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DEGREE GROWTH OF MATRIX INVERSION: BIRATIONAL MAPS OF SYMMETRIC, CYCLIC MATRICES

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Abstract. We consider two (densely defined) involutions on the space of $q \times q$ matrices; $I(x_{ij})$ is the matrix inverse of (x_{ij}), and $J(x_{ij})$ is the matrix whose ijth entry is the reciprocal x_{ij}^{-1}. Let $K = I \circ J$. The set $\mathcal{SC}_q$ of symmetric, cyclic matrices is invariant under K. In this paper, we determine the degrees of the iterates $K^n = K \circ \ldots \circ K$ restricted to $\mathcal{SC}_q$.

1. **Introduction.** Our interest is the dynamics of birational mappings in higher dimension. A lot has been accomplished already in dimension 2 (cf. [15], [5], [17], [16]), but little is known in higher dimension (see the survey in Guedj [G]). The family of mappings defined below has attracted our interest because it exhibits a rich blowup/blowdown behavior which cannot occur in dimension 2.

Let $\mathcal{M}_q$ denote the space of $q \times q$ matrices, and let $\mathbf{P}(\mathcal{M}_q)$ denote its projectivization. For a matrix $x = (x_{ij})$ we consider two maps. One is $J(x) = (x_{ij}^{-1})$ which takes the reciprocal of each entry of the matrix, and the other is the matrix inverse $I(x) = (x_{ij})^{-1}$. The involutions I and J, and thus the mapping $K = I \circ J$, arise as basic symmetries in Lattice Statistical Mechanics (see [13], [9]). This leads to the problem of determining the iterated behavior of K on $\mathbf{P}(\mathcal{M}_q)$ (see [1], [2], [4], [10]). A basic question is to know the degree complexity

$$\delta(K) := \lim_{n\to\infty} (\deg(K^n))^{1/n} = \lim_{n\to\infty} (\deg(K \circ \cdots \circ K))^{1/n}$$

of the iterates of this map. The quantity $\log \delta$ is also called the algebraic entropy (see [10]). We note that $\mathbf{P}\mathcal{M}_q$ has dimension $q^2 - 1$, I has degree $q - 1$, and J has degree $q^2 - 1$. Thus a computer cannot directly evaluate the composition $K^2 = K \circ K$ (or even $K = I \circ J$) unless q is small.

The $q \times q$ matrices correspond to the coupling constants of a system in which each location has q possible states. In more specific models, there may be additional symmetries, and such symmetries define a K-invariant subspace $S \subset \mathbf{P}(\mathcal{M}_q)$ (see [3]). In general, the degree of the restriction $K|S$ will be lower than the degree of K, and the

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corresponding question in this case is to know $\delta(K|S) = \lim_{n\to\infty}(\deg(K^n|S))^{1/n}$. An example of this, related to Potts models, is the subspace $\mathcal{C}_q$ of cyclic matrices, i.e., matrices (x_{ij}) for which x_{ij} depends only on $j-i$ (mod q). A cyclic matrix is thus determined by numbers $x_0, \ldots, x_{q-1}$ according to the formula

$$C(x_0,\ldots,x_{q-1}) = \begin{pmatrix} x_0 & x_1 & & x_{q-1} \\ x_{q-1} & \ddots & \ddots & \\ & \ddots & \ddots & x_1 \\ x_1 & & x_{q-1} & x_0 \end{pmatrix} \tag{1.1}$$

The degree growth of $K|\mathcal{C}_q$ was determined in [10]. Another case of evident importance is $\mathcal{SC}_q$, the symmetric, cyclic matrices. The degree growth of $K|\mathcal{SC}_q$ was determined in [4] for prime q. In this paper we determine $\delta(K|\mathcal{SC}_q)$ for all q. In doing this, we expose a general method of determining δ, which we believe will also be applicable to the study of $\delta(K|S)$ for more general spaces S.

Main Theorem. *The dynamical degree $\delta(K|\mathcal{SC}_q) = \rho^2$, where ρ is the spectral radius of an integer matrix M. When q is odd, M is defined by (4.3–7); when $q = 2\times$odd, M is defined by (5.6–12); and when q is divisible by 4, M is defined by (6.5–12).*

The algorithm of the Main Theorem computes δ_q starting with the prime factorization of q. In the Appendix we show how to carry out the algorithm efficiently in the cases $q = 30, 45$, and 60.

The mappings $K|\mathcal{C}_q$ and $K|\mathcal{SC}_q$ lead to maps of the form $f = L\circ J$ on $\mathbf{P}^N$, where L is linear, and $J = [x_0^{-1} : \cdots : x_N^{-1}]$. In the case of $K|\mathcal{C}_q$, we have $L = F$, the matrix representation of the finite Fourier transform, and the entries are qth roots of unity. By the internal symmetry of the map, the exceptional hypersurfaces $\Sigma_i = \{x_i = 0\}$ all behave in the same way, and δ for these maps is found easily by the method of regularization described below. The family of "Noetherian maps" was introduced in [12] and generalized to "elementary maps" in [7]. These maps have the feature that all exceptional hypersurfaces behave like

$$\Sigma_i \to * \to \cdots \to e_i \rightsquigarrow V_i, \tag{1.2}$$

which means that Σ_i blows down to a point $*$, which then maps forward for finite time until it reaches a point of indeterminacy e_i, which blows up to a hypersurface V_i. The reason for $\deg(f^n) < (\deg(f))^n$ comes from the existence of exceptional hypersurfaces like Σ_i, called "degree lowering" in [18], which are mapped into the indeterminacy locus.

As we pass from $K|\mathcal{C}_q$ to $K|\mathcal{SC}_q$, a number of symmetries are added. Because of these additional symmetries, the dimension of the representation $f = L\circ J$ on $\mathbf{P}^N$ changes from $N = q-1$ to $N = \lfloor q/2\rfloor$. The new matrix L, however, is more difficult to work with explicitly; its entries have changed from roots of unity to more general cyclotomic numbers. The exceptional hypersurfaces all blow down to points, but their subsequent behaviors are richly varied, showing phenomena connected to properties of the cyclotomic numbers.

If $f : \mathbf{P}^N \dashrightarrow \mathbf{P}^N$ is a rational map, then there is a well-defined pullback map on cohomology $f^* : H^{1,1}(\mathbf{P}^N) \to H^{1,1}(\mathbf{P}^N)$. The cohomology of projective space is generated by the class of a hypersurface H, and the connection between cohomology and degree is given by the formula

$$(f^n)^* H = (\deg f^n)\cdot H. \tag{1.3}$$

In our approach, we construct a new complex manifold $\pi : X \to \mathbf{P}^N$, which will be obtained by performing certain (depending on f) blow-ups over $\mathbf{P}^N$. This construction induces a rational map $f_X : X \dashrightarrow X$ which has the additional property that

$$(f_X^n)^* = (f_X^*)^n \quad \text{on} \quad H^{1,1}(X), \tag{1.4}$$

which we call 1-regular or 1-stable. Once we have our good model X, we find $\delta(f) = \delta(f_X)$ by computing the spectral radius of the mapping f_X^*.

Diller and Favre [15] showed that such a construction of X with (1.4) is always possible for birational maps in dimension 2. This method for determining δ then gives a tool for deciding whether f is integrable (which happens when $\delta = 1$) or has positive entropy (in which case $\delta(f) > 1$). This was used in the integrable case in [11] and in both cases in [8], [21], and [22] .

We note that the space X which is constructed by this procedure is useful for understanding further properties of f. For instance, it has proved useful in analyzing the pointwise dynamics of f on real points (see [6]).

An important difference between the cases of dimension 2 and dimension > 2, as well as a reason why the maps $K|\mathcal{SC}_q$ do not fall within the scope of earlier approaches, is that exceptional hypersurfaces cannot always be removed from the dynamical system by blow-ups. In fact, the new map f_X can have more indeterminate components and exceptional hypersurfaces than the original map.

Our method proceeds as follows. After choosing subspaces $\lambda_0, \ldots, \lambda_j$ as centers of blowup, we construct X. The blowup fibers Λ_i over λ_i, $i = 0, \ldots, j$, together with H, provide a convenient basis for $Pic(X)$. A careful examination of f^{-1} lets us determine $f_X^{-1}H$ and $f_X^{-1}\Lambda_i$, and thus we can determine the action of the linear map f_X^* on $Pic(X)$. In order to see whether (1.4) holds, we need to track the forward orbits f^nE for each exceptional hypersurface E. By Theorem 1.1, the condition that $f^nE \not\subset \mathcal{I}_X$ for each $n \geq 0$ and each E is sufficient for (1.4) to hold. We develop two techniques to verify this last condition for our maps $K|\mathcal{SC}_q$. One of them, called a "hook," is a subvariety $\alpha_E \not\subset \mathcal{I}_X$ such that $f_X\alpha_E = \alpha_E$, and $f^jE \supset \alpha_E$. The simplest case of this is a fixed point. The other technique uses the fact that $f = L \circ J$ is defined over the cyclotomic numbers, and we cannot have $f_X^nE \subset \mathcal{I}_X$ for number theoretic reasons. This brings us to a second difference between the cases of dimension 2 and dimension > 2: The map $K|\mathcal{SC}_q$ in case q is not prime, cannot be regularized to satisfy (1.4) by the method of point blowups alone.

Let us describe the contents of this paper. In §2 we discuss blowups and the map J. We show how to write blowups in local coordinates, how to describe J_X, and how to determine J_X^*. We also give sufficient conditions for (1.4).

In §3, we show how this approach may be applied to $K|\mathcal{C}_q$. In this case, the exceptional orbits are of the form (1.2). We construct the space X by blowing up the points of the exceptional orbits. After these blowups, the induced map f_X has no exceptional hypersurfaces, which implies that (1.4) holds. A calculation of f_X^* and its spectral radius leads to the same number $\delta(K|\mathcal{C}_q)$ that was found in [10].

In §4, we give the setup of the symmetric, cyclic case. When q is prime, the map $K|\mathcal{SC}_q$ exhibits the same general phenomenon: the orbits of all exceptional hypersurfaces are of the form (1.2). As before, we construct X by blowing up the point orbits, and we find that the new map f_X has no exceptional hypersurfaces. Thus we recapture the $\delta(K|\mathcal{SC}_q)$ from [4].

When q is not prime, however, the map $K|\mathcal{C}_q$ develops a new kind of symmetry as we pass to $\mathcal{SC}_q$. Now there are exceptional orbits

$$\Sigma_i \to * \to \cdots \to p_i \rightsquigarrow W_i \rightsquigarrow \cdots \rightsquigarrow V_i, \tag{1.5}$$

where p_i blows up to a variety W_i of positive dimension but too small to be a hypersurface, yet W_i blows up further and becomes a hypersurface V_i.

In §5, we work with the case where q is a general odd number. We construct our a blowup space $\pi : X \to \mathcal{SC}_q$, and we obtain an induced map f_X. If i is relatively prime to q, then the orbit of Σ_i has the form (1.2), and after blowing up the singular orbit, Σ_i will no longer be exceptional. On the other hand, if i is not relatively prime to q, then the exceptional orbit has the form (1.5). Let r divide q, and let $\hat{r} = q/r$, and define the sets $S_r = \{1 \le j \le (q-1)/2 : \gcd(j,q) = r\}$. We will see below that if $i \in S_r$ and $j \in S_{\hat{r}}$, then there is an interaction between the (exceptional) orbits of Σ_i and Σ_j (see Figure 2). After blowing up along certain linear subspaces, we find a 2-cycle hook $\alpha_r \leftrightarrow \alpha_{\hat{r}}$ for all hypersurfaces Σ_i, $i \in S_r \cup S_{\hat{r}}$.

In §6, we consider the case $q = 2 \times \text{odd}$. We construct a new space by blowing up along various subspaces. We find that for each odd divisor $r > 1$ of q, the exceptional varieties Σ_i, $i \in S_r \cup S_{2r}$ act like the case where q is odd. As before, we construct a hook $\alpha_r \leftrightarrow \alpha_{\hat{r}}$ for all $i \in S_r \cup S_{2r} \cup S_{\hat{r}} \cup S_{2\hat{r}}$. However, there is also a new phenomenon, which we call the "wringer" (see Figure 3), which consists of an f-invariant 4-cycle of blowup fibers. All of the exceptional hypersurfaces Σ_i, $i \in S_1 \cup S_2$ enter the wringer. We find hooks for all of these hypersurfaces, which shows that (1.4) holds for f_X.

In §7, we consider the case where q is divisible by 4. Again, we construct X and obtain a new map f_X. In this case, f_X has some exceptional hypersurfaces with hooks. Yet a number of exceptional hypersurfaces remain to be analyzed. These hypersurfaces are of the form $\Sigma_i \to c_i \to \cdots$: they blow down to points, and we must show that no point of this orbit blows up, i.e., $f_X^n c_i \notin \mathcal{I}_X$ for all $n \ge 0$. The complication of one such orbit is shown in Figure 4. We approach this problem now by taking advantage of cyclotomic properties of the coefficients of f. We show that we can work over the integers modulo μ, for certain primes μ, and the orbit $\{f_X^n c_i : n \ge 0\}$ is pre-periodic to an orbit which is disjoint from $\mathcal{I}_X$ and periodic in this reduced number ring.

In each of these cases, we regularize f by constructing an X such that (0.4) holds, and we write down f_X^* explicitly. Thus $\delta(K|\mathcal{SC}_q)$ is the spectral radius of this linear transformation, which is given as modulus of the largest zero of the characteristic polynomial of f_X^*. We write down general formulas for the characteristic polynomials in the cases q =odd and $q = 2\times$odd.

We give some Appendices to show how our Theorems may be used to calculate $\delta(f)$ in an efficient manner.

The structures of the sets of exceptional hypersurfaces are both complicated and different for the various cases of q. So at the beginning of each section, we give a visual summary of the exceptional hypersurfaces and their orbits.

2. **Complex manifolds and their blow-ups.** Recall that complex projective space $\mathbf{P}^N$ consists of complex $N+1$-tuples $[x_0 : \cdots : x_N]$ subject to the equivalence condition $[x_0 : \cdots : x_N] \equiv [\lambda x_0 : \cdots : \lambda x_N]$ for any nonzero $\lambda \in \mathbf{C}$. A rational map $f = [F_0 : \cdots : F_N] : \mathbf{P}^N \to \mathbf{P}^N$ is given by an $N+1$-tuple of homogeneous polynomials of the same degree d. Without loss of generality we may assume that these polynomials have no common factor. The *indeterminacy locus* $\mathcal{I} = \{x \in \mathbf{P}^N :$

$F_0(x) = \cdots = F_N(x) = 0\}$ is the set of points where f does not define a mapping to $\mathbf{P}^N$. Since the F_j have no common factor, $\mathcal{I}$ has codimension at least 2. Clearly f is holomorphic on $\mathbf{P}^N - \mathcal{I}$, but if $x_0 \in \mathcal{I}$, then f cannot be extended to be continuous at x_0.

If $S \subset \mathbf{P}^N$ is an irreducible algebraic subvariety with $S \not\subset \mathcal{I}$, then we define the *strict image*, written $f(S)$, as the closure of $f(S - \mathcal{I})$. Thus $f(S)$ is an algebraic subvariety of $\mathbf{P}^N$, which is also called strict or proper transform. We say that S is *exceptional* if the dimension of $f(S)$ is strictly less than the dimension of S.

Let Γ_f denote the closure of the graph $\{(x, y) \in (\mathbf{P}^N - \mathcal{I}) \times \mathbf{P}^N : f(x) = y\}$, and let $\pi_j : \Gamma_f \to \mathbf{P}^N$ be the coordinate projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. For $x \in \mathcal{I}$, we have $f(x) = \pi_2\pi_1^{-1}(x) = \bigcap_{\epsilon>0} \operatorname{closure} f(B(x,\epsilon) - \mathcal{I})$. For a set S we define the *total image* $f_*(S) := \pi_2\pi_1^{-1}(S)$. If S is a subvariety, we have $f_*(S) \supset f(S)$.

A linear subspace is defined by a finite number of linear equations

$$\lambda = \{x \in \mathbf{P}^N : \ell_j(x) = 0,\ 0 \le j \le M\}$$

where $\ell_j(x) = \sum c_{jk}x_k$ and $M \ge 1$. After a linear change of coordinates, we may assume $\lambda = \{x_0 = \cdots = x_M = 0\}$. Thus λ is naturally equivalent to $\mathbf{P}^{N-M-1}$. As a global manifold, $\mathbf{P}^N$ is covered by $N+1$ coordinate charts $U_j = \{x_j \ne 0\} \cong \mathbf{C}^N$. On the coordinate chart U_N we have coordinates $\zeta_j = x_j/x_N$, $0 \le j \le N-1$, so

$$\lambda \cap U_N = \{(\zeta_0, \ldots, \zeta_{N-1}) \in \mathbf{C}^N : \zeta_0 = \cdots = \zeta_M = 0\}.$$

We define the *blowup of* $\mathbf{P}^N$ *over* λ in terms of a complex manifold X and a holomorphic projection $\pi : X \to \mathbf{P}^N$. (See also [19].) Working inside the coordinate chart U_N, we set

$$\pi^{-1}(U_N) \cap X := \{(\zeta, \xi) \in \mathbf{C}^N \times \mathbf{P}^M : \zeta_j\xi_k - \zeta_k\xi_j = 0,\ \forall 0 \le j, k \le M\}$$

and $\pi(\zeta, \xi) = \zeta$. We see that $\pi^{-1} : \mathbf{C}^N - \lambda \to X$ is well-defined and holomorphic, but for $\zeta \in \lambda$ we have $\pi^{-1}(\zeta) = \mathbf{P}^M$. We write a fiber point $\xi \in \pi^{-1}(\zeta)$ as $(\zeta; \xi)$ or $\zeta; \xi$. Abusing notation slightly, we may consider the curve

$$\gamma_\xi : t \mapsto \zeta + t\xi \in \mathbf{C}^N, \tag{2.1}$$

and we say that γ lands at $\zeta; \xi \in X$ when we mean that $\lim_{t\to 0} \pi^{-1}\gamma(t) = \zeta; \xi$. It is convenient for future computations that the exceptional hypersurface $\Lambda := \pi^{-1}\lambda = \mathbf{P}^{N-M-1} \times \mathbf{P}^M$ is a product. Namely, given $z \in \mathbf{P}^{N-M-1}$ and $\xi \in \mathbf{P}^M$, we can represent the line $\overline{z\xi} = \{z + t\xi : t \in \mathbf{C}\}$. This line is independent of choice of representatives z and ξ; and the fiber point $z; \xi$ is the limit in X of the point $z + t\xi$ as $t \to 0$. The fiber of Λ over a point $x \in \lambda$ is illustrated in Figure 1.

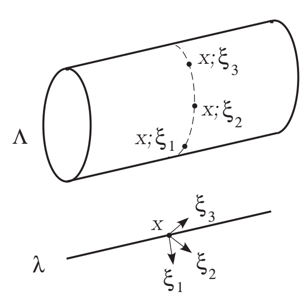

FIGURE 1. Blowup of a Linear Subspace.

For future reference, we give a local coordinate system at a point $p \in \Lambda$. Without loss of generality, we may suppose that $p = (\zeta;\xi)$, where $\zeta = (0,0) \in \mathbf{C}^{M+1} \times \mathbf{C}^{N-M-1}$ and $\xi = [1 : 0 : \cdots : 0] \in \mathbf{P}^M$. Thus we set $\xi_0 = 1$ and define coordinates $(\zeta_0, \xi_1, \ldots, \xi_M, \zeta_{M+1}, \ldots, \zeta_{N-1})$ for the point

$$(\zeta;\xi) = ((\zeta_0, \zeta_0\xi_1, \ldots, \zeta_0\xi_M, \zeta_{M+1}, \ldots, \zeta_N); [1 : \xi_1 : \cdots : \xi_M]) \in X. \tag{2.2}$$

The blowing-up construction is clearly local, so we may use it to blow up a smooth submanifold of a complex manifold. Suppose that $f : \mathbf{P}^N \to \mathbf{P}^N$ is locally biholomorphic at a point p, that λ_1 is a smooth submanifold containing p, and that $\lambda_2 = f\lambda_1$. Let $\pi : Z \to \mathbf{P}^N$ denote the blowup of λ_1 and λ_2. Then for $p;\xi$ in the fiber over $p \in \lambda_1$, we have $f_Z(p;\xi) = fp; df_p\xi$.

If we wish to blow up both a point p and a submanifold λ which contains p, we first blow up p, and then we blow up the strict transform of λ. In the sequel, we will also perform blowups of submanifolds which intersect but do not contain one another. For example, let us consider the x_1-axis $X_1 := \{x_2 = x_3 = 0\} \subset \mathbf{C}^3$ and the x_2-axis $X_2 := \{x_1 = x_3 = 0\} \subset \mathbf{C}^3$. Let $\pi_1 : M_1 \to \mathbf{C}^3$ be the blowup of X_1. The fibers over points of X_1 have the form $\pi_1^{-1}(x_1, 0, 0) = \{(x_1, 0, 0); [0 : \xi_2 : \xi_3]\} \cong \mathbf{P}^1$. These may be identified with the landing points of arcs which approach X_1 normally as in (2.1). Let us set $E_1 := \pi_1^{-1}0$, and let X_2 denote the strict transform of X_2 inside M_1, i.e., $X_2 = \pi_1^{-1}(X_2^*)$. Thus $X_2 \cap E_1 = (0; [0 : 1 : 0])$. Now let $\pi_{12} : M_{12} \to M_1$ denote the blow up of $X_2 \subset M_1$, and set $\pi' : \pi_1 \circ \pi_{12} : M_{12} \to \mathbf{C}^3$. It follows that $(\pi')^{-1}$ is holomorphic on $\mathbf{C}^3 - (X_1 \cup X_2)$. Since π_{12} is invertible over points of $M_1 - X_2 \supset \pi_1^{-1}(X_1 - \{0\})$, the fiber points over $X_1 - \{0\}$ may still be identified with the landing points of arcs approaching X_1 normally. Similarly, we may identify points of $\pi_{12}^{-1}(X_2 - \pi_1^{-1}0)$ as landing points of arcs approaching X_2 normally.

In a similar fashion, we may construct the blow-up space $\pi'' := \pi_2 \circ \pi_{21} : M_{21} \to \mathbf{C}^3$ by blowing up X_2 first and then X_1. We say that a map $h : X_1 \to X_2$ is a *pseudo-isomorphism* if it is biholomorphic outside a subvariety of codimension ≥ 2. Thus (π', M_{12}) and (π'', M_{21}) are pseudo-isomorphic, since $(\pi'')^{-1} \circ \pi'$ extends to a biholomorphism between $M_{12} - (\pi')^{-1}0$ and $M_{21} - (\pi'')^{-1}0$. In our discussion of degree growth, we will be concerned only with divisors, and in this context pseudo-isomorphic spaces are equivalent. Thus when we perform multiple blowups, we will not be concerned about the order in which they are performed since the spaces obtained will be pseudo-isomorphic.

Next we discuss the map $J : \mathbf{P}^N \dashrightarrow \mathbf{P}^N$ given by $J[x_0 : \cdots : x_N] = [x_0^{-1} : \cdots : x_N^{-1}] = [x_{\hat{0}} : \cdots : x_{\hat{N}}]$, where we write $x_{\hat{k}} = \prod_{j \neq k} x_j$. The behaviors we will discuss occur when $N \geq 3$. For a subset $T \subset \{0, \ldots, N\}$ we use the notation

$$\Pi_T = \{x \in \mathbf{P}^N : x_t = 0 \ \forall t \notin T\}, \quad \Pi_T^* = \{x \in \Pi_T : x_t \neq 0 \ \forall t \in T\}$$

$$\Sigma_T = \{x \in \mathbf{P}^N : x_t = 0 \ \forall t \in T\}, \quad \Sigma_T^* = \{x \in \Sigma_T : x_t \neq 0 \ \forall t \notin T\}.$$

A point x is indeterminate for J exactly when two or more coordinates are zero. That is to say

$$\mathcal{I}(J) = \bigcup_{\#T \geq 2} \Sigma_T.$$

The total image of an indeterminate point is given by

$$\Pi_T^* \ni p \mapsto J_*p = \Sigma_T, \quad \text{and} \quad \Sigma_T^* \ni p \mapsto J_*p = \Pi_T. \tag{2.3}$$

The exceptional hypersurfaces for J are exactly the hypersurfaces Σ_i for $0 \leq i \leq N$, and we have $f(\Sigma_i) = e_i := [0 : \cdots : 0 : 1 : 0 : \cdots]$. Let $\pi : X \to \mathbf{P}^N$ denote the blowup of the point e_i, and let $E_i := \pi^{-1} e_i \cong \mathbf{P}^{N-1}$. We introduce the notation $x' = [x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_N]$ and $J'x' = [x_0^{-1} : \cdots : x_{i-1}^{-1} : 0 : x_{i+1}^{-1} : \cdots : x_N^{-1}]$. Thus near Σ_i we have

$$J[x_0 : \cdots : x_{i-1} : t : x_{i+1} : \cdots : x_N] = e_i + tJ'x'. \tag{2.4}$$

Letting $t \to 0$, we find that the induced map $J_X : X \dashrightarrow X$ is given by

$$J_X : \Sigma_i \ni x' \mapsto e_i; J'x' \in E_i. \tag{2.5}$$

The effect of passing to the blowup X is that Σ_i is no longer exceptional. Since J is an involution, we also have

$$J_X : E_i \ni e_i; \xi' \mapsto J'\xi' \in \Sigma_i. \tag{2.6}$$

Let $T \subset \{0, \ldots, N\}$ be a subset with $i \notin T$ and $\#T \geq 2$, and let Σ_T denote its strict transform inside X. We see that $\Sigma_T \cap E_i$ is nonempty and indeterminate for J_X, and the union of such sets gives $E_i \cap \mathcal{I}(J_X)$.

Now let us discuss the relationship between blowups and the indeterminate strata of J. For $T \subset \{0, \ldots, N\}$, $\#T \geq 2$, we have $\Sigma_T \subset \mathcal{I}$, and $J_* : \Sigma_T^* \ni p \mapsto \Pi_T$. Let $\pi : X \to \mathbf{P}^N$ be the blowup of $\mathbf{P}^N$ along the subspaces Σ_T and Π_T. Let $S_T = \pi^{-1}\Sigma_T$ and $P_T = \pi^{-1}\Pi_T$ denote the exceptional fibers. The induced map $J_X : X \dashrightarrow X$ acts to interchange base and fiber coordinates:

$$J_X : S_T \dashrightarrow P_T, \qquad S_T \cong \Sigma_T \times \Pi_T \ni (x; \xi) \mapsto (J''\xi; J'x) \in \Pi_T \times \Sigma_T \cong P_T, \tag{2.7}$$

where $J''(\xi) = \xi^{-1}$ on Π_T, and $J'(x) = x^{-1}$ on Σ_T. In particular, J_X is a birational map which interchanges the two exceptional hypersurfaces, and acts again like J, separately on the fiber and base, and interchanges fiber and base.

Now let $\pi : X \to \mathbf{P}^N$ be a complex manifold obtained by blowing up a sequence of smooth subspaces. If $r = p/q$ is a rational function (quotient of two homogeneous polynomials of the same degree), we will say that $\pi^* r := r \circ \pi$ is a rational function on X. We consider the group $Div(X)$ of integral divisors on X, i.e. the finite sums $D = \sum n_j V_j$, where $n_j \in \mathbf{Z}$, and V_j is an irreducible hypersurface in X. We say that divisors D, D' are linearly equivalent if there is a rational function on X such that $D - D'$ is the divisor of r. We define $Pic(X)$ to be the set of divisors on X modulo linear equivalence.

For a rational map $f : X \dashrightarrow Y$, there is an induced map $f^* : Pic(Y) \to Pic(X)$: if $D \in Pic(Y)$, its preimage $f^{-1}(D)$ is well defined as a divisor on $X - \mathcal{I}$ because f is holomorphic there. Taking its closure inside X, we obtain $f^* D = (f^{-1})_* D$, the total transform of D under f^{-1}. Let $H = \{\ell = 0\}$ denote a linear hypersurface in $\mathbf{P}^N$. The group $Pic(\mathbf{P}^N)$ is generated by H. If $f : \mathbf{P}^N \dashrightarrow \mathbf{P}^N$ is a rational map, then $f^* H = \deg(f) H$. Let $H_X = \pi^* H$ be the divisor of $\pi^*\ell = \ell \circ \pi$ in X. A basis for $Pic(X)$ is given by H_X, together with the (finitely many) irreducible components of exceptional hypersurfaces for π. We may choose an ordered basis $H_X, E_1, \ldots, E_s$ for $Pic(X)$ and write f^* with respect to this basis as an integer matrix M_f. It follows that $\deg(f)$ is the (1,1) entry of M_f.

Let us consider the blowup $\pi : Y \to \mathbf{P}^N$ of $\Sigma_{0,\ldots,M} = \{x_0 = \cdots = x_M = 0\}$, with $M < N$. We write $\mathcal{F}(x) := \pi^{-1} x$ for the fiber over $x \in \Sigma_{0,\ldots,M}$, and we let $\Lambda := \pi^{-1}\Sigma_{0,\ldots,M}$ denote the exceptional divisor of the blowup. It follows that H_Y and Λ give a basis of $Pic(Y)$. Let $J_Y : Y \dashrightarrow Y$ denote the map induced by J. For $j > M$, the induced map $J_Y|\Sigma_j : \Sigma_j \dashrightarrow \mathcal{F}(e_j)$ may be written in coordinates in a fashion similar to (2.5) and is seen to be a dominant map. Since $\mathcal{F}(e_j) \cong \mathbf{P}^{N-M-1}$, we see that Σ_j is exceptional.

We have noted that $\Sigma_{0,\ldots,M} \subset \mathcal{I}$ and that $\Sigma_{0,\ldots,M} \ni p \mapsto J_*p = \Pi_{0,\ldots,M}$. The indeterminacy locus $\mathcal{I}_Y$ of J_Y has codimension 2 and thus does not contain Λ. In fact,

$$J_Y|\mathcal{F}(p) : \mathcal{F}(p) \dashrightarrow \Pi_{0,\ldots,M} \tag{2.8}$$

can be written in coordinates similar to (2.6) and is thus seen to be birational. Now let L be an invertible linear map of $\mathbf{P}^N$, let $f := L \circ J$, and let f_Y be the induced birational map of Y. We write $L = (\ell_0, \ldots, \ell_N)$ for the columns of L. Thus $f\Sigma_j = \ell_j$. We now determine $f_Y^* : Pic(Y) \to Pic(Y)$ in terms of the basis $\{H_Y, \Lambda\}$. Let Γ_L denote the dimension $M+1$ subvariety such that $f\Gamma_L = \Sigma_{0,\ldots,M}$. Assuming that $\Gamma_L \not\subset \Sigma_{0,\ldots,M}$, we may take its strict transform in Y to have

$$f_Y^{-1}\Lambda = \Gamma_L \cup \bigcup_{\ell_j \in \Sigma_{0,\ldots,M}} \Sigma_j, \quad \text{or} \quad f_Y^*\Lambda = \sum_{\ell_j \in \Sigma_{0,\ldots,M}} \Sigma_j. \tag{2.9}$$

We see that we have multiplicity 1 for the divisors Σ_j because the linear factor t in (2.4), we mean that the pullback of the defining function will vanish to first order. Now let us write the class of $\Sigma_j \in Pic(Y)$ in terms of the basis $\{H_Y, \Lambda\}$. First, we see that $\Sigma_j = \{x_j = 0\} = H$ is the class of a general hypersurface in $Pic(\mathbf{P}^N)$, so $\pi^*\Sigma_j = H_Y$. Since we have $\Sigma_{0,\ldots,M} \subset \Sigma_j$ if and only if $j \le M$, we have

$$\Sigma_j = H_Y - \Lambda \ \text{ if } \ j \le M, \qquad \Sigma_j = H_Y \ \text{ otherwise.} \tag{2.10}$$

For instance, if we have $\ell_0, \ell_N \in \Sigma_{0,\ldots,M}$ and $\ell_j \notin \Sigma_{0,\ldots,M}$ for $1 \le j \le N-1$, then we have

$$J_Y^*\Lambda = 2H_Y - \Lambda. \tag{2.11}$$

Finally, we determine $f_Y^*H_Y$. We start by noting that in $\mathbf{P}^N$ we have $H = \{\varphi = 0\}$, and on $\mathbf{P}^N$ we have $f^*H = J^*L^*H = J^*H = J^{-1}\{\varphi = 0\} = N \cdot H$. Now we want to use π^* to pull this equation back to $Pic(Y)$, but in general $\pi^*J^* \ne (J \circ \pi)^*$. We have seen that J_Y maps $\Lambda - \mathcal{I}$ to the strict transform of $\Pi_{0,\ldots,M}$ which is not contained in a general hyperplane. Thus $f_Y^{-1}\{\varphi = 0\}$ will not contain Λ. Pulling back by π^*, we have

$$\pi^*J^*H = N \cdot H_Y = J_Y^*H_Y + m\Lambda \tag{2.12}$$

for some integer m. Writing $\varphi = \sum c_j x_j$, we have $J^*(\varphi) = \sum c_j \widehat{x_j}$, which vanishes to order M on $\Sigma_{0,\ldots,M}$, so $m = M$. To summarize the case where only ℓ_0 and ℓ_N belong to $\Sigma_{0,\ldots,M}$, we may represent f_Y^* with respect to the basis $\{H_Y, \Lambda\}$ as the matrix

$$M_{f_Y} = \begin{pmatrix} N & 2 \\ -M & -1 \end{pmatrix}. \tag{2.13}$$

If $(M_f)^n = M_{f^n}$, then the matrix M_f allows us to determine the degrees of the iterates of f, since the degree of f^n is given by the (1,1)-entry of M_{f^n}. The following result gives a sufficient condition for this to hold. Fornæss and Sibony [18] showed that when $X = \mathbf{P}^N$, this condition is actually equivalent to (2.14). Theorem 2.1 is a special case of Propositions 1.1 and 1.2 of [7].

Theorem 2.1. *Let $f : X \dashrightarrow X$ be a rational map. We suppose that for all exceptional hypersurfaces E there is a point $p \in E$ such that $f^np \notin \mathcal{I}$ for all $n \ge 0$. Then it follows that*

$$(M_f)^n = M_{f^n} \quad \text{for all } n \ge 0. \tag{2.14}$$

Proof. Condition (2.14) is clearly equivalent to condition (1.4). Thus we need to show that $(f^*)^2 = (f^2)^*$ on $Pic(X)$. If D is a divisor, then f^*D is the divisor on X which is the same as $f^{-1}D$ on $X - \mathcal{I}(f)$, since $\mathcal{I}(f)$ has codimension at least 2. Now $\mathcal{I}(f^2) = \mathcal{I}(f) \cup f^{-1}\mathcal{I}(f)$, and we have $(f^2)^*D = f^*(f^*D)$ on $X - \mathcal{I}(f) - f^{-1}\mathcal{I}(f)$. By our hypothesis, $f^{-1}\mathcal{I}(f)$ has codimension at least 2. Thus we have $(f^2)^*D = (f^*)^2D$ on X. □

We note that if there is a point $p \in E$ such that $f^n p \notin \mathcal{I}$ for all $n \geq 0$, then the set $E - \bigcup_{n\geq 0} \mathcal{I}(f^n)$ has full measure in E. Thus the forward pointwise dynamics of f is defined on almost every point of E. The following three results are direct consequences of Theorem 2.1.

Corollary 2.2. *If for each irreducible exceptional hypersurface E, we have $f^n E \not\subset \mathcal{I}$ for all $n \geq 1$, then condition (2.14), or equivalently (1.4), holds .*

Proposition 2.3. *Let $f : X \dashrightarrow X$ be a rational map. Suppose that there is a subvariety $S \subset X$ such that $S, fS \ldots, f^{j-1}S \not\subset \mathcal{I}$, and $f^j S = S$. If E is an exceptional hypersurface such that $E, f^2E, \ldots, f^{\ell-1}E \not\subset \mathcal{I}$, and $f^\ell E \supset S$, then there is a point $p \in E$ such that $f^n p \notin \mathcal{I}$ for all $n \geq 0$.*

In this situation, we will say that S is a *hook* for E. Sometimes, instead of specifying $fS = S$, we will say that $f : S \to S$ is a *dominant map*, which means that the generic rank of $f|S$ is the same as the dimension of the target space S.

Theorem 2.4. *Let $f : X \dashrightarrow X$ be a rational map. If there is a hook for every exceptional hypersurface, then (1.4) and (2.14) hold.*

3. Cyclic (circulant) matrices.

$$\Sigma_i \to F_i \to E_i$$

Let ω denote a primitive qth root of unity, and let us write $F = (\omega^{jk})_{0\leq j,k\leq q-1}$, i.e.,

$$F = \big(f_0, \ldots, f_{q-1}\big) = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \ldots & \omega^{q-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \ldots & \omega^{2(q-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{q-1} & \omega^{2(q-1)} & \omega^{3(q-1)} & \ldots & \omega^{(q-1)^2} \end{pmatrix}.$$

Given numbers $x_0, \ldots, x_{q-1}$, we have the diagonal matrix

$$D = D(x_0, \ldots, x_{q-1}) = \begin{pmatrix} x_0 & & \\ & \ddots & \\ & & x_{q-1} \end{pmatrix}.$$

A basic property (cf. [D, Chapter 3]) is that F conjugates diagonal matrices to cyclic matrices. Specifically,

$$C(x_0, \ldots, x_{q-1}) = F^{-1} D(x'_0, \ldots, x'_{q-1}) F,$$

where $(x'_0, \ldots, x'_{q-1}) = F(x_0, \ldots, x_{q-1})$. Thus the map $x \mapsto F^{-1}D(Fx)F$ gives an isomorphism between $\mathcal{C}_q$ and $\mathbf{P}^{q-1}$. The map $I : \mathcal{C}_q \to \mathcal{C}_q$ may now be represented as

$$C(x_0, \ldots, x_{q-1})^{-1} = F^{-1} D(J(F(x_0, \ldots, x_{q-1}))F.$$

Thus $K = I \circ J : \mathcal{C}_q \to \mathcal{C}_q$ is conjugate to the mapping

$$F^{-1} \circ J \circ F \circ J : \mathbf{P}^{q-1} \dashrightarrow \mathbf{P}^{q-1},$$

where $F : \mathbf{P}^{q-1} \to \mathbf{P}^{q-1}$ denotes the matrix multiplication map $x \mapsto Fx$. A computation (see [D, p. 31]) shows that F^2 is q times the permutation matrix corresponding to the permutation $x_j \leftrightarrow x_{q-j}$ for $1 \le j \le q-1$, so F^4 is a multiple of the identity matrix. On projective space, F^2 simply permutes the coordinates, so we have $F^2 \circ J = J \circ F^2$. From this and the identity $F^{-1} = F^3$ we conclude that $(F^{-1}JFJ)^n = A(FJ)^{2n}$, where $A = I$ if n is even and $A = F^2$ if n is odd. Thus we have

$$\delta(K|\mathcal{C}_q) = (\delta(FJ))^2.$$

Following the discussion in §2, we know that the exceptional divisors of $f := F \circ J$ are $\Sigma_j = \{x_j = 0\}$ for $0 \le j \le q-1$. It is evident that $J(f_j) = \bar{f}_j = f_{q-j}$, so

$$\Sigma_j \to f_j \to e_j \rightsquigarrow F\Sigma_j.$$

We let $\pi : X \to \mathbf{P}^{q-1}$ denote the complex manifold obtained by blowing up the orbits $\{f_j, e_j\}$, $0 \le j \le q-1$. Let F_j and E_j denote the blow-up fibers in X over f_j and e_j. It follows that

$$f_X^* : \quad E_j \mapsto F_j \mapsto \Sigma_j = H_X - \sum_{k \ne j} E_k \tag{3.1}$$

Further, by §1 or [7] we have that f_X satisfies (1.4), and

$$f_X^* H_X = (q-1)H_X - (q-2)\sum_{k=0}^{q} E_k. \tag{3.2}$$

We take $\{H_X, E_0, F_0, \ldots, E_{q-1}, F_{q-1}\}$ as an ordered basis for $H^{1,1}(X)$. Thus the linear transformation f_X^* is completely defined by (3.1) and (3.2), and we may write it in matrix form as:

$$f_X^* = \begin{pmatrix} q-1 & 0 & 1 & & 0 & 1 \\ -q+2 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & & 0 \\ -q+2 & & -1 & \ldots & & -1 \\ 0 & & 0 & \ldots & & 0 \\ & & & \ldots & & \\ -q+2 & & -1 & \ldots & 0 & 0 \\ 0 & & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{3.3}$$

It follows that $\deg(f^n)$ is the upper left hand entry of the nth power of the matrix (3.3). Using row and column operations, we find that the characteristic polynomial of (3.3) is

$$(x-1)^q(x+1)^{q-1}(x^2 + (2-q)x + 1).$$

Summarizing our discussion, we obtain the degree complexity numbers which were found earlier in [10]:

Theorem 3.1. *$\delta(K|\mathcal{C}_q)$ is ρ^2, where ρ is the largest zero of $x^2 + (2-q)x + 1$.*

4. **Symmetric, cyclic matrices: prime q.**

$$\Sigma_0 \to A_0 \to E_0$$
$$\Sigma_i \to A_i \to V_i \to AV_i \to E_i$$

To work with symmetric, cyclic matrices, we consider separately the cases of q even and odd. In §3 and §4 we will assume that

$$q \text{ is odd, and we define } p := (q-1)/2.$$

If the matrix in (1.1) is symmetric, it has the form

$$M(x_0, x_1, \ldots, x_p, x_p, \ldots, x_1) = M(\iota x), \tag{4.1}$$

where $\iota(x_0, \ldots, x_p) = (x_0, x_1, \ldots, x_p, x_p, \ldots, x_1)$. Thus, in analogy with §3, we have an isomorphism

$$\mathbf{P}^p \ni x \mapsto F^{-1}D(F\iota x)F \in \mathcal{SC}_q.$$

With this isomorphism, we transfer the map $F \circ J : \mathcal{SC}_q \dashrightarrow \mathcal{SC}_q$ to a map

$$f := A \circ J : \mathbf{P}^p \dashrightarrow \mathbf{P}^p$$

where A is a $(p+1) \times (p+1)$ matrix which will we now determine. It is easily seen that the 0th column a_0 is the same as the 0th column $f_0 = (1, \ldots, 1)$. For $1 \le j \le p$, the symmetry of ιx means that the jth column of A is the sum of the jth and $(q-j)$th columns of F. Thus we have

$$A = \big(a_0, \ldots, a_p\big) = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 1 & \omega_1 & \omega_2 & \ldots & \omega_p \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_p & \omega_{2p} & \ldots & \omega_{p^2} \end{pmatrix},$$

where we define

$$\omega_j = \omega^j + \omega^{q-j}.$$

Immediate properties are

$$\omega_j = \omega_{-j}, \quad \omega_j = \omega_{j+q}, \quad \omega_{p+j+1} = \omega_{p-j}, \quad \omega_j\omega_k = \omega_{j+k} + \omega_{j-k}. \tag{4.2}$$

Summing over roots of unity, we find

$$1 + \sum_{t=1}^{p} \omega_{st} = 0 \quad \text{if} \quad s \not\equiv 0 \bmod q. \tag{4.3}$$

By (4.2), the (j,k) entry of A^2 is $\sum \omega_{jt}\omega_{tk} = (1+\sum_{t=1}^p \omega_{(j+k)t}) + (1+\sum_{t=1}^p \omega_{(j-k)t})$. Thus, by (4.3), $A^2 = qI$, so A acts as an involution on projective space.

As in the general cyclic case, we see that we have the orbit

$$\Sigma_0 \to a_0 \to e_0.$$

Now we consider the orbit of Σ_i for $i \neq 0$. Let us define $v_1 = [1 : t_1 : \cdots : t_p] \in \mathbf{P}^p$ to be the point whose entries are ± 1 and which is given by

$$\begin{aligned} t_{2n} = t_{2n+1} = (-1)^n \quad &\text{if } p \text{ is even, so} \quad v_1 = [1 : 1 : -1 : -1 : \cdots] \\ t_{2n-1} = t_{2n} = (-1)^n \quad &\text{if } p \text{ is odd, so} \quad v_1 = [1 : -1 : -1 : 1 : \cdots]. \end{aligned} \tag{4.4}$$

Lemma 4.1. *$Ja_1 = Av_1$.*

Proof. $Ja_1 = [1 : 2/\omega_1 : \cdots : 2/\omega_p] = [t_1 : 2t_1/\omega_1 : \cdots : 2t_1/\omega_p]$. Thus we must show

$$t_1 = 1 + 2\sum_{j=1}^{p} t_j, \quad \text{and} \qquad 2t_1 = \omega_k(1 + \sum_{j=1}^{p} \omega_{kj} t_j), \ \ \forall\ 1 \le k \le p. \tag{4.5}$$

The left hand equality is immediate from (4.4). Let us next consider the right hand equation for $k = 1$. Using (4.2), we may rewrite this as

$$2t_1 = \omega_1 + t_1(\omega_0+\omega_2)+t_2(\omega_1+\omega_3)+t_3(\omega_2+\omega_4)+t_4(\omega_3+\omega_5)+\cdots+t_p(\omega_{p-1}+\omega_{p+1}).$$

In order for the ω_1 term to cancel, we need $t_2 = -1$. For ω_3 to cancel, we must have $t_4 = -t_2$, etc. We continue in this fashion and determine $t_j = -t_{j-2}$ for all even j. Using (4.2), we see that $\omega_{p-1} = \omega_p$, so this equation ends like

$$\cdots + t_{p-1}(\omega_{p-2} + \omega_p) + t_p(\omega_{p-1} + \omega_p).$$

Thus we have $t_p = -t_{p-1}$. Now we can come back down the indices and determine $t_{j-2} = -t_j$ for all odd j. We see that these values of t_j are consistent with (4.4), which shows that the right hand equation holds for $k = 1$. □

Now for general k, we have

$$2t_1 = \omega_k + t_1(\omega_0 + \omega_{2k}) + t_2(\omega_k + \omega_{3k}) + t_3(\omega_{2k} + \omega_{4k}) + t_4(\omega_{3k} + \omega_{5k}) + \cdots$$
$$\cdots + t_p(\omega_{(p-1)k} + \omega_{(p+1)k}),$$

and we can repeat the argument that was used for $k = 1$.

We will make frequent use of the sets

$$S_r := \{1 \le j \le p : \ \gcd(j, q) = r\}.$$

Thus S_1 consists of all the numbers $\le p$ which are relatively prime to q. This means that $S_1 = \{1, 2, \ldots, p\}$ if and only if p is prime. Now let us fix $k \in S_1$. The numbers $\omega_1, \ldots, \omega_p$ are distinct, and by the middle equations in (4.2), there is a permutation π of the set $\{1, \ldots, p\}$ such that

$$\{\omega_k, \omega_{2k}, \ldots, \omega_{pk}\} = \{\omega_{\pi(1)}, \ldots, \omega_{\pi(p)}\}$$

provided that p is prime. Let us define

$$v_k = [1 : t'_1 : \cdots : t'_p], \quad t'_{\pi(j)} = t_j$$

with t_j as in (4.4), so v_k is obtained from v_1 by permuting the coordinates.

Lemma 4.2. *If $k \in S_1$, then $Ja_k = Av_k$.*

Proof. As in Lemma 4.1, we will show that $\omega_{ik}(1+\sum_J \omega_{Ji} t'_J) = 2t'_k$ for all $1 \le i \le p$. By Lemma 4.1, we have $\omega_I(1 + \sum \omega_{Ij} t_j) = 2t_1$ for all $1 \le I \le p$. First observe that $\pi(1) = k$, so $t'_k = t_1$. Now set $I = \pi(i)$ and $J = \pi(j)$. It follows that the second equation is obtained from the first one by substitution of the subscripts, which amounts to permuting various coefficients. □

Theorem 4.3. *If $k \in S_1$, then f maps:*

$$\Sigma_k \to a_k \to v_k \to Av_k \to e_k.$$

Proof. We have $fa_k = AJa_k = A^2 v_k$ by Lemma 4.2, and this is equal to v_k since A is an involution. Next, $fv_k = AJv_k = Av_k$, since $Jv_k = v_k$. Finally, $fAv_k = AJAv_k = AJJa_k = Aa_k = e_k$. The second equality follows from Lemma 4.2, and the third equality follows because A is an involution. □

To conclude this Section, we suppose that q is prime. This means that $S_1 = \{1, \dots, p\}$. Let X be the complex manifold obtained by blowing up the points a_j and e_j for $0 \le j \le p$ as well as v_j and Av_j for $1 \le j \le p$. Let $f_X : X \dashrightarrow X$ be the induced birational map. It follows from §1 that f_X has no exceptional divisors and is thus 1-regular, i.e. f_X satisfies (1.4). We note that $\{\Sigma_0\}_X$ denotes the class generated by the strict transform of Σ_0 in $Pic(X)$. To write this in terms of our basis, we observe that of all the blowup points, the only ones contained in Σ_0 are e_i for $i \in S_1$. On the other hand, none of the blowup subspaces $\Pi_{\langle 0 \bmod r\rangle}$ is contained in Σ_0. Thus H_X is equal to $\{\Sigma_0\}_X$ plus E_j for $j \in S_1$, which gives the first line of (4.6). By Theorem 4.3, then, we have:

$$\begin{aligned} f_X^* : & E_0 \mapsto A_0 \mapsto \Sigma_0 = H_X - \sum_{j\neq 0} E_j \\ & E_k \mapsto U_k \mapsto V_k \mapsto A_k \mapsto \Sigma_k = H_X - \sum_{j\neq k} E_j \\ & H_X \mapsto pH_X - (p-1)\sum_{j=0}^{p} E_j. \end{aligned} \tag{4.6}$$

The linear map f_X^* is determined by (4.6). Thus we may use (4.6) to write f_X^* as a matrix and compute its characteristic polynomial. We could do this directly, as we did in §3. In this case, simply observe that Theorem 4.3 implies that $f = AJ$ satisfies (1.2) and thus is an elementary map. A formula for the degree growth of any elementary map was given in Theorem A.1 in [7]. By that formula we recapture the numbers obtained in [4]:

Theorem 4.4. *If q is prime, then $\delta(K|\mathcal{SC}_q) = \rho^2$, where ρ is the largest root of $x^2 - px + 1$.*

5. Symmetric, cyclic matrices: odd q.

$$\begin{aligned} &\Sigma_0 \to A_0 \to E_0 \\ i \in S_1,\ &\Sigma_i \to A_i \to V_i \to AV_i \to E_i \\ i \in S_r,\ &\Sigma_i \to A_i \to \mathcal{F}_i \subset P_r \to \Lambda_r \end{aligned}$$

We observe that in the odd case, we have

$$\{i/r : i \in S_r\} = \{j : \ \gcd(j, q/r) = 1\}. \tag{5.1}$$

We will use this observation to bring ourselves back to certain aspects of the "relatively prime" case. Let $1 < r < q$ be a divisor of q, and set $\tilde q = q/r$, $\tilde p = (\tilde q - 1)/2$. Let us fix an element $k \in S_r$ and set $\tilde k = k/r$. It follows from (5.1) that $\gcd(\tilde k, \tilde q) = 1$. The number $\tilde\omega := \omega^r$ is a primitive $\tilde q$th root of unity. Let $\tilde A$ denote the $\tilde p \times \tilde p$ matrix constructed like A but using the numbers $\tilde\omega_j = \tilde\omega^j + \tilde\omega^{\tilde q - j}$. Let $\tilde v_1 = [1 : \tilde t_1 : \cdots : \tilde t_{\tilde p}]$ denote the vector (4.4). Let

$$\eta_r = [1 : 0 : \cdots : 0 : \tilde t_1 : 0 : \cdots] \in \Pi_{\langle 0 \bmod r\rangle} \subset \mathbf{P}^p$$

be obtained from $\tilde v_1$ by inserting $r-1$ zeros between every pair of coordinates.

Lemma 5.1. *Let $1 < r < q$ be a divisor of q. Then $Ja_r = A\eta_r$, and $fa_r = v_r$.*

Proof. As in the proof of Lemma 4.1 we note that $Ja_r = [1 : 2/\omega_r : 2/\omega_{2r} : \cdots : 2/\omega_{pr}]$. Applying Lemma 4.1 to $\tilde p$, $\tilde q$, and $\tilde\omega$, we have $2\tilde t_1 = \tilde\omega_\kappa(1 + \sum \tilde\omega_{\kappa j}\tilde t_j)$ for all positive κ. Now by the definition of $\tilde\omega_j$ we have $2\tilde t_1 = \omega_{\kappa r}(1 + \sum \omega_{\kappa j r}\tilde t_j)$, which means that equation (4.5) holds for all positive k which are multiples of r. This completes the proof. □

Lemma 5.2. *If $k \in S_r$, then $\eta_k := fa_k$ is obtained from v_r by permuting the nonzero entries.*

Proof. This Lemma follows from Lemma 5.1 exactly the same way that Lemma 4.2 follows from Lemma 4.1. □

Let us construct the complex manifold $\pi_X : X \to \mathbf{P}^p$ by a series of blow-ups. First we blow up e_0 and all the a_j. We also blow up the points v_j, Av_j and e_j for all $j \in S_1$. Next we blow up the subspaces $\Pi_{\langle 0 \bmod r\rangle}$ for all divisors r of q. If r_1 and r_2 both divide q, and r_2 divides r_1, then we blow up $\Pi_{\langle 0 \bmod r_1\rangle}$ before $\Pi_{\langle 0 \bmod r_2\rangle}$. As we observed in §1, we get different manifolds X, depending on the order of the blowups of linear subspaces that intersect, but the results in any case will be pseudo-isomorphic, and thus equivalent for our purposes. We will denote the exceptional blowup fibers over a_j, v_j, Av_j, and e_j by A_j, V_j, AV_j and E_j. We use the notation P_r for the exceptional fiber over $\Pi_{\langle 0 \bmod r\rangle}$.

Now let us discuss the exceptional locus of the induced map $f_X : X \dashrightarrow X$. As in §4, we have

$$\begin{aligned} f_X : \Sigma_0 &\to A_0 \to E_0 \to A\Sigma_0 \\ \Sigma_j &\to A_j \to V_j \to AV_j \to E_j \to A\Sigma_j \quad \forall j \in S_1. \end{aligned} \tag{5.2}$$

Since A is invertible, f_X is locally equivalent to J_X, so by (2.5) and (2.6) we see that none of these hypersurfaces is exceptional for f_X.

$Pic(X)$ is generated by $H = H_X$, the point blow-up fibers, and the P_r's. By (5.2) we have

$$\begin{aligned} f_X^* : &E_0 \mapsto A_0 \mapsto \{\Sigma_0\}_X = H - \hat{E}, \quad \text{where we write } \hat{E} = \sum_{i \in S_1} E_i \\ &E_i \mapsto AV_i \mapsto V_i \mapsto A_i \mapsto \{\Sigma_i\}_X = \\ &\quad = H - E_0 - (\hat{E} - E_i) - \hat{P}, \quad \forall i \in S_1, \text{ where } \hat{P} = \sum_r P_r. \end{aligned} \tag{5.3}$$

The left hand part of the first line follows from (5.2). The right hand side of the same line was seen already in (4.6). For the second line, we have $H_X = \{\Sigma_i\}_X + \cdots$, where the dots represent all the blowup fibers lying over subsets of Σ_i. The the sums of the E's correspond to all the blowup points contained in Σ_i, and for the $\hat{P}$ term recall that if $i \in S_1$ and r divides q, then $i \not\equiv 0 \bmod r$, and thus $\Pi_{\langle 0 \bmod r\rangle} \subset \Sigma_i$.

If $j \notin S_1$, then $j \in S_r$ for $r = \gcd(j, q)$. For $\eta \in \Pi_{\langle 0 \bmod r\rangle}$ we let $\mathcal{F}(\eta)$ denote the P_r fiber over η. For the special points η_j, we write simply $\mathcal{F}_j := \mathcal{F}(\eta_j)$. For each η, the induced map

$$f_X : \mathcal{F}(\eta) \dashrightarrow \Lambda_r := A\Sigma_{\langle 0 \bmod r\rangle} \tag{5.4}$$

is birational by (2.8). Since all the fibers map to the same space Λ_r, it follows that P_r is exceptional. In particular, we have

$$f_X : \Sigma_j \dashrightarrow A_j \dashrightarrow \mathcal{F}_j \dashrightarrow \Lambda_r. \tag{5.5}$$

Thus by (2.5) Σ_j is not exceptional. A similar calculation shows that $A_j \dashrightarrow \mathcal{F}_j$ is dominant, and in particular, the A_j are exceptional for $j \in S_r$.

Since each $\mathcal{F}_j$ is contained in P_r when $j \in S_r$, we have

$$f_X^* : P_r \mapsto \sum_{j \in S_r} A_j. \tag{5.6}$$

The multiplicities of A_j is 1 since f is locally invertible at a_j. Also, for $j \in S_r$, we have

$$f_X^* : A_j \mapsto \Sigma_j = H - E_0 - \hat{E} - (\hat{P} - \sum_{s \in I_r} P_s). \tag{5.7}$$

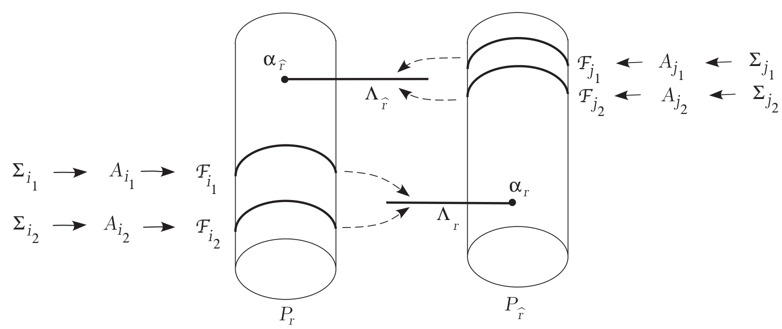

FIGURE 2. Exceptional Orbits: Hooks.

In the sequel we will repeatedly use the notation $\hat{r} := q/r$, where $1 < r < q$ divides q. Thus $\hat{\hat{r}} = r$. Let us define the point $\tau_r := [r-1:0:\cdots:0:-1:0:\cdots:0:-1:0:\cdots] \in \Pi_{\langle 0 \bmod \hat{r}\rangle}$, and let us define $\xi_r := [0:1:\cdots:1:0:1:\cdots:1:0:1:\cdots] \in \Sigma_{\langle 0 \bmod \hat{r}\rangle}$. We define $\alpha_r \in P_{\hat{r}}$ to be the point whose base coordinates are τ_r and whose fiber coordinates are ξ_r.

Now to show that $(f_X^n)^* = (f_X^*)^n$ we will follow the procedure which is sketched in Figure 2. That is, we suppose that $i_1, i_2 \in S_r$ and $j_1, j_2 \in S_{\hat{r}}$, so the orbits are as in (5.4). We will show that there is a 2-cycle $\alpha_r \leftrightarrow \alpha_{\hat{r}}$ with $\alpha_r \in \Lambda_r - \mathcal{I}$ and $\alpha_{\hat{r}} \in \Lambda_{\hat{r}} - \mathcal{I}$. This 2-cycle will serve as a hook for P_r and for all A_j with $j \in S_r$ (see Proposition 2.3).

Lemma 5.3. *$f_X(\alpha_r) = \alpha_{\hat{r}}$, and $\alpha_r \in P_{\hat{r}} \cap \Lambda_r$.*

Proof. Following the discussion in §2, we have $J(\tau_r;\xi_r) = (J'\xi_r; J''\tau_r) = (\xi_r;\tau_r'')$, where τ_r'' has the same coordinates as τ_r, except that the 0th coordinate is $1/(r-1)$. Now

$$\begin{aligned} f_X(\alpha_r) = AJ(\alpha_r) &= \left(\sum_{j \not\equiv \bmod \hat{r}} a_j; \frac{r}{r-1} a_0 - \sum_{j \equiv 0 \bmod \hat{r}} a_j \right) \\ &= \left(\sum a_j - A^{(0)}; \frac{r}{r-1} a_0 - A^{(0)} \right), \end{aligned}$$

where $A^{(0)} = \sum_{j \equiv 0 \bmod \hat{r}} a_j$. Since A is an involution (see §4), we have $AA_0 = \sum_j a_j = qe_0 = (1+2p)e_0$. Since $\hat{r}$ is a divisor of q, we have

$$(x^q - 1) = ((x^{\hat{r}})^r - 1) = (x^{\hat{r}} - 1)(1 + x^{\hat{r}} + x^{2\hat{r}} + \cdots + (x^{\hat{r}})^{r-1}).$$

It follows that $1+\sum_{k=1}^{(r-1)/2}\omega_{k(j\hat{r})}=0$ if $j\not\equiv 0 \bmod r$; and the sum is equal to r otherwise. Thus we have $A^{(0)}=r[1:0:\cdots:0:1:0:\cdots]$. Taking the difference $\sum a_j - A^{(0)}$ and using $2p+1=r\cdot\hat{r}$ we find that the base point of $f_X(\alpha_r)$ is $\tau_{\hat{r}}$. $\square$

Similarly, $r/(r-1)a_0 - A^{(0)} = r/(r-1)\xi_{\hat{r}} + (r/(r-1)-r)A^{(0)}$. Since the fiber of $P_r \cong \Sigma_{\langle 0 \bmod r\rangle}$ we have that the fiber point of $f_X(\alpha_r)$ is $\xi_{\hat{r}}$. Thus by (5.4) $\alpha_{\hat{r}} = f_X(\alpha_r)\in\Lambda_{\hat{r}}$. Replacing r by $\hat{r}$, we complete the proof.

Theorem 5.4. *The action on cohomology f_X^* is given by:*

$$
\begin{aligned}
f_X^* : E_0 \mapsto A_0 \mapsto H - \hat{E}, \quad & P_r \mapsto \sum_{j\in S_r} A_j,\\
& E_i \mapsto AV_i \mapsto V_i \mapsto A_i \mapsto H - E_0 - (\hat{E}-E_i) - \hat{P}, \quad \forall i \in S_1,\\
& A_j \mapsto \Sigma_j = H - E_0 - \hat{E} - (\hat{P} - \sum_{s\in I_r} P_s)\\
& H \to pH - (p-1)E_0 - (p-1)\hat{E} - \sum_r (p - (\lfloor \frac{q-1}{2r}\rfloor + 1))P_r.
\end{aligned}
$$

where $\hat{E}=\sum_{i\in S_1}E_i$, and $\hat{P}=\sum_r P_r$.

Proof. Everything except the last line is a consequence of (5.4), (5.6) and (5.7). It remains to determine f_X^*H, which is the same as J_X^*H. We recall from §2 that J_X^*H is equal to $N\cdot H$ minus a linear combination of the exceptional blowup fibers over the indeterminate subspaces that got blown up. Here $N=p$, the dimension of the space X. The multiples of the exceptional blowup fibers are, according to (2.12) and (2.13), given by $-M$, where M is one less than the codimension of the blowup base. This gives the numbers in the last line of the formula above. $\square$

Let us consider the prime factorization $q = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$. For each divisor $r>1$ of q, we set $\mu_r := \lfloor\frac{q-1}{2r}\rfloor + 1$, $\kappa_r = \#S_r$, and $\kappa = \frac{q-1}{2} - \sum_r \kappa_r$. We define

$$
\begin{aligned}
T_{p_i}(x) &= \kappa_{p_i}\prod_{r\neq p_i}(x^2-\kappa_r), \quad T_0(x) = \prod_r (x^2-\kappa_r) + \sum_r T_r(x),\\
T_r(x) &= \frac{\kappa_r}{x^2-\kappa_r}\left(\sum_{s\in I_r - \{r\}} T_s(x)\right) + \kappa_r\prod_{s\neq r}(x^2-\kappa_s), \text{ for } r\neq p_i.
\end{aligned}
\tag{5.8}
$$

Theorem 5.5. *The map f_X satisfies (1.4), and the dynamical degree $\delta(K|\mathcal{SC}_q)$ is ρ^2, where ρ is the largest root of*

$$
\begin{aligned}
&(x-p)(x^4-1)\prod_r(x^2-\kappa_r) + \kappa(x-1)\prod_r(x^2-\kappa_r)\\
&\quad + (x-1)(x^2+1)T_0(x) + \sum_r (x-\mu_r)(x^4-1)T_r(x).
\end{aligned}
\tag{5.9}
$$

Proof. We have found hooks for all the exceptional hypersurfaces of f_X, so (1.4) holds by Theorem 2.4. The proof that formula (5.9) gives characteristic polynomial of f_X^* is given in Appendix E. $\square$

6. Symmetric, cyclic matrices: $q = 2\times$odd.

$$\begin{aligned} &\Sigma_{0/p} \to A_{0/p} \to E_{0/p} \\ i \in S_1 \cup S_2,\ \ &\Sigma_i \to A_i \to \text{Wringer} \\ i \in S_r \cup S_{2r},\ \ &\Sigma_i \to A_i \to \mathcal{F}_i (\subset P_{e/o,r}) \to \Lambda_{e/o,r} \end{aligned}$$

For the rest of this paper we consider the case of even q. Let us set $p = q/2$ and $\iota(x_0, \ldots, x_p) = (x_0, \ldots, x_{p-1}, x_p, x_{p-1}, \ldots, x_1)$. For even q, the matrix in (1.1) is symmetric if and only if it has the form $M(\iota(x_0, \ldots, x_p))$. As in §3, we have an isomorphism

$$\mathbf{P}^p \ni x \mapsto F^{-1} D(F\iota x) F \in \mathcal{SC}_q.$$

With this isomorphism we transfer the map $F \circ J$ to the map

$$f := A \circ J : \mathbf{P}^p \dashrightarrow \mathbf{P}^p.$$

Matrix transposition corresponds to the involution $x_j \leftrightarrow x_{p-j}$ for $1 \le j \le p-1$. Thus the elements x_0 and x_p have special status. In particular, the 0th column of $A = (a_0, \ldots, a_p)$ is equal to the 0th column of F, i.e., $a_0 = f_0 = (1, \ldots, 1)$, and the pth column is $a_p = f_p = (1, -1, 1, -1, \ldots)$. For $1 \le j \le p-1$

$$a_j = f_j + f_{p-j} = (\omega_{j0}, \ldots, \omega_{jp})$$

where $\omega_j = \omega^j + \omega^{q-j}$. In particular, since $q = 2\times$odd, we have $\omega_{jp} = +2$ if j is even and $\omega_{jp} = -2$ if j is odd, and

$$\omega_{p-j} = \omega_{p+j} = -\omega_j. \tag{6.1}$$

Since q is even, we have

$$A = (a_0, \ldots, a_p) = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 & 1 \\ 1 & \omega_1 & \omega_2 & \ldots & \omega_{p-1} & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \omega_{p-1} & \omega_{2p-2} & \ldots & \omega_{(p-1)^2} & 1 \\ 1 & -2 & 2 & \ldots & 2 & -1 \end{pmatrix}. \tag{6.2}$$

It is evident that

$$f : \Sigma_0 \to a_0 \to e_0, \qquad \Sigma_p \to a_p \to e_p. \tag{6.3}$$

Arguing as in §4, we see that A is an involution on projective space. Since p is odd, every divisor r of p satisfies

$$S_{2r} = \{1 \le j \le p : (j, q) = 2r\} = \{j \text{ even} : (j/2, p/r) = 1\} = \{p - j : j \in S_r\}. \tag{6.4}$$

We will use the notation $\eta_i := f(a_i)$ and

$$\Pi_{\text{even}} := \Pi_{\langle 0 \bmod 2\rangle}, \quad \Pi_{\text{odd}} := \Pi_{\langle 1 \bmod 2\rangle}.$$

Lemma 6.1. *If $i \in S_1$, then $\eta_i \in \Pi_{\text{odd}}$. If $i \in S_2$, then $\eta_i \in \Pi_{\text{even}}$.*

Proof. Let us consider first the case $i = 2 \in S_2$. We will show that $v_2 = [1 : 0 : \pm 1 : 0 : \pm 1 : 0 : \cdots]$, which evidently belongs to Π_{even}. Note that $\tilde{\omega} := \omega^2$ is a primitive pth root of unity, and since p is odd, $-\tilde{\omega}$ is a primitive pth root of -1. We will solve the equation $Ja_2 = Av_2$ with $v_2 = [1 : 0 : t_2 : 0 : t_4 : 0 : \cdots]$. Since $q = 2p$, we have $Ja_2 = [1 : 2/\omega_2 : 2/\omega_4 : \cdots : 2/\omega_{2p-2} : 1]$. Thus the equation $Ja_2 = Av_2$ becomes the system of equations $\omega_{2i}(1 + \sum_{j=1}^{(p-1)/2} \omega_{2ij} t_{2j}) = 2t_2$ for $0 \le i \le p$. Now we repeat the proof of Lemma 5.1 with q replaced by p and with ω replaced by $\tilde{\omega}$, and we find solutions $t_{2j} = \pm 1$. This yields $v_2 \in \Pi_{\text{even}}$, as desired. Finally, we pass from the case $i = 2$ to the case of general $i \in S_2$ by repeating the arguments of Lemma 4.2. $\square$

Now consider $i = 1 \in S_1$. We have $Ja_1 = [1 : 2/\omega_1 : 2/\omega_2 : \cdots : 2/\omega_{p-1} : -1]$. Since $p-1 \in S_2$, we have $\eta_{p-1} = [1 : 0 : t_2 : 0 : \cdots : t_{p-1} : 0] \in \Pi_{\mathrm{even}}$. The equation satisfied by v_{p-1} is $\omega_{k(p-1)}(1 + \sum t_{2j}\omega_{2jk}) = t_{p-1}$ for $0 \le k \le p$. Using (6.1), we convert this equation to

$$\begin{aligned} \omega_k\Big(\sum t_{2j}\omega_{p-2jk} - 1\Big) &= t_{p-1}, \quad \text{if } k \text{ is odd} \\ \omega_k\Big(\sum t_{2j}\omega_{2jk} + 1\Big) &= t_{p-1}, \quad \text{if } k \text{ is even.} \end{aligned}$$

By (4.2) and (6.1) we have $\omega_{p-2j(2\ell+1)} = \omega_{p-2j(2\ell+1)+2\ell p}$ and $\omega_{2j\cdot 2\ell} = \omega_{2\ell p - 2\ell 2j}$. Now setting $k = 2\ell+1$ when k is odd and $k = 2\ell$ when k is even, we have

$$\begin{aligned} \omega_{2\ell+1}\Big(\sum t_{2j}\omega_{(2\ell+1)(p-2j)} - 1\Big) &= 2t_{p-1} \\ \omega_{2\ell}\Big(\sum t_{2j}\omega_{(2\ell)(p-2j)} + 1\Big) &= 2t_{p-1} \end{aligned}$$

It follows that $\eta_1 = [0 : t_{p-1} : 0 : t_{p-3} : \cdots : t_2 : 0 : 1] \in \Pi_{\mathrm{odd}}$. For general $i \in S_1$, we use the argument of Lemma 4.2.

Lemma 6.2. *Let r be an odd divisor of q. For $j \in S_r$, we have $\eta_j := fa_j \in \Pi_{\langle r \bmod 2r\rangle}$, and $\eta_{2j} := fa_{2j} \in \Pi_{\langle 0 \bmod 2r\rangle}$.*

Proof. First we consider $i = 2r \in S_{2r}$. Since $\tilde\omega = \omega^{2r}$ is a primitive (p/r)th root of unity, and p/r is odd, we repeat the proof of Lemma 5.1 to show that $fa_{2r} = \eta_{2r}$ where $\eta_{2r} = [1 : 0 : \cdots : 0 : \pm 1 : 0 : \cdots] \in \Pi_{\langle 0 \bmod 2r\rangle}$. The same reasoning as in Lemma 4.2 shows that for general $i \in S_{2r}$ we have $fa_i = \eta_i \in \Pi_{\langle 0 \bmod 2r\rangle}$ $\square$

Now consider $i = r \in S_r$. Since p/r is odd $\tilde\omega = \omega^r$ is a primitive p/rth root of -1. As before $Ja_r = [1 : 2/\omega_r : 2/\omega_{2r} : \cdots : 2/\omega_{(p-1)r} : -1]$. With the same argument in the proof of Lemma 6.1 we have

$$\begin{aligned} \tilde\omega_k\Big(\sum \tilde t_{2j}\tilde\omega_{k(p/r-2j)} - 1\Big) &= 2\tilde t_{p/r-1}, \quad \text{if } k \text{ is odd} \\ \tilde\omega_k\Big(\sum \tilde t_{2j}\tilde\omega_{k(p/r-2j)} + 1\Big) &= 2\tilde t_{p/r-1}, \quad \text{if } k \text{ is even.} \end{aligned}$$

By the definition of $\tilde\omega_k$ we have

$$\begin{aligned} \omega_{kr}\Big(\sum \tilde t_{2j}\omega_{kr(p/r-2j)} - 1\Big) &= 2\tilde t_{p/r-1}, \quad \text{if } k \text{ is odd} \\ \omega_{kr}\Big(\sum \tilde t_{2j}\omega_{kr(p/r-2j)} + 1\Big) &= 2\tilde t_{p/r-1}, \quad \text{if } k \text{ is even} \end{aligned}$$

which means $fa_r = \eta_r \in \Pi_{\langle r \bmod 2r\rangle}$. For general $i \in S_r$, we use the argument of Lemma 4.2.

Lemma 6.3. *We have:*

$$\begin{aligned} A\Pi_{\mathrm{odd}} &= \{x_0 = -x_p, x_1 = -x_{p-1}, \ldots, x_{(p-1)/2} = -x_{(p+1)/2}\} \\ A\Pi_{\mathrm{even}} &= \{x_0 = x_p, x_1 = x_{p-1}, \ldots, x_{(p-1)/2} = x_{(p+1)/2}\}, \end{aligned}$$

and $fA\Pi_{\mathrm{odd}} = \Pi_{\mathrm{odd}}$, $fA\Pi_{\mathrm{even}} = \Pi_{\mathrm{even}}$.

Proof. Let us first consider the case $A\Pi_{\mathrm{odd}}$. A linear subspace $A\Pi_{\mathrm{odd}}$ is spanned by column vectors $\{a_1, a_3, \ldots, a_p\}$. When j is odd, $a_j = [2 : \omega_j : \omega_{2j} : \cdots : \omega_{(p-1)j} : -2]$. By (6.1) we have $\omega_{(p-k)j} = \omega_{pj-kj} = -\omega_{kj}$ for all $1 \le k \le p-1$. It follows that

$A\Pi_{\rm odd} \subset \{x_0 = -x_p, x_1 = -x_{p-1}, \ldots, x_{(p-1)/2} = -x_{(p+1)/2}\}$. Since A is invertible $\{a_1, a_3, \ldots, a_p\}$ is linearly independent. It follows that

$$\dim A\Pi_{\rm odd} = \frac{p-1}{2} = \dim \{x_0 = -x_p, x_1 = -x_{p-1}, \ldots, x_{(p-1)/2} = -x_{(p+1)/2}\}.$$

With the fact that $\omega_{(p-k)j} = \omega_{kj}$ for even j, the proof for $A\Pi_{\rm even}$ is similar. $\square$

With this formula for $A\Pi_{\rm odd}$, we see that it is invariant under J. Now since A is an involution, we have $fA\Pi_{\rm odd} = \Pi_{\rm odd}$.

Let us construct the complex manifold $\pi : X \to \mathbf{P}^p$ by a series of blow-ups. First we blow up the points e_0, e_p and a_j for all j. Next we blow up the subspaces $\Pi_{\rm even}$, $\Pi_{\rm odd}$, $A\Pi_{\rm even}$, and $A\Pi_{\rm odd}$. Then we blow up the subspaces $\Pi_{\langle 0 \bmod 2r\rangle}$, $\Pi_{\langle r \bmod 2r\rangle}$ and $\Pi_{\langle 0 \bmod r\rangle}$ for all $r \notin S_1 \cup S_2$. We continue with our convention that if r_2 divides r_1 then we first blow up $\Pi_{\langle 0 \bmod 2r_1\rangle}$, $\Pi_{\langle r_1 \bmod 2r_1\rangle}$, then $\Pi_{\langle 0 \bmod r_1\rangle}$, and then the corresponding spaces for r_2. We will use the following notation for (π-exceptional) divisors of the blowup:

$$\pi : P_e \to \Pi_e, \quad AP_e \to A\Pi_e, \quad P_o \to \Pi_o, \quad AP_o \to A\Pi_o,$$

and for every proper divisor r of p we will write:

$$\pi : P_{e,r} \to \Pi_{\langle 0 \bmod 2r\rangle}, \quad P_{o,r} \to \Pi_{\langle r \bmod 2r\rangle}, \quad P_r \to \Pi_{\langle 0 \bmod r\rangle}.$$

For $1 \le i \le p-1$, we let $\mathcal{F}_i = \mathcal{F}(\eta_i)$ denote the fiber over η_i. We define Λ_r as the strict transform of $A\Sigma_{\langle 0 \bmod r\rangle}$ in X, and $\Lambda_{e/o,r}$ as the strict transforms of $A\Sigma_{\langle 0/r \bmod 2r\rangle}$.

We will do two things in the rest of this Section: we will compute f_X^* on $Pic(X)$, and we will show that $f_X : X \dashrightarrow X$ is 1-regular. It is frequently a straightforward calculation to determine f_X^* and more difficult to show that the map satisfies the condition (1.4). Let us start by computing f_X^*. We will take $H = H_X$, $E_{0/p}$, A_i, $i = 0, \ldots, p$, $P_{e/o}$, $AP_{e/o}$, $P_{e/o,r}$, P_r as a basis for $Pic(X)$. We see that Σ_0 contains e_p as well as $\Pi_{\rm odd}$, as well as $\Pi_{\langle r \bmod 2r\rangle} \subset \Pi_{\rm odd}$; and Σ_0 contains no other centers of blow-up. Thus we have

$$H = \{\Sigma_0\} + E_p + \hat{P}_o, \qquad \text{where } \hat{P}_o = P_o + \sum_r P_{o,r}. \tag{6.5}$$

This gives

$$f_X^* : \ E_0 \mapsto A_0 \mapsto \{\Sigma_0\} = H - E_p - \hat{P}_o, \quad E_p \mapsto A_p \mapsto \{\Sigma_p\} = H - E_0 - \hat{P}_e, \tag{6.6}$$

where $\hat{P}_e = P_e + \sum_r P_{e,r}$. Next, consider a divisor r of $p = q/2$, so r is odd. If $i \in S_r$, then i is odd, and the set Σ_i contains the following centers of blowup: e_0, e_p, $\Pi_{\rm even}$, $\Pi_{\langle s \bmod 2s\rangle}$ and $\Pi_{\langle 0 \bmod s\rangle}$ for all s which divide p but not r. Thus we have

$$H = \Sigma_i + E_0 + E_p + \hat{P}_e - (\hat{P}_o - \sum_{j\in I_r} P_{o,j}) - (\hat{P} - \sum_{j \in I_r} P_j) \tag{6.7}$$

where I_r is the set of numbers $1 \le k \le p-1$ which divide r, and $\hat{P} = \sum_r P_r$. Thus we have

$$\begin{aligned} i \in S_r \quad f_X^* : A_i &\mapsto H - E_0 - E_p - \hat{P}_e - (\hat{P}_o - \sum_{j\in I_r} P_{o,j}) - (\hat{P} - \sum_{j\in I_r} P_j) \\ i \in S_{2r} \qquad A_i &\mapsto H - E_0 - E_p - \hat{P}_o - (\hat{P}_e - \sum_{j\in I_r} P_{e,j}) - (\hat{P} - \sum_{j\in I_r} P_j) \end{aligned} \tag{6.8}$$

By a similar argument, we have

$$\begin{aligned} i \in S_1 \quad f_X^* : A_i &\mapsto H - E_0 - E_p - \hat{P}_e - (\hat{P}_o - P_o) - \hat{P} \\ i \in S_2 \qquad A_i &\mapsto H - E_0 - E_p - \hat{P}_o - (\hat{P}_e - P_e) - \hat{P} \end{aligned} \tag{6.9}$$

If $i \in S_1$, then $fa_i \in \Pi_{\text{odd}}$. Further $fA\Pi_{\text{odd}} = \Pi_{\text{odd}}$ and $f_X \Pi_o = A\Pi_e$. We observe that for every divisor r, we have $P_r \to \Lambda_r$, $P_{e/o,r} \to \Lambda_{e/o,r}$, so AP_o and A_i, $i \in S_1$ are the only exceptional hypersurfaces which is mapped by f_X to $\pi^{-1}(\Pi_{\text{odd}})$. Thus we have

$$f_X^* : P_o \mapsto AP_o + \sum_{i \in S_1} A_i, \quad P_e \mapsto AP_e + \sum_{i \in S_2} A_i, \quad AP_{e/o} \mapsto P_{o/e} \tag{6.10}$$

For a divisor r of p we have

$$f_X^* : P_{e,r} \mapsto \sum_{i \in S_{2r}} A_i, \quad P_{o,r} \mapsto \sum_{i \in S_r} A_i, \quad \text{and} \quad P_r \mapsto 0 \tag{6.11}$$

By §2, we have

$$\begin{aligned} f_X^* : H \mapsto\ & pH - (p-1)(E_0 + E_d) - (p - (p+1)/2)(P_e + P_o) \\ & - \sum_r (p - (p/r + 1)/2)(P_{r,e} + P_{r,o}) - \sum_r (p - p/r - 1) P_r \end{aligned} \tag{6.12}$$

Theorem 6.4. *Equations (6.6–6.12) define f_X^* as a linear map of $Pic(X)$.*

Next we discuss the exceptional locus of the induced map $f_X : X \to X$. As in §4, we have

$$f_X : \Sigma_0 \to A_0 \to E_0 \to A\Sigma_0, \quad \text{and} \quad \Sigma_p \to A_p \to E_p \to A\Sigma_p.$$

Using (2.5), (2.6) and (2.8), we see that $\Sigma_{0/p}$, $A_{0/p}$, and $E_{0/p}$ are not exceptional.

Lemma 6.5. *For $i \in S_1 \cup S_2$, Σ_i is not exceptional for f_X, and $f_X|A_i : A_i \dashrightarrow \mathcal{F}_i \subset P_{e/o}$ is a dominant map; thus A_i is exceptional.*

Lemma 6.6. *The maps $f_X : P_e \dashrightarrow AP_o \dashrightarrow P_o \dashrightarrow AP_e \dashrightarrow P_e$ are dominant. In particular, P_e, AP_o, P_o, and AP_e are not exceptional.*

Proof. Since $A\Pi_{\text{odd}}$ and $A\Pi_{\text{even}}$ are not indeterminate, it is sufficient to show that only for P_e and P_o. We will show the mapping $f_X : P_e \dashrightarrow AP_o$ is dominant. The proof for P_o is similar. The generic point of P_e is written as $x;\xi$ where $x = [x_0 : 0 : x_2 : 0 : \cdots : x_{p-1} : 0]$ and $\xi = [0 : \xi_1 : 0 : \xi_3 : \cdots : 0 : \xi_p]$. It follows that $f_X(x;\xi) = \sum_{i:\ \text{odd}} (1/\xi_i) a_i; \sum_{j:\ \text{even}} (1/x_j) a_j$. It is evident that the mapping is dominant and thus P_e is not exceptional. □

By Lemma 6.6, there is a 4-cycle $\{P_e, AP_o, P_o, AP_e\}$ of hypersurfaces, which we call "the wringer"; this is pictured in Figure 3. For $i \in S_1$, the orbit $f_X : \Sigma_i \dashrightarrow A_i \dashrightarrow \mathcal{F}_i$ enters this 4-cycle, which illustrates Lemma 6.5. The fibers $\varepsilon \subset P_e$ are the fibers $\mathcal{F}(e_j)$ for even j, $1 < j \le p-1$, and the fibers $\varepsilon = \mathcal{F}(e_i) \subset P_o$ correspond to i odd. If, for some $n \ge 0$, we have $f_X^n \mathcal{F}_i \subset \varepsilon \subset \mathcal{I}_X$, then the next iteration will blow up to a hypersurface.

Let us identify Π_e, and Π_o, with $\mathbf{P}^{\tilde{p}}$, $\tilde{p} = (p-1)/2$ as follows:

$$\begin{aligned} i_1 &: [x_0 : 0 : x_2 : 0 : \cdots : x_{p-1} : 0] \in \Pi_e \leftrightarrow [x_0 : x_2 : \cdots : x_{p-1}] \in \mathbf{P}^{\tilde{p}} \\ i_2 &: [0 : x_1 : 0 : x_3 : \cdots : 0 : x_p] \in \Pi_o \leftrightarrow [x_p : x_{p-2} : \cdots : x_1] \in \mathbf{P}^{\tilde{p}} \end{aligned} \tag{6.13}$$

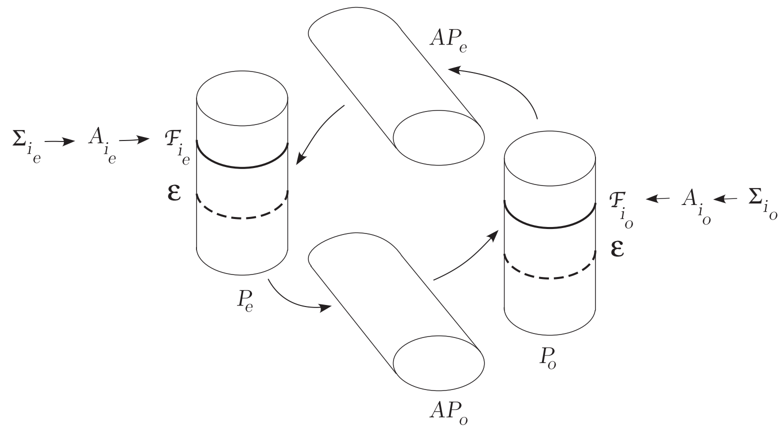

FIGURE 3. Exceptional Orbits: The Wringer.

Thus we may identify $\iota_e := (i_1, i_2) : P_e \cong \Pi_e; \Pi_o \to \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}}$ and $\iota_o := (i_2, i_1) : P_o \cong \Pi_o; \Pi_e \to \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}}$. The number $\tilde{q} = q/2$ is odd, so the map $f_{\tilde{q}} = A_{\tilde{q}} \circ J$ on $\mathbf{P}^{\tilde{p}}$ is one of the maps discussed in §5. Let us define:

$$\begin{aligned} h_1 &:= \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}} \ni (x\,;\,\xi) \mapsto (f_{\tilde{q}}(\xi)\,;\,f_{\tilde{q}}(x)) \in \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}} \\ h_2 &:= \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}} \ni (x\,;\,\xi) \mapsto (f_{\tilde{q}}(x)\,;\,A_{\tilde{q}} \circ \phi_x(\xi)) \in \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}} \end{aligned} \tag{6.14}$$

where for each $v = [v_0 : \cdots : v_{\tilde{p}}] \in \mathbf{P}^{\tilde{p}}$ we set $\phi_v : [w_0 : \cdots : w_{\tilde{p}}] \mapsto [w_0 v_0^{-2} : \cdots : w_{\tilde{p}} v_{\tilde{p}}^{-2}]$. If we set $h := h_2 \circ h_1$, then since i_2 reverses the coordinates, we have

$$f_X^2 = \iota_o^{-1} \circ h \circ \iota_e \text{ on } P_e, \text{ and } \ f_X^2 = \iota_e^{-1} \circ h \circ \iota_o \text{ on } P_o.$$

In other words, ι_e and ι_o conjugate the action of f_X^2 on the wringer to the map h on $\mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}}$.

If $i \in S_2$, then $\tilde{\imath} = i/2$ is relatively prime to $\tilde{q}$, and we write $\tilde{v}_{\tilde{\imath}} \in \mathbf{P}^{\tilde{p}}$ for the vector in Lemma 4.2. Thus we have $\iota_e(\eta_i) = \tilde{v}_{\tilde{\imath}}$, and we have $\iota_e \mathcal{F}_i = \{\tilde{v}_{\tilde{\imath}}\} \times \mathbf{P}^{\tilde{p}}$. Similarly, if $i \in S_1$, $\tilde{\imath} = (p - i)/2$ is relatively prime to $\tilde{q}$, and we have $\iota_o(\eta_i) = \tilde{v}_{\tilde{\imath}}$, and we may identify $\mathcal{F}_i$ with the vertical fiber over $\tilde{v}_{\tilde{\imath}}$.

For $x \in \mathbf{P}^{\tilde{p}}$, let $L(x) \subset \mathbf{P}^{\tilde{p}}$ denote the line containing $a_0 = (1, \ldots, 1)$ and x. Recall that $\tilde{v}_{\tilde{\imath}} = [1 : \pm 1 : \pm 1 : \cdots] = [1 : t_1 : \cdots : t_{\tilde{p}}]$, and define the set $I_{\tilde{\imath}} = \{1 \le k \le \tilde{p} : t_k = -1\}$. It follows that $L(e_{\tilde{\imath}}) = \{x_0 = x_k, k \ne \tilde{\imath}\}$, and

$$L(\tilde{v}_{\tilde{\imath}}) = \{[x_0 : \cdots : x_{\tilde{p}}] : x_0 = x_k, k \notin I_i; x_\ell = x_m, \ell, m \in I_{\tilde{\imath}}\}.$$

Thus $L(e_{\tilde{\imath}}) = \{[x_0 : x_0 : \cdots : x_1 : \cdots : x_0]\}$, where all the entries are x_0, except for one x_1 in the $\tilde{\imath}$ location, and $L(\tilde{v}_{\tilde{\imath}}) = \{[x_0 : \cdots : x_1 : \cdots]\}$, where all the entries are x_0 except for a x_1 in each location in $I_{\tilde{\imath}}$.

If $i \in S_1 \cup S_2$, we write $B_i := L(\tilde{v}_{\tilde{\imath}}) \times L(\tilde{v}_{\tilde{\imath}})$ and $D_i = L(e_{\tilde{\imath}}) \times L(e_{\tilde{\imath}})$.

Lemma 6.7. *$h : B_i \leftrightarrow D_i$.*

Proof. Let us first consider $h(B_i)$. Using defining equations for $L(\tilde{v}_{\tilde{\imath}})$ we have that 1 dimensional linear subspace $L(\tilde{v}_{\tilde{\imath}})$ is invariant under J. Thus $f_{\tilde{q}} L(\tilde{v}_{\tilde{\imath}})$ is a linear subspace containing $f_{\tilde{q}} a_0 = e_0$ and $f_{\tilde{q}} \tilde{v}_{\tilde{\imath}}$. Let us set $f_{\tilde{q}} \tilde{v}_{\tilde{\imath}} = [\alpha_0 : \cdots : \alpha_{\tilde{p}}]$. It follows that $f_{\tilde{q}} L(\tilde{v}_{\tilde{\imath}}) = \{[x_0 : \cdots : x_{\tilde{p}}] : \alpha_k x_1 = \alpha_1 x_k, k = 2, \ldots, \tilde{p}\}$ and $J f_{\tilde{q}} L(\tilde{v}_{\tilde{\imath}}) = \{[x_0 : \cdots : x_{\tilde{p}}] : \alpha_1 x_1 = \alpha_k x_k, k = 2, \ldots, \tilde{p}\}$. Since $J f_{\tilde{q}} L(\tilde{v}_{\tilde{\imath}})$ is again a 1 dimensional linear subspace, we have $f_{\tilde q}^2 L(\tilde v_{\tilde\imath}) = A_{\tilde q}\circ J f_{\tilde q} L(\tilde v_{\tilde\imath})$ is a linear subspace. Note that $e_0 \in J f_{\tilde q} L(\tilde v_{\tilde\imath})$ and $A_{\tilde q} e_0 = a_0$. By the Theorem 4.3, we have $f_{\tilde q}^2 \tilde v_{\tilde\imath} = e_{\tilde\imath}$. Thus we have $f_{\tilde q}^2 L(\tilde v_{\tilde\imath}) = L(e_{\tilde\imath})$. Now consider a generic point in $h_1 L(\tilde v_{\tilde\imath})$. By the previous computation a generic point in $h_1 L(\tilde v_{\tilde\imath})$ is $[y_0 : \cdots : y_{\tilde p}]; [\zeta_0 : \cdots : \zeta_{\tilde p}]$ where $\alpha_k y_1 = \alpha_1 y_k$ and $\alpha_k \zeta_1 = \alpha_1 \zeta_k$ for $k = 2, \dots, \tilde p$. It follows that $\alpha_1(\zeta_1/y_1^2) = \alpha_k(\zeta_k/y_k^2)$. Thus we have $A_{\tilde q}\circ \phi_y(\zeta) \in L(e_{\tilde\imath})$ and therefore $h(B_i) = D_i$.

For $h(D_i)$, we note that $L(e_{\tilde\imath})$ is invariant under J and $A_{\tilde q}$, J are both involutions. Using the previous argument, we have $A_{\tilde q} J A_{\tilde q} L(\tilde v_{\tilde\imath}) = L(e_{\tilde\imath}) = J L(e_{\tilde\imath})$ and therefore $f_{\tilde q}^2 L(e_{\tilde\imath}) = L(\tilde v_{\tilde\imath})$. Recall that $f_{\tilde q} L(e_{\tilde\imath}) = \{[x_0 : \cdots : x_{\tilde p}] : \alpha_1 x_1 = \alpha_k x_k, k = 2, \dots, \tilde p\}$, and with the same reasoning for $f_{\tilde q} L(\tilde v_{\tilde\imath})$, we have $h(D_i) = B_i$. □

By Lemma 6.7, we may simplify notation and write $h|B_i$ and $h|D_i$ in the form

$$h([x_0 : x_1], [y_0 : y_1]) = ([x_0' : x_1'], [y_0' : y_1']).$$

For the following we write h in affine coordinates $h(x, y) = (x', y')$. In order to write $h|B_i$ and $h|D_i$ more explicitly, we will use the following result:

Lemma 6.8. *For $i \in S_1 \cup S_2$, we set $\alpha^{(i)} := \prod_{\ell=1}^{\tilde p} \sum_{j\in I_{\tilde\imath}} \omega_{j\ell}$ and $\beta^{(i)} := \prod_{\ell=1}^{\tilde p} \omega_{\ell\tilde\imath}$. It follows that $(\alpha^{(i)})^2 = (\beta^{(i)})^2 = 1$, and the coefficient $t_{\tilde\imath} = \pm 1$ in $\tilde v_{\tilde\imath}$ satisfies*

$$\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \sum_{j\in I_{\tilde\imath}} \omega_{j\ell} = t_{\tilde\imath}\alpha^{(i)}, \quad \sum_{k=1}^{\tilde p} \omega_k \prod_{\ell\neq k} \sum_{j\in I_{\tilde\imath}} \omega_{j\ell} = (2-p) t_{\tilde\imath}\alpha^{(i)}$$

$$\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \omega_{j\ell\tilde\imath} = \lfloor \frac{\tilde p + 1}{2} \rfloor t_{\tilde\imath}\beta^{(i)}, \quad \sum_{k=1}^{\tilde p} \omega_{2k\tilde\imath} \prod_{\ell\neq k} \omega_{\ell\tilde\imath} = -(1 + 2\lfloor \frac{\tilde p+1}{2} \rfloor) t_{\tilde\imath}\beta^{(i)}.$$

Proof. Recall that for each $i \in S_1 \cup S_2$, we have $\tilde\imath \in S_1(\tilde q)$ and $\tilde v_{\tilde\imath} = [1 : t_1 : \cdots : t_{\tilde p}] = [1 : \pm 1 : \cdots : \pm 1]$ and $A_{\tilde q}\tilde v_{\tilde\imath} = [\alpha_0 : \cdots : \alpha_{\tilde p}]$ where $\alpha_0 = 1 + 2\sum t_j$ and $\alpha_k = 1 + \sum t_j \omega_{jk}$. Since $t_k = \pm 1$ and $1 + \sum \omega_{jk} = 0$ for all $k \neq 0$, it follows that $1 + \sum t_j \omega_{jk} = -2\sum_{j\in I_{\tilde\imath}} \omega_{jk}$. By Lemma 3.2, we have $J a_{\tilde\imath} = A_{\tilde q}\tilde v_{\tilde\imath}$ and $\alpha_0 = t_{\tilde\imath}$. It follows that $[t_{\tilde\imath} : 2t_{\tilde\imath}/\omega_{\tilde\imath} : \cdots : 2t_{\tilde\imath}/\omega_{\tilde p\tilde\imath}] = [t_{\tilde\imath} : -2\sum_{j\in I_{\tilde\imath}} \omega_j : \cdots : -2\sum_{j\in I_{\tilde\imath}} \omega_{\tilde p j}]$ and therefore we have

$$\sum_{j\in I_{\tilde\imath}} \omega_{kj} = -t_{\tilde\imath}/\omega_{k\tilde\imath}. \tag{6.15}$$

Thus we have $\alpha^{(i)} = (-t_{\tilde\imath})^{\tilde p} \prod_{\ell=1}^{\tilde p} 1/\omega_{\ell\tilde\imath}$. Recall that $\omega_j = \omega^j + \omega^{\tilde q - j}$ is real for all j and $t_{\tilde\imath} = \pm 1$. Since $\omega^{\tilde\imath}$ is a $\tilde q$th primitive root of unity, we have $x^{\tilde q} - 1 = (x-1)\prod_{\ell=1}^{\tilde q - 1}(x - \omega^{\ell\tilde\imath})$. By letting $x = -1$ we get

$$|\alpha^{(i)}|^2 = \frac{1}{|\beta^{(i)}|^2} = \prod_{\ell=1}^{\tilde p} \frac{1}{\omega^{\ell\tilde\imath}\cdot \omega^{\tilde q - \ell\tilde\imath}} \prod_{\ell=1}^{\tilde p} \frac{1}{(1+\omega^{\tilde q - \ell\tilde\imath})(1 + \omega^{\ell\tilde\imath})} = 1$$

Notice that $\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \sum_{j\in I_{\tilde\imath}} \omega_{j\ell} = (-t_{\tilde\imath})\alpha^{(i)} \sum_{k=1}^{\tilde p} \omega_{k\tilde\imath} = t_{\tilde\imath}\alpha^{(i)}$. Similarly we have $\sum_{k=1}^{\tilde p} \omega_k \prod_{\ell\neq k} \sum_{j\in I_{\tilde\imath}} \omega_{j\ell} = (-t_{\tilde\imath})\alpha^{(i)} \sum_{k=1}^{\tilde p} \omega_{k\tilde\imath}^2$. Recall that $\omega_{k\tilde\imath}^2 = 2 + \omega_{2k\tilde\imath}$ and $2\tilde\imath$ is relatively prime to $\tilde q$. It follows that $\sum_{k=1}^{\tilde p} \omega_{k\tilde\imath}^2 = 2\tilde p - 1 = p - 2$.

Note that $\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \omega_{\ell\tilde\imath} = \prod_\ell \omega_{\ell\tilde\imath} \sum_{k=1}^{\tilde p} 1/\omega_{k\tilde\imath}$. By (6.15) we have $\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \omega_{\ell\tilde\imath} = (-t_{\tilde\imath}) \prod_\ell \omega_{\ell\tilde\imath} \sum_{j\in I_{\tilde\imath}} \sum_{k=1}^{\tilde p} 1/\omega_{kj}$. Recall (4.4), we have $\# I_{\tilde\imath} = \lfloor (\tilde p + 1)/2 \rfloor$. It follows that $\sum_{k=1}^{\tilde p} \prod_{\ell\neq k} \omega_{\ell\tilde\imath} = t_{\tilde\imath} \lfloor (\tilde p+1)/2 \rfloor \prod_{\ell=1}^{\tilde p} \omega_{\ell\tilde\imath}$. Using (4.2) we have $\omega_{2k\tilde\imath} + 2 = \omega_{k\tilde\imath}^2$. It follows that $\sum_{k=1}^{\tilde p} \omega_{2k\tilde\imath} \prod_{\ell\neq k} \omega_{\ell\tilde\imath} = \prod_\ell \omega_{\ell\tilde\imath} \sum_{k=1}^{\tilde p} \omega_{k\tilde\imath} -$

$2\sum_{k=1}^{\tilde{p}}\prod_{\ell\neq k}\omega_{j\ell\tilde{\imath}}$. By the previous computation, it follows that $\sum_{k=1}^{\tilde{p}}\omega_{2k\tilde{\imath}}\prod_{\ell\neq k}\omega_{\ell\tilde{\imath}} = -(1+2\lfloor\frac{\tilde{p}+1}{2}\rfloor)t_{\tilde{\imath}}\prod_{\ell=1}^{\tilde{p}}\omega_{\ell\tilde{\imath}}$. □

Lemma 6.9. *If $\tilde{p}$ is even, then*

$$h|B_i = \left(\frac{-\tilde{p}+(-\tilde{p}+1)y}{1+y}, \frac{y^2-2xy+\tilde{p}^2(x-1)(y+1)^2+x+\tilde{p}(x-1)(y^2-1)}{2y^2-\tilde{p}(x-1)(y+1)^2-x(y^2+2y-1)}\right)$$
$$h|D_i = \left(\frac{-\tilde{p}y-1}{(\tilde{p}-1)y+1},\right.$$
$$\left.\frac{2\tilde{p}^2(y-1)x^2+x^2-\tilde{p}(x-4)(y-1)x-2yx+3y-2}{(2(1-y)\tilde{p}^2-3(1-y)\tilde{p}+2-y)x^2+(-4y\tilde{p}+4\tilde{p}+2y-4)x-y+2}\right)$$

and a similar formula holds for $\tilde{p}$ odd.

Proof. This is a direct calculation using the definitions of h_1 and h_2 and the identities on Lemma 6.8. □

Lemma 6.10. *If $i \in S_1 \cup S_2$, then the point $(-1,1) \in B_i$ is preperiodic, that is $h(-1,1)$ has period 4. Thus $(-1,1) \in B_i$ is a hook for A_i.*

Proof. The preperiodicity of $(-1,1)$ follows from the formula in Lemma 6.9. To see that $(-1,1)$ is a hook, we argue as follows: Suppose i is even. Then $f_X A_i = \mathcal{F}_i \subset P_e$, and $\mathcal{F}_i$ is the fiber over η_i. We need to show that for all $n \geq 0$, $f_X^n\mathcal{F}_i \not\subset \mathcal{I}_X$. We have identified $\iota_e : P_e \to \mathbf{P}^{\tilde{p}} \times \mathbf{P}^{\tilde{p}}$, and under this identification $\mathcal{F}(\eta_i)$ is taken to $\tilde{v}_{\tilde{\imath}} \times \mathbf{P}^{\tilde{p}}$. Thus $\iota_e(\mathcal{F}_i) \cap B_i$ corresponds to the line $[1:-1]\times\mathbf{P}^{\tilde{p}}$, which contains the point which we represent in affine coordinates as $(-1,1)$. Although it is true that $h_1(-1,1)$ corresponds to a point of indeterminacy of f_X, the rest of $h_1([1:-1]\times\mathbf{P}^1)$ is disjoint from $\mathcal{I}_X$. It follows that $h([1:-1]\times\mathbf{P}^1)$ is a curve in D_i which passes through $h(-1,1)$. Since the 4-cycle $\{h(-1,1), h^2(-1,1), h^3(-1,1), h^4(-1,1)\}$ is disjoint from $\mathcal{I}_X$, our result follows. □

From §2 we have the following:

Lemma 6.11. *When $1 < r < p$ divides p, f_X induces dominant maps $P_{e,r} \dashrightarrow \Lambda_{e,r}$, $P_{o,r} \dashrightarrow \Lambda_{o,r}$, and $P_r \dashrightarrow \Lambda_r$. In particular, the hypersurfaces $P_{e,r}$, $P_{o,r}$, and P_r are exceptional.*

Next we will construct hooks for the subspaces $P_{e,r}$, $P_{o,r}$, and P_r. Let us define $\tau' = [t_0' : \cdots : t_p']$ and $\tau'' = [t_0'' : \cdots : t_p'']$ where $t_0' = -t_p' = t_0'' = t_p'' = -(pr-p)/(p+r)$, $t'_{jp/r} = (-1)^j$, $t''_{jp/r} = 1$ for $1 \leq j \leq r-1$, and $t_i' = t_i'' = 0$ for all other i. We set

$$\tau_{e,r} := \tau' + \tau'' \in \Pi_{\langle 0 \bmod 2\frac{p}{r}\rangle}, \quad \tau_{o,r} := \tau' - \tau'' \in \Pi_{\langle \frac{p}{r} \bmod 2\frac{p}{r}\rangle}.$$

Lemma 6.12. *We have $\tau' = \sum_{i \text{ odd},\, i\not\equiv 0 \bmod r} a_i$ and $\tau'' = \sum_{i \text{ even},\, i\not\equiv 0 \bmod r} a_i$. Thus*

$$\tau_{e,r}, \tau_{o,r} \in A\Sigma_{\langle 0 \bmod 2r\rangle} \cap A\Sigma_{\langle r \bmod 2r\rangle} = A\Sigma_{\langle 0 \bmod r\rangle}.$$

Proof. Since ω is a pth root of -1, we have

$$(\omega^p + 1) = -(\omega + 1)(-1 + \omega - \omega^2 + \cdots + \omega^{p-2} - \omega^{p-1}) = 0.$$

We also have $\omega^{q-k} = \omega^p \cdot \omega^{p-k} = -\omega^{p-k}$, so $\omega^1 - \omega^{p-1} = \omega^1 + \omega^{q-1} = \omega_1$, $\omega^3 - \omega^{p-3} = \omega^3 + \omega^{q-3} = \omega_3, \ldots$ and $-\omega^2 + \omega^{p-2} = \omega^{p+2} + \omega^{p-2} = \omega_{p-2}$, etc. It follows that

$$-1 + \omega - \omega^2 + \cdots + \omega^{p-2} - \omega^{p-1} = \omega_1 + \omega_3 + \cdots + \omega_{p-2} - 1 = 0.$$

Similarly for all odd $k \neq p$, ω^k is a pth root of -1 and $\sum_{i \text{ odd}} \omega_{ki} - 1 = 0$.

Since ω^2 is a pth root of unity, we have

$$((\omega^2)^p - 1) = (\omega^2 - 1)(1 + \omega^2 + \omega^4 + \cdots + \omega^{(p-1)2}) = 0.$$

Since $\omega^{q-2k} = \omega^{2p-2k}$, we have $\omega^2 + \omega^{(p-1)2} = \omega_2$. Similarly, $\omega^4 + \omega^{(p-2)2} = \omega^{q-2(p-2)} + \omega^{(p-2)2} = \omega_{(p-2)2}$, etc. It follows that

$$1 + \omega^2 + \omega^4 + \cdots + \omega^{(p-1)2} = \omega_2 + \omega_6 + \cdots + \omega_{(p-1)2} + 1 = 0.$$

For all even $k \neq 0$ we have $\sum_{i \text{ odd}} \omega_{ki} + 1 = 0$, and we may combine the cases of k even and odd to obtain

$$\sum_{i \text{ odd}} a_i = (p+1)[1 : 0 : \cdots : 0 : -1].$$

Since r is a divisor of p, ω^r is a primitive p/rth root of -1 and $((\omega^r)^{p/r} + 1) = (\omega^r + 1)(1 - \omega^r + \omega^{2r} + \cdots + \omega^{(p/r-1)r})$. Repeating the previous argument with ω^r and p/r, we have

$$\begin{aligned}&\sum_{i \text{ odd},\ i \equiv 0 \text{ mod r}} a_i \\ &= (p/r + 1)[1 : 0 : \cdots : 0 : -1 : 0 : \cdots : 0 : 1 : 0 : \cdots : -1] \in \Pi_{\langle 0 \text{ mod } p/r \rangle}.\end{aligned}$$

Subtracting $\sum_{i \text{ odd}} a_i$ from $\sum_{i \text{ odd},\ i \equiv 0 \text{ mod r}} a_i$, it follows that

$$\tau' = \sum_{i \text{ odd},\ i \not\equiv 0 \text{ mod r}} a_i \in A\Sigma_{\langle 0 \text{ mod } r \rangle}.$$

The proof for τ'' is similar. ∎

Let us define $u'_{e,r} = (u'_i) \in \mathbf{P}^p$ to be the vector such that $u'_i = 1$ if $i \equiv p/r$ mod $2p/r$ and $u'_i = 0$ otherwise. We set $u''_{e,r} = (u''_i)$ where $u''_i = 0$ if $i \equiv 0$ mod p/r and $u''_i = 1$ otherwise. Let us define $u'_{o,r} = (u'_i) \in \mathbf{P}^p$ to be the vector such that $u'_i = 1$ if $i \equiv 0$ mod $2p/r$ and $u'_i = 0$ otherwise. We set $u''_{o,r} = (u''_i)$ where $u''_i = 0$ if $i \equiv 0$ mod p/r and $u''_i = (-1)^i$ otherwise. We let $\ell_{e,r}$ to be the line containing $u'_{e,r}$ and $u''_{e,r}$, and let $\alpha_{e,r}$ be the line in $P_{e,\hat{r}}$ lying over the basepoint $\tau_{e,r}$ and having fiber coordinate in $\ell_{e,r}$. We define $\alpha_{o,r}$ similarly.

Lemma 6.13. *Each of the sets $\alpha_{e,r} \cap \mathcal{I}_X$ and $\alpha_{o,r} \cap \mathcal{I}_X$ consists of 2 points, and $\alpha_{e,r} \cup \alpha_{o,r} \subset \Lambda_r$. $f_X \alpha_{e,r} \subset P_r \cap \Lambda_{e,\hat{r}}$, and $f_X \alpha_{o,r} \subset P_r \cap \Lambda_{o,\hat{r}}$. Finally, $f_X^2 \alpha_{e,r} = \alpha_{e,r}$, and $f_X^2 \alpha_{o,r} = \alpha_{o,r}$.*

Proof. Let us consider the case $\alpha_{e,r}$. By Lemma 6.11, $f_X : P_{e,r} \to \Lambda_{e,r}$ and by Lemma 6.12 $\alpha_{e,r} \subset P_{e,\hat{r}}$. It follows that $f_X \alpha_{e,r} \subset \Lambda_{e,\hat{r}}$. A generic point ζ in $\alpha_{e,r}$

has a form $\tau_{r,o}+\tau_{r,e};[0:1:\cdots:1:x:1:\cdots:1:0:1:\cdots 1:x]$ for some $x\in\mathbf{C}^*$. Applying the map f, we have

$$\zeta \overset{J}{\mapsto} [0:1:\cdots:1:\frac{1}{x}:1:\cdots:1:0:1:\cdots:1:\frac{1}{x}];[\frac{(p+r)}{(pr-p)}:0:\cdots:0:1:0:\cdots]$$
$$\overset{A}{\mapsto} (\tau_{p/r,o}+\tau_{p/r,e}+\frac{1}{x}\sum_{i\equiv p/r \bmod 2p/r} a_i);(\frac{(p+r)}{(pr-p)}a_0+\sum_{i\equiv 0 \bmod 2p/r} a_i).$$

By Lemma 6.12, there exist nonzero constants β_1,β_2, and β_3 such that

$$\begin{aligned}\tau_{p/r,o}+\tau_{p/r,e}+\frac{1}{x}\sum_{i\equiv p/r \bmod 2p/r} a_i \in \Pi_{\langle 0 \bmod r\rangle}\\ =&[\beta_1:0:\cdots:0:\beta_2:0:\cdots:0:\beta_2:0:\cdots:0]\in\Pi_{\langle 0 \bmod 2r\rangle}\\ &+[\frac{\beta_3}{x}:0:\cdots:0:-\frac{\beta_3}{x}:0:\cdots:0:\frac{\beta_3}{x}:0:\cdots]\in\Pi_{\langle 0 \bmod r\rangle}.\end{aligned}$$

It follows that $f_X\alpha_{e,r}\subset P_r\cap\Lambda_{e,\hat{r}}$. Again by Lemma 6.11, we know that $f_X^2\alpha_{e,r}\subset\Lambda_r$. For the fiber for $f_X\zeta$, the j^{th}-coordinate of $\frac{1}{\alpha}a_0+\sum_{i\equiv 0 \bmod 2p/r} a_i$ are all equal for $j\not\equiv 0 \bmod r$. It follows that

$$\begin{aligned}&f_X:\zeta\mapsto f_X\zeta\\ &\quad\mapsto \tau_{r,o}+\tau_{r,e};(\frac{1}{\beta_1+\beta_3/x}a_0+\frac{1}{\beta_2}\sum_{i\equiv r \bmod 2r} a_i+\frac{1}{\beta_2-\beta_3/x}\sum_{i\equiv 0 \bmod 2r} a_i).\end{aligned}$$

Note that both $\alpha_{e,r}$ $f_X^2\alpha_{e,r}$ are 1-dimensional linear subspaces in fiber over $\tau_{e,r}$. Using the computation in Lemma 6.12 we have $f_X^2\alpha_{e,r}=\alpha_{e,r}$. We use a similar argument for $\alpha_{o,r}$. □

Corollary 6.14. *Let $r>1$ be an odd divisor of q. Then for $j\in S_r$, $\alpha_{o,r}$ is a hook for A_j, and $P_{o,r}$ and P_r; and $\alpha_{e,r}$ is a hook for A_{2j}, $P_{e,r}$, and P_r.*

Let us consider the prime factorization $q=2p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$. For each divisor $r>1$ of q, we set $\mu:=\frac{p+1}{2}$, $\kappa=\#S_2=\#S_1$, $\mu_r:=\frac{p/r+1}{2}$, and $\kappa_r=\#S_{2r}=\#S_r$.

Theorem 6.15. *Condition (1.4) holds for f_X, and $\delta(K)=\rho^2$ where ρ is the largest root of*

$$\begin{aligned}(x-p)(x^2-\kappa-1)\prod_r(x^2-\kappa_r)+2\kappa(x-\mu)\prod_r(x^2-\kappa_r)\\ +2(x-1)T_0(x)+2\sum_r(x-\mu_r)(x^2-1)T_r(x)\end{aligned}$$

with the polynomials $T_j(x)$ are defined in (5.8).

Proof. We have determined all the exceptional hypersurfaces for f_X and have found a hook for each of them. Thus by Theorem 2.4, condition (1.4) holds for f_X. Thus $\delta(f)$ is the spectral radius of f_X^*. Consider f_X^* as in Theorem 6.4 and let $\chi(x)$ denote its characteristic polynomial. We may now determine $\chi(x)$ as in Theorem 5.5 (see Appendix E). We find that $\chi(x)$ is the polynomial above times a polynomial whose roots all have modulus one. □

7. Symmetric, cyclic matrices: $q = 2\times$even.

$$\begin{array}{rl} & \Sigma_{0/p} \to A_{0/p} \to E_{0/p} \\ & \Sigma_{\frac{p}{2}} \to A_{\frac{p}{2}} \to A\Pi_{\text{odd}} \to \mathcal{L} \\ i \in S_1 & \Sigma_i \to a_i \to * \in A_{\frac{p}{2}} \\ i \in S_r \cup S_{2r} & \Sigma_i \to A_i \to \mathcal{F}_i \subset P_{e/o,r} \to \Lambda_{e/o,r} \\ i \in S_\rho & \Sigma_i \to \mathcal{F}_i \subset \Gamma_{\check\rho} \to \lambda_i \subset \Gamma_\rho \end{array}$$

In this case we set $p = q/2$, and our mapping is given by $f = A \circ J$, with A as in (6.2). Since q is divisible by 4, we have additional symmetries:

$$\omega_{jp/2} = 0 \text{ if } j \text{ is odd}, \ \omega_{jp/2} = (-1)^{j/2} \text{ if } j \text{ is even, and } \omega_{p/2+j} = -\omega_{p/2-j} \tag{7.1}$$

As before, we have

$$\Sigma_0 \to a_0 \to e_0, \quad \Sigma_p \to a_p \to e_p. \tag{7.2}$$

However, now we encounter the phenomenon that A contains several 0 entries, for instance

$$\Sigma_{p/2} \to a_{p/2} = [1 : 0 : -1 : 0 : 1 : 0 : \cdots] \in \Pi_{\text{even}}. \tag{7.3}$$

We will write $q = 2^m q_{\text{odd}}$ and consider two sorts of divisors ρ and r, which satisfy:

$$\rho|(q/4), \quad \text{and } r = 2^{m-1}r', \quad r'|q_{\text{odd}}. \tag{7.4}$$

We will use the notation $\check\rho := q/(4\rho)$. Note that this is again a divisor of the form ρ.

Lemma 7.1. *Suppose that $r = 2^{m-1}r'$, and r' divides q_{odd}. If $i \in S_r$, then $fa_i \in \Pi_{\langle r \bmod 2r\rangle}$, and if $j \in S_{2r}$, then $fa_j \in \Pi_{\langle 0 \bmod 2r\rangle}$.*

Proof. Since $\tilde\omega = \omega^{2r}$ is a primitive p/rth root of unity and p/r is odd, the proof is the same as Lemma 6.2. □

Lemma 7.2. *Suppose that $1 < \rho < q/4$ divides $q/4$. Then every $i \in S_\rho$ is an odd multiple of ρ, and we have $S_\rho = \{p - j : j \in S_\rho\}$, and $a_i \in \Sigma^*_{\langle\check\rho \bmod 2\check\rho\rangle}$.*

Proof. Since 2ρ is also a divisor of q, every $i \in S_\rho$ is an odd multiple of ρ. Suppose $j \in S_\rho$, then we have $j = k\rho$ where $\gcd(k, q/\rho) = 1$ and $p-j = \rho(p/\rho - k)$. It follows that $\gcd(p/\rho - k, q/\rho) = 1$ and $p - j \in S_\rho$. We observe that $j\check\rho \cdot i = j\check\rho \cdot k\rho = jk \cdot q/4$. By (7.1) it follows that $\omega_{j\check\rho i} = 0$ if j is odd, $\omega_{j\check\rho i} = \pm 2$ if j is even, and $\omega_{ji} \neq 0$ otherwise. □

Lemma 7.3. *If $i \in S_1$, then $a_i \in \Sigma^*_{p/2}$.*

Proof. Since i is relatively prime to q, i is odd and $\omega_{p/2\cdot i} = 0$ by (7.1). ω^i is a qth primitive root of unity, and therefore $\{\omega_0, \omega_1, \ldots, \omega_p\} = \{\omega_{0i}, \omega_{1i}, \ldots, \omega_{pi}\}$ as a set. It follows that each a_i has exactly one zero coordinate. □

Now we construct the space $\pi : X \to \mathbf{P}^p$ by a series of blowups. We blow up a_0, e_0, a_p, e_p, and $a_{p/2}$. For each divisor of the form r in (7.4), we blow up a_i for all $i \in S_r \cup S_{2r}$. As before, A_i denotes the blowup fiber of a_i. We also blow up $\Pi_{\langle 0 \bmod 2r\rangle}$ and $\Pi_{\langle r \bmod 2r\rangle}$; we denote the blowup fibers as $P_{e,r}$ and $P_{o,r}$, respectively. For each divisor of the form ρ in (7.4) (or equivalently $\check\rho$), we blow up $\Sigma_{\langle\rho \bmod 2\rho\rangle}$; we denote the blowup fiber by Γ_ρ. Let $f_X : X \to X$ denote the induced birational map.

Let us take $H = H_X$, $E_{0/p}$, $A_{0/p}$, $A_{\frac{p}{2}}$, A_i, $i \in S_r \cup S_{2r}$, $P_{e/o,r}$, and Γ_ρ as a basis for $Pic(X)$. As in §6, we have

$$f_X^* : \ E_0 \mapsto A_0 \mapsto \{\Sigma_0\} = H - E_p - \hat{P}_o, \quad E_p \mapsto A_p \mapsto \{\Sigma_p\} = H - E_0 - \hat{P}_e, \tag{7.5}$$

where $\hat{P}_{e/o} = \sum_r P_{e/o,r}$. And for a divisor r of q in (7.4), we have

$$\begin{aligned} f_X^* \ &: P_{e,r} \mapsto \sum_{i \in S_{2r}} A_i, \qquad P_{o,r} \mapsto \sum_{i \in S_r} A_i \\ i \in S_r \qquad & A_i \mapsto H - E_0 - E_p - \hat{P}_e - (\hat{P}_o - \sum_{j \in I_r} P_{o,j}) \\ i \in S_{2r} \qquad & A_i \mapsto H - E_0 - E_p - \hat{P}_o - (\hat{P}_e - \sum_{j \in I_r} P_{e,j}) \end{aligned} \tag{7.6}$$

We see that $\Sigma_{p/2}$ contains $e_{0/p}$, $\Pi_{\langle 0 \bmod 2r \rangle}$ and $\Pi_{\langle r \bmod 2r \rangle}$ as well as Γ_ρ. Let us suppose $q = 2^m \cdot \text{odd}$. We set $\hat{\Gamma} = \sum_{\rho: 2^{m-2} \cdot \text{odd}} \Gamma_\rho$. Since $a_j \in \Sigma_{p/2}$ for all odd j, if $p/2$ is odd we have

$$H = \Sigma_{p/2} + E_0 + E_p + \hat{P}_e + \hat{P}_o + \hat{\Gamma} + A_{p/2}. \tag{7.7}$$

Thus we have

$$\begin{aligned} f_X^* : A_{p/2} &\mapsto \{\Sigma_{p/2}\} = H - E_0 - E_p - \hat{P}_e - \hat{P}_o - \hat{\Gamma} - A_{p/2} \quad \text{if } p/2 \text{ is odd} \\ A_{p/2} &\mapsto \{\Sigma_{p/2}\} = H - E_0 - E_p - \hat{P}_e - \hat{P}_o - \hat{\Gamma} \qquad \text{if } p/2 \text{ is even} \end{aligned} \tag{7.8}$$

Let us consider a divisor ρ of q in (7.4). We have

$$f_X^* \ : \ \Gamma_{\check{\rho}} \mapsto \sum_{i \in S_\rho} \{\Sigma_i\}. \tag{7.9}$$

We observe that $\Sigma_{\langle \rho \bmod 2\rho \rangle} \subset \Sigma_{\text{odd} \cdot \rho}$ and $a_{p/2} = [1 : 0 : -1 : 0 : \cdots : \pm 1] \in \Sigma_j$ for all odd j. Thus for $i \in S_\rho$ we have

$$\begin{aligned} \rho \text{ even} \qquad & \{\Sigma_i\} = H - E_0 - E_p - \hat{P}_e - \hat{P}_o - \Gamma_\rho, \\ \rho \text{ odd} \qquad & \{\Sigma_i\} = H - E_0 - E_p - A_{p/2} - \hat{P}_e - \hat{P}_o - \Gamma_\rho. \end{aligned} \tag{7.10}$$

Thus we have

$$\begin{aligned} \rho \text{ even} \qquad f_X^* \ &: \Gamma_\rho \mapsto \sum_{i \in S_{\check{\rho}}} \{\Sigma_i\} = \#S_{\check{\rho}}(H - E_0 - E_p - \hat{P}_e - \hat{P}_o - \Gamma_{\check{\rho}}), \\ \rho \text{ odd} \qquad & \Gamma_\rho \mapsto \sum_{i \in S_{\check{\rho}}} \{\Sigma_i\} = \#S_{\check{\rho}}(H - E_0 - E_p - A_{p/2} - \hat{P}_e - \hat{P}_o - \Gamma_{\check{\rho}}). \end{aligned} \tag{7.11}$$

By §2, we have

$$\begin{aligned} f_X^* : H \mapsto\ & pH - (p-1)(E_0 + E_d) - (p - (p/2+1))A_{p/2} \\ & - \sum_r (p - (p/r+1)/2)(P_{e,r} + P_{o,r}) - \sum_\rho (\rho - 1)\Gamma_\rho \end{aligned} \tag{7.12}$$

This accounts for all of the basis elements of $Pic(X)$, so we have:

Theorem 7.4. *Equations (6.5–12) define f_X^* as a linear map of $Pic(X)$.*

Let us set $\mathcal{L} = \{a_{p/2}; \Pi_{\text{odd}}\} \subset A_{p/2}$.

Lemma 7.5. *$f_X : A_{p/2} \dashrightarrow A\Pi_{\text{odd}} \subset \Sigma_{p/2}$, $A\Pi_{\text{odd}} \not\subset \mathcal{I}_X$, and $f_X : A\Pi_{\text{odd}} \dashrightarrow \mathcal{L}$. In particular, f_X^2 defines a dominant rational map of $\mathcal{L}$ to itself.*

Proof. A generic point of an exceptional divisor $A_{p/2}$ can be expressed as $a_{p/2}; \xi = [1 : 0 : -1 : 0 : \cdots]; [\xi_0 : \xi_1 : \xi_2 : \cdots]$. Thus we have

$$f_X(a_{p/2}; \xi) = A[0 : 1/\xi_1 : 0 : 1/\xi_3 : 0 : \cdots : 1/\xi_{p-1} : 0] = \sum_{i:\text{ odd}} \frac{1}{\xi_i} a_i \in A\Pi_{\text{odd}}.$$

From the computation, it is clear that the rank of $f_X|A_{p/2}$ is equal to the dimension of $A\Pi_{\text{odd}}$. With (7.1) and the same reasoning as in Lemma 6.3, we have

$$A\Pi_{\text{odd}} = \{x_0 = -x_p, x_1 = -x_{p-1}, \ldots, x_{p/2-1} = -x_{p/2+1}, x_{p/2} = 0\}$$

Now the generic point x of $A\Pi_{\text{odd}}$ is $x = [x_0 : x_1 : \cdots : x_{p/2-1} : 0 : -x_{p/2-1} : \cdots : -x_0]$, and $A\Pi_{\text{odd}} \subset \Sigma_{p/2}$. Now

$$f_X(x) = a_{p/2}; A[1/x_0 : \cdots : 1/x_{p/2-1} : 0 : -1/x_{p/2-1} : \cdots : -1/x_0] \in \mathcal{L},$$

and the mapping is dominant. By the previous computation for $A_{p/2}$, $f_X^2 : \mathcal{L} \dashrightarrow \mathcal{L}$ is dominant. □

From §2 we have the following:

Lemma 7.6. *Let r be a divisor of the form (7.4). If $i \in S_r$, we let $\mathcal{F}_i$ denote the fiber of $P_{o,r}$ over fa_i. In this notation, we have dominant maps: $f_X : \Sigma_i \dashrightarrow A_i \dashrightarrow \mathcal{F}_i$. In fact, for every fiber $\mathcal{F}$ of $P_{o,r}$, $f_X : \mathcal{F} \dashrightarrow A\Sigma_{\langle r \bmod 2r\rangle}$ is a dominant map. Similarly, suppose $j \in S_{2r}$. With corresponding notation, we have dominant maps $f_X : \Sigma_i \dashrightarrow A_i \dashrightarrow \mathcal{F}_i \subset P_{e,r}$ and $f_X : \mathcal{F} \dashrightarrow A\Sigma_{\langle 0 \bmod 2r\rangle}$.*

Proposition 7.7. *$A\Pi_{\text{odd}} \subset A\Sigma_{\langle 0 \bmod 2r\rangle} \cap A\Sigma_{\langle r \bmod 2r\rangle}$ is a hook for the spaces: $A_{p/2}$, and $P_{e,r}$, $P_{o,r}$, A_i, $i \in S_r \cup S_{2r}$, for every divisor r in (7.4).*

Lemma 7.8. *$f_X a_1 = a_{p/2}; [0 : p-1 : 0 : 3-p : 0 : p-5 : 0 : \cdots : \pm 1 : 0] \in \mathcal{L}$. If $i \in S_1$, then $f_X a_i$ is obtained from $f_X a_1$ by permuting the nonzero coordinates.*

Proof. Using Lemma 7.3 and 7.5 which show that $f_X a_1 \in \mathcal{L}$, we can set $f_X a_1 = a_{p/2}; [0 : \xi_1 : 0 : \xi_3 : 0 : \cdots : \xi_{p-1} : 0]$. Recall that $a_1 = [2 : \omega_1 : \cdots : \omega_{p/2-1} : 0 : -\omega_{p/2-1} : \cdots : -2]$. Applying f_X we have $\xi_k = 1 + 2\sum_{j=1}^{p/2-1} \omega_{kj} \cdot \frac{1}{\omega_j}$ for $k = 1, 3, \ldots, p-1$. If $k = 1$, we have $\xi_1 = 1 + 2\sum_{j=1}^{p/2-1} \omega_j/\omega_j = p-1$. For $k \geq 1$, we will show that $\xi_k + \xi_{k+2} = (-1)^{(k-1)/2}2$. Let us recall the last equality in (4.2). $\omega_j \cdot \omega_{(k+1)j} = \omega_{(k+1)j-j} + \omega_{(k+1)j+j} = \omega_{kj} + \omega_{(k+2)j}$. It follows that

$$\xi_k + \xi_{k+2} = 2 + 2\sum_{j=1}^{p/2-1} (\omega_{kj} + \omega_{(k+2)j}) \cdot \frac{1}{\omega_j} = 2 + 2\sum_{j=1}^{p/2-1} \omega_{(k+1)j}.$$

When $k+1 \equiv 2 \bmod 4$, $\omega^{(k+1)j}$ is a pth root of unity and therefore $\sum_{j=1}^{p/2-1} \omega_{(k+1)j} = 1 + \sum_{j=1}^{p/2-1} \omega_{(k+1)j} - 1 = 0$. If $k+1 \equiv 0 \bmod 4$, $\omega_{(k+1)p/4+(k+1)j} = (-1)^{(k+1)/4}\omega_{(k+1)j}$ and $\sum_{j=1}^{p/2-1} \omega_{(k+1)j} + 2 = 0$. Thus we get $\xi_k + \xi_{k+2} = 2$ if $k+1 \equiv 2 \bmod 4$, and $\xi_k + \xi_{k+2} = -2$ if $k+1 \equiv 0 \bmod 4$. For general $i \in S_1$, we use the same permutation argument as in Lemma 4.2. In fact, if we set $f_X a_i = a_{p/2}; [0 : \xi_1^{(i)} : 0 : \xi_3^{(i)} : \cdots]$, then $\xi_i^{(i)} = p-1$, and $\xi_k^{(i)} + \xi_{k+2i}^{(i)} = 2$ if $k+i \equiv 2 \bmod 4$, and $\xi_k^{(i)} + \xi_{k+2i}^{(i)} = -2$ if $k+i \equiv 0 \bmod 4$. □

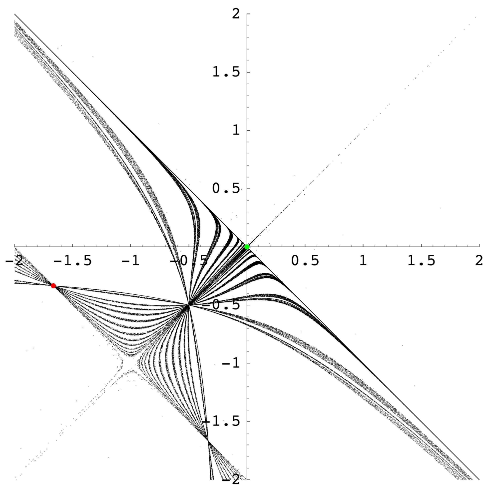

FIGURE 4. An Exceptional Orbit: $q = 12$.

In Figure 4 we consider $q = 12$, $p = 6$. Thus $\mathcal{L}$ has dimension 2, and we plot points of the orbit $f^{2n+1}a_1$, $n \geq 0$, in an affine coordinate chart inside $\mathcal{L}$.

Let us define $i_1 : \Pi_{\text{odd}} \ni [0 : x_1 : 0 : \cdots : x_{p-1} : 0] \mapsto [x_1 : x_3 : \cdots : x_{p-1}] \in \mathbf{P}^{\frac{p}{2}-1}$ and $J_1 := i_1^{-1} \circ J_{\mathbf{P}^{\frac{p}{2}-1}} \circ i_1 : \Pi_{\text{odd}} \to \Pi_{\text{odd}}$. Similarly, let $i_2 : A\Pi_{\text{odd}} \ni [x_0 : x_1 : \cdots : x_{p/2-1} : 0 : -x_{p/2-1} : \cdots : -x_0] \mapsto [x_0 : x_1 : \cdots : x_{p/2-1}] \in \mathbf{P}^{\frac{p}{2}-1}$, and define $J_2 := i_2^{-1} \circ J_{\mathbf{P}^{\frac{p}{2}-1}} \circ i_2 : A\Pi_{\text{odd}} \to A\Pi_{\text{odd}}$. Now we define $\varphi := i_1(AJ_2 \circ AJ_1)i_1^{-1}$ as a $p/2$-tuple of polynomials with coefficients in $\mathbf{Z}[\omega]$. Thus φ is a map of $\mathbf{Z}[\omega]^{p/2}$ to itself. The map φ also induces a map of $\mathbf{P}^{p/2-1}$ to itself, and i_1 conjugates this map of projective space to $f_X^2 : \mathcal{L} \to \mathcal{L}$.

Lemma 7.9. *For $j \in S_1$, there is a polynomial $R_j \in \mathbf{Z}[\omega]$ such that $x_j | R_j$, and*

$$\begin{aligned}\varphi[x_1 : x_3 : x_5 : \cdots : x_{p-1}] &= 2(p/2)^2[p-1 : 3-p : \cdots : \pm 1]\widehat{x_1}^{p/2} + R_1(x) \\ &= V_j \widehat{x_j}^{p/2} + R_j(x)\end{aligned}$$

where V_j is obtained from $2(p/2)^2[p-1 : 3-p : \cdots ; \pm 1]$ by permuting the coordinates.

Proof. Let us set $[y_1 : y_3 : \cdots : y_{p-1}] = \varphi[x_1 : x_3 : \cdots : x_{p-1}]$. A direct computation gives that y_i is equal to $2(\sum_{s:\ \text{odd}} \widehat{x_s}) \prod_{k=1}^{p/2-1} (\sum_{s:\ \text{odd}} \omega_{ks}\widehat{x_s})$ times

$$\prod_{k=1}^{p/2-1} \left(\sum_{s:\ \text{odd}} \omega_{ks}\widehat{x_s} \right) + 2 \left(\sum_{s:\ \text{odd}} \widehat{x_s} \right) \cdot \sum_{\ell=1}^{p/2-1} \left[\omega_{j\ell} \prod_{k \neq \ell} (\sum_{s:\ \text{odd}} \omega_{ks}\widehat{x_s}) \right].$$

Recall that $\widehat{x_s} = 0$ on $\bigcup_{j \neq s}\{x_j = 0\}$ and $\widehat{x_s} \neq 0$ on $\{x_s = 0\} \cup \bigcap_{j \neq s}\{x_j \neq 0\}$. It follows that on Σ_1^*,

$$y_j = (\prod_{k=1}^{p/2-1} \omega_k)^2 \cdot [1 + 2\sum_{\ell=1}^{p/2-1} \omega_{j\ell} \cdot \frac{1}{\omega_\ell}] \cdot \widehat{x_1}^{p/2} \quad \forall j :\ \text{odd}.$$

Let us write $\Omega = \prod_{k=1}^{p/2-1} \omega_k$. With the previous Lemma, it is clear that we have a polynomial R_1 such that x_1 is a divisor of R_1 and $\varphi(x) = \Omega[p-1 : 3-p : \cdots :$

$\pm 1] \cdot \widehat{x_1}^{p/2-1} + R_1(x)$. We want to show that $\Omega = \pm p/2$. Now $\Omega = \prod_{k=1}^{p/2-1} \omega^{-k}(1+\omega^{2k}) = (\prod_{k=1}^{p/2-1} \omega^{-k}) \cdot (\prod_{k=1}^{p/2-1} 1+\omega^{2k})$. Since ω is a primitive qth root of unity and 4 is a divisor of q we see that

$$\prod_{k=1}^{q-1} \omega^k = \pm(\prod_{k=1}^{p-1} \omega^k)^2 = \pm(\prod_{k=1}^{p/2-1} |\omega^k|)^4 = 1.$$

For all $1 \leq k \leq p/2-1$, ω^{2k} is a $p/2$th root of -1 and therefore

$$(x^{p/2}+1) = (x+1)(1-x+x^2-\cdots+x^{p/2-1}) = (x+1)\prod_{k=1}^{p/2-1}(x-\omega^{2k}).$$

Setting $x=-1$, we have $-\frac{p}{2} = \prod_{k=1}^{p/2-1}(1+\omega^{2k})$. For $j \in S_1$, we reason as in the proof of Lemma 4.2. □

Lemma 7.10. *For $i \in S_1$, $f_X^n a_i \notin \mathcal{I}_X$ for all $n \geq 0$.*

Proof. By Proposition 7.7, $i_1 f_X a_1 = [p-1 : 3-p : p-5 : \cdots : \pm 1]$. Let us set $u_1 = (p-1, 3-p, p-5, \cdots, \pm 1)$. It suffices to show that $\varphi^n(u_1) \notin i_1\mathcal{I}_X$ for all $n \geq 0$. For this we need to know that for each n, at most one coordinate of $\varphi^n u_1$ can vanish. Let us choose a prime number $p/2 < \mu \leq p-1$. One of the coordinates of u_1 is equal to $\pm\mu$. Suppose it is the jth coordinate. Then $2j-1$ must be relatively prime to q, so we can apply Lemma 7.9. Working modulo μ, we see that $\varphi u_1 = b_j u_j$, where u_j is obtained from u_1 by permuting the coordinates, and $b_j = 2(p/2)^2((u_1)_{\hat{j}})^{p/2}$. For each $k \neq j$, the kth coordinate of u_1 is nonzero modulo μ. Thus b_j is a unit modulo μ, and so φu_1 is a unit times a permutation of u_1. The permutation preserves the set S_1, so if j_2 denotes the coordinate of $i_1^{-1}\varphi u_1$ which vanishes modulo μ, then $j_2 \in S_1$. Thus we may repeat this argument to conclude that, modulo μ, $\varphi^n u_1$ is equal to a unit times a permutation of u_1. Thus at most one entry of $\varphi^n u_1$ can vanish, even modulo μ. □

From (6.1), (6.2) and (7.1) we have the following:

Lemma 7.11. *Consider a divisor ρ in (7.4). We have*

$$\begin{aligned} A\Pi_{\langle \rho \bmod 2\rho\rangle} = \{ & x_0 = -x_{2\check\rho} = x_{4\check\rho} = -x_{6\check\rho} = \cdots = \pm x_{2\rho\check\rho}, \\ & x_1 = -x_{2\check\rho-1} = -x_{2\check\rho+1} = x_{4\check\rho-1} = x_{4\check\rho+1} = \cdots = \pm x_{2\rho\check\rho-1}, \ldots, \\ & x_{\check\rho-1} = -x_{\check\rho+1} = -x_{3\check\rho-1} = x_{3\check\rho+1} = \cdots = \pm x_{2\rho\check\rho-\check\rho+1}\}. \end{aligned}$$

Proof. By (6.1), (6.2) and (7.1) , it is easy to check that a_j, $j \equiv \rho \bmod 2\rho$ satisfies all the equations. □

Lemma 7.12. *Consider a divisor of the form ρ in (7.4). Then $A\Pi_{\langle \rho \bmod 2\rho\rangle} \subset \Gamma_{\check\rho}$. Let us use the notation $\Lambda_\rho := \pi^{-1}A\Pi_{\langle \rho \bmod 2\rho\rangle}$ for the exceptional fiber over $A\Pi_{\langle \rho \bmod 2\rho\rangle}$. Then we have a dominant mapping $f_X : \Gamma_\rho \dashrightarrow \Lambda_\rho$. Furthermore, $f_X^2 : \Lambda_\rho \dashrightarrow \Lambda_\rho$ is a dominant mapping, so Λ_ρ is a hook for Γ_ρ.*

Proof. A linear subspace $A\Pi_{\langle \rho \bmod 2\rho\rangle}$ is spanned by $a_{k\rho}, k$:odd. For j odd, the $j\check\rho$-th coordinate of $a_{k\rho}$ is $\omega_{j\check\rho\cdot k\rho} = \omega_{jk\cdot p/2} = 0$ by (7.1). If follows that $A\Pi_{\langle \rho \bmod 2\rho\rangle} \subset \Gamma_{\check\rho}$. □

Let us conisder a generic point $x;\xi$ in Γ^*_ρ. Using the previous argument, it is clear that the base of $f_X(x;\xi)$ is in $A\Pi_{\langle \rho \bmod 2\rho\rangle} \subset \Gamma_{\check\rho}$. The fiber point of $f_X(x;\xi)$ is $[0 : \cdots : 0 : \zeta_{\check\rho} : 0 : \cdots : 0 : \zeta_{3\check\rho} : \cdots]$ where $\zeta_{k\check\rho} = 1/x_0 + \sum_{j \not\equiv \rho \bmod 2\rho} \omega_{jk\check\rho} 1/x_j \pm 1/x_p$.

Furthermore, for the generic point $x;\xi$ in $\Lambda_{\check\rho}$, $x \in A\Pi_{\langle \check\rho \bmod 2\check\rho\rangle}$ and $\xi \in \Pi_{\langle \rho \bmod 2\rho\rangle}$. Using Lemma 7.11, we have the fiber point of $f_X(x;\xi) \in \Pi_{\langle \check\rho \bmod 2\check\rho\rangle}$. Replacing ρ by $\check\rho$ we have a dominant mapping from Λ_ρ to Λ_ρ.

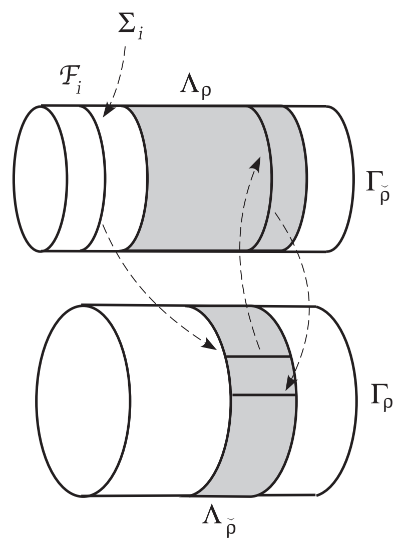

FIGURE 5. Moving fibers.

It remains to track the orbit of Σ_i for $i \in S_\rho$. In this case, $f_X\Sigma_i = \mathcal{F}_i$, which is a fiber of $\Gamma_{\check\rho}$. What happens here is that $f_X : \Gamma_{\check\rho} \leftrightarrow \Gamma_\rho$; as was seen in §2, $\Gamma_{\check\rho}$ and Γ_ρ are both product spaces, and we will show that all subsequent images $f_X^{2n+1}\mathcal{F}_i$ are horizontal sections of $\Lambda_{\check\rho} \cap \Gamma_\rho$. A horizontal section may be written as (base space) $\times \{\varphi_{2n+1}\}$, where φ_{2n+1} is a fiber point (see Figure 5). In order to show that $f_X^{2n+1}\mathcal{F}_i \not\subset \mathcal{I}_X$, we track the "moving fiber" point φ_{2n+1} in the same way we tracked the orbit of f_Xa_i for $i \in S_1$.

Lemma 7.13. *If $i \in S_\rho$, then let $\mathcal{F}_i$ be the fiber in $\Gamma_{\check\rho}$ over a_i. Let $\phi_\rho = \rho[0 : 0 : \cdots : 0 : p/\rho - 1 : 0 : \cdots : 0 : 3 - p/\rho : 0 : \cdots] \in \Pi_{\langle \rho \bmod 2\rho\rangle}$, and set ϕ_i obtained by permuting the nonzero coordinates. $\lambda_i = A\Pi_{\langle \check\rho \bmod 2\check\rho\rangle}; \phi_i \subset \Gamma_\rho$. Then we have dominant maps $f_X : \Sigma_i \dashrightarrow \mathcal{F}_i \dashrightarrow \lambda_i$.*

Proof. Let us first consider the case $i = \rho$. Repeating the argument in previous sections, $f_X : \Sigma_\rho \dashrightarrow \mathcal{F}_\rho$ is a dominant mapping. For a generic point $a_\rho;\xi$, $\xi = [0 : \cdots : \xi_{\check\rho} : 0 : \cdots : 0 : \xi_{3\check\rho} : \cdots] \in \Pi_{\langle \check\rho \bmod 2\check\rho\rangle}$ and $\Pi_{\langle \check\rho \bmod 2\check\rho\rangle}$ is invariant under J. Since A is linear and invertible , the rank of $f_X|\mathcal{F}_\rho$ is the same as the dimension of Λ_ρ. Now we will show that the constant fiber for $f_X\mathcal{F}_\rho$ is ϕ_ρ. Since $\Lambda_\rho \subset \Gamma_\rho$, the fiber coordinate is $[0 : \cdots : 0 : \xi_\rho : 0 : \cdots : 0 : \xi_{3\rho} : 0 : \cdots]$, and $\xi_{k\rho} = 1 + \sum_{j \not\equiv \check\rho \bmod 2\check\rho} \omega_{kj\rho} \cdot 1/\omega_{j\rho}$ for an odd k. If $k = 1$, we have

$$\xi_\rho = 1 + \sum_{j \not\equiv \check\rho \bmod 2\check\rho} \omega_{j\rho} \cdot 1/\omega_{j\rho} = 1 + (p-1) - r = r(p/r - 1).$$

For a general k, using (4.2)

$$\xi_{k\rho}+\xi_{(k+2)\rho}=2+\sum_{j\not\equiv\check\rho \bmod 2\check\rho}(\omega_{kj\rho}+\omega_{(k+2)j\rho})\cdot 1/\omega_{j\rho}$$

$$=2+\sum_{j\not\equiv\check\rho \bmod 2\check\rho}\omega_{(k+1)j\rho}=\rho(2+\sum_{j=1}^{\check\rho-1}\omega_{(k+1)j\rho}).$$

Following the same reasoning as in Lemma 7.8, we have $\xi_{k\rho}+\xi_{(k+2)\rho}=2$ if $k+1\equiv 2 \bmod 4$, and $\xi_{k\rho}+\xi_{(k+2)\rho}=-2$ if $k+1\equiv 0 \bmod 4$. For general $i\in S_\rho$, we follows the discussion in Lemma 4.2. □

For each divisor ρ in (7.4), let us set $\mathcal{L}_\rho = A\Pi_{\langle\check\rho \bmod 2\check\rho\rangle};\Pi_{\langle\rho \bmod 2\rho\rangle}$. Let us identify $\mathcal{L}_\rho\cong \mathbf{P}^{p/(2\rho)-1}$ by a projection $\pi:(x;\xi)\to\xi$, and let $\varphi_\rho:\mathbf{P}^{p/(2\rho)-1}\dashrightarrow \mathbf{P}^{p/(2\rho)-1}$ be the induced map corresponding to $f_X^2:\mathcal{L}_\rho\dashrightarrow\mathcal{L}_\rho$. As we saw before Lemma 7.9, we may choose the coordinates of φ_ρ to be homogeneous polynomials with coefficients in $\mathbf{Z}[\omega]$.

Lemma 7.14. *For $j\in S_\rho$, there is a polynomial $R_{\rho,j}\in\mathbf{Z}[\omega^\rho]$ such that $x_j|R_j$, and*

$$\varphi_\rho[x_1,x_3,x_5,\dots,x_{p/\rho-1}]=(\check\rho)^{2\rho-2}(p/2)^2[p/\rho-1:3-p/\rho:\cdots;\pm1](x_{\hat 1})^{\check\rho}+R_{\rho,1}(x)$$
$$=V_j(x_{\hat j})^{\check\rho}+R_{\rho,j}(x)$$

where V_j is obtained from $(\check\rho)^{2\rho-2}(p/2)^2[p/\rho-1:3-p/\rho:\cdots;\pm1]$ by permuting the coordinates.

Proof. Following the discussion in Lemma 7.9. Let us set $\varphi_\rho[x_\rho:x_{3\rho}:\cdots:x_{p/\rho-1}]=[y_\rho:y_{3\rho}:\cdots:y_{p/\rho-1}]$. On $\Sigma_\rho^*\subset\{x_\rho=0\}$, we have

$$y_{j\rho}=\rho\cdot(\prod_{k=1}^{\check\rho-1}\omega_{k\rho})^{2\rho}\cdot[1+2\sum_{\ell=1}^{\check\rho-1}\omega_{j\ell\rho}\cdot\frac{1}{\omega_\ell}]\cdot(x_{\hat 1})^{\check\rho}\quad\forall j:\ \text{odd}$$

and $\prod_{k=1}^{\check\rho-1}\omega_k=$ a unit in $\mathbf{Z}[\omega^\rho]\cdot p/(2\rho)$. Combining Lemma 7.13 followed by the same discussion in Lemma 4.2 we have the desired result. □

Lemma 7.15. *For $j\in S_\rho$, $f_X^n\lambda_j\not\subset\mathcal{I}_X$ for all $n\ge 0$.*

Proof. We apply Lemma 7.14 modulo μ following the line of argument of Lemma 7.10. □

To summarize: in this Section we have constructed the space X and determined f_X^* on $Pic(X)$. Further, we have shown that for every exceptional hypersurface E of f_X, we have $f^nE\not\subset\mathcal{I}$ for all $n\ge0$. Thus we can apply Theorems 2.1 and 2.4 to conclude:

Theorem 7.16. *The map f_X satisfies (1.4), and $\delta(f)$ is the spectral radius of the linear transformation f_X^*, which is defined in (7.5–7.12).*

Appendix A. $\mathbf{q=45=3^2\cdot 5}$. Let us carry out the algorithm implicit in Theorems 5.4 and 5.5. If $q=45$, then $p=22$. The divisors are $r=3,5,9,15$, and we have $S_1=\{1,2,4,7,8,11,13,14,16,17,19,22\}$, $S_3=\{3,6,12,21\}$, $S_5=\{5,10,20\}$, $S_9=\{9,18\}$, and $S_{15}=\{15\}$. Let us define $E^{(1)}=\sum_{i\in S_1}E_i$, $AV^{(1)}=\sum_{i\in S_1}AV_i$,

$V^{(1)} = \sum_{i \in S_1} V_i$. And for each divisor r, we set $A^{(r)} = \sum_{i \in S_r} A_i$. By the symmetries of the equations defining f_X^* we see that we may rewrite them in terms of the new, consolidated basis elements as

$$E_0 \mapsto A_0 \mapsto H - E_0 - E^{(1)}$$

$$E^{(1)} \mapsto AV^{(1)} \mapsto V^{(1)} \mapsto A^{(1)} \mapsto 12H - 12E_0 - 11E^{(1)} - 12\hat{P}$$

$$P_3 \mapsto A^{(3)} \mapsto 4H - 4E_0 - 4E^{(1)} - 4\hat{P} + 4P_3$$

$$P_5 \mapsto A^{(5)} \mapsto 3H - 3E_0 - 3E^{(1)} - 3\hat{P} + 3P_5$$

$$P_9 \mapsto A^{(9)} \mapsto 2H - 2E_0 - 2E^{(1)} - 2\hat{P} + 2P_3 + 2P_9$$

$$P_{15} \mapsto A^{(15)} \mapsto H - E_0 - E^{(1)} - \hat{P} + P_3 + P_5 + P_{15}$$

$$H \mapsto 22H - 21E_0 - 21E^{(1)} - 14P_3 - 17P_5 - 19P_9 - 20P_{15}.$$

The characteristic polynomial of this linear transformation is $(x+1)(x-1)^2$ times $24 - 264x - 290x^2 + 310x^3 + 559x^4 + 109x^5 - 410x^6 - 300x^7 + 136x^8 + 144x^9 - 20x^{10} - 21x^{11} + x^{12}$, which gives a spectral radius $\rho \approx 21.6052$, and $\delta(K|\mathcal{SC}_{45}) \approx 466.784$.

Appendix B. **Spectral radius for $\mathbf{q = 45}$.** Let us demonstrate how to use the formula in Theorem 5.5. For $q = 3^2 \cdot 5$ we have $\kappa_3 = 4$, $\kappa_5 = 3$, $\kappa_9 = 2$, $\kappa_{15} = 1$, and $\kappa = 12$. $\mu_3 = 8$, $\mu_5 = 5$, $\mu_9 = 3$, and $\mu_{15} = 2$. For prime divisors we have

$$T_3(x) = 4(x^2-3)(x^2-2)(x^2-1), \quad T_5(x) = 3(x^2-4)(x^2-2)(x^2-1).$$

For non-prime divisors we have

$$T_9 = \frac{2}{x^2-2} T_3(x) + 2(x^2-4)(x^2-3)(x^2-1) = 2x^2(x^2-3)(x^2-1)$$
$$T_{15} = \frac{1}{x^2-1}[T_3(x) + T_5(x)] + (x^2-4)(x^2-3)(x^2-2) = (x^4-12)(x^2-2).$$

Thus we get

$$T_0 = (x^2-4)(x^2-3)(x^2-2)(x^2-1) + \sum_r T_r(x) = -72 + 150x^2 - 76x^4 + 8x^6 + x^8.$$

Finally, plugging into the formula (5.9) gives us $(x-1)(24 - 264x - 290x^2 + 310x^3 + 559x^4 + 109x^5 - 410x^6 - 300x^7 + 136x^8 + 144x^9 - 20x^{10} - 21x^{11} + x^{12})$.

Appendix C. $\mathbf{q = 30 = 2 \cdot 3 \cdot 5}$. Now let us demonstrate how to use the algorithm in Theorem 6.15. If $q = 30$, then $p = 15$, and the odd divisors are $r = 3$ and 5. Thus $S_1 = \{1, 7, 11, 13\}$, $S_2 = \{2, 4, 8, 14\}$, $S_3 = \{3, 9\}$, $S_6 = \{6, 12\}$, $S_5 = \{5\}$, and $S_{10} = \{10\}$. The linear transformation f_X^* has symmetries under $e \leftrightarrow o$ and $j \leftrightarrow p - j$. Further, since $P_r \mapsto 0$, we do not need to consider P_r for the purpose of computing the spectral radius. Thus we define the symmetrized elements

$$E = E_0 + E_{15}, \quad A = A_0 + A_{15}, \quad A^{(r)} = \sum_{j \in S_r \cup S_{2r}} A_j$$

$$P_w = P_o + P_e, \quad AP_w = AP_o + AP_e, \quad P_{w,r} = P_{o,r} + P_{e,r},$$

where r denotes a divisor of p. We see that we may take all of these elements, together with H, as the basis of an f_X^*-invariant subspace of $H^{1,1}(X)$. We have:

$$\begin{aligned}
&E \mapsto A \mapsto 2H - E - P_w - P_{w,3} - P_{w,5} \\
&AP_w \mapsto P_w \mapsto A^{(1)} + AP_w \\
&A^{(1)} \mapsto 8H - 8E - 4P_w - 8P_{w,3} - 8P_{w,5} \\
&A^{(3)} \mapsto 4H - 4E - 2P_w - 2P_{w,3} - 4P_{w,5} \\
&A^{(5)} \mapsto 2H - 2E - P_w - 2P_{w,3} - P_{w,5} \\
&P_{w,3} \mapsto A^{(3)}, \quad P_{w,5} \mapsto A^{(5)} \\
&H \mapsto 15H - 14E - 7P_w - 12P_{w,3} - 13P_{w,5}.
\end{aligned}$$

We may also define anti-symmetric elements $E' = E_0 - E_{15}$, $P'_w = P_e - P_o$, $AP'_w = AP_e - AP_o$, etc., as well as $\sum_{j\in S_r} t_j A_j - \sum_{k\in S_{2r}} t'_k A_k$, for any odd divisor r and $\sum t_j = \sum t'_k$. By the symmetries of f_X^*, the anti-symmetric elements define a complementary invariant subspace. The spectral radius, however, is given by the transformation above. Its characteristic polynomial is $x(x+1)(x-1)^2$ times $-6-16x+11x^2+32x^3-6x^4-14x^5+x^6$, which gives a spectral radius $\rho \approx 14.26$, and $\delta(K|\mathcal{SC}_{30}) \approx 203.347$.

Appendix D. $\mathbf{q = 60 = 2^2 \cdot 3 \cdot 5.}$ Finally, let us illustrate the algorithm of Theorem 7.16 for $q = 60$. In this case, $p = 30$, and in (7.4) notation, the divisors are $r = 2, 6, 10$, and $\rho = 3, 5$. We have $S_1 = \{1, 7, 11, 13, 17, 19, 23, 29\}$, $S_2 = \{2, 14, 22, 26\}$, $S_3 = \{3, 9, 21, 27\}$, $S_4 = \{4, 8, 16, 28\}$, $S_5 = \{5, 25\}$, $S_6 = \{6, 18\}$, $S_{10} = \{10\}$, $S_{12} = \{12, 24\}$, $S_{20} = \{20\}$. As before, we work with the symmetrized elements

$$A = A_0 + A_{30}, \quad E = E_0 + E_{30}$$
$$A^{(r)} = \sum_{j\in S_r\cup S_{2r}} Aj, \quad P_r = P_{o,r} + P_{e,r}.$$

Thus f_X^* maps these symmetrized elements as:

$$\begin{aligned}
A_{15} &\mapsto H - E - \Gamma_5 - \Gamma_3 - P_2 - P_6 - P_{10} \\
E &\mapsto A \mapsto 2H - E - P_2 - P_6 - P_{10} \\
P_2 &\mapsto A^{(2)} \mapsto 8H - 8E - 4P_2 - 8P_6 - 8P_{10} \\
P_6 &\mapsto A^{(6)} \mapsto 4H - 4E - 2P_2 - 2P_6 - 4P_{10} \\
P_{10} &\mapsto A^{(10)} \mapsto 2H - 2E - P_2 - 2P_6 - P_{10} \\
\Gamma_5 &\mapsto 4H - 4E - 4A_{15} - 4P_2 - 4P_6 - 4P_{10} - 4\Gamma_3 \\
\Gamma_3 &\mapsto 2H - 2E - 2A_{15} - 2P_2 - 2P_6 - 2P_{10} - 2\Gamma_5 \\
H &\mapsto 30H - 29E - 14A_{15} - 22P_2 - 27P_6 - 28P_{10} - 2\Gamma_5 - 4\Gamma_3.
\end{aligned}$$

The spectral radius of this transformation is the largest root of $512 + 256x - 1760x^2 - 720x^3 + 2304x^4 + 756x^5 - 1494x^6 - 256x^7 + 441x^8 - 5x^9 - 29x^{10} + x^{11}$, which is ≈ 28.6503. Thus $\delta(K|\mathcal{SC}_{60}) \approx 820.841$.

Appendix E. **Characteristic polynomial for $\mathbf{q}$ = odd.** Here we give a sketch of the proof of Theorem 5.5. We set

$$D(a) = \begin{pmatrix} -x & a \\ 1 & -x \end{pmatrix}, \quad U(a) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$M_n(a_1,\dots,a_n)=\begin{pmatrix} D(a_1) & U(a_2) & \dots & U(a_n) \\ & D(a_2) & & \vdots \\ & & \ddots & U(a_n) \\ & & & D(a_n)\end{pmatrix},$$

where the empty spaces are filled by zeros.

Lemma E.1. $\det(M_n(a_1,\dots,a_n))=\prod_{j=1}^n(x^2-a_j)$. *Any of the blocks* $U(a_j)$ *may be replaced by* 2×2 *blocks of zeros without changing the determinant.*

Proof. By adding $1/x\cdot(2i-1)$th row to $2i$th row for all $1\le i\le n$, we obtain the diagonal matrix with diagonal entries $-x, -x+a_1/x, -x, -x+a_2/x, \dots, -x+a_n/x$. The result follows immediately. $\square$

Let us define $H(a)=\begin{pmatrix}0 & a\end{pmatrix}$,

$$B=\begin{pmatrix}0 & 0\\ 1 & -x\end{pmatrix},\quad\text{and}\quad M_n'(a_1,\dots,a_n)=\begin{pmatrix}M_n(a_1,\dots,a_{n-1}) & C(a_n)\\ E(a_1) & B\end{pmatrix},$$

where $C(a_n)$ is the $2(n-1)\times 2$ column matrix obtained by stacking $(n-1)$ copies of $U(a_n)$ vertically, and $E(a_1)$ is the $2\times 2(n-1)$ matrix obtained by starting on the left with $U(a_1)$ and following with zeros.

Lemma E.2. $\det(M_2'(a_1,a_2))=-a_1a_2$, *and*

$$\begin{aligned}&\det(M_n'(a_1,\dots,a_n))\\ =&a_1\left[\sum_{k=2}^{n-1}\prod_{j=2}^{k-1}(x^2-a_j)\cdot\det M_{n-k+1}'(a_k,\dots,a_n)-a_n\prod_{j=2}^{n-1}(x^2-a_j)\right].\end{aligned}$$

Proof. We first expand in minors along the next to last row which contains a_1 in the second slot and then expand in minors along the second row which has only one entry 1 in the first slot. It follows that $\det(M_n'(a_1,\dots,a_n))=a_1\cdot\det(M_{n-1}''(a_2,\dots,a_n))$ where $B'=\begin{pmatrix}1 & -x\end{pmatrix}$ and

$$M_{n-1}''(a_2,\dots,a_n)=\begin{pmatrix}H(a_2) & H(a_3) & \dots & H(a_n)\\ D(a_2) & U(a_3) & & \vdots\\ & & \ddots & U(a_n)\\ & & & B'\end{pmatrix}.$$

Now we use the first row to compute minors. It is not hard to see that each minor can be computed from the matrix of the form

$$\begin{pmatrix}M_{k-3}(a_2,\dots,a_{k-2}) & *\\ 0 & M_{n-k+1}''(a_K,\dots,a_n)\end{pmatrix}.$$

The result follows using Lemma E.1 and its proof. $\square$

Proof of Theorem 5.5. We use the symmetry of M_f noted in Appendix A and work with a symmetrized basis for $Pic(X)$: $H=H_X$, P_r, $A^{(r)}$, E_0, A_0, $E^{(1)}$, $AV^{(1)}$, $V^{(1)}$, $A^{(1)}$. We order the basis so that if $r_1|r_2$ then P_{r_1}, $A^{(r_1)}$ appears before P_{r_2}, $A^{(r_2)}$; thus we we start with the prime factors of q. To compute the characteristic polynomial, we consider a matrix M_f-xI. For a simpler format, we add first row to the row corresponding to P_r, E_0 and $E^{(1)}$. After the series of row operations, we have the determinant of $(M_f - xI)$ is equal to the deteminant of

$$\begin{pmatrix} p-x & H(a_1) & H(a_2) & \cdots & H(a_\kappa) & H(1) & H(0) & H(1) \\ V(b_1-x) & D(a_1) & U(a_2) & \cdots & U(a_\kappa) & U(1) & & \\ V(b_2-x) & & D(a_2) & \ddots & U(a_\kappa) & U(1) & & \\ \vdots & & & \ddots & U(a_\kappa) & U(1) & & \\ V(b_\kappa-x) & & & & D(a_\kappa) & U(1) & & \\ V(1-x) & & & & & D(1) & & \\ V(1-x) & & & & & & D(0) & U(1) \\ 0 & & & & & & U(1) & D(0) \end{pmatrix}$$

where the empty spaces are filled by zeros and each a_j b_j is determined by a proper divisor of q and κ is the number of proper divisors, and $V(a) = \begin{pmatrix} a \\ 0 \end{pmatrix}$. Now we expand in the minors along the first column. For the $(j,1)$-minor we move the first row to the jth row and then expand in minors along the jth row. The rest of the computation follows using Lemmas E.1 and E.2. $\square$

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