Semi-Parabolic Bifurcations in Complex Dimension Two

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§0. Introduction. Parabolic bifurcations in one complex dimension demonstrate a wide variety of interesting dynamical phenomena [D, DSZ, L, Mc, S]. Consider for example the family of dynamical systems $f_\epsilon(z) = z + z^2 + \epsilon^2$. When $\epsilon = 0$ then 0 is a parabolic fixed point for f_0. When $\epsilon \neq 0$ the parabolic fixed point bifurcates into two fixed points. (The use of the term ϵ^2 in the formula allows us to distinguish these two fixed points.)

We can ask how the dynamical behavior of f_ϵ varies with ϵ. One way to capture this is to consider the behavior of dynamically significant sets such as the Julia set, J, and the filled Julia set, K, as functions of the parameter.

Theorem. *([D,L]) The functions $\epsilon \mapsto J(f_\epsilon)$ and $\epsilon \mapsto K(f_\epsilon)$ are discontinuous at $\epsilon = 0$ when viewed as mappings to the space of compact subsets of $\mathbf{C}$ with the Hausdorff topology.*

In this paper we consider semi-parabolic bifurcations of families of diffeomorphisms in two complex dimensions. We let M denote a complex manifold of dimension 2, usually $M = \mathbf{C}^2$, and we consider a family of holomorphic diffeomorphisms $F_\epsilon : M \to M.$ such that

$$F_\epsilon \text{ is holomorphic in } \epsilon^2, \quad \text{and} \quad F_\epsilon(x, y) = (x + x^2 + \epsilon^2 + \cdots, b_\epsilon y + \cdots) \tag{0.1}$$

where $|b_\epsilon| < 1$, and the '$\cdots$' term in the first coordinate has order ≥ 3 in (ϵ, x, y), and in the second coordinate it has order ≥ 2. When $\epsilon = 0$ this map has the origin as a fixed point, and the eigenvalues of the derivative at the origin are 1 and b_0. We say that F_0 is semi-parabolic at the origin. In [U1,2] it is shown that the set of points attracted to O in forward time can be written as $\mathcal{B} \cup W^{ss}(O)$, where $\mathcal{B}$ is a two complex dimensional basin of locally uniform attraction and $W^{ss}(O)$ is the one complex dimensional strong stable manifold corresponding to the eigenvalue b_0. The point O is not contained in the interior of its attracting set, and we describe this by saying that the point is semi-attracting. The set of points attracted to O in backward time can be written as $\Sigma \cup O$ where Σ is a one complex dimensional manifold called the asymptotic curve.

The principal case we consider here is where $M = \mathbf{C}^2$, and F_ϵ is a polynomial diffeomorphism. Friedland and Milnor [FM] have classified these into two classes. One of them has dynamical behavior which is too simple for there to be a semi-parabolic fixed point. Thus a family satisfying (0.1) must belong to the other class which, up to conjugacy, consists of compositions of generalized Hénon mappings. We refer to [BS1], [FS] and [HO1] for general discussions of Hénon maps.

Polynomial diffeomorphisms have constant Jacobian, and to be consistent with (0.1), we assume that the Jacobian is less than one in absolute value. Analogs of the filled Julia set are the sets K^+ and K^-, consisting of points p so that $F^n(p)$ remains bounded as $n \to \pm\infty$. Dynamically interesting sets also include $J^\pm = \partial K^\pm$. We also consider $K = K^+ \cap K^-$ and $J = J^+ \cap J^-$. It is a basic fact that the one variable Julia set J is the closure of the set of expanding periodic points. We define J^* to be the closure of the set of periodic saddle points. It has a number of other interesting characterizations: it is the Shilov boundary of K and is the support of the unique measure of maximal entropy (see [BS1] and [BLS]).

Both the sets J and J^* can be considered analogs of the Julia set in one variable. The set J^* is contained in J, but it is still not known whether these two sets are always equal.

If $\{X_\epsilon\}$ is a family of closed sets, we will say that a point p belongs to $\liminf_{\epsilon\to 0} X_\epsilon$ iff for every neighborhood V of p we have $X_\epsilon \cap V \neq \emptyset$ for all sufficiently small ϵ. We say that p belongs to $\limsup_{\epsilon\to 0} X_\epsilon$ iff for every neighborhood V of p we have $X_\epsilon \cap V \neq \emptyset$ for infinite sequence of ϵ's tending to 0. We say that $\epsilon \mapsto X_\epsilon$ is upper semicontinuous if $\limsup_{\epsilon\to 0} X_\epsilon \subset X_0$, and lower semicontinuous if $\liminf_{\epsilon\to 0} X_\epsilon \supset X_0$. Certain semicontinuity properties hold generally (see [BS1]): *For a continuous family $\epsilon \mapsto F_\epsilon$ of Hénon maps, the set-valued mapping $\epsilon \mapsto X(F_\epsilon)$ is upper semicontinuous if $X = K^+$, K^- or K; and it is lower semicontinuous if $X = J^+$, J^-, or J^*.*

We will show that at a semi-parabolic fixed point we have additional information:

Theorem 1. *Suppose that F_ϵ as in (0.1) is a family of polynomial diffeomorphisms of $\mathbf{C}^2$. Then for $X = J^*$, J, J^+, K or K^+ the function $\epsilon \mapsto X(F_\epsilon)$ is discontinuous at $\epsilon = 0$. For $X = J^-$ or K^- the function $\epsilon \mapsto X(F_\epsilon)$ is continuous.*

Our approach follows the outlines of the approach of the corresponding result in one variable. In the one variable case we work with $f_\epsilon(z) = z + z^2 + \epsilon^2 + \cdots$; the first step is to analyze certain sequences of maps $f_{\epsilon_j}^{n_j}$, where the parameter ϵ_j and the number of iterates n_j are both allowed to vary. The idea is the following. Let p be a point in the basin of 0 for f_0. When ϵ is small but non-zero the fixed point at 0 breaks up into two fixed points. As n increases, the point $f_\epsilon^n(p)$ will come close to 0 and may pass between these two fixed points and exit on the other side. Following standard terminology we refer to this behavior as "passing through the eggbeater". When ϵ is small the point moves more slowly and more iterations are required for it to pass through the eggbeater. It is possible to choose sequences ϵ_j going to 0 and n_j going to infinity so that $f_{\epsilon_j}^{n_j}(p)$ will converge to some point on the other side of the eggbeater, in particular some point other than 0. The limit maps which arise this way have a convenient description in terms of Fatou coordinates of the map f_0 (see [Mi]). A Fatou coordinate is a $\mathbf{C}$-valued holomorphic map φ defined on an attracting or repelling petal which satisfies the functional equation $\varphi(f(p)) = \varphi(p) + 1$. There is an "incoming" Fatou coordinate φ^ι on the attracting petal and an "outgoing" Fatou coordinate φ^o on the repelling petal. Let $\tau_\alpha(\zeta) = \zeta + \alpha$ be the translation by α, acting on $\mathbf{C}$, and let $t_\alpha := (\varphi^o)^{-1} \circ \tau_\alpha \circ \varphi^\iota$ be the transition map that maps the incoming (attracting) petal to the outgoing (repelling) petal.

Theorem. *(Lavaurs) If $\epsilon_j \to 0$ and $n_j \to \infty$ are sequences such that $n_j - \pi/\epsilon_j \to \alpha$, then* $\lim_{j\to\infty} f_{\epsilon_j}^{n_j} = t_\alpha$.

We will define ϵ_j to be an *α-sequence* if $\epsilon_j \to 0$ and $n_j \to +\infty$ can be chosen so that $n_j - \pi/\epsilon_j \to \alpha$. Sometimes for clarity we will refer to (n_j, ϵ_j) as the α-sequence.

In Section 3 of this paper we prove the analogous result in two complex dimensions. Shishikura [S] gives a careful proof of this Theorem in one dimension using the Uniformization Theorem. The concept of uniformization is much more delicate in two variables so, by way of preparation for our two dimensional result, we re-prove the 1-dimensional result without using the Uniformization Theorem in Section 2.

The existence of the Abel-Fatou functions, or Fatou coordinates, in the semi-parabolic case was established in [U1,2]. Let $\varphi^\iota : \mathcal{B} \to \mathbf{C}$ denote the Fatou coordinate on the attracting basin and $\varphi^o : \Sigma \to \mathbf{C}$ the Fatou coordinate on the asymptotic curve (or repelling leaf). Note that the function φ^ι has a two complex dimensional domain of definition, while φ^o is defined on the Riemann surface Σ. In fact the map φ^ι is a submersion and defines a fibration whose leaves are described in Theorem 1.2. Our principal object of study will be the *transition map*

$$T_\alpha := (\varphi^o)^{-1} \circ \tau_\alpha \circ \varphi^\iota : \mathcal{B} \to \Sigma \tag{0.2}$$

where $\alpha \in \mathbf{C}$ is given, and τ_α denotes the translation by α. We have

$$F \circ T_\alpha = T_{\alpha+1} = T_\alpha \circ F.$$

Fatou coordinates could also be defined by adding constants to $\varphi^{\iota/o}$. The family $\{T_\alpha\}$ covers all such possibilities.

Theorem 2. *Let F_ϵ be a family of polynomial diffeomorphisms as in (0.1), and let ϵ_j be an α-sequence. Then $\lim_{j\to\infty} F_{\epsilon_j}^{n_j} = T_\alpha$ uniformly on compact subsets of $\mathcal{B}$.*

We note that we are taking very high iterates of a dissipative diffeomorphism, so it is reasonable that the limiting map has a one-dimensional image. A version of Theorem 2 holds if F_ϵ is only locally defined at O; see the discussion of the local basin at the end of §1.

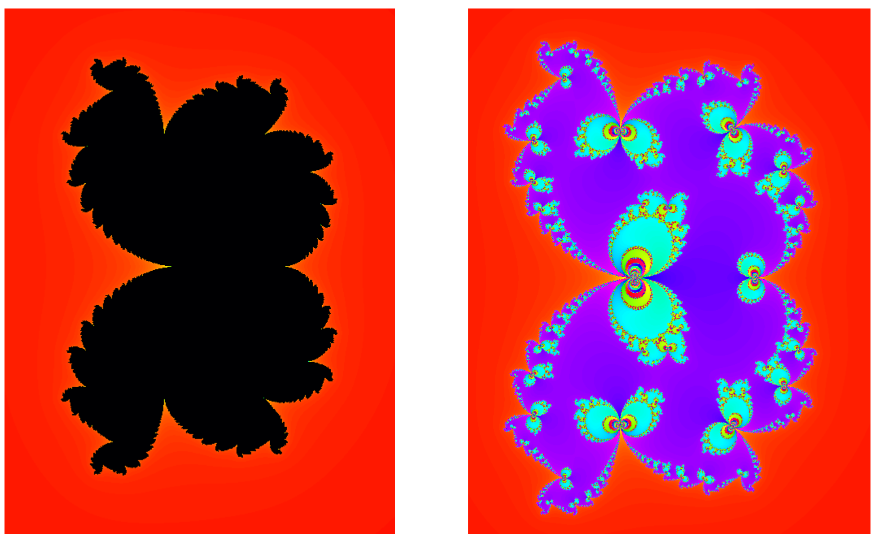

Figure 1. The discontinuity of the map $\epsilon \mapsto K^+(F_{a,\epsilon})$ illustrated by showing complex linear slices in $\mathbf{C}^2$ for two nearby parameter values. $F_{a,\epsilon}$ is given by equation (0.3) with $a = .3$; $\epsilon = 0$ (left), $a = .3$, $\epsilon = .05$ (right).

When ϵ is small, a point may pass through the eggbeater repeatedly. We may use the map T_α to model this behavior. In case $T_\alpha(p)$ happens to lie in $\mathcal{B}$, we may define the iterate $T_\alpha^2(p)$. More generally we can view T_α as a partially defined dynamical system. A point for which T_α^n can be defined for n iterations corresponds to a point which passes through the eggbeater n times.

Following the approach of [D, L] in one dimension we may introduce sets $J^*(F_0, T_\alpha)$ and $K^+(F_0, T_\alpha)$ which play the role of J^* and K^+ for the partially defined map T_α. (See Definitions 4.2 and 4.3.)

Theorem 3. *Suppose that F_ϵ is a family of polynomial diffeomorphisms of $\mathbf{C}^2$ as in (0.1), and let ϵ_j be an α-sequence. Then*

$$\liminf_{j\to\infty} J^*(F_{\epsilon_j}) \supset J^*(F_0, T_\alpha).$$

Though we have stated Theorem 3 for polynomial diffeomorphisms, the definition of the set $J^*(F_\epsilon)$ as the closure of the periodic saddle points makes sense for a general holomorphic diffeomorphism, and Theorem 3 is true in this broader setting.

Theorem 4. *Suppose that F_ϵ is a family of polynomial diffeomorphisms of $\mathbf{C}^2$ as in (0.1), and let ϵ_j be an α-sequence. Then we have*

$$\mathcal{B} \cap \limsup_{j\to\infty} K^+(F_{\epsilon_j}) \subset K^+(F_0, T_\alpha).$$

If the function $\epsilon \mapsto J^*(F_\epsilon)$ were continuous at $\epsilon = 0$, then the limit of $J^*(F_{\epsilon_j})$ along an α-sequence would be independent of α and would be equal to $J^*(F_0)$. Theorem 3 implies that $J^*(F_0)$ would have to contain every set $J^*(F_0, T_\alpha)$. Theorem 4 implies that $J^*(F_0)$ would have to be contained in every set $K^+(F_0, T_\alpha)$. The following shows that these conditions are incompatible.

Theorem 5. *If F_0 is a polynomial diffeomorphism with a semi-parabolic fixed point O, then (i) there exists $\alpha \in \mathbf{C}$ such that $\mathcal{B} \cap J^*(F_0, T_\alpha) \neq \emptyset$, and (ii) for each $p \in \mathcal{B}$ there exists an α' such that $p \notin K^+(F_0, T_{\alpha'})$.*

We can use Theorem 5 to prove the discontinuity statement of the maps $\epsilon \mapsto J^*(F_\epsilon)$ and $\epsilon \mapsto K^+(F_\epsilon)$, but in fact the same argument shows the discontinuity of any dynamically defined set X which is sandwiched between J^* and K^+. Using this idea, we now give a proof of Theorem 1.

Proof of Theorem 1. We begin by proving the statement concerning discontinuity. Let X be one of the sets J, J^*, J^+, K or K^+. Assume that the function $\epsilon \mapsto X(F_\epsilon)$ is continuous at $\epsilon = 0$. By Theorem 5 we may choose $p \in \mathcal{B}$ and α, α' to that $p \in \mathcal{B} \cap J^*(F_0, T_\alpha)$, but $p \notin K^+(F_0, T_{\alpha'})$. Let ϵ_j be an α-sequence and let ϵ'_j be an α'-sequence. Since

$$J^*(F_\epsilon) \subset X(F_\epsilon) \subset K^+(F_\epsilon)$$

we have that

$$J^*(F_0, T_\alpha) \subset \liminf_{j\to\infty} J^*(F_{\epsilon_j}) \subset \liminf_{j\to\infty} X(F_{\epsilon_j}) = X(F_0)$$

by Theorem 3. Further, by Theorem 4, we have

$$K^+(F_0, T_{\alpha'}) \supset \mathcal{B} \cap \limsup_{j\to\infty} K^+(F_{\epsilon'_j}) \supset \mathcal{B} \cap \limsup_{j\to\infty} X(F_{\epsilon'_j}) = \mathcal{B} \cap X(F_0).$$

By the properties of α, α', and p, we see that the lim inf and lim sup cannot be the same.

The fact that J^- and K^- vary continuously follows from the fact that for polynomial diffeomorphisms which contract area the sets J^- and K^- are equal (see [FM]). We combine this with the facts that J^- varies lower semi-continuously and K^- varies upper semi-continuously. □

To give a 2D analogue of the family f_ϵ we start with the quadratic Hénon maps $H_{a,c}(x,y) = (x^2+c-ay, x)$. $H_{a,c}$ has a semi-parabolic fixed point when $c = (a+1)^2/4$. We fix a and consider the perturbation $c = (a+1)^2/4+\delta$. Now we conjugate with a translation to move the fixed point to the origin and then a linear map to diagonalize the differential. If we use the parameter $\epsilon^2 = \delta/(1-a)^2$, we arrive at the family (0.3).

$$F_{a,\epsilon} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + (x+ay)^2 + \epsilon^2 \\ ay - (x+ay)^2 - \epsilon^2 \end{pmatrix} \tag{0.3}$$

If $0 < |a| < 1$, the family (0.3) is in the form (0.1). When $\epsilon = 0$, the origin O is the unique fixed point and has multipliers 1 and a, and T = the x-axis is the eigenspace for the multplier 1. Figure 1 shows the slice $K^+ \cap T$, where the points are colored according to the value of the Green function G^+. The set $K^+ = \{G^+ = 0\}$ is colored black. It is hard to see black in the right hand of Figure 1 because the set $T \cap K^+$ is small. In this case, there are points of $T \cap K^+$ at all limit points of infinite color changes. Figure 1 illustrates the behaviors described in Theorems 3 and 4. In the perturbation shown in Figure 1, there is not much change to the "outside" of K^+, whereas the "inside" shows the effect of an "implosion." Further discussion of the figures in this paper is given at the end of §1.

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§**1. Fatou coordinates and transition functions.** Let M be a 2-dimensional complex manifold, and let F be a holomorphic automorphism of M, corresponding to the case $\epsilon = 0$ in (0.1), so O is a fixed point which is semi-attracting and semi-parabolic. We will give a brief sketch of the results in [U1].

We choose local holomorphic coordinates (x, y) with center $O = (0, 0)$ so that the local strong stable manifold is given by $W^{ss}_{loc}(O) = \{x = 0, |y| < 1\}$. Further, for any i, j we may change coordinates so that F has the form

$$\begin{aligned} x_1 &= x + a_2x^2 + a_3x^3 + \ldots + a_ix^i + a_{i+1}(y)x^{i+1} + \ldots \\ y_1 &= by + b_1xy + \ldots + b_jx^jy + b_{j+1}(y)x^{j+1} + \ldots \end{aligned} \tag{1.1}$$

We will suppose that $a_2 \neq 0$, and by scaling coordinates, we may assume $a_2 = 1$. (In the case where $a_2 = \ldots = a_m = 0$, $a_{m+1} \neq 0$, the results analogous to [U1] have been treated by Hakim [H].) For $r, \eta > 0$, we set

$$B^\iota_{r,\eta} = \{|x + r| < r, |y| < \eta\} = \left\{\Re\left(\frac{-1}{x}\right) > \frac{1}{2r}, |y| < \eta\right\}. \tag{1.2}$$

If we take r and η small, then the iterates $F^n\overline{B}^\iota_{r,\eta}$ of the set $\overline{B}^\iota_{r,\eta}$ shrink to O as $n \to \infty$. Further, $B^\iota_{r,\eta}$ plays the role of the "incoming petal" and is a base of convergence, which is to say that $\mathcal{B} := \bigcup_{n\geq 0} F^{-n} B^\iota_{r,\eta}$ is the set of points where the forward iterates converge locally uniformly to O.

With a_3 as in (1.1), we set $q = a_3 - 1$ and choosing the principal branch of the logarithm, we set

$$w^\iota(x,y) := -\frac{1}{x} - q\log(-x). \tag{1.3}$$

It follows that for $p \in \mathcal{B}$ the limit

$$\varphi^\iota(p) = \lim_{n\to\infty} (w^\iota(F^n(p)) - n)$$

is an analytic function $\varphi^\iota : \mathcal{B} \to \mathbf{C}$ satisfying the property of an Abel-Fatou coordinate: $\varphi^\iota \circ F = \varphi^\iota + 1$. Further,

$$\varphi^\iota(x,y) = w^\iota(x,y) + B(x,y) \tag{1.4}$$

where B is bounded on $B^\iota_{r,\eta}$ and tends to 0 when $\Re(-1/x) \to +\infty$ (see [U1, Section 8.1]).

Theorem 1.1. *There is a holomorphic function $\Phi_2(x,y)$ on $\mathcal{B}$ such that $\Phi = (\varphi^\iota, \Phi_2)$ gives a biholomorphic isomorphism between $\mathcal{B}$ and $\mathbf{C}^2$. The map F is conjugated via Φ to the translation of $\mathbf{C}^2$ given by $(X,Y) \mapsto (X+1, Y)$.*

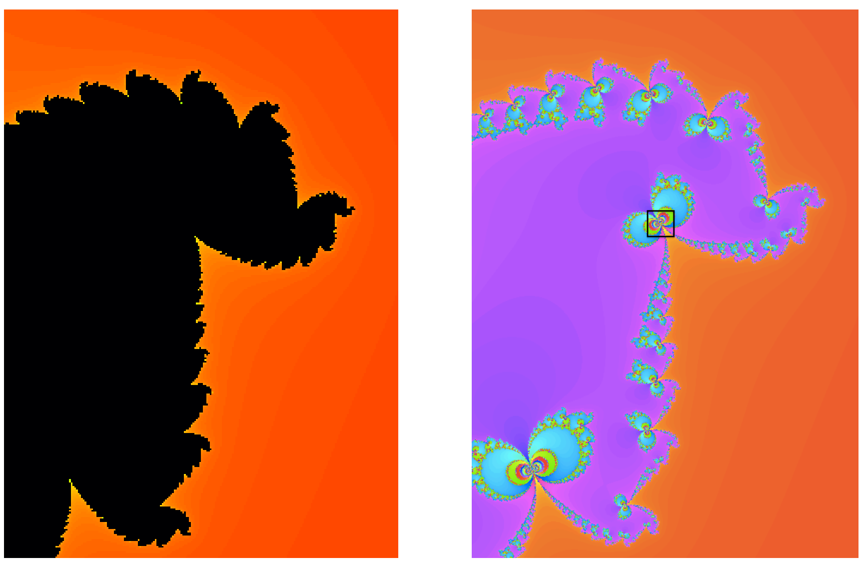

Figure 2a. Slices of K^+ by an unstable manifold for $F_{a,\epsilon}$. $a = .3$, $\epsilon = 0$ (left); $a = .3$, $\epsilon = .05$ (right).

By Theorem 1.1, the sets $\{\varphi^\iota = const\}$ define a holomorphic fibration $\mathcal{F}_{\varphi^\iota}$ of $\mathcal{B}$ whose fibers are closed complex submanifolds which are holomorphically equivalent to $\mathbf{C}$. The intersection of this fibration with $B^\iota_{r,\eta}$ may be thought of consisting of vertical disks as was observed in [U1, §8]. More precisely, for each $p \in B^\iota_{r,\eta}$, there is an analytic function $\psi_p : \{|y| < \eta\} \to \mathbf{C}$ such that the fiber of $\mathcal{F}_{\varphi^\iota}$ through p is contained in a graph:

$$\{(x,y) \in B^\iota_{r,\eta} : \varphi^\iota(x,y) = \varphi^\iota(p)\} \subset \{(x,y) : x = \psi_p(y), |y| < \eta\}$$

These graphs converge to the (vertical) local strong stable manifold $W^{ss}_{loc}(O)$ as $p \to W^{ss}_{loc}(O)$ through $B^\iota_{r,\eta}$.

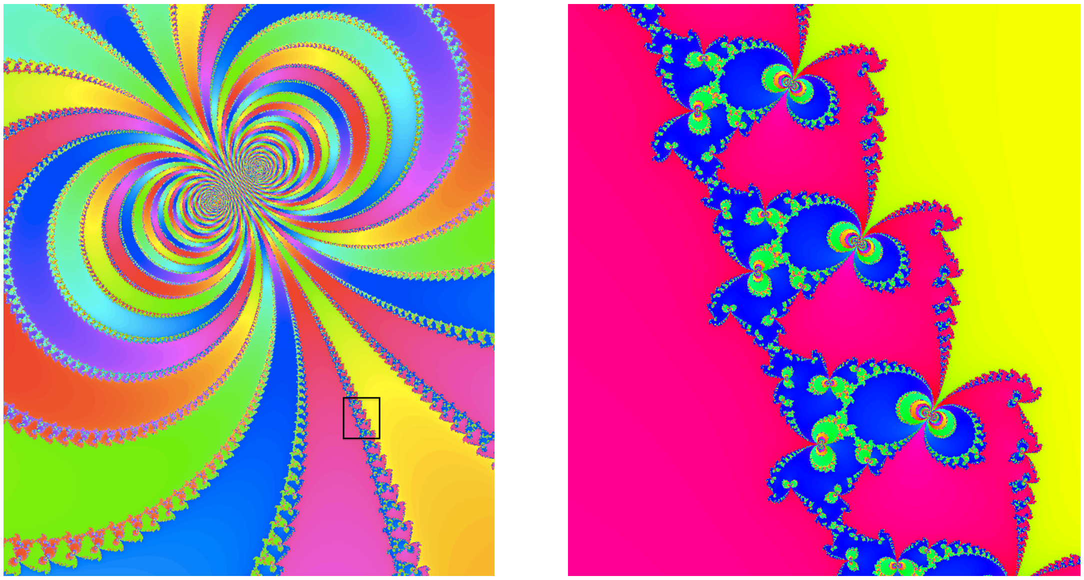

Figure 2b. Zoom of Figure 2a (right). Further zoom.

The fibers of $\mathcal{F}_{\varphi^\iota}$ can be distinguished by their intrinsic dynamical properties. In fact they are strong stable manifolds in the sense that the distance between points in a given fiber converges to zero exponentially, whereas for points in distinct fibers the convergence is quadratic (cf. [Mi, Lemma 10.1]).

Theorem 1.2. *For $p_1, p_2 \in \mathcal{B}$ such that $p_1 \neq p_2$ and $\varphi^\iota(p_1) = \varphi^\iota(p_2)$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log \operatorname{dist}(F^n p_1, F^n p_2) = \log |b|. \tag{1.5}$$

On the other hand, if $\varphi^\iota(p_1) \neq \varphi^\iota(p_2)$, then $\lim_{n\to+\infty} \left(n^2 \cdot \operatorname{dist}(F^n p_1, F^n p_2)\right) \neq 0$.

Proof. The forward orbit of a point of $\mathcal{B}$ will enter the set $B^\iota_{r,\eta}$, so we may assume that $p_1, p_2 \in B^\iota_{r,\eta_0}$. If $\varphi^\iota(p_1) = \varphi^\iota(p_2)$ we may assume that they are contained in a graph $\{x = \psi_\xi(y) : |y| < \eta_0\}$. The behavior in the y-direction is essentially a contraction by a factor of $|b| + o(1)$ where the $o(1)$ refers to a term which vanishes as the orbit tends to O. Since the graphs become vertical as they approach O, the distance from $F^n p_1$ to $F^n p_2$ is essentially contracted by $|b|^n$, which gives the first assertion.

For the second assertion, we use (1.3) and (1.4) so that

$$\varphi^\iota = -\frac{1}{x} - q\log(-x) + o(1)$$

since $\Re(-1/x) \to +\infty$ along forward orbits. From this we obtain

$$-x - \frac{1}{\varphi^\iota} = x^2 q \log(-x) + o(x^2 \log(-x)).$$

Thus

$$-x = (\varphi^\iota)^{-1} = O(x^{3/2})$$

holds on $B^\iota_{r,\rho}$ as $\Re(-1/x) \to +\infty$. For $i = 1, 2$, we write $p_i = (x_i, y_i)$ and $\varphi^\iota(x_i, y_i) = c_i$. The orbit is denoted $(x_{i,n}, y_{i,n}) = F^n(x_i, y_i)$. We have $\varphi^\iota(F^n(x_i, y_i)) = c_i + n$, so we have $|x_{i,n}| \le 2/n$. Thus it follows that

$$x_{1,n} - x_{2,n} = -\left(\frac{1}{c_1 + n} + O(n^{-3/2})\right) + \left(\frac{1}{c_2 + n} + O(n^{-3/2})\right) = \frac{c_1 - c_2}{n^2} + O(n^{-5/2})$$

which gives the second assertion in the Theorem. □

We may also define the asymptotic curve

$$\Sigma := \{p \in M \setminus \{O\} : F^{-n}(p) \to O \text{ as } n \to \infty\} \tag{1.6}$$

In [U2] this was shown to be the image of an injective holomorphic map $H : \mathbf{C} \to \Sigma$ which satisfies $H(\zeta + 1) = F(H(\zeta))$. The outgoing Fatou coordinate $\varphi^o : \Sigma \to \mathbf{C}$ is defined to be the inverse of H, so $\varphi^o(F) = \varphi^o + 1$.

Let us define

$$B^o_{r,\eta} := \{|x - r| < r, |y| < \eta\} = \left\{\Re\left(-\frac{1}{x}\right) < -\frac{1}{2r}, |y| < \eta\right\} \tag{1.7}$$

Then there is an $r > 0$ and a component Σ_0 of $\Sigma \cap B^o_{r,\eta}$ that can be expressed as a graph

$$\Sigma_0 := \{y = \psi(x), |x - r| < r\} \tag{1.8}$$

where ψ is holomorphic, and $\psi(x) \to 0$ when $\Re(-1/x) \to -\infty$. This is the analogue of the "outgoing petal".

We have two Fatou coordinates, φ^ι and φ^o, defined on the set $\mathcal{B} \cap \Sigma$. It is natural to compare them, so we set $\Omega := \varphi^o(\mathcal{B} \cap \Sigma) \subset \mathbf{C}$ and define the *horn map*

$$h := \varphi^\iota \circ (\varphi^o)^{-1} = \varphi^\iota \circ H : \Omega \to \mathbf{C}$$

The critical points of h correspond to the points ζ_c where $(\varphi^o)^{-1}(\zeta_c)$ is a point of tangency between the asymptotic curve Σ and the strong stable fibration $\mathcal{F}_{\varphi^\iota}$.

We recall the following from [U2]. The map h satisfies $h(\zeta + 1) = h(\zeta) + 1$. For $R > 0$, let us write $\Omega^\pm_R := \{\zeta \in \mathbf{C} : \pm\Im(\zeta) > R\}$, and choose R large enough that $\Omega^\pm_R \subset \Omega$. On $\Omega^\pm_R$ we have

$$h(\zeta) = \zeta + c^\pm_0 + \sum_{n>0} c^\pm_n e^{\pm 2n\pi i\zeta}.$$

In particular h is injective on $\Omega^\pm_R$ if R is sufficiently large. Since h is periodic, it defines a map of the cylinder $\mathbf{C}/\mathbf{Z}$; we see that the upper (resp. lower) end of the cylinder will be attracting if $\Im(c^+_0) > 0$ (resp. $\Im(c^-_0) < 0$).

In the construction of the Fatou coordinates we have

$$\varphi^\iota(x,y) = -\frac{1}{x} - q\log(-x) + o(1)$$
$$\varphi^o(x,y) = -\frac{1}{x} - q\log(x) + o(1)$$

where we interpret $o(1)$ in the first case to mean that $\mathcal{B} \ni (x,y) \to (0,0)$, and x not tangential to the positive real axis, and in the second case $\Sigma \ni (x,y) \to (0,0)$, and x is not tangential to the negative real axis. Thus if we compare the values of log at the upper and lower ends of the cylinders and use these values in the formula for h_α, we find $c_0^\pm = \pm\pi i q$, which gives the normalization $c_0^+ + c_0^- = 0$. For comparison, we note that Shishikura [S] uses the normalization $c_0^+ = 0$. In the case of the semi-parabolic map (0.3) with $\epsilon = 0$, we find that if we use the normalization in equation (1.1) with $a_2 = 1$, then we have $a_3 = 2a/(a-1)$ and $q = a_3 - 1$, so

$$c_0^+ = \pi i \frac{a+1}{a-1} \tag{1.9}$$

We comment that this construction is in fact local. In case F is defined in a neighborhood U of O, we may define the local basin

$$\mathcal{B}_{loc} := \{p : f^n p \in U \ \forall n \geq 0, f^n p \to O \text{ locally uniformly as } n \to \infty\},$$

as well as the local asymptotic curve Σ_{loc}. Similarly, we have Fatou coordinates φ^ι and φ^o on $\mathcal{B}_{\text{loc}}$ and Σ_{loc}. In this case there is an R such that

$$\varphi^\iota(\mathcal{B}_{\text{loc}}) \supset \{\zeta \in \mathbf{C} : -\Re\zeta + R < |\Im\zeta|\}, \quad \varphi^o(\Sigma_{\text{loc}}) \supset \{\zeta \in \mathbf{C} : \Re\zeta + R < |\Im\zeta|\}.$$

We define $W_R := \{\zeta \in \mathbf{C} : |\Re\zeta| + R < |\Im\zeta|\}$, so for R sufficiently large,

$$\varphi^{\iota/o}(\mathcal{B}_{\text{loc}} \cap \Sigma_{\text{loc}}) \supset W_R,$$

and possibly choosing R even larger, $h = \varphi^\iota \circ (\varphi^o)^{-1}$ is defined as a map of W_R to $\mathbf{C}$. Note that we have $h(\zeta+1) = h(\zeta) + 1$ for $\zeta \in W_R$ such that both sides of the equation are defined. If we shrink the domain U of F, we may need to increase R, but the germ of h at infinity is unchanged. Let $h^\bullet$ denote the germ at infinity of h on W_R. It is evident that:

Theorem 1.3. *If F and F' are locally holomorphically conjugate at O, then the germs $h^\bullet$ and $h'^\bullet$ are conjugate.*

Up to this point, the discussion has applied to a general complex manifold M. Let us suppose for the rest of this section that M is a Stein manifold. (See [FG] for the definition and properties). For instance $\mathbf{C}^2$ is Stein. One of the properties of Stein manifolds is that they have no compact complex submanifolds.

If M is Stein, then Σ does not continue through O. More precisely, we have:

Proposition 1.4. *If M is Stein, then Σ_0, as defined in (1.8), cannot be extended analytically past O.*

Proof. If ψ extended analytically to a neighborhood of $x = 0$, then $\Sigma \cong \mathbf{C}$ would be contained in a strictly larger complex manifold. But it is known that the Riemann sphere $\mathbf{P}^1$ is the only complex manifold that strictly contains $\mathbf{C}$. But this is compact and thus cannot be contained in M. □

Proposition 1.5. *If M is Stein, every component of $\mathcal{B} \cap \Sigma$ is simply connected. In particular, if $M = \mathbf{C}^2$ and F is a polynomial automorphism, then each component of $\mathcal{B} \cap \Sigma$ is conformally equivalent to a disk.*

Proof. We consider the sequence of holomorphic mappings $F^n \circ H : \mathbf{C} \to M$, $n = 1, 2, 3, \ldots$. Since any Stein manifold can be embedded into some $\mathbf{C}^N$, we can regard $F^n \circ H$ as a mapping into $\mathbf{C}^N$. Now $H^{-1}(\Sigma \cap \mathcal{B})$ is the set of points $\zeta \in \mathbf{C}$ for which $\{F^n \circ H(\zeta)\}$ converges to O locally uniformly. Hence by the maximum principle, $H^{-1}(\Sigma \cap \mathcal{B})$ is simply connected. □

Theorem 1.6. *Assume that M is Stein, and $\Omega \neq \mathbf{C}$. If Ω' is a connected component of Ω, then the function $h|_{\Omega'}$ cannot be continued analytically over any boundary point of Ω'. In particular, since $\Omega^\pm \neq \mathbf{C}$, the derivative h' is nonconstant on both Ω^+ and Ω^-, and there exist points in both of these sets where $|h'| < 1$ and where $|h'| > 1$.*

Proof. Suppose there is a disk $\Delta \subset \mathbf{C}$ with center at a point of $\partial\Omega$ and a connected component W of $\Omega \cap \Delta$ such that the restriction of h_0 to W extends to a holomorphic function on Δ. Then, by shrinking Δ, we can assume that $h = \varphi^\iota \circ (\varphi^o)^{-1}$ is bounded on W.

If $\Phi = (\varphi^\iota, \Phi_2)$ is the map from Theorem 1.1, we have

$$||\Phi((\varphi^o)^{-1}(\zeta))||^2 = |\varphi^\iota((\varphi^o)^{-1}(\zeta))|^2 + |\Phi_2((\varphi^o)^{-1}(\zeta))|^2 \to \infty$$

as $\zeta \in \Delta_0$ approaches $\partial W \cap \Delta$. Hence $|\Phi_2((\varphi^o)^{-1}(\zeta))| \to \infty$ as $\zeta \to \partial W \cap \Delta$. By Radó's Theorem (see [N]) $\Phi_2 \circ (\varphi^o)^{-1}$ can be extended to a meromorphic function on Δ, with poles on $\Delta - W$. This shows that $\Delta - W$ consists of isolated points, and contradicts the fact that every connected component of Ω is simply connected. □

We note that in all dynamically interesting cases that we have encountered, the manifold M is Stein. However, there are cases where M is not Stein.

Examples. The first example is the product $M_0 = \mathbf{P}^1 \times \mathbf{C}$. Let F act as translation on $\mathbf{P}^1 \times \{0\}$ with fixed point $O = (\infty, 0) \in \mathbf{P}^1 \times \{0\}$, and let F multiply the factor $\mathbf{C}$ by b. Then $\Sigma = (\mathbf{P}^1 \times \{0\}) - O = \mathbf{C} \times \{0\}$. All points of M_0 are attracted to O. The convergence is locally uniform on $\mathcal{B} = \mathbf{C} \times \mathbf{C}$, so we see that $\Sigma \subset \mathcal{B}$.

For the second example, we start with the linear map $L(x, y) = (b(x + y), by)$ on $\mathbf{C}^2$, so the x-axis, $X = \{y = 0\}$, is invariant. The origin is an attracting fixed point, and we let M_1 denote $\mathbf{C}^2$ blown up at the origin. The strict transform of X inside M_1 will be again denoted as X. Thus L lifts to a biholomorphic map of M_1. We write the exceptional fiber as E and note that E is equivalent to $\mathbf{P}^1$, and L is equivalent to translation on E. The fixed point of $L|_E$ is $E \cap X$, which is the only fixed point on M_1. We have $\Sigma = E - X$, and X is the strong stable manifold of $E \cap X$. All points of M_1 are attracted to $E \cap X$, but the convergence is not locally uniform in a neighborhood of any point of X. We have $\mathcal{B} = M_1 - X$, and so $\mathcal{B}$ contains Σ. The second example is different from the first because E has negative self-intersection (see [FG] for the definition of self-intersection and the fact that it is negative in this case).

Both M_0 and M_1 fail to be Stein because they contain compact holomorphic curves. Similar examples can be constructed for all of the Hirzebruch surfaces.

Graphical representation. In Figure 1, we saw slices of K^+ by a plane T which is not invariant. If we wish to make the picture invariant, we may slice by unstable manifolds of periodic saddle points. If Q is a periodic point of period p and of saddle type, then the unstable manifold $W^u(Q)$ may be uniformized by $\mathbf{C}$ so that Q corresponds to $0 \in \mathbf{C}$. The restriction of F^p to $W^u(Q)$ corresponds to a linear map of $\mathbf{C}$ in the uniformizing coordinate, so the slice picture is self-similar. The unstable slice picture cannot be taken at the fixed point O when $\epsilon = 0$ because it is not a saddle. Instead, we can use the unique 2-cycle $\{Q, F(Q)\}$, which remains of saddle type throughout the bifurcation.

The left hand side of Figure 2a shows this picture for $a = .3$; the point Q corresponds to the tip at the rightmost point, and the factor for self-similarity is approximately 8. The two pictures, Figure 1 (left) and Figure 2a (left), are slices at different points O and Q of $\partial\mathcal{B}$. However, the "tip" shape of the slice $W^u(Q) \cap \mathcal{B}$ at Q appears to be repeated densely at small scales in the slice $T \cap \mathcal{B}$ as well as in $W^u(Q) \cap \mathcal{B}$. This might be explained by the existence of transversal intersections between the stable manifold $W^s(Q)$ and T at a dense subset of $T \cap \partial\mathcal{B}$. Similarly the "cusp" at O of the slice $T \cap \mathcal{B}$ appears to be repeated at small scales in the slice $W^u(Q) \cap \mathcal{B}$ as well as in $T \cap \mathcal{B}$. A phenomenon which is closely related to what we have just described has in fact been proved to hold in the hyperbolic case (see [BS7]).

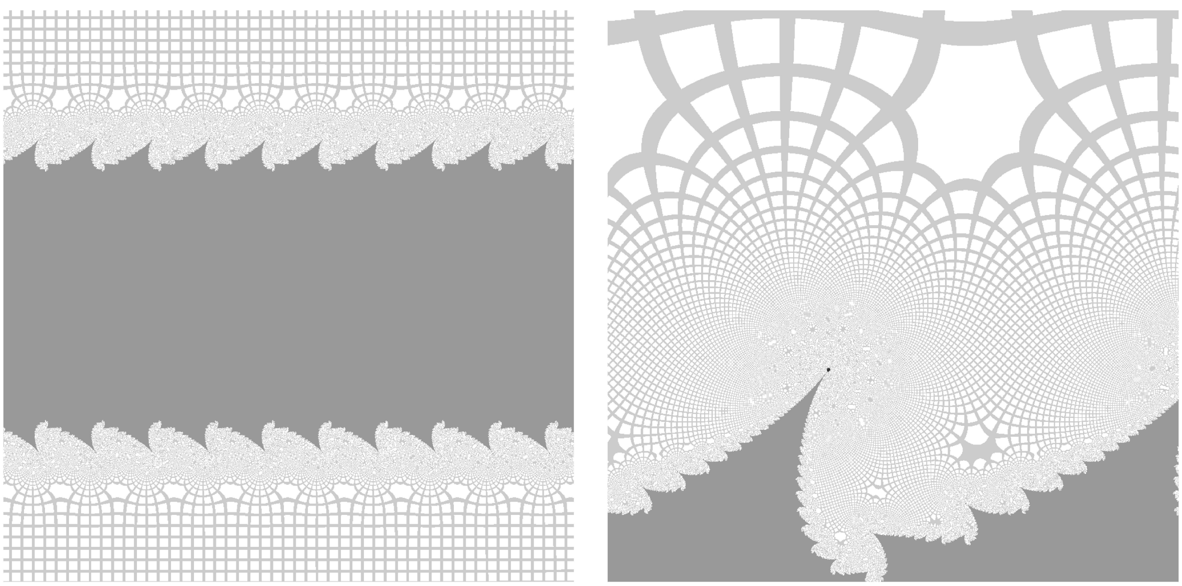

Figure 3. Basin $\mathcal{B}$ in Fatou coordinates; $a = .3$; 10 periods (left) and detail (right).

In general, the set K^+ consisting of points with bounded forward orbits, coincides with the zero set of the Green function $\{G^+ = 0\}$. The set $\mathcal{B}$ is contained in K^+ but the set K^+ is of dynamical interest, especially when $\mathcal{B} = \emptyset$. In Figures 1 and 2, $\mathcal{B}$ seems to have "imploded" leaving $K^+ = \partial K^+ = J^+$ without interior when $\epsilon \neq 0$. On the other hand, the computer detail in Figure 2b persists as $\epsilon \to 0$. This means that $\epsilon \mapsto J^+(F_\epsilon)$ will appear to have "exploded" a little bit as $\epsilon = 0$ changes to $\epsilon \neq 0$. We will see a marked similarity between Figures 2b (right) and 5 (right), which correspond to Theorems 3 and 4.

We may use φ^ι and φ^o to represent $\mathcal{B} \cap \Sigma$ graphically. We use φ^o to parametrize Σ. In Figure 3 we have drawn part of the slice $\mathcal{B} \cap \Sigma$. By the periodicity, we see that there are at least two components, which must be simply connected by Proposition 1.4. Figure 3 also shows level sets of the real and imaginary parts of φ^ι. At most places, the two families of level sets form curvilinear quadrilaterals, but the places where they form curvilinear octagons indicate the presence of critical points inside. One critical point for h (as well as its complex conjugate and translates) is clearly evident on the left hand picture, and at least two more critical points are evident on the right. Figures 1 and 3 give invariant slices of the same basins and share certain features. But Figure 3, which is specialized to parabolic basins, has more focus on the interior; and the two pictures are localized differently.

§**2. "Almost Fatou" coordinates: dimension 1.** We consider a family of holomorphic maps $f_\epsilon(x)$ defined in a neighborhood V of $x = 0$ in $\mathbf{C}$ depending holomorphically on the parameter ϵ. We assume that f_0 has the form

$$f_0(x) = x + x^2 + O(x^3)$$

and that f_ϵ has fixed points at $x = \pm\epsilon$. Then f_ϵ has the form

$$f_\epsilon(x) = x + (x^2 + \epsilon^2)\alpha_\epsilon(x) \tag{2.1}$$

with

$$\alpha_\epsilon(x) = 1 + p\epsilon + (q+1)x + O(|x|^2 + |\epsilon|^2) \tag{2.2}$$

holomorphic in a neighborhood of $(x, \epsilon) = (0, 0)$. We may assume without loss of generality that $p = 0$, by changing to the coordinates $(\hat{x}, \hat{\epsilon})$ given by $x = (1 - p\hat{\epsilon})\hat{x}$ and $\epsilon = \hat{\epsilon} - p\hat{\epsilon}^2$.

We are interested in analyzing the behavior of the iterates $f_\epsilon^n(x)$ when $n \to \infty$, and $\epsilon = \epsilon_n$ is an α-sequence, and thus ϵ tends to 0 tangentially to the real axis and satisfies

$$0 < \Re(\epsilon), \quad |\Im(\epsilon)| \le c|\epsilon|^2, \text{ or equivalently, } \left|\Im\left(\frac{1}{\epsilon}\right)\right| \le c. \tag{2.3}$$

As the first step, we introduce a change of coordinates, depending on ϵ, in which f_ϵ is close to the translation by 1. For ϵ with $\Re(\epsilon) > 0$, we denote $\mathbf{C} - (L_\epsilon^+ \cup L_\epsilon^-)$, where $L_\epsilon^\pm = \{\pm i\epsilon t : t \ge 1\}$ are half lines with endpoints $\pm i\epsilon$, and we define

$$u_\epsilon = \frac{1}{\epsilon}\arctan\frac{x}{\epsilon} = \frac{1}{2i\epsilon}\log\frac{i\epsilon - x}{i\epsilon + x} \qquad \text{for } \; x \in \mathbf{C} - \left(L_\epsilon^+ \cup L_\epsilon^-\right). \tag{2.4}$$

where we choose the single-valued branch of logarithm so that $u_\epsilon(0) = 0$. Then $u_\epsilon(x)$ maps $\mathbf{C} - (L_\epsilon^+ \cup L_\epsilon^-)$ conformally onto the strip in the u-plane given by $|\Re(\epsilon u)| < \frac{\pi}{2}$. The inverse is given by $x = \epsilon\tan(\epsilon u)$. We note that any line $\Re(\epsilon u) = a$ corresponds to a circular arc with endpoints $\pm\epsilon i$ and passing through the point $\epsilon\tan a$ in the x-plane.

We denote by $S_{0,r}^{\iota/o}$ the disk of radius r with center $\pm r$. For nonzero ϵ, we let $S_{\epsilon,r}$ be the union of two disks of radius r with centers at $\pm\frac{\epsilon}{|\epsilon|}\sqrt{r^2 - |\epsilon|^2}$, as pictured in Figure 4. Let $H_\epsilon^{\iota/o}$ denote the half-space to the left/right of the line $\epsilon i\mathbf{R}$. Since ϵ is almost real, $H_\epsilon^{\iota/o}$ is approximately the left/right half plane. The spaces of "incoming"/"outgoing" points are $S_{\epsilon,r}^{\iota/o} := H_\epsilon^{\iota/o} \cap S_{\epsilon,r}$. Finally, we write $D_\epsilon := \{|x| < \epsilon\}$.

Figure 4 shows the boundary of H^ι_ϵ (dashed) and its image under u_ϵ (also dashed). The image $u_\epsilon(S^\iota_{\epsilon,r})$ is bounded by two parallel lines: one of them passes through $-\frac{\pi}{2\epsilon}+\rho$, where $\rho = \frac{1}{2\epsilon}\arctan\frac{|\epsilon|}{\sqrt{r^2-|\epsilon|^2}}$, and the other is inside the shaded strip. The set D_ϵ is the shaded region on the left; its image is the shaded vertical strip on the right hand side.

We will take r small enough that the 'O' term in (2.2) is small. Note that the proportions in Figure 4 may be misleading because r will be fixed while $\epsilon \to 0$, so $S_{\epsilon,r}$ will be of a fixed diameter while $D_\epsilon := \{|x| < \epsilon\}$ shrinks. Also note that if r is fixed, then ρ stays bounded as $\epsilon \to 0$.

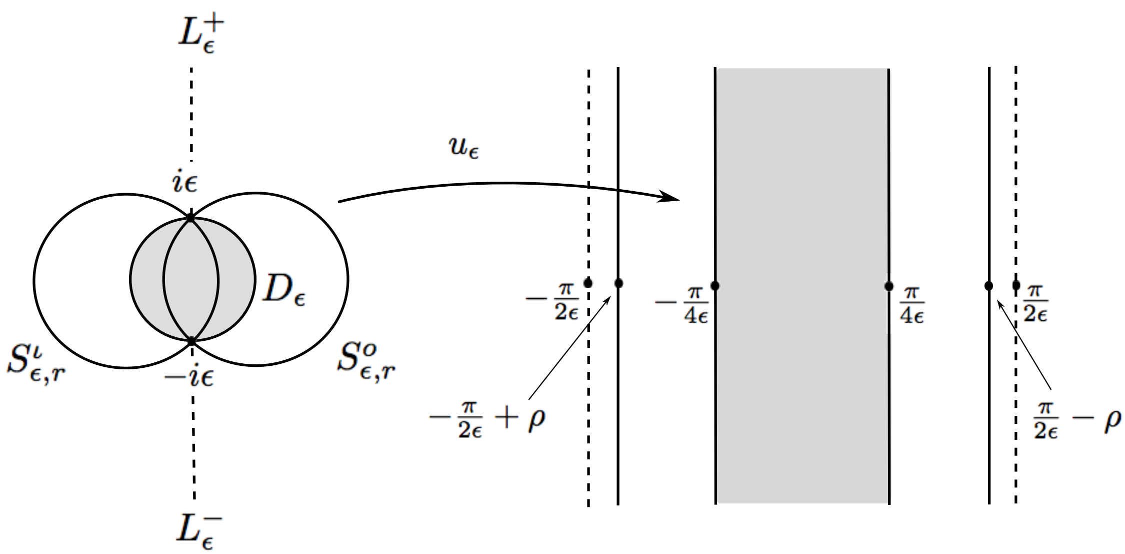

Figure 4. Mapping of the slit region by u_ϵ for $\epsilon > 0$.

To describe the behavior of the mapping f_ϵ in terms of u_ϵ, we note

$$\frac{i\epsilon - f_\epsilon(x)}{i\epsilon + f_\epsilon(x)} = \frac{(i\epsilon - x)\{1 + (x + i\epsilon)\alpha_\epsilon(x)\}}{(i\epsilon + x)\{1 + (x - i\epsilon)\alpha_\epsilon(x)\}} = \frac{(i\epsilon - x)(1 + i\epsilon\gamma_\epsilon(x))}{(i\epsilon + x)(1 - i\epsilon\gamma_\epsilon(x))}$$

where we have put

$$\gamma_\epsilon(x) = \frac{\alpha_\epsilon(x)}{1 + x\alpha_\epsilon(x)} = 1 + qx + \cdots. \tag{2.5}$$

Thus we have

$$\begin{aligned} u_\epsilon(f_\epsilon(x)) - u_\epsilon(x) &= \frac{1}{2i\epsilon}\log\frac{1 + i\epsilon\gamma_\epsilon(x)}{1 - i\epsilon\gamma_\epsilon(x)} \\ &= \frac{1}{i\epsilon}\left\{i\epsilon\gamma_\epsilon(x) + \frac{1}{3}(i\epsilon\gamma_\epsilon(x))^3 + \cdots\right\} \\ &= \gamma_\epsilon(x) - \frac{\epsilon^2}{3}\gamma_\epsilon(x)^3 + \cdots \\ &= 1 + qx + O(|\epsilon|^2 + |x|^2). \end{aligned} \tag{2.6}$$

We note that, although $u_\epsilon(x)$ and $u_\epsilon(f_\epsilon(x))$ are not defined in a neighborhood of $(x,\epsilon) = (0,0)$, their difference is a well-defined single-valued holomorphic function in a neighborhood of $(x,\epsilon) = (0,0)$. By shrinking the domain V, we have

$$|u_\epsilon(f_\epsilon(x)) - u_\epsilon(x) - 1| < \frac{1}{4}$$

for $x \in V - (L_\epsilon^+ \cup L_\epsilon^-)$.

We note that the mapping u_ϵ does not converge when $\epsilon \to 0$; we define

$$u_\epsilon^\iota(x) := u_\epsilon(x) + \frac{\pi}{2\epsilon}, \quad u_\epsilon^o(x) := u_\epsilon(x) - \frac{\pi}{2\epsilon}.$$

Proposition 2.1. *We have convergence*

$$u_\epsilon^\iota(x) \to u_0^\iota(x) := -\frac{1}{x}, \qquad \Re(x) < 0$$

$$u_\epsilon^o(x) \to u_0^o(x) := -\frac{1}{x}, \qquad \Re(x) > 0$$

on compact subsets as $\epsilon \to 0$.

Proof. We have

$$u_\epsilon^\iota(x) = \frac{1}{2i\epsilon}\left(\log\frac{i\epsilon - x}{i\epsilon + x} + i\pi\right) = \frac{1}{2\pi}\log\frac{x - i\epsilon}{x + i\epsilon} =$$

$$= \frac{1}{2i\epsilon}\log\frac{1 - i\epsilon/x}{1 + i\epsilon/x} = -\frac{1}{i\epsilon}\left(\frac{i\epsilon}{x} + \frac{1}{3}\left(\frac{i\epsilon}{x}\right)^2 + \cdots\right)$$

where, in the first function in the second line, we choose the single-valued branch on $\mathbf{C} - [-i\epsilon, i\epsilon]$ that vanishes at $x = \infty$. This converges uniformly to $-1/x$ on compact sets. The proof is similar for $u_\epsilon^o(x)$. $\square$

Proposition 2.2. *For any compact subset* $C \subset S_{0,r}^\iota$*, there are positive constants* ϵ_0*,* C_0 *and* K_0 *such that for* $|\epsilon| < \epsilon_0$ *and* $x \in C$*, the following hold:*

(i) $f_\epsilon^j(x) \in S_{\epsilon,r}^\iota \cup D_\epsilon$*, for* $0 \le j \le \frac{3\pi}{5|\epsilon|} - K_0$

(ii) $|f_\epsilon^j(x)| \le C_0 \max\left\{\frac{2}{j}, |\epsilon|\right\}$*, for* $0 \le j \le \frac{3\pi}{5|\epsilon|} - K_0$

(iii) $f^j(x) \in D_\epsilon$ *for* $\frac{\pi}{3|\epsilon|} \le j \le \frac{3\pi}{5|\epsilon|} - K_0$.

Proof. By Proposition 2.1,

$$u_\epsilon(x) + \frac{\pi}{2\epsilon} \to -\frac{1}{x} \quad (\epsilon \to 0)$$

uniformly on compact subsets of $S_{0,r}^\iota$, so there is $K_0 > 0$ such that

$$-\frac{\pi}{2|\epsilon|} < \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(x)\right) < -\frac{\pi}{2|\epsilon|} + K_0$$

on C. We know that $u_\epsilon(x) \mapsto u_\epsilon(f_\epsilon(x))$ is approximately translation by adding $+1$, and since ϵ satisfies (2.3), it is approximately real, so we can multiply by $\epsilon/|\epsilon|$ and have

$$\frac{3}{4} < \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(f_\epsilon(x))\right) - \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(x)\right) < \frac{5}{4}$$

and so it follows by induction on j that

$$-\frac{\pi}{2|\epsilon|} + \frac{3j}{4} < \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(f_\epsilon^j(x))\right) < -\frac{\pi}{2|\epsilon|} + \frac{5j}{4} + K_0. \tag{2.7}$$

If $0 \le j < 3\pi/(5|\epsilon|) - K_0$, then

$$\Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(f_\epsilon^j(x))\right) < \frac{\pi}{4|\epsilon|}$$

and hence $f_\epsilon^j(x) \in S^\iota_{\epsilon,r} \cup D_\epsilon$, which proves (i).

For (ii) we use (2.4) and have

$$-\frac{\pi}{2} \le \Re(\epsilon u_\epsilon) \le -\frac{\pi}{4} \Rightarrow |x| \le |\epsilon| \tan|\Re(\epsilon u_\epsilon)| < \frac{|\epsilon|}{\frac{\pi}{2} + \Re(\epsilon u_\epsilon)}$$

and $|\Re(\epsilon u_\epsilon)| \le \frac{\pi}{4}$ implies that $|x| \le |\epsilon|$. Now by (2.7) we have

$$\frac{\pi}{2|\epsilon|} + \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(f_\epsilon^j(x))\right) \ge \frac{3j}{4}.$$

For (iii), we note that if $\pi/(3|\epsilon|) \le j < 3\pi/(5|\epsilon|) - K_0$, then by (2.7)

$$-\frac{\pi}{4|\epsilon|} < \Re\left(\frac{\epsilon}{|\epsilon|}u_\epsilon(f_\epsilon^j(x))\right) < \frac{\pi}{4|\epsilon|}$$

and hence $f_\epsilon^j(x) \in D_\epsilon$. □

Next, with q as in (2.2), we define

$$\begin{aligned} w_\epsilon(x) &= u_\epsilon(x) - \frac{q}{2}\log\left(\epsilon^2 + x^2\right) \\ &= \frac{1}{2i\epsilon}\log\frac{i\epsilon - x}{i\epsilon + x} - \frac{q}{2}\log(\epsilon^2 + x^2) \end{aligned} \tag{2.8}$$

The corresponding incoming and outgoing versions are obtained by adding terms that depend on ϵ but do not depend on x:

$$\begin{aligned} w_\epsilon^{\iota/o}(x) &:= w_\epsilon(x) \pm \frac{\pi}{2\epsilon} \\ &= \frac{1}{\epsilon}\left(\pm\frac{\pi}{2} + \arctan\frac{x}{\epsilon}\right) - \frac{q}{2}\log(\epsilon^2 + x^2) \end{aligned} \tag{2.9}$$

We set $w_0^{\iota/o}(x) = -\frac{1}{x} - q\log(\mp x)$. With this notation we have

Lemma 2.3. *$\lim_{\epsilon\to 0} w^\iota_\epsilon = w^\iota_0$ on $S^\iota_{0,r}$ and $\lim_{\epsilon\to 0} w^o_\epsilon = w^o_0$ on $S^o_{0,r}$.*

Let us define $A_\epsilon(x) := w_\epsilon(f_\epsilon(x)) - w_\epsilon(x) - 1$, which measures how far $w_\epsilon(x)$ is from being a Fatou coordinate. Although $w_\epsilon(f_\epsilon)$ and w_ϵ are defined on a domain that varies with ϵ, the difference A_ϵ is defined on a uniformly large neighborhood of $(\epsilon, x) = (0, 0)$.

Proposition 2.4. $A_\epsilon(x) = O(|\epsilon|^2 + |x|^2)$.

Proof. First we observe that

$$\begin{aligned}\epsilon^2 + f_\epsilon(x)^2 &= \epsilon^2 + \{x + (\epsilon^2 + x^2)\alpha_\epsilon(x)\}^2\\ &= \epsilon^2 + x^2 + 2x(\epsilon^2 + x^2)\alpha_\epsilon(x) + (\epsilon^2 + x^2)^2\alpha_\epsilon(x)^2\\ &= (\epsilon^2 + x^2)\left(1 + 2x\alpha_\epsilon(x) + (\epsilon^2 + x^2)\alpha_\epsilon(x)^2\right)\\ &= (\epsilon^2 + x^2)(1 + 2x + O(|\epsilon|^2 + |x|^2)).\end{aligned}$$

It follows that

$$\begin{aligned}&w_\epsilon(f_\epsilon(x)) - w_\epsilon(x) =\\ &\quad= (u_\epsilon(f_\epsilon(x)) - u_\epsilon(x)) - \frac{q}{2}\left(\log\left(\epsilon^2 + f_\epsilon(x)^2\right) - \log\left(\epsilon^2 + x^2\right)\right)\\ &\quad= 1 + qx + O(|\epsilon|^2 + |x|^2) - \frac{q}{2}\left(2x + O(|\epsilon|^2 + |x|^2)\right)\\ &\quad= 1 + O(|\epsilon|^2 + |x|^2)\end{aligned}$$

which gives the desired result. □

We note, too, that

$$A_\epsilon(x) = A_0(x) + \epsilon\tilde{A}(x) + O(\epsilon^2) \tag{2.10}$$

where $A_0(x) = O(x^2)$ and $\tilde{A}(x) = O(x)$.

Corollary 2.5. $w^{\iota/o}_\epsilon(f_\epsilon(x)) - w^{\iota/o}_\epsilon(x) - 1 = O(|\epsilon|^2 + |x|^2)$

Lemma 2.6. *There exists $K_0 > 0$ such that: If $x, f_\epsilon(x), \ldots, f^n_\epsilon(x) \in S_{\epsilon,r}$, then*

$$|w_\epsilon(f^n_\epsilon(x)) - w_\epsilon(x) - n| \le K_0$$

and hence

$$|w^o_\epsilon(f^n_\epsilon(x)) - w^\iota_\epsilon(x) + \frac{\pi}{\epsilon} - n| \le K_0$$

Proof. We have

$$w_\epsilon(f^n_\epsilon(x)) - w_\epsilon(x) - n = \sum_{j=0}^{n-1} A_\epsilon(f^j_\epsilon(x)).$$

Choose $0 < n_1 < n_2 < n$ such that

$$\begin{aligned}&f^j_\epsilon(x) \in S^\iota_{\epsilon,r} - D_\epsilon, && 0 \le j \le n_1 - 1\\ &f^j_\epsilon(x) \in D_\epsilon, && n_1 \le j \le n_2 - 1\\ &f^j_\epsilon(x) \in S^o_{\epsilon,r} - D_\epsilon, && n_2 \le j \le n\end{aligned}$$

Then $n_2 - n_1 \le const/|\epsilon|$, and

$$\begin{aligned}|A_\epsilon(f_\epsilon^j(x))| &\le const/j^2, \quad 0 \le j \le n_1 - 1\\ |A_\epsilon(f_\epsilon^j(x))| &\le const\,|\epsilon|^2, \quad n_1 \le j \le n_2 - 1\\ |A_\epsilon(f_\epsilon^j(x))| &\le const/(n-j)^2, \quad n_2 \le j \le n\end{aligned}$$

This proves the Lemma. □

We will use the following condition:

$$\{m_j, \epsilon_j\} \text{ is a sequence such that } \frac{\pi}{2\epsilon_j} - m_j \text{ is bounded} \tag{2.11}$$

Recall that $\{n_j, \epsilon_j\}$ is an *α-sequence* if $\epsilon_j \to 0$, and $n_j - \frac{\pi}{\epsilon_j} \to \alpha$ as $j \to \infty$. For instance, (j, ϵ_j) with $\epsilon_j = \frac{\pi}{j-\alpha}$ is an α sequence. If $\{n_j, \epsilon_j\}$ is an α-sequence, then ϵ_j eventually satisfies (2.3), and $\{n_j/2, \epsilon_j\}$ satisfies (2.11).

We define an almost Fatou coordinate in the incoming direction:

$$\varphi^\iota_{\epsilon,n}(x) = w^\iota_\epsilon(f^n_\epsilon(x)) - n = w^\iota_\epsilon(x) + \sum_{j=0}^{n-1} A_\epsilon(f^j_\epsilon(x)).$$

We recall that $\mathcal{B}$ denotes the parabolic basin of points where the iterates f_0^j converge locally uniformly to $O = (0,0)$.

Theorem 2.7. *If (2.11) holds, then on $\mathcal{B}$ we have*

$$\lim_{j\to\infty} \varphi^\iota_{\epsilon_j,n_j} = \varphi^\iota.$$

Proof. If $x \in \mathcal{B}$, we may assume that $x \in S^\iota_{\epsilon,r}$, where ϵ and r are as above. If we set $\varphi^\iota_{0,n} = w^\iota_0 + \sum_{j=0}^{n-1} A_0(f_0^j(x))$, we have $\varphi^\iota = \lim_{n\to\infty} \varphi^\iota_{0,n}$. We consider

$$\varphi^\iota_{\epsilon,n} - \varphi^\iota_{0,n} = w^\iota_\epsilon(x) - w^\iota_0(x) + \sum_{j=0}^{n-1} \left(A_\epsilon(f^j_\epsilon(x)) - A_0(f^j_0(x)) \right).$$

We will show that this difference vanishes as $\epsilon = \epsilon_j \to 0$ and $n = n_j \to \infty$. We have $w^\iota_\epsilon - w^\iota_0 \to 0$ by Lemma 2.3. The summation is estimated by

$$\left|\sum\right| \le \sum \left|A_0(f^j_\epsilon(x)) - A_0(f^j_0(x))\right| + \sum \left|A_\epsilon(f^j_\epsilon(x)) - A_0(f^j_\epsilon(x))\right| = \sum\nolimits_I + \sum\nolimits_{II}$$

For the first sum, we recall that $A_0(x) = O(x^2)$, and so by Proposition 2.2 we have that if $(n-1)|\epsilon| \in (\frac{\pi}{2}, \frac{3\pi}{5})$, then the following estimate holds:

$$\sum\nolimits_I \le \sum_{j=1}^{n-1} \left(\left|A_0(f^j_\epsilon)\right| + \left|A_0(f^j_0)\right| \right) \le K \sum_{j=1}^{n-1} \left(\frac{1}{j^2} + |\epsilon|^2 \right) \le K \left(\frac{3\pi|\epsilon|}{5} + \sum_{j=1}^{\infty} \frac{1}{j^2} \right) \le B$$

as $n \to \infty$ and $\epsilon \to 0$. For $\delta > 0$ we choose J such that $\sum_J^\infty j^{-2} < \delta$. If we write $\sum_I \le \sum_1^J + \sum_{J+1}^\infty$, then we see that $\sum_{J+1}^\infty \le \pi K|\epsilon|/2 + K\delta$. On the other hand, for fixed j we have $A_0(f_\epsilon^j) \to A_0(f_0^j)$ as $\epsilon \to 0$, so we conclude that

$$\sum_1^J = \sum_1^J \left| A_0(f_\epsilon^j) - A_0(f_0^j) \right| \to 0$$

as $\epsilon \to 0$. In conclusion, we see that $\lim_{\epsilon\to 0} \sum_I \le K\delta$ for all δ, so that $\sum_I \to 0$.

For the second part, we use (2.10) so that

$$\sum\nolimits_{II} \le \sum_{j=0}^{n-1} \left| A_\epsilon(f_\epsilon^j) - A_0(f_\epsilon^j) \right| \le \sum_{j=0}^{n-1} \left| \epsilon \tilde{A}(f_\epsilon^j) \right| + \sum_{j=0}^{\frac{c}{|\epsilon|}} K|\epsilon|^2$$
$$\le K'|\epsilon| + K'|\epsilon| \sum_{j=1}^{\frac{c}{|\epsilon|}} \frac{1}{j} \le K''|\epsilon| \log\left(\frac{c}{|\epsilon|}\right)$$

and this last term vanishes as $\epsilon \to 0$, which completes the proof. □

We may also define almost Fatou coordinates in the outgoing direction:

$$\varphi_{\epsilon,n}^o := w_\epsilon^o(f_\epsilon^{-n}(x)) + n.$$

The direct analogue of Theorem 2.7 also holds for the outgoing direction:

Corollary 2.8. *If (2.11) holds, then* $\lim_{j\to\infty} \varphi_{\epsilon_j,n_j}^o$ *converges to* φ^o *uniformly on compact subsets of* $S_{0,r}^o$.

For any $x_0 \in S_{0,r}^\iota$ and $x_0' \in S_{0,r}^o$, we set $\alpha = \varphi^o(x') - \varphi^\iota(x)$. Then setting $\tau_\alpha(\zeta) = \zeta + \alpha$, we can define the transition map

$$T_\alpha(x) = (\varphi^o)^{-1} \circ \tau_\alpha \circ \varphi^\iota(x)$$

so that $T_\alpha(x_0) = x_0'$, and T_α extends holomorphically to a neighborhood of x_0.

Theorem 2.9. *Let* x_0, x_0' *and* α *be as above. For any* α*-sequence* (n_j, ϵ_j), *we have*

$$\lim_{j\to\infty} f_{\epsilon_j}^j(x) = T_\alpha(x)$$

uniformly in a neighborhood of x_0.

Proof. Choose m_j and m_j' so that $n_j = m_j + m_j'$, and (2.11) holds for $\{m_j\}$ and $\{m_j'\}$. By Theorem 2.6, $\varphi_{\epsilon_j,m_j}^\iota$ converges uniformly to φ^ι in a neighborhood of x_0, and $\varphi_{\epsilon_j,m_j'}^o$ converges to φ^o uniformly in a neighborhood of x_0'. Since φ^o is invertible, it follows that $(\varphi_{\epsilon_j,m_j}^o)^{-1}$ converges to $(\varphi^o)^{-1}$ uniformly in a neighborhood of $\varphi^o(x_0')$. For j sufficiently large, $\varphi_{\epsilon_j,m_j}^\iota$ maps a small neighborhood of x_0 to a small neighborhood of $\varphi^\iota(x_0)$. If we set $\alpha_j := n_j - \frac{\pi}{\epsilon_j}$, then $\alpha_j \to \alpha$, so for j sufficiently large, $\tau_{\alpha_j} \circ \varphi^{\iota}_{\epsilon_j,m_j}$ maps a small neighborhood of x_0 into the domain of $(\varphi^{o}_{\epsilon_j,m'_j})^{-1}$. We conclude that

$$x_j := (\varphi^{o}_{\epsilon_j,m'_j})^{-1} \circ \tau_{\alpha_j} \circ \varphi^{\iota}_{\epsilon_j,m_j}(x) \to T_\alpha(x)$$

as $j \to \infty$. Applying $\varphi^{o}_{\epsilon_j,m'_j}$, we find

$$\varphi^{o}_{\epsilon_j,m'_j}(x_j) = \varphi^{\iota}_{\epsilon_j,m_j}(x) + \alpha_j$$

The left hand side of this equation is

$$w^{o}_{\epsilon_j}(f^{-m'_j}_{\epsilon_j}(x_j)) + m'_j = w_{\epsilon_j}(f^{-m'_j}_{\epsilon_j}(x_j)) + \frac{1}{2\epsilon_j} + m'_j$$

and the right hand side is

$$w^{\iota}_{\epsilon_j}(f^{m_j}_{\epsilon_j}(x)) - m_j = w_{\epsilon_j}(f^{m_j}_{\epsilon_j}(x)) - m_j - \frac{1}{2\epsilon_j} + \alpha_j.$$

Thus $w_{\epsilon_j}(f^{-m'_j}_{\epsilon_j}(x_j)) = w_{\epsilon_j}(f^{m_j}_{\epsilon_j}(x))$, so we conclude that $x_j = f^{n_j}_{\epsilon_j}(x)$, and so we obtain the desired convergence. □

Now let us assume that f is defined on all of $\mathbf{C}$.

Proposition 2.10. *If f is defined on all of $\mathbf{C}$, then $(\varphi^o)^{-1}$ extends to an entire function $H : \mathbf{C} \to \mathbf{C}$.*

Proof. The quantities $\varphi^{\iota/o} - w^{\iota/o}$ are bounded, so the image $\varphi^o(S^o_{0,r})$ contains $\{\Re(\zeta) < -K\}$ for some large K. Further, φ^o is invertible on $S^o_{0,r}$, and thus $(\varphi^o)^{-1}$ is defined on $\{\Re(\zeta) < -K\}$. Now by the identity $(\varphi^o(\zeta+1))^{-1} = f((\varphi^o)^{-1}(\zeta))$, we may extend this function from $\{\Re(\zeta) < -K\}$ to a function H defined on all of $\mathbf{C}$. □

If f if globally defined, then we let H be as in Proposition 2.10; thus $T_\alpha := H \circ \tau_\alpha \circ \varphi^\iota : \mathcal{B} \to \mathbf{C}$ is defined on the whole basin $\mathcal{B}$. It is evident that we have: $f \circ T_\alpha = T_\alpha \circ f = T_{\alpha+1}$. Using this functional relation, we obtain a more global version of Theorem 2.9:

Theorem 2.11. *Suppose that f is defined on all of $\mathbf{C}$. If (n_j, ϵ_j) is an α-sequence, then $\lim_{j\to\infty} f^{n_j}_{\epsilon_j} = T_\alpha$ on $\mathcal{B}$.*

Proof. We note that for an α-sequence (n_j, ϵ_j) and an integer k, the sequence $(n_j - k, \epsilon_j)$ is an $(\alpha - k)$-sequence. For given $x_0 \in \mathcal{B}$ and α we choose an integer k so that $T_{\alpha-k}(x_0)$ is in $S^o_{0,r}$. By Theorem 2.9, it follows that $f^{n_j}_{\epsilon_j}(x_0) = f^k_{\epsilon_j} \circ f^{n_j-k}_{\epsilon_j}(x_0)$ converges to $f^k \circ T_{\alpha-k}(x_0) = T_\alpha(x_0)$. □

§3. Two-dimensional case: Convergence of the "Almost Fatou" coordinate. We consider a one-parameter family F_ϵ of holomorphic diffeomorphisms of a complex manifold M, varying analytically in ϵ, such that $F_0(x,y) = (x + x^2 + \cdots, by + \cdots)$. The fixed point $O = (0,0)$ has multiplicity 2 as a solution of the fixed point equation, and we will assume that for $\epsilon \neq 0$ the fixed point O will split into a pair of fixed points. We parametrize so that the fixed points are $(\pm i\epsilon, 0) + O(\epsilon^2)$. We consider here only fixed points of multiplicity two. We suspect that perturbations of fixed points of higher multiplicity might be quite complicated, since this is already the case in dimension 1, as was shown by Oudkerk [O1,2].

Theorem 3.1. *By changing coordinates and reparametrizing ϵ, we may suppose that our family of maps has the local form*

$$F_\epsilon(x,y) = \big(x + (x^2+\epsilon^2)\alpha_\epsilon(x,y), b_\epsilon(x)y + (x^2+\epsilon^2)\beta_\epsilon(x,y)\big) \tag{3.1}$$

where $\alpha_\epsilon = 1 + (q+1)x + sy + O(|x|^2+|y|^2+|\epsilon|^2)$ and $b_0(0) = b$. In particular, the points $(\pm i\epsilon, 0)$ are fixed, the lines $\{x = \pm i\epsilon\}$ are local strong stable manifolds, where the map is locally linear. Further, the multipliers at the fixed points are $(1 \pm 2i\epsilon + O(\epsilon^2), b_\epsilon(\pm i\epsilon))$.

Proof. By a change of variables, we may assume that the fixed points are $(\pm i\epsilon, 0)$. Each fixed point will have eigenvalues $1 + O(\epsilon)$ and $b + O(\epsilon)$. There will be local strong stable manifolds $W^{ss}_{\rm loc}$ corresponding to the eigenvalue $b+O(\epsilon)$, and these are proper in a uniformly large domain in (x,y)-space for $|\epsilon| < \epsilon_0$. That is, we may rescale coordinates so that we have graphs

$$W^{ss}_{loc}(\pm i\epsilon, 0) = \{x = \psi^\pm(\epsilon, y) : |y| < 1\},$$

where $\psi^\pm$ is analytic in y. Further, the local strong stable manifolds vary holomorphically in ϵ for $\epsilon \neq 0$, and they converge to the y-axis when $\epsilon \to 0$. Thus $\psi^\pm(\epsilon, y)$ is jointly analytic in both ϵ and y.

Let us consider new coordinates X, Y defined by $x = \chi_0(\epsilon, y) + X\chi_1(\epsilon, y)$, $Y = y$, where we set $\chi_0 = \frac{1}{2}(\psi^+ + \psi^-)$ and $\chi_1 = \frac{1}{2i\epsilon}(\psi^+ - \psi^-)$. Since $\psi^\pm$ are uniquely determined and analytic in ϵ, we have $\lim_{\epsilon\to 0}(\psi^+(\epsilon,y) - \psi^-(\epsilon,y)) = 0$, from which we conclude that χ_1 is analytic in (ϵ, y).

In order for F_ϵ to have the desired form in the y-coordinate, we need to change coordinates so that $y \mapsto F_\epsilon(\pm i\epsilon, y)$ is linear in y. We set $b^\pm_\epsilon = \frac{\partial (F_\epsilon)_2}{\partial y}(\pm i\epsilon, 0)$. There is a unique function $\xi^\pm_\epsilon(y) = y + O(y^2)$ such that $F_\epsilon(\pm i\epsilon, \xi^\pm_\epsilon(y))_2 = \xi^\pm_\epsilon(b^\pm_\epsilon y)$. We note that $\xi^\pm_\epsilon$ is holomorphic in ϵ, and $\xi^+_0 = \xi^-_0$. Thus $\epsilon \mapsto (\xi^-_\epsilon - \xi^+_\epsilon)/\epsilon$ is analytic, and we may define a new coordinate system (X, Y) with $X = x$ and $Y = [(i\epsilon - x)\xi^-_\epsilon(y) + (x + i\epsilon)\xi^+_\epsilon(y)]/(2i\epsilon)$. F has the desired form in the new coordinate system.

Our map now has the form (3.1) with $\alpha_\epsilon = 1 + p\,\epsilon + (q+1)x + sy + O(|x|^2+|y|^2+|\epsilon|^2)$. We now pass to the new coordinate system $(\hat{x}, \hat{\epsilon})$ given by $x = (1 - p\hat{\epsilon})\hat{x}$ and $\epsilon = \hat{\epsilon} - p\hat{\epsilon}^2$, and now we have $p = 0$. The remaining statements in the Theorem are easy consequences of (3.1). □

One motivation for the normalization in (3.1) is that for the map $f_\epsilon : z \mapsto z + z^2 + \epsilon^2$, the fixed points are $\pm i\epsilon$, and the multipliers are $1 \pm 2i\epsilon$. We comment that the expression in (3.1) is strictly local. If $F(x,y)$ is a polynomial diffeomorphism which satisfies (3.1) locally, then the vertical lines $\{x = \pm i\epsilon\}$ are contained in the local, and thus global, strong stable manifolds. However, for a Hénon map, every stable manifold of a saddle point is dense in J^+ and thus not closed in $\mathbf{C}^2$ (see [BS1]). Thus the polynomial expression of a Hénon map can never take the form (3.1).

Now let us define $D = \{\epsilon \in \mathbf{C} : |\epsilon| < \epsilon_0\}$ and let $\tilde{F} : D \times M \to D \times M$ be defined by $\tilde{F}(\epsilon, p) := (\epsilon, F_\epsilon(p))$. The point $(0, O)$ is fixed, and the eigenvalues of the differential of $\tilde{F}$ at this fixed point are 1, 1, and b_0. We will now make use of the Center Manifold Theorem (see [HPS] or [ST]). In particular, as a direct consequence of Theorem 2 of [ST] we have:

Theorem 3.2. *Given k, there is a neighborhood $\tilde{U}$ of $(0,O)$ and a submanifold $\tilde{W}^c$ of $\tilde{U}$ which is C^k smooth and has the properties: $(0,O) \in \tilde{W}^c$, the tangent space $T_{(0,O)}\tilde{W}^c$ is the (ϵ,x)-plane. Further, $\tilde{W}^c$ is forward and backward invariant under $\tilde{F}$ in the following sense: if $p \in W^c$, and $f(p) \in \tilde{U}$, then $f(p) \in \tilde{W}^c$; similarly, if $f^{-1}(p) \in \tilde{U}$, then $f^{-1}(p) \in \tilde{W}^c$.*

We set $W^c_\epsilon := \tilde{W}^c \cap (\{\epsilon\} \times M)$ to be the slice of $\tilde{W}^c$ for fixed ϵ. By abuse of notation, we will suppose that W^c_ϵ is a submanifold of $U := \tilde{U} \cap (\{0\} \times M)$.

Proposition 3.3. *Let us fix a compact $U_0 \subset U$. There are constants $\beta < 1$ and $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ and $p \in U_0$, we have $\mathrm{dist}(F^j_\epsilon(p), W^c_\epsilon) \le \beta^j \mathrm{dist}(p, W^c_\epsilon)$ for $1 \le j \le j_0$ if $F^j_\epsilon(p) \in U$ for $1 \le j \le j_0$.*

Proof. The y-axis is normal to W^c_0 at O, and the y-derivative of the second coordinate of F_0 at O is b with $|b| < 1$, so the derivative of F_ϵ normal to W^c_ϵ near O will be less than some $\beta < 1$. This uniform contraction in the direction normal to W^c_ϵ, means that the distance to W^c_ϵ decreases by a factor of β with each iteration. □

Proposition 3.4. *If $|\epsilon| < \epsilon_0$, $p \in U$, and $(x_j, y_j) := F^j_\epsilon(p) \in U$ for $1 \le j \le j_0$, then*

$$|y_j| \le C(|x_j^2 + \epsilon^2| + \beta^j)$$

for $1 \le j \le j_0$.

Proof. The strong stable manifold W^{ss} for F_0 passing through O is transverse to W^c_0. For $\epsilon \ne 0$, there are two saddle points $(\pm i\epsilon, 0)$ for F_ϵ. The stable manifold $W^s(F_\epsilon)$ for these two saddle points is transverse to W^c_ϵ, so $W^c_\epsilon \cap W^s(F_\epsilon) \ne \emptyset$. It follows that W^c_ϵ must contain both saddle points $(\pm i\epsilon, 0)$. Since $\tilde{W}^c$ is C^2 smooth, it follows that the curvature is bounded by some constant C, and we have $|y| \le C|x^2 + \epsilon^2|$ for $(x,y) \in W^c_\epsilon$.

Now let $\eta : U \to W^c_\epsilon$ be the projection in the x-direction, so that if $p = (x,y) \in U$, $\eta(p) = (x, \hat{y}) \in W^c_\epsilon$. By Proposition 3.3, we know that $|y - \hat{y}|$, the y-distance to W^c_ϵ, decreases by a factor of β with iteration. Further, by the previous discussion, we have $|\hat{y}| \le C|x^2 + \epsilon^2|$, which gives the desired conclusion. □

Now we want to re-do the estimates of §2 in the context of $\mathbf{C}^2$, so as in (2.5) we define

$$\gamma_\epsilon(x,y) = \frac{\alpha_\epsilon(x,y)}{1 + x\alpha_\epsilon(x,y)} = 1 + qx + sy + \cdots. \tag{3.2}$$

If we define $\tilde{u}_\epsilon(x,y) = u_\epsilon(x)$ and assume that F has the form (3.1), we find the analogue of (2.6):

$$\begin{aligned}\tilde{u}_\epsilon(F_\epsilon(x,y)) - \tilde{u}_\epsilon(x,y) &= \gamma_\epsilon(x,y) - \frac{\epsilon^2}{3}\gamma_\epsilon(x,y)^3 + \cdots \\ &= 1 + qx + sy + O(|\epsilon|^2 + |x|^2 + |y|^2)\end{aligned} \tag{3.3}$$

We use the notation $(x_{\epsilon,j}, y_{\epsilon,j}) := F^j_\epsilon(x,y)$, and we let r and ϵ be as in §2 just before Proposition 2.2. We recall the sets $B^\iota_{r,\eta} = S^\iota_{0,r} \times \{|y| < \eta\}$ from (1.2).

Proposition 3.5. *For any compact subset $\tilde{C} \subset B^{\iota}_{r,\eta}$, there are positive constants ϵ_0, C_0 and K_0 such that for $|\epsilon| < \epsilon_0$ and $x \in \tilde{C}$, the following hold:*
(i) $x_{\epsilon,j} \in S^{\iota}_{\epsilon,r} \cup D_\epsilon$ *for* $0 \le j \le \frac{3\pi}{5|\epsilon|} - K_0$
(ii) $|x_{\epsilon,j}| \le C_0 \max\left\{\frac{2}{j}, |\epsilon|\right\}$, *for* $0 \le j \le \frac{3\pi}{5|\epsilon|} - K_0$
(iii) $x_{\epsilon,j} \in D_\epsilon$ *for* $\frac{\pi}{3|\epsilon|} \le j \le \frac{3\pi}{5|\epsilon|} - K_0$.

Proof. This result is very similar to Proposition 2.2. While Proposition 2.2 was proved using (2.6), we will use (3.3), which is the 2-dimensonal analogue. The principal difference between (2.6) and (3.3) is the presence of the term sy. In Proposition 3.4, the term $y_{\epsilon,j}$ was bounded by $C(|x_{\epsilon,j}|^2 + |\epsilon|^2 + \beta^j)$, and this term is smaller than, and easily absorbed into, the right hand side of item (ii). Similarly, items (i) and (iii) remain valid. □

Our next step is to define an analogue of w_ϵ, where we adjust it slightly by adding a multiple of y:

$$\tilde{w}_\epsilon(x,y) = w_\epsilon(x) - sy/(b-1), \quad \tilde{w}^{\iota/o}_\epsilon(x,y) = w^{\iota/o}_\epsilon(x) - sy/(b-1).$$

The addition of the multiple of y causes the y-terms to cancel in the expression

$$\begin{aligned} &\tilde{w}_\epsilon(F_\epsilon(x,y)) - \tilde{w}_\epsilon(x,y) = \\ &\quad = \tilde{u}_\epsilon(F_\epsilon(x,y)) - \tilde{u}_\epsilon(x,y) - \frac{q}{2}\left(\log(\epsilon^2 + x^2_{\epsilon,1}) - \log(\epsilon^2 + x^2)\right) - \frac{s}{b-1}(y_{\epsilon,1} - y) \\ &\quad = 1 + O(\epsilon^2 + |x|^2 + |y|^2) \end{aligned}$$

As in §2, we note that although $\tilde{u}_\epsilon(F_\epsilon)$ and $\tilde{u}_\epsilon$ are defined in regions that vary with ϵ, their difference is holomorphic in a uniform neighborhood of $(x,y) = (0,0)$. Thus, as in Corollary 2.5, we have:

Proposition 3.6.

$$\tilde{w}^{\iota/o}_\epsilon(F_\epsilon(x,y)) - \tilde{w}^{\iota/o}_\epsilon(x,y) - 1 = O(|\epsilon|^2 + |x|^2 + |y|^2)$$

holds for $|\epsilon| < \epsilon_0$ *and* (x,y) *in a neighborhood of* $(0,0)$.

We define the incoming almost Fatou coordinate:

$$\varphi^{\iota}_{\epsilon,n}(x,y) := \tilde{w}^{\iota}_\epsilon(F^n_\epsilon(x,y)) - n$$

Theorem 3.7. *If $\epsilon_j \to 0$, and if n_j satisfies (2.11), then $\lim_{j\to\infty} \varphi^{\iota}_{\epsilon_j,n_j} = \varphi^{\iota}_0$ locally uniformly on $\mathcal{B}$.*

Proof. We define $\tilde{A}_\epsilon(x,y) := \tilde{w}_\epsilon(F_\epsilon(x,y)) - \tilde{w}_\epsilon(x,y) - 1$, so that

$$\varphi^{\iota}_{\epsilon,n}(x,y) = \tilde{w}^{\iota}_\epsilon(x,y) + \sum_{j=0}^{n-1} \tilde{A}_\epsilon(F^j_\epsilon(x,y))$$

By Proposition 3.6, $\tilde{A}_\epsilon(x,y) = O(|x|^2 + |y|^2 + |\epsilon|^2)$, so we may repeat the proof of Theorem 2.7 and obtain our result. □

Let us define $W^c_{\epsilon,r} := W^c_\epsilon \cap B^o_{r,\eta}$. The sets $W^c_{\epsilon,r}$ are smooth disks which vary smoothly with ϵ: there is a family of smooth functions $\psi_\epsilon : S^o_{0,r} \to \mathbf{C}$ depending smoothly on ϵ and x such that

$$W^c_{\epsilon,r} = \{y = \psi_\epsilon(x) : x \in S^o_{0,r}\}$$

We recall the set Σ_0 defined in (1.6). By the following result, Σ_0 is part of the smooth family $\{W^c_{\epsilon,r} : |\epsilon| < \epsilon_0\}$.

Proposition 3.8. $W^c_{0,r} = \Sigma_0$.

Proof. It will suffice to show that $W^c_{0,r} \subset \Sigma_0$. In [U2] it was shown that

$$\Sigma_0 = \{p \in B^o_{r,\eta} : F^{-n}(p) \in B^o_{r,\eta}, \forall n \geq 0\}.$$

Now let us start with $(x_0, y_0) \in W^c_{0,r}$ and iterate it backwards. As in Proposition 3.5, the coordinate x_{-j} stays inside $S^o_{0,r}$ and tends to zero. By the center manifold property of W^c_0, (x_{-j}, y_{-j}) remains in $W^c_{0,r}$, and thus $(x_{-j}, y_{-j}) \to O$, and for $(x,y) \in W^c_{0,r}$, we have $|y| \leq C|x|^2$. This means that $F^{-n}(x_0, y_0) \in B^o_{r,\eta}$, and thus $(x_0, y_0) \in \Sigma_0$. □

We define the almost Fatou coordinate $\varphi^o_{\epsilon,n}$ on $W^c_{\epsilon,r}$ by setting

$$\varphi^o_{\epsilon,n}(p) = \tilde{w}^o_\epsilon(F^{-n}(p)) + n.$$

for $p \in W^c_r$. Although for distinct ϵ, the outgoing almost Fatou coordinates $\varphi^o_{\epsilon,n}$ are defined on distinct sets, they converge to the outgoing Fatou coordinate in the following sense:

Proposition 3.9. *Suppose that the sequence (ϵ_j, n_j) satisfies (2.11) and that $|x - r| < r$. Then*

$$\lim_{j\to\infty} \varphi^o_{\epsilon_j,n_j}(x, \psi_{\epsilon_j}(x)) = \varphi^0(x, \psi_0(x)).$$

Proof. The proof is parallel to the proof of Theorem 3.6, except that now the starting points are $(x, \psi_{\epsilon_j}(x))$, so that the y-coordinates converge as $j \to \infty$. □

The following is one of our principal results and concerns convergence to the mapping T_α, defined in (1.10).

Theorem 3.10. *Suppose that $p \in B^\iota_{r,\eta}$ and that $T_\alpha(p) \in \Sigma_0$. If $\{\epsilon_j\}$ is an α-sequence, then $F^{n_j}_{\epsilon_j}(p)$ converges to $T_\alpha(p)$.*

Proof. Shrinking U if necessary, we may choose β and $\hat\beta$ such that $\beta < 1$, $\beta^2\hat\beta < 1$ and such that: in the vertical direction, F contracts with a factor of β; and $\mathrm{dist}(F^{-1}(q_1), F^{-1}(q_2)) \leq \hat\beta\,\mathrm{dist}(q_1, q_2)$. Now let us write $q = F^{n_j}_{\epsilon_j}(p)$, and let $q' := (\pi(q), h_{\epsilon_j}(\pi(q)))$ denote the projection to M_{ϵ_j}. By Proposition 3.7, we have $\mathrm{dist}(q, q') = O(\beta^{n_j})$. Now we write $n_j = m'_j + m''_j$, where m'_j and m''_j are both essentially $n_j/2$. We have

$$F^{m'_j}_{\epsilon_j}(p) = F^{-m''_j}_{\epsilon_j}(q) = F^{-m''_j}_{\epsilon_j}(q') + O(\hat\beta^{m''_j}\beta^{n_j}) = F^{-m''_j}_{\epsilon_j}(q') + o(1)$$

Adding and subtracting $\pi/(2\epsilon_j)$ and n_j to $w_{\epsilon_j} F^{m'_j}_{\epsilon_j}(p) = w_{\epsilon_j} F^{-m''_j}(q') + o(1)$, we have

$$w^\iota_{\epsilon_j} F^{m'_j}(p) - m'_j = w^o_{\epsilon_j} F^{-m''_j}_{\epsilon_j}(q') + m''_j + \left[\frac{\pi}{\epsilon} - n_j + o(1)\right]$$

As we let $j \to \infty$, the left hand side will converge to $\varphi^\iota p$. The term in brackets will converge to $-\alpha$. Thus we conclude that $w^o_{\epsilon_j} F^{-m''_j}_{\epsilon_j}(q') + m''_j$ will converge to $\varphi^\iota(p) + \alpha$. By hypothesis, we have $T_\alpha(p) \in \Sigma_0$, and φ^o is a coordinate on Σ_0. Thus φ^o is a coordinate on W^c_ϵ for ϵ small. By Proposition 3.9, $\hat{q} \mapsto w^o_{\epsilon_j} F^{-m''_j}_{\epsilon_j}(\hat{q}) + m''_j$ gives a uniform family of coordinates on $W^c_{\epsilon_j}$, so we conclude that q' must converge to a point $q_0 \in \Sigma$. By the condition that $\varphi^\iota(p) = \varphi^o(q_0) - \alpha$, we conclude that $q_0 = T_\alpha(p)$. □

Proof of Theorem 2. We recall that $\mathcal{B}$ and Σ are invariant in both forward and backward time. Further $T_{\alpha+1} = T_\alpha \circ F$. Thus for an arbitrary point $p \in \mathcal{B}$ and arbitrary $\alpha \in \mathbf{C}$ we may map p and add an integer to α so that the projection $\pi(p) = x$ satisfies $|x+r| < r$, $p \in B^\iota_{r,\eta}$, and $T_\alpha(p) \in \Sigma_0$. The hypotheses of Theorem 3.10 are now satisfied, so $\lim_{j\to\infty} F^{n_j}_{\epsilon_j}(p) = T_\alpha(p)$. □

§**4. Semi-continuity of Julia sets.** In this section we give the proofs of Theorems 3, 4 and 5. Recall the domain $\Omega := \varphi^o(\mathcal{B} \cap \Sigma) \subset \mathbf{C}$. For $\alpha \in \mathbf{C}$, we define the map $h_\alpha : \Omega \to \mathbf{C}$ which is given by

$$\begin{aligned} h_\alpha &:= \varphi^o \circ T_\alpha \circ (\varphi^o)^{-1} \\ &= \tau_\alpha \circ \varphi^\iota \circ (\varphi^o)^{-1} \\ &= \tau_\alpha \circ \varphi^\iota \circ H. \end{aligned}$$

We say that $\zeta_0 \in \mathbf{C}$ is a *periodic point* for h_α if $\zeta_j := h^j_\alpha(\zeta_0) \in \Omega$ is defined for all j, and $h^n_\alpha(\zeta_0) = \zeta_0$. By the chain rule we have $(h^n_\alpha)'(\zeta_0) = \prod_{j=0}^{n-1} h'_\alpha(\zeta_j)$. We say that ζ_0 is a *repelling* (resp. *attracting* periodic point if $|(h^n_\alpha)'(\zeta_0)| > 1$ (resp. < 1).

Theorem 4.1. *Let ζ_0 be a repelling (resp. attracting) periodic point of period μ for h_α, and let $p_0 = (\varphi^o)^{-1}(\zeta_0) \in \mathcal{B} \cap \Sigma$ be its image. Then there exists j_0 such that for $j \ge j_0$, there is a point p_j near p_0, which has period ν_j for F_{ϵ_j}, with $\epsilon_j = \frac{\pi}{j+\alpha}$ and which is a saddle (resp. sink). Further, ν_j divides $j\mu$, and $\nu_j \to \infty$.*

Proof. We will prove the repelling case; the attracting case is similar and easier. Let ζ_0 be a repelling periodic point, and let $\Delta_0 \subset \mathbf{C}$ be a small disk about ζ_0. Now we transfer this picture to M and write $p_0 = (\varphi^o)^{-1}(\zeta_0) = H(\zeta_0)$, and write Δ_0 again for its image under H. We may choose a local holomorphic coordinate system such that $p_0 = (0,0)$, and Δ_0 lies inside the x-coordinate axis. We let Δ' be a small disk about 0 inside the y-coordinate axis, and we consider a product neighborhood $\Delta_0 \times \Delta'$ of p_0. Since T_α is defined on $\mathcal{B}$, it is defined in a neighborhood of the closure of $\Delta_0 \times \Delta'$ in M. Since ζ_0 is repelling for h^μ_α, we see that T^μ_α has the following properties:

(*i*) $T^\mu_\alpha(\partial\Delta_0 \times \overline{\Delta'}) \cap \overline{\Delta_0 \times \Delta'} = \emptyset$ (because of the expansion on Δ_0), and

(*ii*) $\overline{T^\mu_\alpha(\Delta_0 \times \Delta')} \cap (\overline{\Delta_0} \times \partial\Delta') = \emptyset$ (because the range of T_α is in Σ).

By Theorem 3.10, the sequence $F^j_{\epsilon_j}$ converges uniformly on $\overline{\Delta_0 \times \Delta'}$ to T^μ_α. It follows that $F^{\mu n_j}_{\epsilon_j}$ also satisfies properties (*i*) and (*ii*) for j sufficiently large. In the terminology of [HO2], $F^{\mu n_j}_{\epsilon_j}$ is a crossed mapping of $\Delta_0 \times \Delta'$ to itself with degree 1. A property of crossed self-maps of degree 1 is that they have a unique saddle fixed point p_j (see [BSsym]). This point p_j is periodic for $F^{n_j}_{\epsilon_j}$, and we denote its period by ν_j. The period ν_j of p_j must divide $j\mu$. Since $\Delta_0 \times \Delta'$ can be taken arbitrarily small, we see that the p_j must converge to p_0. Finally, ν_j cannot have a bounded subsequence, or else p_0 would be periodic for F_0. But this is impossible since $p_0 \in \mathcal{B}$. □

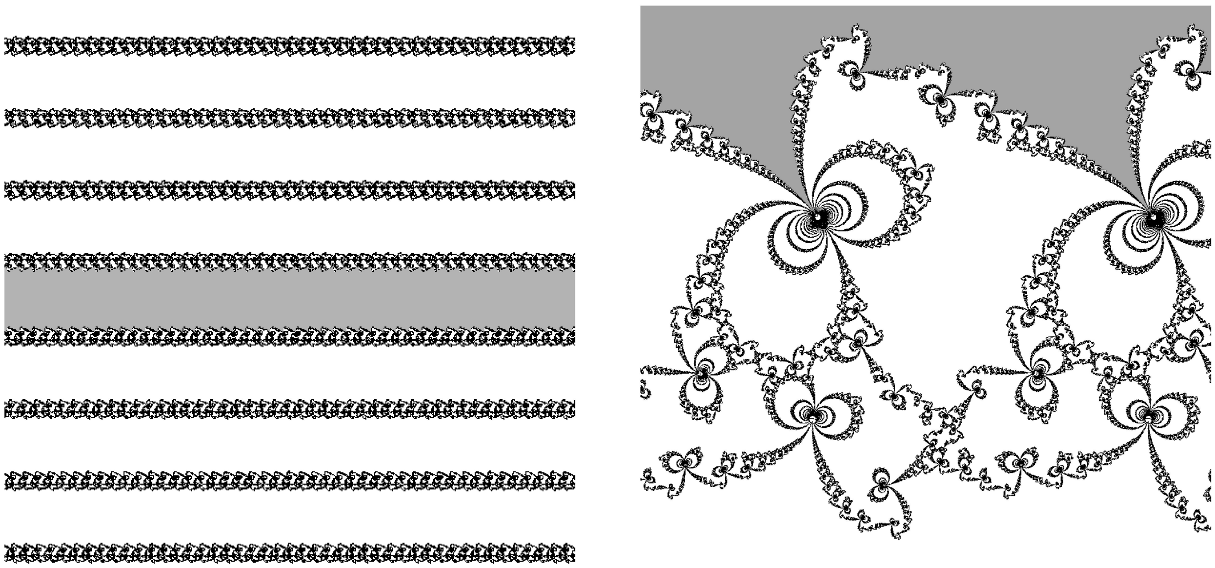

Figure 5. Slice $K^+(F, T_\alpha) \cap \Sigma$, $a = .3$, $\alpha = 0$: 44 periods, $|\Im\zeta| < 22$ (left); detail (right).

Recall that for an automorphism F of a manifold M we defined $J^* = J^*(F)$ to be the closure of the set of saddle periodic points of F. In general the sets $J^*(F_\epsilon)$ are lower semicontinuous as a function of ϵ. It is evident that $J^*(F) \cap \mathcal{B} = \emptyset$.

Definition 4.2. *Let $\mathcal{R}_\alpha$ denote the set of repelling periodic points of h_α. By $J^*(F, T_\alpha)$ we denote the closure in M of $(\varphi^o)^{-1}(\mathcal{R}_\alpha)$.*

Proof of Theorem 3. We must show that $\liminf_{j\to\infty} J^*(F_{\epsilon_j}) \supset J^*(F_0, T_\alpha)$. Let p_0 be a periodic point in $J^*(F, T_\alpha)$. It will suffice to show that for every neighborhood V of p_0 there is a j_0 such that for $j \geq j_0$ there is a saddle point p_j for F_{ϵ_j} on V. This property is given by Theorem 4.1, which completes the proof. □

Now let us suppose that $M = \mathbf{C}^2$ and $F : M \to M$ is a polynomial automorphism. We define an analogue of a "filled Julia-Lavaurs set".

Definition 4.3. *Let $K^+(F, T_\alpha)$ to be the set of points $p \in K^+(F)$ which satisfy one of the following two properties:*

(i) Either $p \in K^+(F) - \mathcal{B}$ or there is an integer $n \geq 1$ such that $T_\alpha^k(p) \in \mathcal{B}$ for $k \leq n-1$, and $T_\alpha^n(p) \in K^+ - \mathcal{B}$.

(ii) $T_\alpha^n(p)$ is defined and belongs to $\mathcal{B}$ for all $n \geq 0$.

Thus the complement of $K^+(F, T_\alpha)$ consists of the points satisfying the condition: there is an $n \geq 0$ such that $T_\alpha^k(p) \in \mathcal{B}$ for $k \leq n-1$ and that $T_\alpha^n(p) \notin K^+$. It is immediate from the definition that

$$K^+(F, T_\alpha) - \mathcal{B} = K^+(F) - \mathcal{B} \subset K^+(F, T_\alpha) \subset K^+(F).$$

Proposition 4.4. *$K^+(F,T_\alpha) = F(K^+(F,T_\alpha)) = K^+(F,T_{\alpha+1})$, so $K^+(F,T_\alpha)$ depends only on the equivalence class of α modulo $\mathbf{Z}$. Further, $K^+(F,T_\alpha)\cap\mathcal{B}$ is a union of fibers $\{\varphi^\iota = const\}$, and $K^+(F,T_\alpha)\cap\mathcal{B}\neq\mathcal{B}$.*

Proof. The last statement is the only one that is not immediate from the definitions. For this, we recall that $\Sigma\not\subset K^+$. So choose a point $p_0 = (\varphi^o)^{-1}(\zeta_0)\notin K^+$. It follows that the fiber $\{\varphi^\iota = \zeta_0-\alpha\}$ is mapped to p_0. Thus this fiber is outside of $K^+(F,T_\alpha)$. □

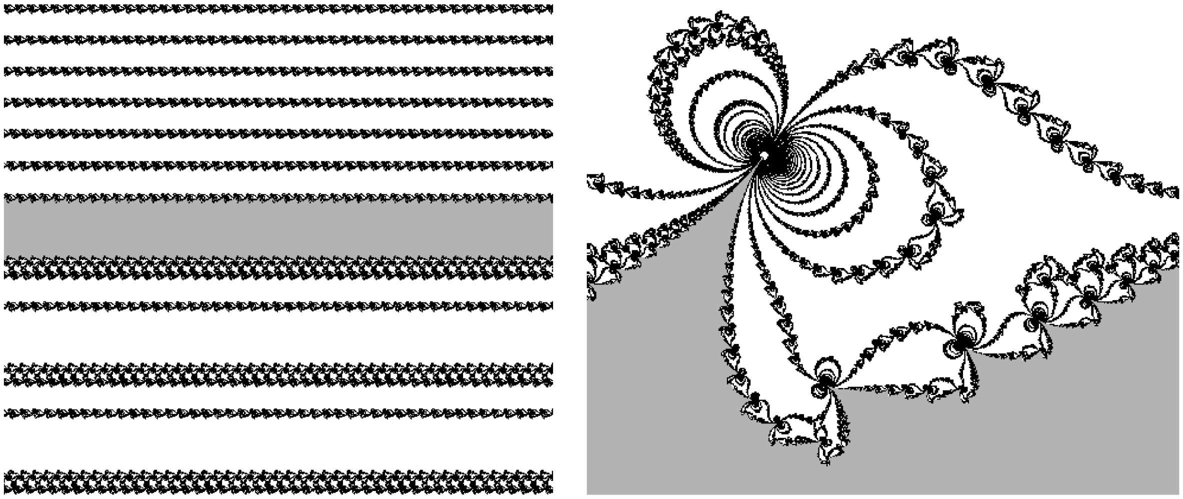

Figure 6a. Slices $K^+(F,T_\alpha)\cap\Sigma$; $a=.3$, $\alpha=\pi i$; 44 periods, $|\Im\zeta|<22$ (left), detail (right).

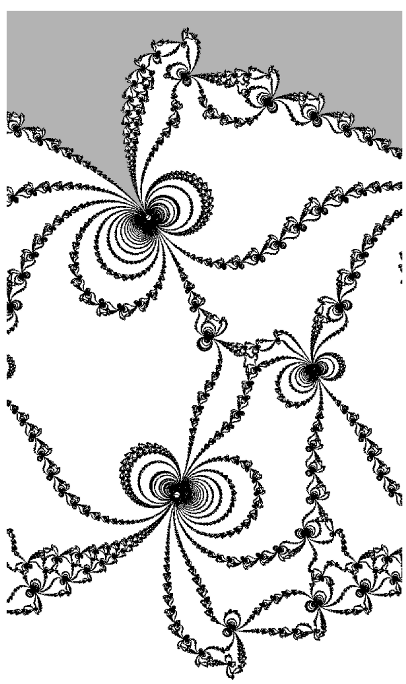

Figure 6b. Detail of 6a: one period from top of the bottom component of $\Sigma\cap\mathcal{B}$.

By Theorem 1.1, the fibration $\mathcal{F}_{\varphi^\iota}$ is trivial, and it follows from Proposition 4.4 that $K^+(F,T_\alpha)\cap\mathcal{B}$ is biholomorphic to a product $S^+\times\mathbf{C}$, where $S^+ := S^+(F,T_\alpha)\subset\mathbf{C}$ is a closed subset. Now let $p\in\Sigma\cap K^+(F,T_\alpha)\cap\mathcal{B}$ be a point that is not critical for h_α. (Critical points are a discrete subset of $\mathcal{B}\cap\Sigma$.) This means that Σ is transverse to the fibration $\mathcal{F}_{\varphi^\iota}$ at p. Thus S^+ is locally homeomorphic to a neighborhood of p inside the slice $\Sigma\cap K^+(F,T_\alpha)$.

Proof of Theorem 4. We must show that $\mathcal{B}\cap\limsup_{j\to\infty}K^+(F_{\epsilon_j})\subset K^+(F_0,T_\alpha)$. Let us choose a point $p\in\mathcal{B}-K^+(F,T_\alpha)$. Thus there exists an m such that $T_\alpha^m(p)\notin K^+(F_0)$. It will suffice to show that $p\notin K^+(F_{\epsilon_j})$ for large j. By Theorem 2, it follows that $F_{\epsilon_j}^{n_j m}p$ is approximately $T_\alpha^m(p)$, and thus $F_{\epsilon_j}^{n_j m}p\notin K^+(F_0)$. By the semicontinuity of K^+, it follows that $F_{\epsilon_j}^{n_j m}p\notin K^+(F_{\epsilon_j})$. Thus $p\notin K^+(F_{\epsilon_j})$. □

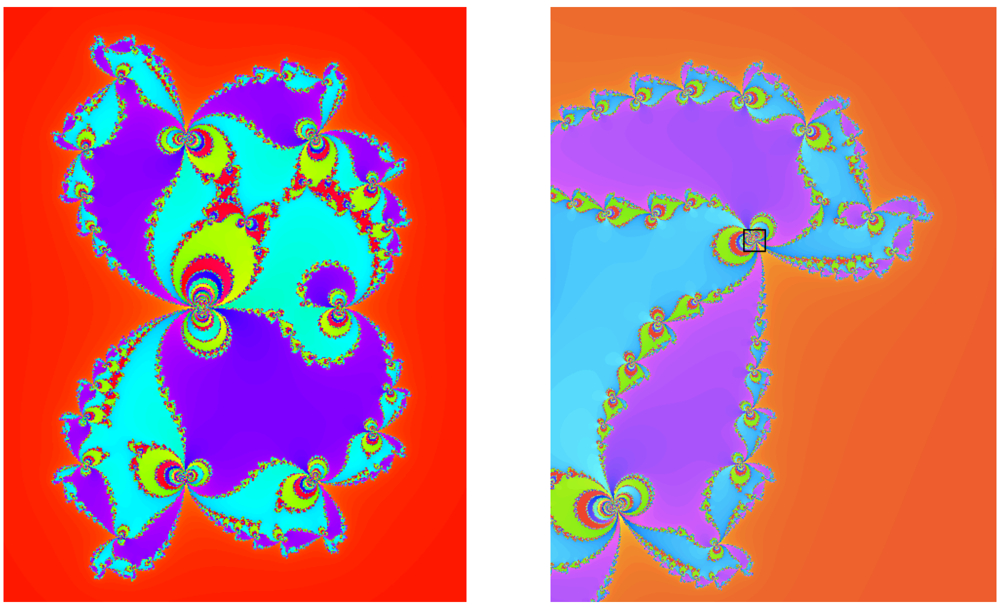

Figure 7a. Slices of K^+ for Hénon map $F_{a,\epsilon}$ as in (0.3), with $a=.3$, $\epsilon=\pi/(n-i\alpha)$, $n=1000$, $\alpha=4.3$: Linear slice $K^+\cap T$ (left), unstable slice $K^+\cap W^u(q)$ (right)

Proof of Theorem 5. We must show two things: (i) There exists an $\alpha\in\mathbf{C}$ such that $\mathcal{B}\cap J^*(F_0,T_\alpha)\neq\emptyset$. By Theorem 1.5, there exists a $\zeta_0\in\Omega=\varphi^o(\mathcal{B}\cap\Sigma)$ with $|h_0'(\zeta_0)|>1$. We choose $\alpha\in\mathbf{C}$ so that $h_\alpha(\zeta_0)=h_0(\zeta_0)+\alpha=\zeta_0$. Since $h_\alpha'(\zeta_0)=h_0'(\zeta_0)$, the point ζ_0 is a repelling fixed point of h_α. Thus $p=(\varphi^o)^{-1}(\zeta_0)$ is a fixed point of T_α and $p\in\mathcal{B}\cap J^*(F_0,T_\alpha)$.

(ii) For each $p\in\mathcal{B}$ there exists an α' such that $p\notin K^+(F_0,T_{\alpha'})$. For any pair of points $p\in\mathcal{B}$ and $q\in\Sigma$ we can choose $\alpha'\in\mathbf{C}$ so that $T_{\alpha'}(p)=q$. Since F_0 is Hénon, we can choose $q\in\Sigma$ so that $q\notin K^+(F_0)$. Thus $p\notin K^+(F_0,T_{\alpha'})$. □

Definition 4.5. *We use the notation:*

$$K(F,T_\alpha) := J^-(F)\cap K^+(F,T_\alpha) = K^-(F)\cap K^+(F,T_\alpha),$$
$$J(F,T_\alpha) := J^-(F)\cap\partial K^+(F,T_\alpha).$$

Thus we have $J^*(F,T_\alpha)\subset J(F,T_\alpha)$ and:

Corollary 4.6. *If F_ϵ satisfies (0.1), and if $\{\epsilon_j\}$ is an α-sequence, then*

$$\mathcal{B} \cap \limsup_{j\to\infty} K(F_{\epsilon_j}) \subset K(F, T_\alpha).$$

To illustrate $K^+(F,T_\alpha)$ graphically, we return to the Hénon family defined in (0.3). The pictures in Figure 5 correspond to those in Figure 2. That is, they are slices of $K^+(F,T_\alpha) \cap \Sigma$, with the values $a = .3$ and $\alpha = 0$, which corresponds to real ϵ. The gray region is the complement of $K^+(F)$, the set $K^+(F,T_\alpha)$ is black, and $K^+(F) - K^+(F,T_\alpha)$ is white. All pictures are invariant under the translation $\zeta \mapsto \zeta + 1$. The viewboxes on the left hand sides of Figures 5 and 6 are taken to be symmetric around the real axis $\{\Im\zeta = 0\}$; the viewboxes are taken to have side $= 44$ in order to show what happens when $\Im\zeta$ is large. We see a number of horizontal "chains" in the left hand pictures in Figures 5 and 6. In the upper half of each of these pictures, the map h_α acts approximately as a vertical translation, moving each chain to the one below it, until it reaches the chain just above the gray region, which corresponds to the complement of $\mathcal{B}$. By (1.9) the amount of vertical translation in the upper region is approximately $c_0^+ \approx -5.83$, and there are $8 \approx 44/5.83$ horizontal strips in Figure 5. In Figure 6, the vertical translation in the upper part is $c_0^+ + \Im\alpha \approx -2.69$. In the chains bordering the complement of the basin, the map is not like a translation and is more complicated. The bottom half of the left hand side of Figure 5 and 6 is analogous, with the approximate translation near the bottom of the figures being approximately $c_0^- + \Im\alpha$. In fact, the symmetry in Figure 5 comes because h_α commutes with complex conjugation. The parameter $\alpha = \pi i$ in Figure 6 is nonreal, and we see that the symmetry of complex conjugation is lost. The pictures on the right of Figures 5, 6a, and 6b give a detail from the edge of the gray region, spanning a little more than 1 period. The implosion phenomenon corresponding to Figure 6 is given in Figures 7a,b,c. By the time we have zoomed in as far as Figure 7c, we begin to see the "detail" images from Figures 6a,b.

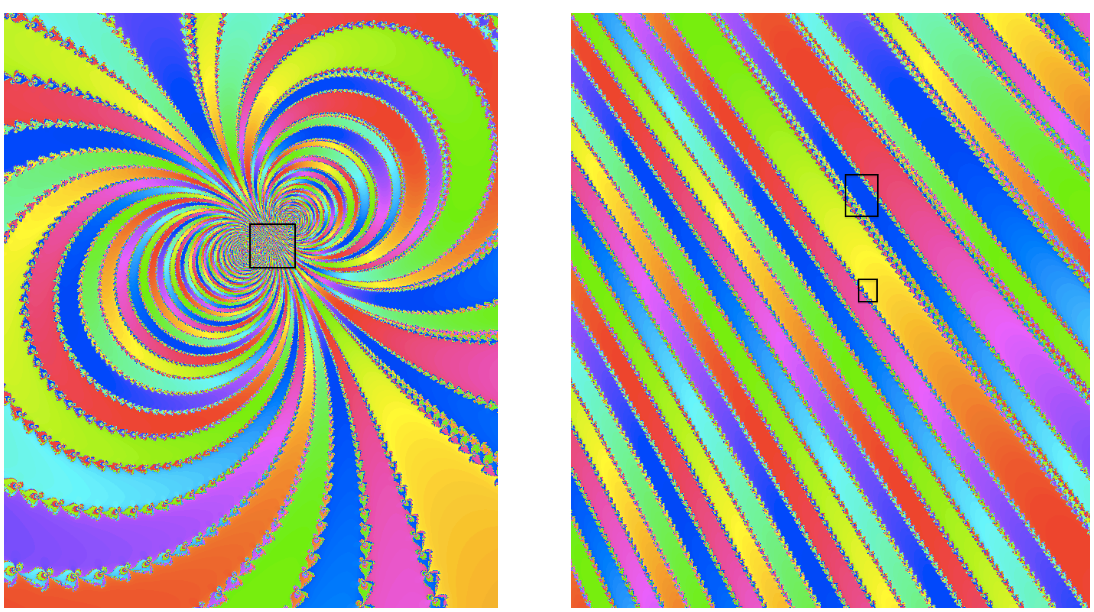

Figure 7b. On left, a zoom of Figure 7a (right). On right: a further zoom.

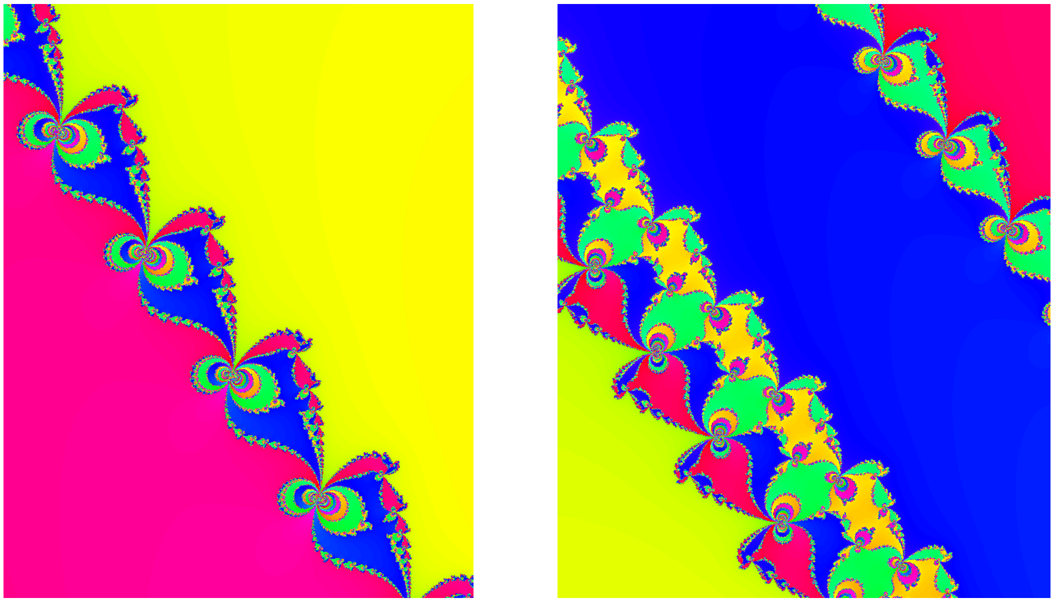

Figure 7c. Zooms of the right hand image of Figure 7b.

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