 - \mathrm{m.s.}(T_Q)\ .$$

The reference to T is implicit here. Since m.s.$(T_{Q,n}) \to$ m.s.(T_Q), the defect is the asymptotic proportion of bad components over Q. One has the following properties:

Proposition 3.3.

(i) *k, j being fixed,* $\sum_{Q \in \mathcal{Q}_k^j} \mathrm{dft}(Q) \leq C$*;*

(ii) *if $Q' \subset Q$ then* $\mathrm{dft}(Q') \leq \mathrm{dft}(Q)$.

Proof. For $Q \in \mathcal{Q}_k^j$, one has the estimate

$$\lim_{n \to \infty} \langle d_n^{-1}[C_n] - T_{Q,n}, \mathbf{1}_{Q \times \mathbb{C}} \pi^*(idz \wedge d\bar{z}) \rangle = \mathrm{dft}(Q) r_k^2\ ,$$

and (i) is a consequence of estimate (3.2).

To prove the second point, note that good components over Q are good components over Q', so m.s.$(T_{Q,n}) \geq$ m.s.$(T_{Q',n})$, and let $n \to \infty$. $\square$

Definition 3.4. For $x \in Q_0$, we let $\mathrm{dft}(x) = \lim \mathrm{dft}(Q_p)$, where (Q_p) is any decreasing sequence of squares such that $\{x\} = \cap_p Q_p$.

The fiber $\{x\} \times \mathbb{C}$ is regular, if $\mathrm{dft}(x) = 0$, singular (resp. ε-singular) otherwise (resp. if $\mathrm{dft}(x) \geq \varepsilon$).

One easily checks the definition of $\mathrm{dft}(x)$ is non ambiguous using property (ii) of the preceding proposition.

Proposition 3.5. *There are at most countably many singular fibers; moreover,*

$$\sum_{x \in Q_0} \mathrm{dft}(x) \leq 3C \ .$$

Proof. j being fixed, the number of squares $Q \in \mathcal{Q}_k^j$ where $\mathrm{dft}(Q) \geq \varepsilon$ is less than C/ε, which implies the bound C/ε on the number of ε-singular fibers. The result follows. □

Remark 3.6. We have no result on the structure of singular fibers. If a diffuse strongly approximable current has Lelong Number $\geq \varepsilon$ at some point p, generic slices through p have a Dirac mass ε at p and the fiber is ε-singular. On the other hand, the invariant currents associated to polynomial automorphisms of $\mathbb{C}^2$ have no singular fibers. Here is a rough argument: Take some complex line L in $\mathbb{C}^2$ and iterate L backwards. It is known that the iterates $d^{-n}[f^{-n}(L)]$ converge to the stable current T^+ which is laminar; since the wedge product $T^+ \wedge T^-$ is geometric (i.e., described by intersection of disks, see [1], or [12]), this implies $f^{-n}(L)$ intersects many disks of T^- and so does L. In particular, L is not a singular fiber associated to T^-.

The following proposition is the basic link between defect and analytic continuation.

Proposition 3.7. *Let Q, Q' be two squares such that $Q \cap Q' \neq \emptyset$, $\mathrm{dft}(Q) \leq \alpha$, $\mathrm{dft}(Q') \leq \alpha'$, with $\alpha + \alpha' < 1$. Then there exists a uniformly laminar current $T_{Q\cup Q'}$ in $(Q \cup Q') \times \mathbb{C}$, such that*

$$T_{Q\cup Q'}|_Q \leq T_Q, \ T_{Q\cup Q'}|_{Q'} \leq T_{Q'}$$

and

$$\mathrm{m.s.}(T_{Q\cup Q'}) \geq 1 - \alpha - \alpha' \ .$$

Proof. As $n \to \infty$, $\lim \mathrm{m.s.}(T_{Q,n}) \geq 1 - \alpha$, and $\mathrm{m.s.}(T_{Q,n}) = \frac{1}{d_n} \# \mathcal{G}(Q, n)$ is the number of good components over Q. So for $n \geq n(\varepsilon)$ there are at least $(1 - \alpha - \varepsilon)d_n$ (resp. $(1 - \alpha' - \varepsilon)d_n$) good components over Q (resp. Q'). As the total number of components over $Q \cap Q'$ is bounded by d_n, at least $(1 - \alpha - \alpha' - 2\varepsilon)d_n$ components match over $Q \cap Q'$ for n large, giving rise to as many global good components over $Q \cup Q'$. Extracting a convergent subsequence of the sequence

$$\frac{1}{d_n} \sum_{\Gamma \in \mathcal{G}(Q\cup Q', n)} [\Gamma]$$

gives the desired $T_{Q\cup Q'}$. □

We inductively use this proposition to construct analytic continuation of disks along paths in Q_0. We assume all paths are continuous.

Definition 3.8. Let $\gamma : [0, 1] \to Q_0$ be an injective path. T is said to have almost analytic continuation property up to ε along γ, if there exists an open set $V_\varepsilon \supset \gamma$ and a uniformly laminar current $T_{V_\varepsilon} \leq T$ made up of graphs over V_ε, and such that $\mathrm{m.s.}(T_{V_\varepsilon}) \geq 1 - \varepsilon$.

T has the analytic continuation property along γ, if it has the almost analytic continuation property up to ε for all $\varepsilon > 0$.

Proposition 3.7 has the following corollary. The proof is left to the reader.

Corollary 3.9. *Let* $\gamma : [0, 1] \to Q_0$ *be an injective path. Suppose there exists* $\varepsilon > 0$, *and a covering of* γ *by a family of squares* $\mathcal{F}_\varepsilon$ *satisfying*

$$\sum_{Q \in \mathcal{F}_\varepsilon} \operatorname{dft}(Q) \leq \varepsilon .$$

Then T *has almost analytic continuation property up to* ε *along* γ.

By definition the total defect of γ is the lower bound of the sums $\sum_{Q \in \mathcal{F}} \operatorname{dft}(Q)$ for families $\mathcal{F}$ of squares in $\mathcal{Q}_k^j$ covering γ. The next proposition is the crucial technical point in the proof of the theorem.

Proposition 3.10. *Let* x_1 *and* x_2 *be two points in* Q_0. *For every* $\varepsilon > 0$ *there exists a path* γ_ε *such that* T *has almost continuation property along* γ_ε *up to* $(\operatorname{dft}(x_1) + \operatorname{dft}(x_2) + \varepsilon)$.

In particular, if x_1 and x_2 are regular, T has the ε-almost continuation property. For ease of reading we use the following notation: $a \approx b$ means $c^{-1}a \leq b \leq ca$ and $a \lesssim b$ means $a \leq cb$, with c a constant independent of r (size of the squares). The idea of the proof is to construct enough essentially disjoint paths joining x_1 and x_2 in the subdivisions and apply Proposition 3.3.

Proof. Assume first x_1 and x_2 lie on the same horizontal line. Consider "big" squares $Q_1 \ni x_1$ and $Q_2 \ni x_2$ of size $\approx \sqrt{r}$, and join Q_1 and Q_2 by $N \approx 1/\sqrt{r}$ disjoint horizontal paths γ_i with mutual distance $\geq 10r$. Each path γ_i is covered by a family of "small" squares of size r, $\mathcal{F}_i \subset \mathcal{Q}^0 \cup \mathcal{Q}^1 \cup \mathcal{Q}^2$. The families $\mathcal{F}_i$ are disjoint. Complete the paths γ_i by adding affine pieces so that the paths join x_1 and x_2.

We now evaluate the total defect of the family of paths, using Proposition 3.3

$$\sum_{i=1}^{N} \operatorname{dft}(\gamma_i) \leq \sum_{i=1}^{N} \left(\operatorname{dft}(Q_1) + \operatorname{dft}(Q_2) + \sum_{Q \in \mathcal{F}_i} \operatorname{dft}(Q) \right) \leq N(\operatorname{dft}(Q_1) + \operatorname{dft}(Q_2)) + 3C .$$

As $N \approx 1/\sqrt{r}$, the average defect is

$$\frac{1}{N} \sum_{i=1}^{N} \operatorname{dft}(\gamma_i) \leq \operatorname{dft}(Q_1) + \operatorname{dft}(Q_2) + c\sqrt{r} ,$$

so at least one of the paths has total defect $\leq \operatorname{dft}(Q_1) + \operatorname{dft}(Q_2) + c\sqrt{r}$.

In the general case consider an affine isometry h such that $h(x_1)$ and $h(x_2)$ lie on the same horizontal line, and remark that, if Q is one of the squares of the preceding construction $h(Q)$ is included in a square of size at most twice that of Q. □

We will prove Theorem 3.1 through the following reformulation, which is of independent interest.

Proposition 3.11. *Let $U \subset Q_0$ be a connected open subset. Let S be a uniformly laminar current with vertically compact support in $U \times \mathbb{C}$, made up of graphs over U, and such that $S \leq T$ in $U_1 \times \mathbb{C}$, for some open $U_1 \subset U$.*

Then $S \leq T$ in $U \times \mathbb{C}$.

The conclusion of the proposition is that the relation $S \leq T$ propagates to the domain of definition of S; this is a continuation result in terms of disks subordinate to T.

We first prove that this proposition implies the theorem. Recall we localized the problem so that $\mathcal{L}$ is a vertically compact lamination whose leaves are graphs Δ_α over Q_0. Then one has

$$T|_{\mathcal{L}} = \int_X p_\alpha [\Delta_\alpha]\, d\mu_X(\alpha)$$

where μ_X is a positive measure on the global transversal X and for every α, p_α is a.e. equal to a lower semicontinuous non negative function. We have to prove the functions p_α are constant for a.e. α. The idea is as follows: If p_α takes the value p_0 at some point, then the preceding proposition forces $p_\alpha \geq p_0$ on Δ_α.

Indeed, consider the measurable function

$$X \ni \alpha \longmapsto \inf(p_\alpha)\ ,$$

where inf denotes essential infimum. Then for $\varepsilon > 0$,

$$\Delta_\alpha^\varepsilon = \{x \in \Delta_\alpha,\ p_\alpha(x) > \inf(p_\alpha) + \varepsilon\}$$

is either empty, or an open subset up to a set of zero area. We prove $\Delta_\alpha^\varepsilon$ has area zero for a.e. α.

Assume the contrary. There exists $Y \subset X$ of positive transverse measure such that $\Delta_\alpha^\varepsilon$ has positive area. Hence, the current

$$T_\varepsilon = \int_Y \mathbf{1}_{\Delta_\alpha^\varepsilon} p_\alpha [\Delta_\alpha]\, d\mu_X(\alpha)$$

is a laminar current of positive mass. By a monotone convergence argument and the Fubini theorem (see e.g., [1, Proposition 6.2]) there exists a square R and a set Y_R of positive transverse measure such that for $\alpha \in Y_R$, $\pi^{-1}(R) \cap \Delta_\alpha \subset \Delta_\alpha^\varepsilon$ up to a set of zero area. In particular,

$$0 < S = \left(\int_{Y_R} (\inf(p_\alpha) + \varepsilon)\, d\mu_X(\alpha)\right) |_{R\times\mathbb{C}} \leq T_\varepsilon|_{R\times\mathbb{C}} \leq T|_{R\times\mathbb{C}}\ .$$

Since S is uniformly laminar, by Proposition 3.11, this relation propagates to $Q_0 \times \mathbb{C}$, contradicting the definition of $\inf p_\alpha$.

Proof of Proposition 3.11. Without loss of generality, assume U_1 is a disk. We use the representation (2.3) over families of disjoint disks $T = \int_{\mathcal{A}} p_a [D_a]\, d\mu(a)$. Using Proposition 2.4 one gets

$$S = \int_{\mathcal{A}|_{U_1}} q_a [D_a]\, d\mu(a)$$

where $\mathcal{A}|_{U_1}$ is the set of restrictions to $U_1 \times \mathbb{C}$ of the disks of $\mathcal{A}$, and q_a is a constant on every $D_a \in \mathcal{A}|_{U_1}$ since S is uniformly laminar; moreover $q_a \leq p_a$ a.e. For a square Q let $\mathcal{A}_Q \subset \mathcal{A}|_Q$ be the set of disks that are graphs over Q. The uniformly laminar currents T_Q previously considered have the form

$$T_Q = \int_{\mathcal{A}_Q} p_{a,Q}[D_a]\, d\mu(a)$$

where $p_{a,Q}$ is a constant function.

Lemma 3.12. *For almost every $x \in U_1 \times \mathbb{C}$ there exists a decreasing sequence of squares Q_p, $\cap_{p\geq 0} Q_p = \{x\}$, and for every p a uniformly laminar current $S_{Q_p} \leq S$, in $Q_p \times \mathbb{C}$, such that $S_{Q_p} \leq T_{Q_p}$ et m.s.$(S_{Q_p}) \to m =$ m.s.(S) as $p \to \infty$.*

Proof. Recall that, if $\mathcal{Q}_k$ is one of the three3 sequences of subdivisions $\mathcal{Q}_k^j$, the sequence $T_{\mathcal{Q}_k} = \sum_{Q\in\mathcal{Q}_k} T_Q$ increases to T. Let

$$S_{\mathcal{Q}_k} = \sum_{Q\in\mathcal{Q}_k} \int_{\mathcal{A}_Q} \inf(q_a, p_{a,Q})[D_a]\, d\mu(a) = \int_{\mathcal{A}} \inf\left(q_a, \sum_{Q\in\mathcal{Q}_k} \mathbf{1}_{Q\times\mathbb{C}}\, p_{a,Q}\right)[D_a]\, d\mu(a)\, .$$

The current $S_{\mathcal{Q}_k}$ is uniformly laminar in each $Q \times \mathbb{C}$, $Q \in \mathcal{Q}_q$, and since

$$p_{a,\mathcal{Q}} = \sum_{Q\in\mathcal{Q}_k} \mathbf{1}_{Q\times\mathbb{C}}\, p_{a,Q}$$

increases $\|T\|$ a.e. to $p_a \geq q_a$, one gets $\inf(q_a, p_{a,\mathcal{Q}_k}) \nearrow q_a$ and the sequence of currents $S_{\mathcal{Q}_k}$ increases to S as $k \to \infty$. From this one easily deduces the conclusion of the lemma. □

We continue with the proof of Proposition 3.11. The basic idea is to transport the relation $S_Q \leq T_Q$ by using analytic continuation along paths. Fix $\varepsilon > 0$ and $x_0 \in U_1$ such that the conclusion of the lemma is satisfied and $\mathrm{dft}(x_0) = 0$. For $x_0 \in Q \in \mathcal{Q}_k$, k large enough, one has a uniformly laminar $S_Q \leq T_Q$, such that m.s.$(S_Q) \geq$ m.s.$(S) - \varepsilon = m - \varepsilon$.

On the other hand, by Proposition 3.10 for every $x_1 \in U$, there exists a path γ_ε joining x_0 and x_1, such that T has $(\mathrm{dft}(x_1) + \varepsilon)$-almost analytic continuation along γ_ε, i.e., there exists $V \supset \gamma_\varepsilon$, and $T_V \leq T$ uniformly laminar, such that m.s.$(T_V) \geq 1 - \mathrm{dft}(x_1) - \varepsilon$. Applying Proposition 3.7 to Q, V, and the square $Q_1 \in \mathcal{Q}_k$ containing x_1 yields the existence of a uniformly laminar current T_{Q,Q_1}, simultaneously subordinate to T_Q, T_V and T_{Q_1}, with

$$\text{m.s.}(T_{Q,Q_1}) \geq 1 - \mathrm{dft}(Q) - \mathrm{dft}(Q_1) - \varepsilon - \mathrm{dft}(x_1) \geq 1 - \mathrm{dft}(Q) - 2\mathrm{dft}(Q_1) - \varepsilon\, .$$

By construction the graphs of T_{Q,Q_1} over Q_1 are the analytic continuations along γ_ε of those over Q.

We then prove the sum of T_{Q,Q_1}, with varying Q_1, approximate T in $U \times \mathbb{C}$:

$$\begin{aligned}\left\langle T - \sum_{Q_1\in\mathcal{Q}_k} T_{Q,Q_1}, \mathbf{1}_{U\times\mathbb{C}}\, \pi^* i dz \wedge d\bar{z}\right\rangle &\leq \sum_{Q_1\in\mathcal{Q}_k} (\mathrm{dft}(Q) + \varepsilon + 2\mathrm{dft}(Q_1))\mathrm{area}(Q_1)\\ &\leq (\mathrm{dft}(Q) + \varepsilon) + 2\left(\sum_{Q_1\in\mathcal{Q}_k} \mathrm{dft}(Q_1)\right) r_k^2\end{aligned}$$

and the right-hand side is less than 3ε, if k is large and Q small.

We now claim there exists for all Q_1 a current S_{Q,Q_1} such that in $Q \times \mathbb{C}$

$$S_{Q,Q_1} \leq T_{Q,Q_1} \leq T,\ S_{Q,Q_1} \leq S_Q \leq S, \text{ and } \text{m.s.}(S_{Q,Q_1}) \geq m - \mathrm{dft}(Q) - 2\mathrm{dft}(Q_1) - 2\varepsilon. \quad (3.3)$$

Let us see first why this implies the proposition: The current S being uniformly laminar, we can use the holonomy to extend the current S_{Q,Q_1}, which is originally defined in $Q \times \mathbb{C}$, to $Q_1 \times \mathbb{C}$, and get a current we still denote by S_{Q,Q_1}, subordinate to both S and T in $Q_1 \times \mathbb{C}$ and satisfying the last estimate in (3.3). So we get as before the following estimate in $U \times \mathbb{C}$

$$\left\langle \sum_{Q_1 \in \mathcal{Q}_k} S_{Q,Q_1}, \mathbf{1}_{U\times\mathbb{C}}\, \pi^* i dz \wedge d\bar{z} \right\rangle \geq \left\langle S, \mathbf{1}_{U\times\mathbb{C}}\, \pi^* i dz \wedge d\bar{z} \right\rangle - 4\varepsilon ,$$

that is, $\sum_{Q_1 \in \mathcal{Q}_k} S_{Q,Q_1}$ increases to S. On the other hand, $\sum_{Q_1 \in \mathcal{Q}_k} S_{Q,Q_1} \leq T$ and we conclude that $S \leq T$ in $U \times \mathbb{C}$.

It remains to prove our claim. The data are

$$S_Q \leq T_Q,\ \text{m.s.}(S_Q) \geq m - \varepsilon \text{ and } \ T_{Q,Q_1} \leq T_Q,\ \text{m.s.}(T_{Q,Q_1}) \geq 1 - \text{dft}(Q) - 2\text{dft}(Q_1) - \varepsilon ,$$

all these currents being uniformly laminar in $Q \times \mathbb{C}$. Fix a global transversal $\{c\} \times \mathbb{C}$, $c \in Q$, and consider the respective slices m_{S_Q}, m_{T_Q}, $m_{T_{Q,Q_1}}$ of S_Q, T_Q and T_{Q,Q_1}. By the Radon-Nikodym Theorem there exists a function $0 \leq f_{S_Q} \leq 1$ (resp. $0 \leq f_{T_{Q,Q_1}} \leq 1$) such that $m_{S_Q} = f_{S_Q} m_{T_Q}$ (resp. $m_{T_{Q,Q_1}} = f_{T_{Q,Q_1}} m_{T_Q}$).

Let $f = \inf(f_{S_Q}, f_{T_{Q,Q_1}})$, then one has the estimate

$$\int f\, dm_{T_Q} \geq m - \text{dft}(Q) - \text{dft}(Q_1) - 2\varepsilon .$$

Define S_{Q,Q_1} as the uniformly laminar current in $Q \times \mathbb{C}$ subordinate to S, and having transverse measure $f dm_{T_Q}$ in $\{c\} \times \mathbb{C}$. S_{Q,Q_1} has the required properties (3.3). □

Remark 3.13. The definition of laminar currents may be relaxed to let the disks intersect. One obtains the class of web-laminar currents, considered by Dinh [9], which seems to be of interest. For instance, the cluster values of a sequence of curves in $\mathbb{P}^2$ with degree d_n and geometric genus $O(d_n)$ and no assumption on the singularities are of this form—such a statement may easily be extracted from [11], and is explicit in [9]. Moreover, such currents are strongly approximable in the sense that estimate (2.4) holds, with the $T_{Q,n}$ being sums of intersecting graphs (web-uniformly laminar currents).

One may then define disks subordinate to a web-laminar current, and prove an analytic continuation theorem in the strongly approximable case, in the same way as above.

Remark 3.14. The estimate (2.4) plays of course an important role in this section. However, a careful reading of Proposition 3.10 shows it can be relaxed by replacing $O(r^2)$ by $O(r^{1+\varepsilon})$; in particular the analytic continuation statement holds in this case.

4. Non self intersection

In this section we prove the second part of Theorem 1.1, which asserts that disks subordinate to a diffuse strongly approximable T are compatible. Due to our Definition 2.3 of disks subordinate to a laminar current, this is equivalent to saying that uniformly laminar currents subordinate to T do not intersect non trivially.

In contrast to the preceding section, the result here is purely local, and only uses the mass estimate of Proposition 2.7. We first recall the statement.

Theorem 4.1. *Let T be a strongly approximable and diffuse laminar current in $\Omega \subset \mathbb{C}^2$. Then two disks subordinate to T are compatible, i.e., their intersection is either empty, or open in the disk topology.*

A few comments are in order here. First there are simple examples of laminar currents with intersecting subordinate disks, given by sums of uniformly laminar currents with transversals of zero area (see e.g., [12, Example 2.2]). Thus, such currents cannot be strongly approximable.

On the other hand, if one weakens the definition of disks subordinate to T to the following "a disk is subordinate to T, if it is the union of disks appearing in the laminar representation (2.2), up to a set of zero measure," then disks subordinate (in this sense) to a strongly approximable T may intersect. For example, a pencil of lines with any transverse measure satisfies (2.5). We nevertheless believe this is not a workable definition of disks subordinate to T.

In case T has continuous potential, the theorem is a consequence of the results of [12]. Indeed, we proved that $T \wedge T = 0$ in this case, so if S_1 and S_2 are uniformly laminar currents subordinate to T in $\Omega' \subset \Omega$, one has $S_1 \wedge S_2 = 0$ (currents dominated by T also have continuous potential). One interesting point here is that no potential is involved; the result may thus appear as a "geometric version" of the equation $T \wedge T = 0$.

Proof. The proof is by contradiction. Therefore, assume that S_1 and S_2 are uniformly laminar currents in $\Omega' \subset \Omega$, with non trivial intersection, and such that $S_i \leq T$. It is no loss of generality to assume $\Omega' = \Omega$. Most intersections between the leaves of the associated laminations $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ are transverse by [1, Lemma 6.4], so focusing on a neighborhood of such a transverse intersection point and reducing Ω, S_1 and S_2, if necessary, we assume S_1 and S_2 are made up of almost parallel disks and that any leaf of S_1 is a global transversal to $\mathcal{L}(S_2)$.

Next, recall that T is the increasing limit of sums of uniformly laminar currents in cubes $\sum T_Q$ given by Proposition 2.7. The approximation is increasing, so if a disk subordinate to, say, S_1 appears at some stage of the approximation, it will persist in all finer subdivisions. Moreover, the approximating currents are uniformly laminar, so in the approximation, disks do not ever intersect non trivially.

In what follows the notation $\mathcal{Q}$ denotes subdivisions by families of affine *cubes* in Ω, as given by Proposition 2.7. Recall also from this proposition that subdivisions may be translated since only the projections π_1 and π_2 are fixed, and estimate (2.5) still holds.

There are two mutually disjoint cases.

- Either at some stage of the approximation, one obtains a current T_Q, with $S_1' \leq T_Q \leq S_1$, such that $\mathrm{Supp}(S_1') \cap \mathrm{Supp}(S_2) \neq \emptyset$—the case where 1 and 2 are swapped is similar. In this case, the disks subordinate to S_1' persist in finer subdivisions, and the corresponding intersecting disks subordinate to S_2 never appear.
- Or such a current never appears.

In both cases, some disks subordinate to S_2 will never appear in the approximation process. More precisely, these correspond to the set of disks in subdivisions by cubes of size r, subordinate to S_2, and intersecting some fixed $S_1' \leq S_1$. We wish to prove that this contradicts estimate (2.5). Without loss of generality, we put $S_1' = S_1$, we also renormalize the transverse measures so that the measure induced by S_2 on the leaves of S_1 is of mass 1, and make an affine transformation so that the projections π_1 and π_2 become orthogonal.

For a given subdivision by affine cubes $\mathcal{Q}$, and $Q \in \mathcal{Q}$, we denote by $\frac{Q}{2}$ the image of Q by scaling of factor $1/2$ with respect to its center.

Lemma 4.2. *For every $r > 0$, there exists a subdivision $\mathcal{Q}$ by cubes of size r, and $N(r)$ leaves $(L_i)_{i=1}^{N(r)}$ of $\mathcal{L}(S_1)$, with mutual distance $\geq 5r$, such that if $m_i = S_2 \wedge [L_i]$ denotes the transverse measure induced by S_2 on L_i, one has*

$$\left(\sum_{i=1}^{N(r)} m_i\right)\left(\bigcup_{Q\in\mathcal{Q}} \frac{Q}{2}\right) \geq \frac{N(r)}{32} \underset{r\to 0}{\longrightarrow} \infty .$$

Let us see first why the lemma implies the theorem. By the reductions made so far, we know that no disk of T_Q traced on a leaf of $\mathcal{L}(S_2)$ intersects the $N(r)$ leaves L_i. Moreover, for every cube Q of size r, there exists a constant c such that any subvariety of Q intersecting $\frac{Q}{2}$ has area at least cr^2 (Lelong's Theorem). Thus, if L is any leaf of $\mathcal{L}(S_1)$, the total mass of the uniformly laminar current subordinate to S_2, made up of the disks through $\frac{Q}{2}$ is at least $(S_2 \wedge [L])(\frac{Q}{2})cr^2$. Since leaves at $5r$ distance cannot hit the same (affine) cube of size r, the preceding lemma provides us with a sum of uniformly laminar currents, subordinate to S_2, with mass greater than $\frac{c}{32}N(r)r^2$, that will never appear in the approximation process. This is a contradiction since $N(r) \to \infty$. □

Proof of the lemma. First recall that the holonomy of $\mathcal{L}(S_1)$ is Hölder continuous, so if a transverse section of $\mathcal{L}(S_1)$ is fixed, for appropriate constants C and τ, points mutually distant of Cr^τ in the transversal give rise to leaves distant of $5r$ in Ω. Pick $N(r)$ such points in the transversal; as $r \to 0$, $N(r) \to \infty$ since S_1 is diffuse.

For the associated leaves L_i, let m_i be the measure induced by S_2, and $m = \sum_i m_i$, which is a measure of mass $N(r)$ by the normalization done before. It is an easy consequence of the translation invariance of Lebesgue measure and the Fubini Theorem (see [12, Lemma 4.5]) that there exists a translate of $\mathcal{Q}$ such that the mass of m concentrated in $\cup\frac{Q}{2}$ is larger than

$$\frac{1}{2}\frac{\text{volume}\left(\frac{Q}{2}\right)}{\text{volume(Q)}}N(r) = \frac{N(r)}{32} ,$$

which yields the desired conclusion. □

5. Measured laminations

In this section we will reinterpret the preceding results in a more geometric fashion, by constructing a weak measured lamination associated to a strongly approximable current in $\mathbb{P}^2$. This has the advantage of clarifying the question of representation of laminar currents, since the measured lamination is the "largest" possible representation. We emphasize that this construction is more generally valid for currents satisfying the conclusions of Theorem 1.1. In analogy with Cantat [5], we define those as the strongly laminar currents. Theorem 1.1 paraphrases then as "strongly approximable currents are strongly laminar."

Next we relate closed currents subordinate to T and invariant transverse measures on its associated weak lamination (Theorem 5.7); this result really needs the mass estimate (2.5).

Weak laminations

We first define a notion of weak lamination adapted to our setting. The definitions are ad hoc so we assume the ambient space is a two-dimensional complex manifold. We fix a diffuse strongly laminar current T.

Definition 5.1. A flow box for T is the (closed) support of a lamination $\mathcal{L}$ embedded in $U \simeq \mathbb{D}^2$, such that in this coordinate chart $\mathcal{L}$ is biholomorphic to a lamination by graphs over the unit disk, and moreover satisfying $T|_{\mathcal{L}} > 0$ and $\mathrm{Supp}(T|_{\mathcal{L}}) = \mathcal{L}$.

The regular set $\mathcal{R}$ is the union of the disks subordinate to T, or equivalently the union of flow boxes.

The condition on the support of $T|_{\mathcal{L}}$ insures that we do not consider disks not subordinate to T. By definition, for any laminar current, the regular set has full measure in T.

Definition 5.2. Two flow boxes are said to be compatible, if the associated disks intersect in a compatible way.

A weak lamination is a union of a family of compatible flow boxes. We say it is σ-compact, if there are countably many flow boxes.

It turns out that this definition fits well with the theory of laminar currents, where no transverse topology is *a priori* involved; see, however, the density topology below. After this article was written, we realized that a similar definition already appears in Zimmer [25]. Given a weak lamination, one easily define leaves, as in the usual case; if the weak lamination is σ-compact then the leaves are σ-compact (see e.g., [6], or [13]).

We say a closed set $\tau \ni x$ is a local transversal to the weak lamination at x, if it is a local transverse section in a flow box. Due to the compatibility condition, this is independent of the choice of the flow box containing x. One then defines holonomy maps between transversals as in the usual case, and one may speak of holonomy invariant transverse measures, that is, a collection of measures on all transversals, invariant by holonomy (see Sullivan [23], Ghys [15]).

The following proposition is a reformulation of Theorem 1.1.

Proposition 5.3. *Let T be a laminar current satisfying the conclusions of Theorem 1.1—a strongly laminar current. Then the regular set $\mathcal{R}$ has the structure of a weak lamination, and T induces a holonomy invariant transverse measure on $\mathcal{R}$.*

Proof. The only non trivial statement is the existence of the invariant transverse measure. Note, first that T induces an invariant transverse measure on each flow box $\mathcal{L}$, since $T|_{\mathcal{L}}$ is uniformly laminar. Now if two flow boxes $\mathcal{L}_1$ and $\mathcal{L}_2$ have non trivial intersection—compatible due to the non intersection of disks subordinate to T—, the transverse measures coincide on common transversals: Just construct a flow box $\mathcal{L}$ from this transversal, subordinate to both $\mathcal{L}_1$ and $\mathcal{L}_2$, and apply the analytic continuation theorem again. □

Notice that this result gives a natural representation of T as an integral over families of disks, since the definition of $\mathcal{R}$ does not involve any choice: We take all disks subordinate to T. This means the class of strongly laminar currents should be a reasonable intermediate class between general and uniformly laminar currents.

The following intuitive proposition asserts the transverse mass of a flow box is computed using wedge products. We know that, if $\mathcal{L}$ is a flow box, $T|_{\mathcal{L}}$ is uniformly laminar, so if τ is any global holomorphic transversal, the transverse mass of $\mathcal{L}$ is given by $m = \mathbf{M}(T|_{\mathcal{L}} \wedge [\tau])$, which is easily proved to be a well-defined wedge product. The expected thing is that $m = (T \wedge [\tau])(\mathcal{L} \cap \tau)$ provided the wedge product is well defined, which is almost true: This is the content of the next proposition. Note, that using the techniques of Section 6 one may replace the smooth uniformly laminar currents in the proof by uniformly laminar currents not charging pluripolar sets.

Proposition 5.4. *Let $(\tau_\lambda)_{\lambda\in\mathbb{D}}$ be a smooth family of disjoint global holomorphic transversals to $\mathcal{L}$. Then for almost every $\lambda \in \mathbb{D}$, $T \wedge [\tau_\lambda]$ is a well defined positive measure and the transverse mass of $\mathcal{L}$ is $(T \wedge [\tau_\lambda])(\mathcal{L} \cap \tau_\lambda)$.*

Proof. Let ψ be any positive test function in $\mathbb{D}$ and let S be the smooth uniformly laminar current $S = \int_{\mathbb{D}}[\tau_\lambda]\psi(\lambda)\, d\lambda$. Since S is smooth, the wedge product $S \wedge T$ is well defined and described by the geometric intersection of disks constituting S and T (for more details on this topic see [12]), i.e., there is a laminar representation of T, $T = \int_{\mathcal{A}}[D_a]\, d\mu(a)$ such that

$$S \wedge T = \int_{\mathcal{A}\times\mathbb{D}} [\tau_\lambda \cap D_a]\psi(\lambda)\, d\lambda\, d\mu(a)\ .$$

Indeed, since S is smooth one has

$$T \wedge S = \int_{\mathcal{A}} ([D_a] \wedge S)\, d\mu(a) = \int_{\mathcal{A}} \left(\int_{D_a} S\right) d\mu(a)\ ,$$

and S being uniformly laminar, $[D_a] \wedge S$ is a geometric intersection.

It is a classical fact that for a.e. λ, the wedge product $T \wedge [\tau_\lambda]$ is well defined, and by the preceding argument it is geometric. We conclude by using the fact that disks subordinate to T are compatible, so through every point in $\mathcal{L} \cap \tau_\lambda$, the only disk subordinate to T is the corresponding leaf of $\mathcal{L}$. □

Su's construction

In the specific case of strongly approximable currents in $\mathbb{P}^2$ one has a little bit more information, since the slices by generic lines give probability measures, yielding the notion of defect. These "reference" measures allow one, following Meiyu Su [23], to produce a topology—the *density topology*—in which $\mathcal{R}$ becomes a genuine lamination. This actually does not give more structure on the weak lamination, since the topology is canonically associated to the measurable structure. We do not give full details, the reader is referred to [23] and [13] instead.

We fix a diffuse strongly approximable T in $\mathbb{P}^2$, and a linear projection π such that the condition on projections described in Definition 2.5 holds for π and the set of vertical disks has zero measure. Therefore, we can define the defect function as before, and the slice m^z of T by the fiber $\pi^{-1}(z)$ for every fiber of zero defect (by an increasing limit process; note that in general slicing is only defined for fibers outside a polar set). By definition of the defect, the regular set $\mathcal{R}$ has full transverse measure in regular fibers.

Now pick a regular fiber $\pi^{-1}(z)$, and A a measurable subset of $\pi^{-1}(z)$. Recall w is a density point of A, relative to m^z, iff

$$\frac{m^z(A \cap B(w,r))}{m^z(B(w,r))} \underset{r\to 1}{\longrightarrow} 1\ ;$$

Lebesgue's Theorem asserts that, if $m^z(A) > 0$, almost every point in A is a density point; it holds in the case of Radon measures in Euclidean space, see [17].

We can now define the density topology by specifying its open sets.

Definition 5.5. A subset $A \in \pi^{-1}(z)$ is d-open, if it is empty, or if A is measurable, $m^z(A) > 0$, and every $w \in A$ is a density point.

One easily checks this defines a topology in $\pi^{-1}(z)$. Given a flow box made of graphs for the projection π, we define the density topology on the flow box as the product of the usual topology along the leaves and the (restriction of the) density topology on a given vertical transversal. Since holonomy maps are continuous and preserve the measures m^z restricted to the flow box (thus preserve density points), this is independent of the transversal chosen. Moreover, the density topologies on intersecting flow boxes coincide by compatibility and invariance of the transverse measure.

We collect the following simple facts pertaining to the d-topology:

- Sets of measure zero are d-closed sets of empty interior. In particular, removing (possibly countably) many sets of zero measure does not affect the d-open property.
- Open sets of positive measure are d-open, so that the d-topology refines the ambient topology in flow boxes.
- The d-topologies induced by distinct generic linear projections π coincide.

We can now formulate Su's Theorem in our setting.

Theorem 5.6. *There exists a lamination $\mathcal{L}$ and an injection $i : \mathcal{L} \hookrightarrow \operatorname{Supp} T$ continuous along the leaves and respecting the laminar structure, with image of full measure, and full transverse measure on each transversal.*

The lamination $\mathcal{L}$ has an invariant transverse measure, and if $T(\mathcal{L})$ denotes the associated foliated cycle, one has $i_*(T(\mathcal{L})) = T$.

Proof. We construct a d-open subset $\mathcal{R}' \subset \mathcal{R}$ of full transverse measure, and saturated with respect to the weak lamination on $\mathcal{R}$. Let $(B_m)_{m\geq 0}$ be a covering of $\mathcal{R}$ by flow boxes of positive measure. The d-interior of B_m is denoted by d-int (B_m). For each box B_k, one removes from d-int (B_k) all the plaques corresponding to leaves containing plaques of $\cup_{m\geq 0} B_m \setminus$d-int (B_m) (k itself is included in the union since a leaf may intersect B_k several times). The set of removed plaques has zero transverse measure so it remains a d-open subset $B_k' \subset B_k$. Let $\mathcal{R}' = \cup B_k'$, which is d-open and saturated by construction.

Now for every $x \in \mathcal{R}'$, $x \in B_k'$ for some k, and B_k' is a foliated d-neighborhood of x, so $\mathcal{R}'$ supports a natural lamination $\mathcal{L}$, which has the desired properties. □

Subordinate transverse measures

In this paragraph we relate subordination at the levels of transverse measures and closed laminar currents.

Theorem 5.7. *Let T be a strongly approximable current on $\mathbb{P}^2$, and μ be the induced transverse measure on its associated weak lamination $\mathcal{R}$. Then to every invariant transverse measure $\mu' \leq \mu$ on $\mathcal{R}$, the foliation cycle induced by μ' on $\mathcal{R}$ corresponds to a closed strongly laminar current $T' \leq T$ in $\mathbb{P}^2$.*

Corollary 5.8. *If T is an extremal current, the transverse measure μ is ergodic.*

Recall that "ergodic" means every saturated set $\mathcal{R}' \subset \mathcal{R}$ has either zero, or full measure. This corollary has non trivial dynamical consequences that will be developed in further work. The theorem means that there is a good (one way) correspondence between closed positive currents on the weak lamination $\mathcal{R}$ and closed positive currents on the ambient space. We do conjecture the converse also holds, that is, *every closed positive current $T' \leq T$ is the foliation cycle of some* *invariant transverse measure* $\mu' \leq \mu$. This would imply for instance that T is extremal, if and only if its transverse measure is ergodic. This conjecture seems to raise rather delicate problems of analytic type.

Proof. The result is local in some open Ω. Pick two linear projections π_1 and π_2 satisfying Definition 2.5, and for each basis of projection, consider three overlapping subdivisions by squares of size r, as in Section 3. We then form nine overlapping subdivisions $\mathcal{Q}_1, \dots, \mathcal{Q}_9$ by affine cubes of size r from the projections and the squares, as in Proposition 2.7. For each $\mathcal{Q}_i$, one gets a current $T_{\mathcal{Q}_i}$, uniformly laminar in each cube, and such that $\mathbf{M}(T - T_{\mathcal{Q}_i}) = O(r^2)$.

We are given a measure μ' on the weak lamination $\mathcal{R}$, with $\mu' \leq \mu$. This means that for each local transversal τ (in a flow box by definition of the transversals), there exists a function f_τ, $0 \leq f_\tau \leq 1$, with $\mu' = f_\tau \mu$. Since μ' is invariant by holonomy, the functions f_τ patch together to a global $0 \leq f \leq 1$ on $\mathcal{R}$, constant along the leaves. The associated foliated cycle is the current fT. We have to prove it is closed.

So let ϕ be a test 1-form in Ω. We consider a partition of unity $(\theta_Q)_{Q \in \mathcal{Q}}$ subordinate to the covering $\mathcal{Q} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_9$ of Ω. It is easily seen that since the cubes have size r, the functions θ_Q may be chosen to have derivatives uniformly bounded by C/r, with C independent of r. Now,

$$\langle fT, d\phi \rangle = \left\langle fT, d\Big(\sum_{Q \in \mathcal{Q}} \theta_Q \phi\Big)\right\rangle = \sum_{i=1}^{9} \left\langle fT, d\Big(\sum_{Q \in \mathcal{Q}_i} \theta_Q \phi\Big)\right\rangle ,$$

and for each of the nine terms of the sum, we replace fT by $fT_{\mathcal{Q}_i} + f(T - T_{\mathcal{Q}_i})$. The important fact is that since f is constant along the leaves, for each $Q \in \mathcal{Q}_i$ the current $fT_{\mathcal{Q}_i}|_Q$ is closed in Q, so we get

$$\left\langle fT_{\mathcal{Q}_i}, d\Big(\sum_{Q \in \mathcal{Q}_i} \theta_Q \phi\Big)\right\rangle = \sum_{Q \in \mathcal{Q}_i} \langle fT_{\mathcal{Q}_i}|_Q, d(\theta_Q \phi)\rangle = 0 ,$$

since θ_Q has compact support in Q. On the other hand,

$$\left| \left\langle f(T - T_{\mathcal{Q}_i}), d\Big(\sum_{Q \in \mathcal{Q}_i} \theta_Q \phi\Big)\right\rangle \right| \leq \mathbf{M}(f(T - T_{\mathcal{Q}_i})) \sup_{Q \in \mathcal{Q}_i} \|d(\theta_Q \phi)\| \leq \mathbf{M}(T - T_{\mathcal{Q}_i}) O\Big(\frac{1}{r}\Big) = O(r) .$$

This implies $\langle fT, d\phi \rangle = 0$ and the theorem follows. □

6. Pluripotential theory and laminar currents

We give in this paragraph a few applications of the foregoing study. More precisely, we first prove that the potential of a strongly approximable laminar current T is either harmonic, or identically $-\infty$ on almost all disks subordinate to T (leaves of the induced measured lamination). We also exhibit a decomposition of a strongly approximable laminar current into a sum of two closed laminar currents, one essentially supported on a pluripolar set and the other not charging pluripolar sets.

Some results on uniformly laminar currents

We first collect some useful results on uniformly laminar currents. The proofs only use a few simple ideas from 1-variable classical potential theory. Our first goal is the following proposition, although the intermediate lemmas may be of independent interest.

Proposition 6.1. *Let T be a uniformly laminar current, given as the integral of holomorphic graphs in the bidisk, $T = \int [\Gamma_\alpha]\, d\mu(\alpha)$. Assume T does not give mass to pluripolar sets. Then T can be written as a countable sum $T = \sum T_j$, where the $T_j = \int [\Gamma_\alpha]\, d\mu_j(\alpha)$ have continuous potential and disjoint support.*

The main ingredient of the proof is the following 1-variable result, which may be found for instance in Hörmander's book [16, Theorem 3.4.7].

Proposition 6.2. *Let μ be a positive measure with compact support in $\mathbb{C}$. Assume the logarithmic potential $G_\mu(z) = \int \log|z-\zeta|\, d\mu(\zeta)$ satisfies $G_\mu > -\infty$ μ-a.e.; this is true in particular if μ does not charge polar sets. Then there exists a sequence of disjoint compact subsets (K_j) such that $\mu = \sum \mu|_{K_j}$ and for each j, $\mu|_{K_j}$ has continuous potential.*

The proposition is a consequence of Lusin's Theorem, together with the "continuity principle" for logarithmic potential.

We proceed, in several steps, to the proof of Proposition 6.1. We denote by $\mathcal{L}$ the lamination by horizontal graphs in the bidisk, associated to T. The family of (vertical) disks $\{z\} \times \mathbb{D}$ is a family of global transversals to the lamination. Let $h^z = h^{0,z}$ be the holonomy map from $\mathcal{L} \cap (\{0\} \times \mathbb{D})$ to $\mathcal{L} \cap (\{z\} \times \mathbb{D})$, and similarly $h^{z,z'}$. We identify an abstract transverse measure μ on $\mathcal{L}$ with its image in $\{0\} \times \mathbb{D}$, so that the parameter α is identified with the point $(0, \alpha)$, and let $\mu^z = (h^z)_*\mu$ be the push forward of μ in $\{z\} \times \mathbb{D}$.

Lemma 6.3. *Let $T = \int [\Gamma_\alpha]\, d\mu(\alpha)$ as above. Then the function*

$$u_T : (z, w) \longmapsto \int_{\{z\}\times\mathbb{D}} \log|w-\zeta|\, d\mu^z(\zeta)$$

is a plurisubharmonic potential for T.

Proof. Classical, we include it for completeness. Let $w = \varphi_\alpha(z)$ be the equation of the graph Γ_α. Then

$$u(z, w) = \int \log|w - \varphi_\alpha(z)|\, d\mu(\alpha) \tag{6.1}$$

is a potential for T. Now the holonomy map h^z maps $(0, \alpha)$ to $(z, \varphi_\alpha(z))$, so for any continuous function F on $\{z\} \times \mathbb{D}$

$$\int_{\{z\}\times\mathbb{D}} F(\alpha)\, d\mu^z(\alpha) = \int_{\{0\}\times\mathbb{D}} F(\varphi_\alpha(z))\, d\mu(\alpha)$$

and writing $\log|w - \cdot|$ as a decreasing sequence of continuous functions, we get $u = u_T$. □

There is a good correspondence between continuity properties of the potentials of the transversal measures and the current itself: This is the content of the next lemma.

Lemma 6.4. *Assume μ has continuous potential as a measure on $\{0\} \times \mathbb{D}$. Then for every z, μ^z has continuous potential. Moreover, the above defined potential u_T is continuous.*

Proof. The first assertion is a consequence of the following: The class of plane measures with continuous potentials is preserved by bi-Hölder continuous homeomorphisms. Indeed, let μ be a plane positive measure with compact support, and

$$k_\mu(z,r) = \left| \int_{B(z,r)} \log|z-\zeta|\, d\mu(\zeta) \right| .$$

Let $c \geq 2$; using the fact that $B(w, c\,|z-w|) \subset B(w,(c+1)\,|z-w|)$ and the mean value inequality for the logarithm one easily gets

$$\left|G_\mu(z) - G_\mu(w)\right| \leq \frac{1}{c} + k(z, c\,|z-w|) + k(w,(c+1)\,|z-w|) .$$

Taking for example $c = |z-w|^{-1/2}$, one deduces the following result: If $k(z,r) \to 0$ locally uniformly in z as $r \to 0$, then G_μ is continuous. Using similar estimates it is proven by Shvedov [20] that the converse is also true.

Assume now μ has continuous potential in $\{0\} \times \mathbb{D}$. Then $k_\mu(x,r) \to 0$ uniformly by the Shvedov result; moreover, the holonomy map h^z associated to $\mathcal{L}$ is Hölder continuous, as well as its inverse, say of exponent α, and we get $k_{\mu^z}(w,r) \leq Ck_\mu((h^z)^{-1}(w), Cr^\alpha)$. This implies μ^z also has continuous potential; note that the modulus of continuity is uniform.

It remains to prove continuity of u_T as a function of (z,w). First, we extend the lamination $\mathcal{L}$ to a neighborhood of $\mathrm{Supp}(T)$ using Slodkowski's Theorem. Now it follows from formula (6.1) that the potential u_T is harmonic, or identically $-\infty$ along the leaves. Under the hypothesis of the theorem we know that the restrictions to the slices are continuous and using the Hölder property of holonomy again, one easily gets that u_T is bounded. Then we split

$$u_T(z,w) - u_T\left(z',w'\right) = \left(u_T(z,w) - u_T\left(z, h^{z',z}\left(w'\right)\right)\right) + \left(u_T\left(z, h^{z',z}\left(w'\right) - u_T\left(z',w'\right)\right)\right),$$

where the first term on the right-hand side is small because of the continuity of $\zeta \mapsto u_T(z,\zeta)$, and the second because $z \mapsto u_T(z, h^{z',z}(w'))$ is a uniformly bounded harmonic function, hence uniformly Lipschitz. □

Recall that $X \subset \mathbb{C}$ (resp. $X \subset \mathbb{C}^2$) is polar (resp. pluripolar) if $X \subset \{u = -\infty\}$ where u is a subharmonic (resp. plurisubharmonic) function, not identically equal to $-\infty$.

Lemma 6.5. *Let X be a subset of $\{0\} \times \mathbb{D}$, and $\widehat{X}$ the set saturated from X by the lamination $\mathcal{L}$ (i.e., the set of leaves through X). Then X is polar iff $\widehat{X}$ is pluripolar.*

Proof. First, note that the holonomy map preserves the class of closed polar subsets of the fibers $\{z\} \times \mathbb{D}$. A way to prove this is to use the following characterization of polar sets (transfinite diameter zero, see Tsuji [24]): X is polar iff

$$\lim_{n\to\infty} \sup\left\{\prod \left|x_i - x_j\right|^{2/n(n-1)},\ x_1, \ldots, x_n \in X\right\} = 0 .$$

This condition is stable under bi-Hölder homeomorphisms. Another method is to use Lemma 6.4 and the fact that a non polar compact set carries a measure with continuous potential [16, Theorem 3.4.5].

If X is not closed (polar sets are G_δ sets in general), use the fact that X is polar iff for every compact $K \subset X$, K is polar, and rather transport the compact subsets.

Now assume $\widehat{X}$ is pluripolar. Then $\widehat{X} \subset \{u = -\infty\}$ for some non degenerate p.s.h. function in $\mathbb{D}^2$. Hence, for almost every slice $\{z\} \times \mathbb{D}$, $u|_{\{z\}\times\mathbb{D}} \not\equiv -\infty$ and $\big(u|_{\{z\}\times\mathbb{D}}\big)\big(\widehat{X} \cap (\{z\} \times \mathbb{D})\big) \equiv -\infty$, so $\widehat{X} \cap (\{z\} \times \mathbb{D})$ is polar. The preceding observation implies $X = \widehat{X} \cap (\{0\} \times \mathbb{D})$ is polar.

Conversely, assume X is polar. Then by [16, Theorem 3.4.2] there exists a positive measure μ supported on X such that $X \subset \left\{G_\mu = -\infty\right\}$. Consider the following plurisubharmonic function

$$u(z, w) = \int_{\{z\}\times\mathbb{D}} \log|w - \zeta|\, d\mu^z(\zeta)\,.$$

On each leaf, u is harmonic, or identically $-\infty$. We thus get $\widehat{X} \subset \{u = -\infty\}$. □

From these lemmas one easily deduces the proof of Proposition 6.1. Assume T does not charge pluripolar sets. Then the transverse measure does not charge polar subsets of $\{0\} \times \mathbb{D}$ by Lemma 6.5. Write $\mu = \sum \mu_j$ as given by Proposition 6.2, and $T = \sum T_j$ according to this decomposition. By Lemma 6.4, T_j has continuous potential.

The next proposition gives a decomposition of a uniformly laminar current in the bidisk as a sum of two parts, one giving mass to pluripolar sets, the other not. It will be used in Theorem 6.8.

Proposition 6.6. *Let T be a uniformly laminar current, integral of holomorphic graphs in the bidisk, $T = \int [\Gamma_\alpha]\, d\mu(\alpha)$. Then T admits a unique decomposition as a sum $T = T' + T''$ of uniformly laminar currents, with T' not charging pluripolar sets, and T'' giving full mass to a pluripolar set.*

Proof. Uniqueness is obvious: If $T = T_1' + T_1'' = T_2' + T_2''$, just write $T_1' - T_2' = T_2'' - T_1''$. We first decompose the transverse measure, and then apply the preceding lemmas. Let μ^0 be the slice of T by $\{0\} \times \mathbb{D}$, as before, and $u_0 = u_T(0, \cdot)$ be the logarithmic potential of μ^0. Then

$$\mu^0 = \mu' + \mu'' = \mu^0|_{\{u_0 > -\infty\}} + \mu^0|_{\{u_0 = -\infty\}}\,.$$

Let v be the logarithmic potential of μ'. Since $\mu' \leq \mu^0$, $u_0 - v$ is subharmonic, so $v \geq u_0 + O(1)$, and v is finite μ'-a.e. By Proposition 6.2 above, μ' does not charge polar sets; moreover μ' is a sum of measures with continuous potential. On the other hand, μ'' gives full mass to the polar set $\{u_0 = -\infty\}$.

Now decompose $T = T' + T''$ according to this decomposition of μ^0. By Lemma 6.5 above, T'' has full measure on a pluripolar set. Moreover, since μ' is a sum of measures with continuous potential, we get an analogous decomposition for T' by Lemma 6.4, and T' does not charge pluripolar sets. □

The potential along the leaves

Recall that a disk Δ is subordinate to T, if it is subordinate to a uniformly laminar $S \leq T$ in $\Omega' \subset \Omega$. Notice in the following theorem that the condition of being harmonic, or $-\infty$ on Δ is clearly independent of potential chosen for T.

Theorem 6.7. *Let $T = dd^c u$ be a diffuse strongly approximable laminar current in Ω. Then for almost every disk Δ subordinate to T, with respect to the transverse measure, either $u|_\Delta$ is harmonic, or $u|_\Delta \equiv -\infty$.*

We remark that there are disks on which u is harmonic, without being subordinate to the current in our sense. For example, if T is a current made up of a measured family of disjoint branched coverings of degree 2 over the unit disk, say branched over 0, and accumulating on the horizontal line (this is called a folded uniformly laminar current in [13]), then the estimate (2.5) is satisfied, and the potential of T is harmonic on the horizontal line, even if there is no laminated set of positive measure containing it.

It would be interesting to understand more about the disks in Supp(T) such that $u|_\Delta$ is harmonic. It seems that such disks should be "tangent" to T in some sense.

Also, we believe the result should be true for *every* disk subordinate to T. Of course this is true if u is continuous.

Proof. We have to prove that, if $S \leq T$ is a uniformly laminar current, for a.e. disk Δ subordinate to S, $u|_\Delta$ is harmonic or $-\infty$. Reducing Ω', if necessary we may assume S is made up of graphs over some disk. We apply Propositions 6.6 and 6.1 to S and get $S = S' + S''$; moreover, $S' = \sum S_j$, with the uniformly laminar currents S_j of disjoint support and continuous potential.

We proved in [12, Remark 4.6] that if S_j is uniformly laminar with continuous potential, the wedge product $S_j \wedge T$ is geometric, i.e., described by the geometric intersection of the disks constituting the current; moreover, by Theorem 1.1, disks subordinate to T do not intersect. So $S_j \wedge T = 0$. This means exactly that u is harmonic on a.e. disk of S_j.

On the other hand, we claim that $u \equiv -\infty$ on S''-a.e. leaf. Indeed, S'' gives full mass to a pluripolar set, so if $u_{S''}$ denotes the logarithmic potential of S'' as in Lemma 6.3, one has $u_{S''} \equiv -\infty$ on a.e. leaf of S'', for if $u_{S''}$ was finite on a set of positive measure on some transverse section, say $\{z\} \times \mathbb{D}$, we could construct a measure with continuous potential subordinate to $(\mu^{z})'' = S'' \wedge [\{z\} \times \mathbb{D}]$, which is impossible. We conclude that $u \equiv -\infty$ on almost every leaf of S'' because $T \geq S''$ implies $u \leq u_{S''} + O(1)$. □

A canonical decomposition

The following result is reminiscent of both the Skoda-El Mir extension theorem and Siu's decomposition theorem for positive closed currents. It takes in our case a particularly complete form.

Theorem 6.8. *Let T be a strongly approximable laminar current in $\mathbb{P}^2$. Then there exists a unique decomposition of T as a sum of positive closed laminar currents $T = T' + T''$, where T' does not charge pluripolar sets, and T'' gives full measure to a pluripolar set. Moreover, T' and T'' correspond to foliation cycles on the weak lamination induced by T.*

In particular, if the current T is extremal, only one of T' and T'' can appear.

Proof. Note, first that uniqueness is obvious. The proof actually implies T'' gives full mass to a countable union of locally pluripolar sets, which is globally pluripolar in the special case of $\mathbb{P}^2$, due to a theorem of Alexander.

The result is an easy consequence of Theorem 5.7. Indeed, we saw in Proposition 6.6 that a uniformly laminar current S in a flow box admits a canonical decomposition $S = S' + S''$; this decomposition corresponds to a decomposition of the transverse measure. So for each transversal τ (as defined in Section 5), the measure μ_τ induced by T has a decomposition $\mu'_\tau + \mu''_\tau$, which is holonomy equivariant. Thus, the transverse measure writes as $\mu = \mu' + \mu''$, and applying Theorem 5.7 gives the result. □

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Institut de Mathématiques de Jussieu, Université Denis Diderot,
Case 7012, 2 place Jussieu, 75251 Paris cedex 05, France
e-mail: dujardin@math.jussieu.fr