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Real polynomial diffeomorphisms with maximal entropy: II. Small Jacobian

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Abstract. Let $\{h_{a,b} : a, b \in \mathbb{R}, b \neq 0\}$ denote the Hénon family of quadratic polynomial diffeomorphisms of $\mathbb{R}^2$, with b equal to the Jacobian of $h_{a,b}$. In this paper, we describe the locus of parameter values (a, b) such that $0 < |b| < 0.06$, and the restriction of $h_{a,b}$ to its non-wandering set is topologically conjugate to the horseshoe map. The boundary of the horseshoe locus is shown to be characterized by a homoclinic tangency which is part of a generic unfolding.

0. *Introduction*

The problem of understanding the dynamical behavior of diffeomorphisms has played a central role in the field of dynamical systems. One way of approaching this question is to ask about generic behavior in the space of diffeomorphisms. Another way to approach it is to ask about behavior in some specific parametrized family. The family of diffeomorphisms of $\mathbb{R}^2$ introduced by Hénon has often played the role of such a test case. This is a two-parameter family given by the formula

$$f_{a,b}(x, y) = (a - x^2 - by, x)$$

for $b \neq 0$. There are regions of parameter space that are well understood. If we fix b, then for $a \ll 0$ the non-wandering set of $f_{a,b}$ is empty. For $a \gg 0$, it is shown in [**DN**] that the restriction of $f_{a,b}$ to its non-wandering set is hyperbolic and topologically conjugate to the full two-shift. Such diffeomorphisms are called 'horseshoes'. How the dynamics change between these two extremes has been the subject of much investigation. The case $b = 0$ is an interesting special case. In this case the map $f_{a,b}$ is not a diffeomorphism; in fact, the dynamical behavior is essentially one-dimensional. The dynamical complexity of $f_{a,0}$ increases monotonically with a (see [**MT**]). For other values of b no such results are known. In fact, in [**KKY**] it was shown that in some respects the behavior should not be expected to be monotone. One way of measuring the topological complexity is through the topological entropy, $h_{\text{top}}(f_{a,b})$. This is a continuous real-valued function of the parameters, which takes on values in the interval $[0, \log 2]$. The case $a \ll 0$ corresponds to $h_{\text{top}} = 0$. The case $a \gg 0$ corresponds to $h_{\text{top}} = \log 2$. In this paper we study the set of parameters (a, b) for which $h_{\text{top}}(f_{a,b})$ takes on its maximal value. We say that $f_{a,b}$ has maximal entropy if $h_{\text{top}}(f_{a,b}) = \log 2$. We analyze the 'maximal entropy locus' when the Jacobian parameter b is small. The following will be a consequence of Theorems 1.1, 1.2, and 4.8.

THEOREM 1. *For each b with $|b| < 0.06$, there is a unique $a = a_b$ so that $h_{\text{top}}(f_{a,b}) < \log 2$ for $a < a_b$ and $h_{\text{top}}(f_{a,b}) = \log 2$ for $a \geq a_b$. Further, we have the following.*

(1) *If $a > a_b$, $f_{a,b}$ is a hyperbolic horseshoe.*

(2) *If $a = a_b$, $f_{a,b}$ has a quadratic tangency between stable and unstable manifolds of fixed points. This tangency is homoclinic when $b > 0$ and heteroclinic when $b < 0$.*

The next result discusses properties of the function $b \mapsto a_b$ defined in Theorem 1.

THEOREM 2. *The function $b \mapsto a_b$ is continuous on the interval $(-0.06, 0.06)$. It is analytic on the subintervals $(-0.06, 0)$ and $(0, 0.06)$ but not differentiable at $b = 0$. Furthermore, there is a generic unfolding of the homoclinic tangency at the parameter (a_b, b), i.e. at the point of tangency, the stable and unstable manifolds move past one another with positive speed with respect to a.*

The terminology 'generic unfolding' will be explained in greater detail in §5.

Part of Theorem 1 follows from a more general analysis of polynomial diffeomorphisms of maximal entropy in degree $d \geq 2$, which was carried out in [**BS1**] and [**BS2**]. In particular we proved in this more general context that a maximal entropy polynomial diffeomorphism is either hyperbolic or has a quadratic tangency between stable and unstable manifolds of periodic points. The contribution of this paper is to describe the set of parameter values corresponding to these two types of behavior.

Although these results are stated for the diffeomorphisms $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ our methods give us very complete information about the corresponding complex extensions $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ for maximal entropy parameter values. In fact, it is the analysis of these complex extensions that allows us to obtain information about the real Hénon diffeomorphisms. In particular, we take advantage of the theory of intersections of complex manifolds to analyze the complex extensions of the real stable and unstable manifolds.

In addition to proving Theorems 1 and 2, a goal of this paper is to develop the technique of crossed mappings as a method of more general applicability in the analysis of families of polynomial diffeomorphisms of $\mathbb{C}^2$. These techniques are explored further in [**BS3**].

We note that this is not the first time that complex methods have been used to address similar questions. Hubbard and Oberste-Vorth [**O**] used complex methods to improve the result of Devaney–Nitecki, and Fornæss and Gavosto [**FG1, FG2**] have used complex methods to show that there is a generic unfolding of a complex tangency for $f_{a,b}$ for certain parameters (a, b).

1. *The quadratic horseshoe locus: first approximation*

We consider mappings of the form

$$f_{a,b}(x, y) = (a - x^2 - by, x) \tag{1.1}$$

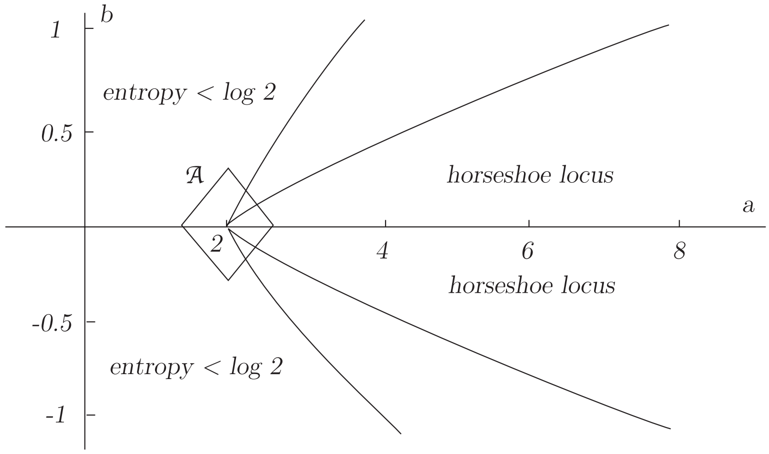

FIGURE 1.

with $b \neq 0$. Note that f may also be written in the form $\chi^{-1} \circ f \circ \chi = (y, y^2 - a - bx)$, where we set $\chi(x, y) = (-y, -x)$. We say that $f_{a,b}$ is a *(complex) horseshoe* if $f_{a,b}$ is hyperbolic on $J = J(f_{a,b})$ and if $f|J$ is topologically conjugate to the full 2-shift. If, in addition, $J \subset \mathbb{R}^2$, we say that $f_{a,b}$ is a *real horseshoe*. Hubbard and Oberste-Vorth have obtained estimates on the (complex) horseshoe locus; see [**O**] and [**MNTU**, Proposition 7.4.6]. These are summarized in the following.

THEOREM 1.1. *If $b \neq 0$, and if $|a| > 2(1 + |b|)^2$, then $f_{a,b}$ is a (complex) horseshoe. If, in addition, $b \in \mathbb{R}$, and $a > 0$, then $f_{a,b}$ is a real horseshoe.*

Since horseshoes have entropy equal to log 2, the following result gives a large region of parameter space where there are no horseshoes. This is the region to the left in Figure 1.

THEOREM 1.2. *Define $\sigma^-(b) = 2 - \frac{13}{8}b - \frac{7}{64}b^2$ and $\sigma^+(b) = 2 + \frac{7}{4}b + \frac{5}{16}b^2$. If $(a, b) \in \mathbb{R}^2$ satisfy $b \neq 0$, $|b| \leq 1$ and $a \leq \max(\sigma^-(b), \sigma^+(b))$, then the entropy of $f|\mathbb{R}^2$ is less than* log 2.

Proof. First we note that a fixed point of $f_{a,b}$ has the form (x, y), where

$$x = y = -\tfrac{1}{2}[b + 1 \pm \sqrt{(b + 1)^2 + 4a}]. \tag{1.2}$$

Now we recall some results from [**BS2**]. If $f_{a,b}$ is a quadratic diffeomorphism of $\mathbb{R}^2$, and if $f|\mathbb{R}^2$ has entropy log 2, then $f_{a,b}$ has two fixed points, which must be saddles. Further, one of these points must be unstably one-sided, and the unstable eigenvalue of *Df* at this fixed point is (strictly) greater than 4. The other fixed point has a negative eigenvalue in the unstable direction, and this eigenvalue must be less than -2.

The differential is given in (x, y)-coordinates as

$$Df_{a,b} = \begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix}.$$

The product of the eigenvalues is b, so we may write them as λ and b/λ. Thus, the trace of the differential is

$$-2x = \lambda + \frac{b}{\lambda}.$$

If $|b| \le 1$, then $\lambda \mapsto \lambda + b/\lambda$ is strictly increasing in λ for $|\lambda| > 1$. Thus, the condition that there is an eigenvalue greater than 4 gives us the inequality

$$-2x > 4 + \frac{b}{4}, \tag{1.3}$$

and the condition that there is an eigenvalue less than -2 gives the inequality

$$-2x < -\left(2 + \frac{b}{2}\right). \tag{1.4}$$

(Note that the inequalities (1.3) and (1.4) refer to different fixed points and thus involve different values of x.)

Now we substitute expression (1.2) into (1.3) and obtain

$$\begin{aligned} b + 1 \pm \sqrt{(b+1)^2 + 4a} &> 4 + \frac{b}{4} \\ \sqrt{(b+1)^2 + 4a} &> 3 - \frac{3}{4}b \\ b^2 + 2b + 1 + 4a &> 9 - \frac{9}{2}b + \frac{9}{16}b^2, \end{aligned}$$

which gives $a > \sigma^-(b)$.

Similarly, we substitute (1.2) into (1.4) and find

$$\begin{aligned} b + 1 \pm \sqrt{(b+1)^2 + 4a} &< -2 - \frac{b}{2} \\ \pm\sqrt{(b+1)^2 + 4a} &< -3 - \frac{3}{2}b \\ b^2 + 2b + 1 + 4a &> 9 + 9b + \frac{9}{4}b^2, \end{aligned}$$

where the last inequality is reversed because the quantities being squared are negative. Thus, $a > \sigma^+(b)$. The case that one of these inequalities fails happens exactly when we have $a \le \max(\sigma^+(b), \sigma^-(b))$, and in this case the entropy is not equal to $\log 2$. □

Figure 1 shows the information on parameter space that is given by Theorems 1.1 and 1.2, together with the region $\mathcal{A}$ which will be defined in §2. We see that the set $\{|b| < 0.06\}$ is contained in the union of the sets $\{\text{entropy} < \log 2\}$ and $\mathcal{A}$, and the horseshoe locus. Figure 1 considers only parameters $|b| \le 1$. In fact, we restrict our attention without loss of generality to the case $|b| \le 1$ throughout this paper. Each of the items discussed in the theorem, maximal entropy, the horseshoe property, and generic unfolding, will hold for f if and only if it holds for f^{-1}. Thus, the fact that $f_{a,b}^{-1}$ is conjugate to $f_{a/b^2,1/b}$ means that the regions that define these dynamical behaviors are invariant under the involution $(a,b) \mapsto (ab^{-2}, b^{-1})$. In particular, this gives versions of these theorems corresponding to the case $|b| > (0.06)^{-1}$.

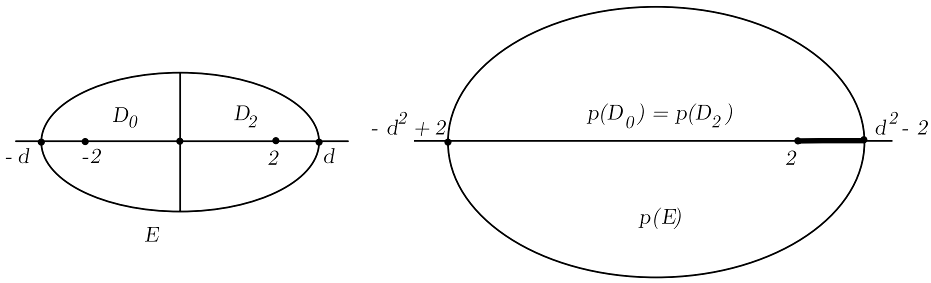

FIGURE 2.

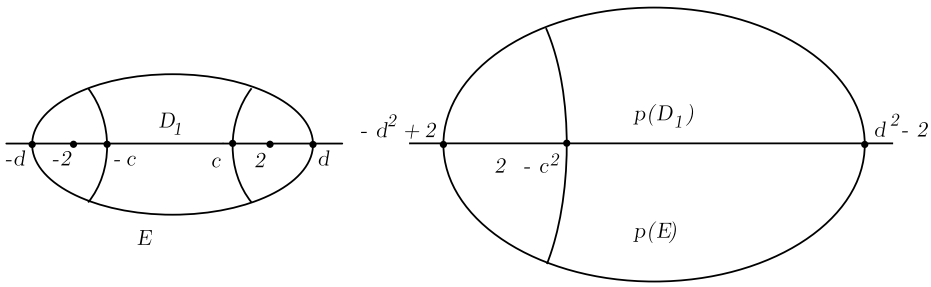

FIGURE 3.

2. *Complex boxes and crossed mappings*

In order to study the system $f|K : K \to K$, we introduce an open cover by 'boxes' B_i and study a family of 'crossed mappings' $f_{i,j} : B_i \to B_j$. We start by working with $p(z) = 2 - z^2$ and a covering of its Julia set $[-2, 2]$. The Green function for $[-2, 2]$ is

$$G(z) = \log\left|\frac{z + \sqrt{z^2 - 4}}{2}\right|.$$

For $\lambda > 0$, p induces a 2-fold branched covering $p : \{G < \lambda\} \to \{G < 2\lambda\}$. The level sets $\{G = \lambda\}$ are ellipses with foci at ± 2, and the gradient lines (the orthogonal trajectories) are given by the family of hyperbolae with foci at ± 2.

Let us fix $c = \frac{1}{2}(\sqrt{17} - 1) \sim 1.5615$ and $d = \frac{1}{2}(1 + \sqrt{7 + 2\sqrt{17}}) \sim 2.4523$. Let $E \subset \mathbb{C}$ be the domain bounded by the ellipse with foci at ± 2 and passing through $\pm d$. It follows that $p(E)$ is the ellipse with foci at ± 2 and passing through $\pm(d^2 - 2)$. We set $D_0 := \{\zeta \in \mathbb{C}^2 : \Re(\zeta) < 0\} \cap E$ and $D_2 := \{\zeta \in \mathbb{C}^2 : \Re(\zeta) > 0\} \cap E$ as in Figure 2. It follows that $p(D_0) = p(D_2) = p(E) - [2, d^2 - 2)$. Let D_1 denote the region in E lying between the hyperbolae with foci at ± 2 and which pass through $\pm c$ as in Figure 3. Thus, $p(D_1)$ is the region of the ellipse $p(E)$ to the right of the hyperbola with foci at ± 2 and passing through $2 - c^2$. We have the following inclusions:

$$D_0 \cup D_1 \subset p(D_0) = p(D_2), \quad D_2 \subset p(D_1).$$

We may also compute certain distances related to these inclusions:

$$\begin{aligned}\operatorname{dist}(\partial p(D_0), D_0) &= d^2 - 2 - d \\ \operatorname{dist}(\partial p(D_0), D_1) &= 2 - c \\ \operatorname{dist}(\partial p(D_1), D_2) &= c^2 - 2.\end{aligned} \tag{2.1}$$

While calculating the distances between ellipses can be difficult in general, these calculations are straightforward because the relevant ellipses are in a confocal family. Thus, the minimal distances between these ellipses are realized by points on the real axis. By the choices of c and d, we have

$$\Delta := d^2 - d - 2 = 2 - c = c^2 - 2 \sim 0.4384. \tag{2.2}$$

Now choose $e > d$ and set $B_j = \{(x, y) \in \mathbb{C}^2 : x \in D_j, |y| < e\} = D_j \times \{|y| < e\}$ for $j = 0, 1, 2$. Thus, $B_0 \cup B_1 \cup B_2 = E \times \{|y| < e\}$. We introduce the set

$$\mathcal{A} := \{(a, b) \in \mathbb{C}^2 :\ b \neq 0,\ |a - 2| + e|b| < \Delta\} \approx \{|a - 2| + 2.4|b| < 0.43\}.$$

PROPOSITION 2.1. *If $(a, b) \in \mathcal{A}$, then $K \subset B_0 \cup B_1 \cup B_2$.*

Proof. By [**MNTU**, p. 238], we know that K is contained in the bidisk $\{|x|, |y| < R\}$, where R is the larger root of the equation $t^2 - (1 + |b|)t - |a| = 0$. By the condition $|a - 2| + e|b| < \Delta$, we conclude that we may take e sufficiently close to d, then we have

$$R \le \frac{1 + \Delta/e + \sqrt{(1 + \Delta/e)^2 + 4(2 + \Delta)}}{2} \sim 2.258\,45.$$

Recall that pE is an ellipse with foci at ± 2 and major axis of length $d^2 - 2 \sim 4.013\,78$. We then compute that its minor axis has length $\sqrt{(d^2 - 2)^2 - 4} \sim 3.48$.

To prove the proposition, we need to show that if $(x, y) \in \{|x|, |y| < R\}$, and if $x \notin E$, then $(x, y) \notin K$. For such (x, y), the x-coordinate x' of $f(x, y)$ satisfies $|x' - p(x)| < |a - 2| + |by| < |a - 2| + R|b| < \Delta$ since $|b| < \Delta/e$. Now let $D := \{\zeta \in pE : \operatorname{dist}(\zeta, \partial(pE)) > 1.034\,65\Delta\}$. Since $x \notin E$, it follows that $px \notin pE$, and so the x-coordinate of $f(x, y)$ does not belong to D. On the other hand, the minor axis of pE is 3.48, so that D contains the disk of radius $3.48 - 1.034\,65\Delta \sim 3.0264 > R$. Thus, $f(x, y) \notin K$. □

The vertical and horizontal components of the boundaries are defined to be

$$\partial_v B_j = (\partial D_j) \times \{|y| \le e\}, \quad \partial_h B_j = \bar{D}_j \times \{|y| = e\}.$$

We set $\mathcal{G} = \{(0, 0), (0, 1), (1, 2), (2, 1), (2, 0)\}$, and we interpret $\mathcal{G}$ as a graph on the vertices $\{B_0, B_1, B_2\}$ as in Figure 4.

PROPOSITION 2.2. *If $(a, b) \in \mathcal{A}$, then $f_{a,b}(\partial_v B_i) \cap \bar{B}_j = \emptyset$ and $f_{a,b}(\bar{B}_i) \cap \partial_h B_j = \emptyset$ for all $(i, j) \in \mathcal{G}$.*

Proof. By estimates (2.1) and (2.2) and the fact that $p(\partial D_0) = p(\partial D_2)$, we have

$$\operatorname{dist}(p(\zeta), \partial D_j) \ge \Delta$$

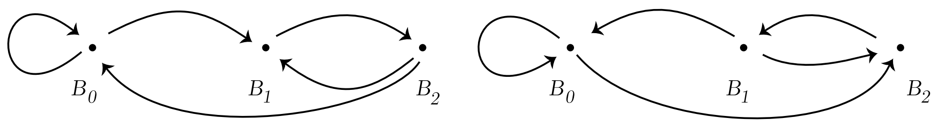

FIGURE 4. Graphs $\mathcal{G}$ for f (on left) and $\mathcal{G}^{-1}$ for f^{-1} (on right).

for $\zeta \in \partial D_i$ and $(i, j) \in \Gamma$. Thus, if $x \in \partial D_i$ and $|y| < e$, the first coordinate of $f_{a,b}(x, y)$ satisfies

$$|a - x^2 - by - p(x)| \leq |a - 2| + |by| < \Delta.$$

This gives $a - x^2 - by \notin \bar{D}_j$, so $f(\partial_v B_i) \cap \bar{B}_j = \emptyset$.

Note that $\partial_h B_j \subset \{|y| = e\}$. The second coordinate of f is x, so the second condition follows from the fact that $\bar{D}_j \cap \{|y| = e\} = \emptyset$, independently of a and b. □

Let $\pi_v(x, y) = x$ and $\pi_h(x, y) = y$ denote the projections in the vertical and horizontal directions. We let $f_{i,j}$ denote the mapping $f : B_i \cap f^{-1}B_j \to B_j$. Following [**HO**], we say that $f_{i,j}$ is a *crossed mapping* if for each $y \in \{|y| < e\}$,

$$\pi_v \circ f : (D_i \times \{y\}) \cap f^{-1}B_j \to D_j \tag{2.3}$$

is proper. Given $(i, j) \in \mathcal{G}$, then it follows from Proposition 2.2 that $f_{i,j}$ is a crossed mapping (see [**D**] or [**HO**]). We say that the degree of $f_{i,j}$ as a crossed mapping is the mapping degree of the map in (2.3) (which is independent of y). Similarly, we say that $f^{-1} : B_j \cap f B_i \to B_i$, which we denote by $f_{j,i}^{-1}$, is a crossed mapping if for each $x \in D_j$,

$$\pi_h \circ f^{-1} : (\{x\} \times \{|y| < e\}) \cap fB_i \to \{|y| < e\} \tag{2.4}$$

is proper. As was observed in [**HO**], $f_{i,j}$ is a crossed mapping if and only if $f_{j,i}^{-1}$ is. The degree of $f_{j,i}^{-1}$ as a crossed mapping is defined as the mapping degree of the map in (2.4) (which is independent of x). This, in turn, is the same as the degree of $f_{i,j}$. We will say that $(\mathcal{B}, \mathcal{G})$ is a system of crossed mappings, if $f_{i,j}$ induces a crossed mapping from B_i to B_j for each $(i, j) \in \mathcal{G}$. The following corollary is a consequence of Proposition 2.2.

COROLLARY 2.3. *If $(a, b) \in \mathcal{A}$, then $(\mathcal{B}, \mathcal{G})$ is a system of crossed mappings.*

Let us say that a sequence $I = i_0 i_1 \cdots i_n$ is *admissible* if $(i_k, i_{k+1}) \in \mathcal{G}$ for all k. We will sometimes also say that a sequence $J = j_0 j_1 \cdots j_m$ is admissible if $(j_k, j_{k+1}) \in \mathcal{G}^{-1}$ for all k. It will be clear from context whether we mean $\mathcal{G}$ or $\mathcal{G}^{-1}$. We define an *orbit* in a system of crossed mappings as a bi-infinite sequence $(p_j, i_j)_{j \in \mathbb{Z}}$ such that for all $j \in \mathbb{Z}$, $p_j \in B_{i_j}$, $(i_j, i_{j+1}) \in \mathcal{G}$, and $f(p_j) = p_{j+1}$. Next we give conditions that guarantee that every f-orbit $(p_j)_{j \in \mathbb{Z}}$ in K can be lifted to an orbit of the system of crossed mappings.

PROPOSITION 2.4. *Suppose that $K \cap (B_0 \cup B_1) \subset f(B_0 \cup B_2)$ and $K \cap B_2 \subset f(B_1)$. Then for $q \in K$ there is an admissible sequence $I = (i_n)_{n \in \mathbb{Z}}$ such that $f^n q \in B_{i_n}$ for all $n \in \mathbb{Z}$.*

Proof. Let us start by making a sequence $J_M = \{j_n : -M \leq n \leq M\}$ of finite length. If we have determined j_n already, then $f^n(q) \in B_{j_n} \cap K$. If $j_n = 0$ or 1, then by hypothesis

$f^{n-1}q \in (B_0 \cup B_2) \cap K$. Thus, we may choose $j_{n-1} \in \{0, 2\}$ such that $f^{n-1}q \in B_{j_{n-1}}$, and in either case we have $(j_{n-1}, j_n) \in \mathcal{G}$. Similarly, if $j_n = 2$, then $f^n q \in B_2 \cap K$, and by hypothesis we have $f^{n-1}q \in B_1$. Thus, we set $j_{n-1} = 1$ and $(j_{n-1}, j_n) = (1, 2) \in \mathcal{G}$. Starting at $n = M$, we continue backwards and generate an admissible sequence J_M.

Now we have admissible sequences $J_1, J_2, \ldots$ of increasing length. For each M, there is a sequence I_M that is a subsequence of infinitely many sequences J_{k_m}. We may make M increasingly large and thus via a diagonal process extract an infinite sequence I. □

PROPOSITION 2.5. *If $a, b \in \mathbb{R} \cap \mathcal{A}$, then*

$$(\bar{B}_{0,r} \cup \bar{B}_{1,r}) \cap f(B_0 \cup B_1 \cup B_2) \subset f(B_0 \cup B_2)$$
$$\bar{B}_{2,r} \cap f(B_0 \cup B_1 \cup B_2) \subset f(B_1).$$

Proof. We note that $(B_0 \cup B_1 \cup B_2 - B_0 \cup B_2) \cap \mathbb{R}^2 = \{0\} \times (-e, e)$. Thus, to prove the first inclusion, it suffices to show that $(\bar{B}_{0,r} \cup \bar{B}_{1,r}) \cap f(\{0\} \times (-e, e)) = \emptyset$. The x-projection of the f-image of this set is

$$\pi_v \circ f(\{0\} \times (-e, e)) = \{a - x^2 - by : x = 0, |y| < e\} \subset (2 - \Delta, 2 + \Delta).$$

On the other hand, $\bar{B}_{0,r} \cup \bar{B}_{1,r} = [-d, c] \times [-e, e]$. Thus, $(\bar{B}_{0,r} \cup \bar{B}_{1,r}) \cap f(\{0\} \times (-e, e)) = \emptyset$ since $c + \Delta = 2$, which proves the first inclusion.

For the second inclusion, we note that

$$(B_0 \cup B_1 \cup B_2 - B_1) \cap \mathbb{R}^2 = ((-d, -c) \cup (c, d)) \times (-e, e).$$

The x-projection of the f-image of this set is

$$\{a - x^2 - by : c < |x| < d, |y| < e\} \subset (2 - d^2 - \Delta, 2 - c^2 + \Delta).$$

On the other hand, the x-projection of $\bar{B}_{2,r}$ is $[c, d]$, which is disjoint from $(-\infty, 2 - c^2 + \Delta)$ since $2 - c^2 + \Delta = c$. □

Let $V \subset B_i$ be a complex subvariety. We say that V is a *horizontal multi-disk* (respectively *vertical multi-disk*) if each component of V is conformally equivalent to a complex disk, and if $\partial_h B_i \cap \bar{V} = \emptyset$ (respectively $\partial_v B_i \cap \bar{V} = \emptyset$). With this terminology the union of horizontal (respectively vertical) multi-disks is again a horizontal (respectively vertical) multi-disk. We denote the set of horizontal (respectively vertical) multi-disks by $\mathcal{D}_h(B_i)$ (respectively $\mathcal{D}_v(B_i)$). If $V \in \mathcal{D}_h(B_0)$ (respectively $V \in \mathcal{D}_v(B_i)$), then $\pi_v : V \to D_i$ (respectively $\pi_h : V \to \{|y| < e\}$) is proper, and we let $\delta(V)$ denote the degree of the corresponding projection. By $\mathcal{D}_h^m(B_i)$ (respectively $\mathcal{D}_v^m(B_i)$) we denote the set of horizontal (respectively vertical) multi-disks V such that for each component W of V, the degree $\delta(W)$ is no greater than m. We note the following:

$$\text{If } V' \in \mathcal{D}_v(B_i) \text{ and } V'' \in \mathcal{D}_h(B_i), \text{ then } \#(V' \cap V'') = \delta(V')\delta(V''). \tag{2.5}$$

If $f_{i,j}$ is a crossed map, and if $V \subset B_i$ is a subvariety for which $\partial_h B_i \cap \bar{V} = \emptyset$, then $\tilde{f}_{i,j}(V) := f(V) \cap B_j$ is closed in B_j and satisfies $\partial_h B_j \cap \bar{\tilde{V}} = \emptyset$, and is thus a horizontal subvariety. If $\deg(f_{i,j})$ denotes the degree as a crossed map, then we have

$$\deg(f_{i,j})\delta(V) = \delta(\tilde{f}_{i,j}(V)). \tag{2.6}$$

PROPOSITION 2.6. *If* $(a, b) \in \mathcal{A}$, *it follows that*

$$\tilde{f}_{1,2} : \mathcal{D}_h^m(B_i) \to \mathcal{D}_h^{2m}(B_j) \quad \textit{and} \quad \tilde{f}_{2,1}^{-1} : \mathcal{D}_v^m(B_j) \to \mathcal{D}_v^{2m}(B_i),$$

and if $(i, j) \in \mathcal{G}$, $(i, j) \neq (1, 2)$, *then*

$$\tilde{f}_{i,j} : \mathcal{D}_h^m(B_i) \to \mathcal{D}_h^m(B_j) \quad \textit{and} \quad \tilde{f}_{j,i}^{-1} : \mathcal{D}_v^m(B_j) \to \mathcal{D}_v^m(B_i).$$

Proof. We will show that $\tilde{f}_{i,j}(V)$ is conformally equivalent to a disk; the degree is given by (2.6). Suppose that V is a horizontal disk in B_i. Then, taking boundary inside $\mathbb{C}^2$, we have $\partial V \subset \partial_v(B_i)$. By Proposition 2.2, $f(\partial V) \cap \partial B_j = \emptyset$. Thus, each component of $f(V) \cap B_j$ is closed in B_j. The second part of Proposition 2.2 implies that $\pi_v : f(V) \cap B_j \to D_j$ is proper. Finally, we need to show that each component W of $f(V) \cap B_j$ is conformally equivalent to the disk. Since V is a disk, there is a conformal equivalence $\varphi : \Delta \to V \subset \mathbb{C}^2$. Now the components of $fV \cap B_j$ correspond to the components of $\{\zeta \in \Delta : f \circ \varphi(\zeta) \in B_j\} = \{\zeta \in \Delta : \pi_v \circ f \circ \varphi(\zeta) \in D_j\}$. Since $\bar{D}_j$ is simply connected, there is a subharmonic function s on $\mathbb{C}$ such that $\bar{D}_j = \{s \leq 0\}$. It follows from the maximum principle that each component of $(\pi_v \circ f \circ \varphi)^{-1}(\bar{D}_j) = \{s \circ f \circ \varphi \leq 0\}$ is simply connected. Finally, since $\pi_v \circ f \circ \varphi$ is an open mapping, each component of $(\pi_v \circ f \circ \varphi)^{-1} D_j$ is simply connected. □

The crossed mapping $f_{0,0}$ from B_0 to itself will be seen in §3 to have degree one, so it follows from [**HO**] that there is a saddle fixed point $p_0 \in B_0$. Let us define $W_0^{s/u}$ to be the connected component of $W^{s/u}(p_0) \cap B_0$, which contains p_0. It follows that $W_0^{s/u} \in \mathcal{D}_{v/h}^1(B_0)$. Note that

$$W_0^u = \bigcap_{n \geq 0} f^n B_0, \quad W_0^s = \bigcap_{n \geq 0} f^{-n} B_0. \tag{2.7}$$

The maps $f_{1,2}$ and $f_{2,1}$ defined crossed mappings of $B_1 \cap B_2$ to itself. In §3, they will be seen to have degree one. Thus by [**HO**] there is a saddle point $p_1 \in B_1 \cap B_2$. We let W_1^u denote the component of $W^u(p_1) \cap B_1$ that contains p_1. We show in Proposition 4.3 that if $(a, b) \in \mathcal{A}$, then it is a horizontal disk of degree one.

For admissible sequences I (for $\mathcal{G}$) and J (for $\mathcal{G}^{-1}$), we use the notation

$$W_I^u = W_{i_0 i_1 \cdots i_n}^u = \tilde{f}_{i_{n-1} i_n} \tilde{f}_{i_{n-2} i_{n-1}} \cdots \tilde{f}_{i_0 i_1}(W_{i_0}^u) \tag{2.8}$$

$$W_J^s = W_{j_0 j_1 \cdots j_n}^s = \tilde{f}_{j_{n-1} j_n}^{-1} \tilde{f}_{j_{n-2} j_{n-1}}^{-1} \cdots \tilde{f}_{j_0 j_1}^{-1}(W_{j_0}^s). \tag{2.9}$$

It follows that $W_0^{s/u}$ are vertical/horizontal disks of degree one in B_0, and W_{02}^s is a vertical disk of degree one in B_2. By Proposition 2.6, W_{01}^u are vertical/horizontal disks of degree one; and W_{012}^u is a horizontal disk of degree two. This last statement includes two possibilities: W_{012}^u might consist of two disjoint disks of degree one or one disk on which π_v has degree two. In either case, W_{012}^u intersects W_{02}^s in B_2 with multiplicity two, which means that either $W_{012}^u \cap W_{02}^s$ consists of two distinct points, or the intersection is tangential.

3. *Mappings of real boxes*

Here we work under the additional condition that $f_{a,b}$ is a real mapping. In this section, we will restrict our attention to the real parameter region

$$\mathcal{A}_r := \mathcal{A} \cap \mathbb{R}^2.$$

Let τ be the involution of $\mathbb{C}^2$ defined by $\tau(x, y) = (\bar{x}, \bar{y})$. The fixed point set of τ is $\mathbb{R}^2$. The condition that $a, b \in \mathbb{R}$ is equivalent to the condition that $f_{a,b}$ commutes with τ. We say that a set $S \subset \mathbb{C}^2$ is *real* if $\tau S = S$. For instance $\tau B_i = B_i$, so in this terminology B_i is real. Let $\mathcal{D}_{h/v,r}(B_i)$ denote the set of horizontal/vertical disks in $\mathcal{D}_{h/v}(B_i)$ that are real. If $(a, b) \in \mathcal{A}_r$, then Proposition 2.6 applies to real disks to yield

$$\tilde{f}_{1,2} : \mathcal{D}^m_{h,r}(B_1) \to \mathcal{D}^{2m}_{h,r}(B_2) \quad \text{and} \quad \tilde{f}^{-1}_{2,1} : \mathcal{D}^m_{v,r}(B_2) \to \mathcal{D}^{2m}_{v,r}(B_1),$$

and

$$\tilde{f}_{i,j} : \mathcal{D}^m_{h,r}(B_i) \to \mathcal{D}^m_{h,r}(B_j) \quad \text{and} \quad \tilde{f}^{-1}_{j,i} : \mathcal{D}^m_{v,r}(B_j) \to \mathcal{D}^m_{v,r}(B_i)$$

for $(i, j) \in \mathcal{G}$, $(i, j) \neq (1, 2)$.

We set $B^r_i := B_i \cap \mathbb{R}^2$, which is a rectangle in $\mathbb{R}^2$ with sides parallel to the axes.

PROPOSITION 3.1. *If $V \in \mathcal{D}_{h,r}(B_i)$, then $V \cap B_{i,r}$ consists of a non-empty, connected, one-dimensional curve. In fact, there is a conformal uniformization $h : \Delta \to V$ such that $h(\bar{\zeta}) = \tau \circ h(\zeta)$.*

Proof. Let $\varphi : \Delta \to V$ be a conformal uniformization of V. It follows that $\kappa : \Delta \ni \zeta \mapsto \varphi^{-1} \circ \tau \circ \varphi(\zeta) \in \Delta$ is an anti-conformal involution of Δ. It follows that κ is an orientation-reversing isometry for the Poincaré metric, so the fixed point set $\gamma := \{\zeta \in \Delta : \kappa(\zeta) = \zeta\}$ is a Poincaré geodesic. Let ψ be a conformal automorphism of Δ that maps the real axis $(-1, 1) \subset \Delta$ to γ. It follows that $\psi^{-1} \circ \kappa \circ \psi$ is an isometric involution of Δ that fixes $(-1, 1)$, so it is simply the map $\zeta \mapsto \bar{\zeta}$. Thus, $h = \varphi \circ \psi$ is the desired uniformization. □

If f is a real map, then for $(i, j) \in \mathcal{G}$, $f_{i,j}$ is a crossed mapping of the pair (B^r_i, B^r_j).

PROPOSITION 3.2. *If $a, b \in \mathcal{A}_r$, then $B_{0,r} \cap f B_0$ lies below $B_{0,r} \cap f B_2$ inside $B_{0,r}$, and $B_{1,r} \cap f B_0$ lies below $B_{1,r} \cap f B_2$ inside $B_{1,r}$. In particular, let $I = 0 i_1 \cdots i_n 00$ and $J = 0 j_1 \cdots j_m 20$ be admissible sequences. Then W^u_I lies below W^u_J inside $B_{0,r}$. Similarly, if $K = 0 k_1 \cdots k_n 01$ and $L = 0 l_1 \cdots l_m 21$ are admissible sequences, then W^u_K lies below W^u_L inside $B_{1,r}$.*

Proof. The y-coordinate of f is $\pi_h \circ f = x$. Since B_0 lies to the left of B_2, it follows that the y-coordinate of $f B_{0,r}$ is less than that of $f B_{2,r}$. Thus, it lies below.

For the assertions about the pieces of unstable manifolds, we note that if I is a sequence that ends in *ij*, then $W^u_I \subset f B_i \cap B_j$. Thus, for a sequence I that ends in 00 and a sequence J that ends in 20, we will have $W^u_I \subset B_{0,r} \cap f B_0$, which lies below $W^u_J \subset B_{0,r} \cap f B_2$. □

If $(i, j) \in \mathcal{G}$, $(i, j) \neq (1, 2)$, then the crossed mapping $f_{i,j}$ has degree one. This means that real, horizontal curves in $B_{i,r}$ that run from left to right are taken to real, horizontal curves in $B_{j,r}$ that run either from left to right or from right to left. If the left-to-right direction is preserved, we assign the symbol $\epsilon_u = +$ to f. Otherwise, we set $\epsilon_u = -$. Similarly, real, vertical curves in $B_{j,r}$ that run from bottom to top are mapped under f^{-1} to real, vertical curves that either run from bottom to top or from top to bottom. If they run from bottom to top, then we assign the symbol $\epsilon_s = +$ to f^{-1}. Otherwise, $\epsilon_s = -$.

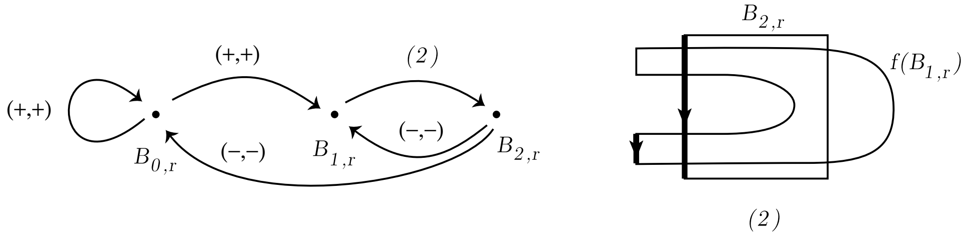

FIGURE 5. Graph induced by f (orientation-preserving).

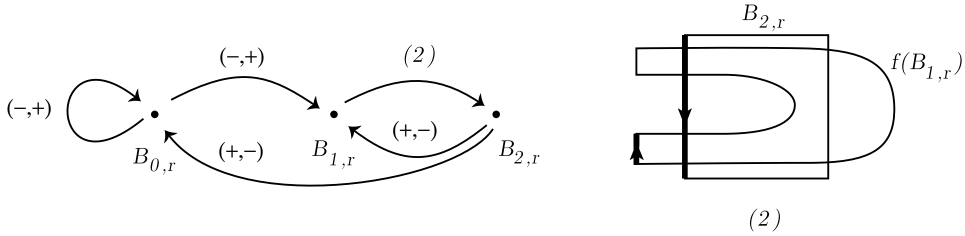

FIGURE 6. Graph induced by f (orientation-reversing).

PROPOSITION 3.3. *If $(a, b) \in \mathcal{A}_r$, then the signs (ϵ_s, ϵ_u) are given as in Figures 5 and 6.*

Proof. First we consider the degenerate case $b = 0$. The map $a - x^2 = \pi_v \circ f_{a,0}(x, y)$ is increasing on $D_0 \cap \mathbb{R} = (-d, 0)$ and decreasing on $D_2 \cap \mathbb{R} = (0, d)$. Thus, we have $\epsilon_u = +$ on $D_0 \cap \mathbb{R}$ and $\epsilon_u = -$ on $D_0 \cap \mathbb{R}$. This condition continues to hold for $b \neq 0$. Thus, we have $\epsilon_u = +$ on D_0 and $\epsilon_u = -$ on D_2. This continues to hold for $b \neq 0$, so the arrows of $\mathcal{G}$ emanating from B_0 should be labeled $(\cdot, +)$, and the arrows emanating from B_2 should be labeled $(\cdot, -)$. In the orientation-preserving case, the only possible labels are $(+, +)$ and $(-, -)$. In the orientation-reversing case, the only possible labels are $(+, -)$ and $(-, +)$. Thus we have the labeling shown in the graphs in Figures 5 and 6. □

The crossed map $f_{1,2}$ has degree two and is less easy to work with. The illustrations on the right-hand sides of Figures 5 and 6 indicate its combinatorial behavior in the following sense. The left-hand side of the vertical boundary of $B_{1,r}$ is $\{-c\} \times [-e, e]$, and the right-hand side is $\{c\} \times [-e, e]$. In the degenerate case $b = 0$, $f_{a,0}$ maps the left boundary to the point $(a - c^2, -c)$, which is to the left of $B_{2,r}$; and the right boundary goes to $(a - c^2, c)$, which is directly above $(a - c^2, -c)$. If $b \neq 0$, then the image of the left boundary will continue to be to the left of $B_{2,r}$ and below the image of the right boundary. The use we make of this combinatorial/topological information is given in Proposition 3.4, whose proof is a straightforward consequence of the preceding discussion.

PROPOSITION 3.4. *Suppose $(a, b) \in \mathcal{A}_r$. Suppose, too, that A_1 and A_2 are curves that cross $B_{1,r}$ from left to right and that A_1 lies below A_2 inside $B_{1,r}$. If f preserves orientation, then the curves $C_1 = \tilde{f}_{1,2}A_1$ and $C_2 = \tilde{f}_{1,2}A_2$ open to the left, and C_2 lies inside C_1 as illustrated in Figure 7. If f reverses orientation, then the relative positions of C_1 and C_2 are exchanged.*

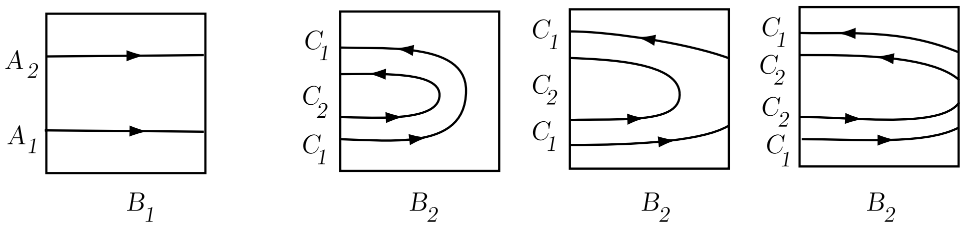

FIGURE 7. Curve C_2 lies inside C_1: three possibilities.

In the sequel, we will work with parameter values in $\mathcal{A}_r$. For these parameter values, the boxes in $\mathcal{B}$ are product sets $D_j \times \Delta$, and the non-empty intersections, i.e. $\bar{D}_0 \cap \bar{D}_1$ and $\bar{D}_1 \cap \bar{D}_2$, are topological disks. The essential properties of the partially-defined mappings $f_{i,j} : B_i \to B_j$ which we will use are:

$$(\mathcal{B}, \mathcal{G}) \text{ is a family of real, crossed mappings, with the topological configurations shown in Figures 5, 6, and 7.} \tag{†}$$

We may summarize the discussion above by the following statement: *If* $(a, b) \in \mathcal{A}_r$*, then* (†) *holds.* We also introduce the two conditions

$$\#(W^s_{02} \cap W^u_{012} \cap B_{2,r}) = 2 \quad \text{if } b > 0 \quad \text{and} \quad \#(W^s_{02} \cap W^u_{12} \cap B_{2,r}) = 2 \quad \text{if } b < 0, \quad (*)$$

$$\#(W^s_{02} \cap W^u_{01212} \cap B_{2,r}) = 4 \quad \text{if } b > 0 \quad \text{and} \quad \#(W^s_{02} \cap W^u_{12012} \cap B_{2,r}) = 4 \quad \text{if } b < 0, \quad (**)$$

where the intersections are counted with multiplicity.

Remark on notation. We have now defined a parameter domain $\mathcal{A}_r$ as well as three conditions that may or may not hold for a given parameter value (a, b). The condition (†) requires the boxes $\mathcal{B}$ to have specified behavior under f and f^{-1}. The conditions $(*)$ and $(**)$ define dynamical characteristics of $f_{a,b}$. It will be shown below that (†) holds for all parameters in $\mathcal{A}_r$ and that $(**)$ implies $(*)$.

PROPOSITION 3.5. *If* (†) *holds, then* $(**) \Rightarrow (*)$.

Proof. We only treat the case $b < 0$ since the case $b > 0$ is similar. Let us suppose that $(*)$ fails. We map W^u_1 forward under $\tilde{f}_{1,2}$ to W^u_{12}. By Proposition 3.1, $W^u_{12} \cap B_{2,r}$ is a non-empty, connected curve, and by Proposition 3.4 it forms a curve that opens to the left, which by hypothesis does not intersect W^s_{02}. This is pictured in the pair of boxes on the left-hand side of Figure 8. Next we map W^u_{12} forward under $\tilde{f}_{2,0}$. Again by Proposition 3.1, $W^u_{120} \cap B_{0,r}$ is a non-empty, connected curve. Since the sign of $f_{2,0}$ is $(\cdot, -)$, the x-direction of the curve is reversed, so $W^u_{120} \cap B_{0,r}$ opens to the right. By Proposition 3.2, W^u_{120} lies above $W^u_0 = W^u_{00}$, which is drawn in gray as a visual aid to the reader, although it is not necessary for the proof. (The gray dot is p_0, and W^u_1 is above W^u_{01} in $B_{0,r}$ by Proposition 3.2.) Since the sign of $f_{2,1}$ is $(+, \cdot)$, the vertical orientation is preserved, so W^u_{121} contains W^u_1 as well as a curve below it. Since the sign of $f_{2,0}$ is $(+, \cdot)$, the vertical orientation is preserved, so the upper part of W^u_{120} with a single hash mark is identified

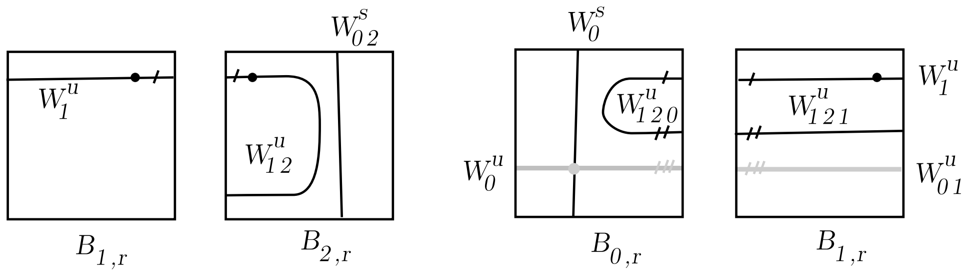

FIGURE 8.

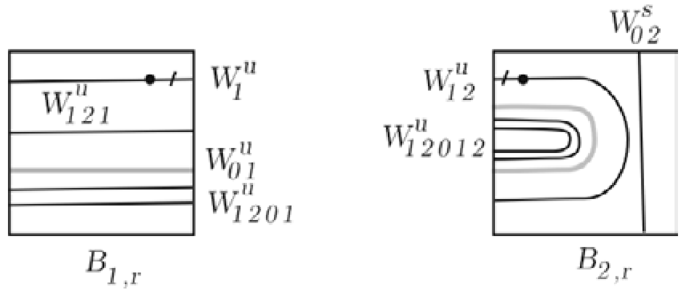

FIGURE 9.

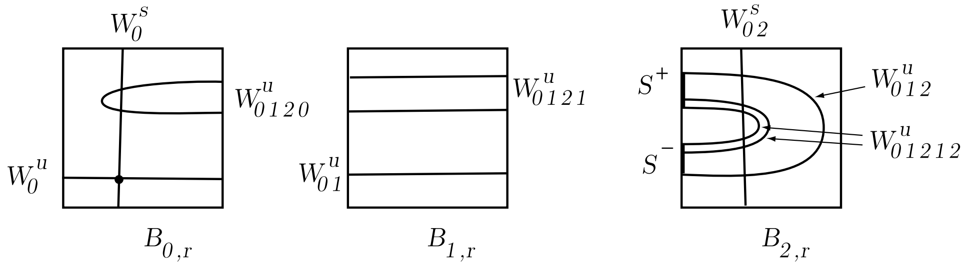

FIGURE 10. Moving W^u_0 forward along the sequence 01212 (case $b > 0$).

with W^u_1 on the set $B_{0,r} \cap B_{1,r}$. Since $W^s_{02} \cap W^u_{12} = \emptyset$, W^u_{120} is disjoint from W^s_0, so we obtain the picture as in the right-hand pair of boxes in Figure 8.

Next we map W^u_{120} forward under $\tilde{f}_{0,1}$. This is shown in the left-hand picture of Figure 9. Since $f_{0,1}$ has signature $(-,\cdot)$, the vertical orientation is reversed, so W^u_{1201} lies below W^u_{121} and W^u_{01} in $B_{1,r}$. Finally, we map forward under $\tilde{f}_{1,2}$ and obtain the picture in the right-hand box of Figure 9. The two arches of W^u_{12012} lie inside W^u_{12} by Proposition 3.4. Thus, W^u_{12012} cannot intersect W^s_{02}, so condition $(**)$ does not hold. □

Figure 10 illustrates conditions $(*)$ and $(**)$ in the case $b > 0$. To understand Figure 10, start in the left-hand box with W^u_0 and W^s_0 passing through the saddle point p_0. We move W^u_0 to box $B_{1,r}$ via the map $\tilde{f}_{0,1}$, and to box $B_{2,r}$ via $\tilde{f}_{0,2}$. The map $f_{1,2}$ has degree two, and $\tilde{f}_{1,2} W^u_{01} = W^u_{012}$ is a curve of degree two that opens to the left by Proposition 3.4.

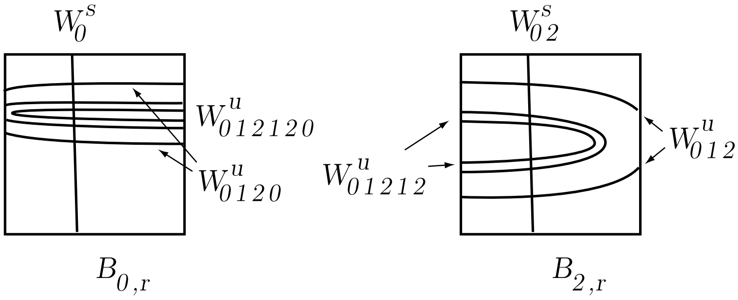

FIGURE 11. Alternative to Figure 10.

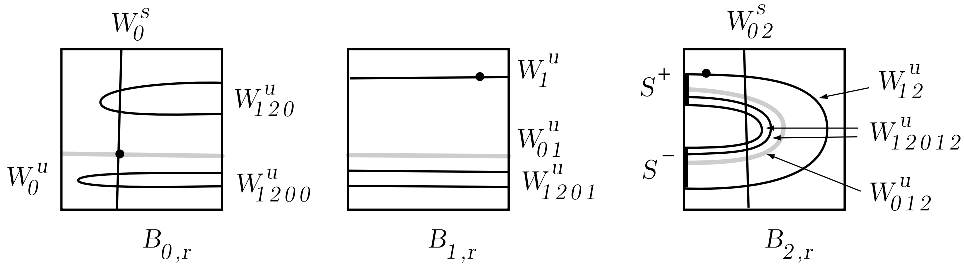

FIGURE 12. Moving W^u_1 forward along the sequences 1200, 1201, and 12012 (case $b < 0$).

By condition $(*)$, W^u_{012} crosses W^s_{02}. The crossed map $f_{2,0}$ has degree one and sign $(\cdot, -)$, so the left-opening, degree two curve W^u_{012} produces a degree two curve $W^u_{0120} = \tilde{f}_{2,0} W^u_{012}$ in $B_{0,r}$ that opens to the right. Condition $(*)$ maps forward under $f_{2,0}$, so W^u_{0120} intersects W^u_0 in two points.

The crossed map $f_{2,1}$ has degree one, so $\tilde{f}_{2,1} W^u_{012} = W^u_{0121}$ has degree two and by Proposition 3.2, it lies above W^u_{01}. Now $(\tilde{f}_{0,2} \cup \tilde{f}_{2,1})(W^u_{012})$ is a curve in $B_0 \cup B_1$ of degree two, and since $W^u_{0120} \cap B_{0,r}$ is connected, it follows that $W^u_{0121} \cap B_{1,r}$ consists of two curves of degree one. By Proposition 3.1, then it follows that the complex variety W^u_{0121} consists of two irreducible components. Now we map $W^u_{0121} \cap B_{1,r}$ under $\tilde{f}_{1,2}$, which has degree two. By Proposition 3.4, $W^u_{01212} = \tilde{f}_{1,2} W^u_{0121}$ lies inside W^u_{012}. By $(**)$, W^u_{01212} intersects W^s_{02}. Note that the arrangement of W^u_{01212} corresponds to one of the possibilities in Figure 7. Another possibility is given in the right-hand side of Figure 11. This picture is mapped forward under $\tilde{f}_{2,0}$, to show one possibility for W^u_{012120} inside $B_{0,r}$.

Figure 12 deals with the orientation-reversing case and shows various unstable pieces W^u_I starting with W^u_1 through p_1 and moving forward along the sequences $I = 1200$, 1201, and 12012. The construction of this picture was explained in large part in the proof of Proposition 3.5, so we do not repeat it here.

During the preceding discussion, we encountered a special case of the following proposition.

PROPOSITION 3.6. *Let Γ be a horizontal disk in B_1 which is real. If* $(\dagger)$ *and* $(*)$ *hold, then $\tilde{f}_{2,1} \tilde{f}_{1,2} \Gamma$ is the union of two horizontal disks in B_1.*

Proof. By Proposition 3.4, $\tilde{f}_{1,2}\Gamma \in \mathcal{D}^2_{h,r}(B_2)$ is a real disk of degree two such that $\tilde{f}_{1,2}\Gamma \cap \mathbb{R}^2$ opens to the left. Applying $(\tilde{f}_{2,0} \cup \tilde{f}_{2,1})$ to $\tilde{f}_{1,2}\Gamma$, we obtain a disk Γ' of degree two, which is horizontal in $B_0 \cup B_1$. There can be at most one critical point for the projection $\pi_v : \Gamma' \to B_0 \cup B_1$, and if there is a critical point, it must be real, since its conjugate is also a critical point.

Since the sign of $f_{2,0} \cup f_{2,1}$ is $(\cdot, -)$, $\Gamma' \cap \mathbb{R}^2$ opens to the right. By Proposition 3.1, $\tilde{f}_{2,0}\tilde{f}_{1,2}\Gamma = \Gamma' \cap B_0$ intersects $B_{0,r}$ in a non-empty real curve. Thus, if there is a critical point, then vertical projection $\pi_v : \Gamma \cap (B_{0,r} \cup B_{1,r}) \to (-d, c)$ has a critical point. Since $(*)$ holds, this critical point must belong to $B_{0,r}$, and since $(a, b) \in \mathcal{A}$, this point cannot belong to $B_{1,r}$. In particular, it follows that π_v has no critical point in $\tilde{f}_{2,1}\tilde{f}_{1,2}\Gamma = \Gamma' \cap B_1$. Thus, $\Gamma' \cap B_1$ consists of two components, which are horizontal disks in B_1. □

When $(**)$ holds, we use Figures 10 and 12 to define $S^{\pm}$ as the closed subintervals of the left-hand component of $\partial_v B_{2,r}$ which meet each component of $\bar{W}^u_{012} \cup \bar{W}^u_{01212}$ if $b > 0$ (respectively each component of $\bar{W}^u_{12} \cup \bar{W}^u_{12012}$ if $b < 0$).

PROPOSITION 3.7. *Suppose that $b > 0$ and that $(\dagger)$ and $(**)$ hold. Let I be an admissible sequence starting with 0 and ending with k, and let Γ be a connected component of W^u_I. Then we have the following.*

- *If $k = 0$, then Γ is disjoint from the component of $B_{0,r} - W^u_0$ lying below W^u_0. If $\delta(\Gamma) \neq 1$, then $\delta(\Gamma) = 2$, and Γ intersects $W^s_0 \cap B_{0,r}$, and $\bar{\Gamma}$ intersects the right-hand component of $\partial_v B_{0,r}$ in two points.*
- *If $k = 1$, then $\delta(\Gamma) = 1$, and Γ is disjoint from the topmost and bottommost components of $B_{1,r} - (W^u_{01} \cup W^u_{0121})$.*
- *If $k = 2$, then Γ is disjoint from the innermost and outermost components of $B_{2,r} - (W^u_{012} \cup W^u_{01212})$. If $\delta(\Gamma) \neq 1$, then $\delta(\Gamma) = 2$, and $\bar{\Gamma}$ intersects both S^+ and S^-.*

Proof. The proof proceeds by induction on the length of the sequence I. First, the case $I = 0$ is clear. Now we suppose that the proposition holds for $I = I'i$. We will show that if $(i, j) \in \mathcal{G}$, then the proposition holds for $I = I'ij$ by considering five cases.

Case $(i, j) = (0, 0)$. Since $f_{0,0}$ has sign $(+, \cdot)$, $f_{0,0}$ maps the component of $B_{0,r} - W^u_0$ above W^u_0 to itself. So $\tilde{f}_{0,0}\Gamma$ is disjoint from the component of $B_{0,r} - W^u_0$ below W^u_0. Now suppose that $\delta(\Gamma) = 2$. $f_{0,0}$ maps W^s_0 into itself, and the sign of $f_{0,0}$ is $(\cdot, +)$, so $\tilde{f}_{0,0}\Gamma$ intersects $W^s_0 \cap B_{0,r}$, and $\bar{\Gamma}$ intersects the right-hand component of $\partial_v B_{0,r}$ in two points.

Case $(i, j) = (0, 1)$. By Proposition 3.2, $f_{0,1}(B_{0,r})$ lies below W^u_{0121}. On the other hand Γ is above W^u_0 and $f_{0,1}$ has sign $(+, \cdot)$, so $\tilde{f}_{0,1}\Gamma$ is above W^u_{01} in $B_{1,r}$. It remains to show that $\tilde{f}_{0,1}\Gamma$ consists of two components of degree one. For this, we may assume that $\delta(\Gamma) = 2$, and $\Gamma \cap W^s_0 \cap B_{0,r} \neq \emptyset$. Consider how $\gamma' = (\tilde{f}_{0,0} \cup \tilde{f}_{0,1})(B_{0,r} \cap \Gamma)$ maps across $B_{0,r} \cup B_{1,r}$: the left-hand side of γ' intersects $W^s_0 \cap B_{0,r}$ and the right-hand side goes across the right-hand boundary of $\partial_v B_{1,r}$. Thus, $\gamma' \cap B_{1,r}$ consists of two curves. By Proposition 3.1, $\tilde{f}_{0,1}\Gamma \cap B_{0,1} = \gamma' \cap B_1$ consists of two disks of degree one.

Case $(i, j) = (1, 2)$. This is a direct consequence of Proposition 3.4.

Cases $(i,j) = (2,1)$ *and* $(i,j) = (2,0)$. Let Γ be as in case $k = 2$. We may assume that $\delta(\Gamma) = 2$. Since $f_{2,0}$ and $f_{2,1}$ have sign $(\cdot, -)$, it follows that $\gamma' := (\tilde{f}_{2,0} \cup \tilde{f}_{2,1})\Gamma \cap (B_{0,r} \cup B_{1,r})$ is a 2-fold curve opening to the right. Since $\bar{\Gamma}$ intersects both S^+ and S^-, we have $\Gamma \cap W^s_{02} \cap B_{2,r} \neq \emptyset$, and it follows that $\gamma' \cap W^s_0 \cap B_{0,r} \neq \emptyset$. By Proposition 3.2, γ' lies above W^u_0. This finishes the case $ij = 20$.

For the case $(i,j) = (2,1)$, we observe that by Proposition 3.6 $\tilde{f}_{2,1}\Gamma$ consists of two components of degree one. □

In the following, we let $B^+_{0,r}$ denote the right-hand component of $B_{0,r} - W^s_0$.

PROPOSITION 3.8. *Suppose that* $b < 0$, *and that* $(\dagger)$ *and* $(**)$ *hold. Let* I *be an admissible sequence starting with 1 and ending with* k, *and let* Γ *be a connected component of* W^u_I. *Then we have the following.*

- *If* $k = 0$, *then* Γ *is disjoint from the topmost and bottommost components of* $B^+_{0,r} - (W^u_{120} \cup W^u_{1200})$. *If* $\delta(\Gamma) \neq 1$, *then* $\delta(\Gamma) = 2$, *and* Γ *intersects* $W^s_0 \cap B_{0,r}$, *and* $\bar{\Gamma}$ *intersects the right-hand component of* $\partial_v B_{0,r}$ *in two points.*
- *If* $k = 1$, *then* $\delta(\Gamma) = 1$, *and* Γ *is disjoint from the topmost and bottommost components of* $B_{1,r} - (W^u_1 \cup W^u_{1201})$.
- *If* $k = 2$, *then* Γ *is disjoint from the innermost and outermost components of* $B_{2,r} - (W^u_{12} \cup W^u_{12012})$. *If* $\delta(\gamma) \neq 1$, *then* $\delta(\Gamma) = 2$, *and* $\bar{\Gamma}$ *intersects both* S^+ *and* S^-.

Proof. This proof is analogous to the proof of Proposition 3.6; we omit the details. □

PROPOSITION 3.9. *Suppose that* $(\dagger)$ *and* $(**)$ *hold. Let* I *be an admissible sequence of the form* $I = 0i_1 \cdots i_n 2$ *if* $b > 0$ *or* $I = 1i_1 \cdots i_n 2$ *if* $b < 0$. *Then for each component* Γ *of* W^u_I, *we have* $\#(W^s_{02} \cap \Gamma \cap B_{2,r}) = \delta(\Gamma)$. *In particular, if the intersection in the definition of* $(**)$ *is not tangential, then there is no tangency between* W^s_0 *and* W^u_I.

Proof. This follows from the case $k = 2$ in Propositions 3.7 and 3.8. The only case to consider is $\delta(\Gamma) = 2$. Now if Γ is not one of the curves W^u_I in condition $(**)$, $\Gamma \cap B_{2,r}$ is trapped between an inner and an outer curve. Since its closure intersects both S^+ and S^-, it must cross W^s_{02} at least twice. These two intersections account for the total intersection number, and so these intersections must be simple (non-tangential), and there can be no further intersections. □

PROPOSITION 3.10. *Suppose that* $(\dagger)$ *and* $(**)$ *hold. Let* I *be an admissible sequence of the form* $I = 0i_1 \cdots i_n 0$ *if* $b > 0$ *or* $I = 1i_1 \cdots i_n 0$ *if* $b < 0$. *Then for each component* Γ *of* W^u_I, $\#(W^s_0 \cap \Gamma \cap B_{0,r}) = \delta(\Gamma)$. *In particular, if the intersection in the definition of* $(**)$ *is not tangential, then there is no tangency between* W^s_0 *and* W^u_I.

Proof. This follows by applying the map $\tilde{f}_{2,0}$, which has degree one, to the result of Proposition 3.9. □

In particular, we see from Proposition 3.10 that all intersections are real. This allows us to characterize the mappings of maximal entropy.

THEOREM 3.11. *Suppose that* $(\dagger)$ *holds. If the real map* $f_{a,b}$ *has entropy equal to* $\log 2$, *then* $(**)$ *holds. Conversely, if* $S \subset \{(a,b) \in \mathbb{R}^2 : b \neq 0\}$ *is a connected set such that* $(**)$

holds for all $(a, b) \in S$*, and if* f_{a_0,b_0} *has entropy* $\log 2$ *for some* $(a_0, b_0) \in S$*, then* $f_{a,b}$ *has entropy* $\log 2$ *for all* $(a, b) \in S$*.*

Proof. The proof will be based on the following criterion from [**BLS**]: $f_{a,b}$ has (maximal) entropy $\log 2$ if and only if for all saddle points p and q, all (complex) intersection points of $W^s(p) \cap W^u(q)$ belong to $\mathbb{R}^2$.

We suppose first that the entropy of $f_{a,b}$ is $\log 2$. If $b > 0$, we take $p = q = p_0$. By (2.6), $\delta(W^s_{02}) = 1$ and $\delta(W^u_{01212}) = 4$. If $b < 0$, we take $p = p_0$ and $q = p_1$. Again by (2.6), we have $\delta(W^u_{12012}) = 4$. By (2.5) we have $\#(W^s_{02} \cap \Gamma) = \delta(W^s_{02})\delta(\Gamma) = 4$ with $\Gamma = W^u_{01212}$ if $b > 0$ and $\Gamma = W^u_{12012}$ if $b < 0$. By the criterion above, all (complex) intersections between W^s_{02} and Γ must belong to $\mathbb{R}^2$, so it follows that $(**)$ holds.

Now let us suppose that $(**)$ holds for all $(a, b) \in S$. Consider the subset S_0 of points $(a, b) \in S$ such that the entropy of $f_{a,b}$ is equal to $\log 2$. Since $(a, b) \mapsto \text{entropy}(f_{a,b})$ is continuous (see [**N**]), it follows that S_0 is a closed subset of S. Since S is connected, it suffices to show that S_0 is an open subset of S. Let us fix a point $(a_0, b_0) \in S_0$. By Proposition 2.5 there is an open set $U_0 \subset \mathbb{C}^2$ such that $\bar{B}_{0,r} \cup \bar{B}_{1,r} \cup \bar{B}_{2,r} \subset U_0$, and

$$U_0 \cap f_{a,b}(B_0 \cup B_1 \cup B_2) \subset f_{a,b}(B_0 \cup B_2)$$
$$U_0 \cap f_{a,b}(B_0 \cup B_1 \cup B_2) \subset f_{a,b}(B_1)$$

holds for $(a, b) = (a_0, b_0)$. Thus, it holds for (a, b) in a small neighborhood of (a_0, b_0). Thus, we also have that $K_{a_0,b_0} \subset \mathbb{R}^2$ since f_{a_0,b_0} has maximal entropy. By Proposition 2.1, then $K_{a_0,b_0} \subset B_{0,r} \cup B_{1,r} \cup B_{2,r} \subset U_0$. Since $(a, b) \mapsto K_{a,b}$ is upper semicontinuous, it follows that for (a, b) sufficiently close to (a_0, b_0) we have $K_{a,b} \subset U_0$, and thus $f_{a,b}$ satisfies the hypotheses of Proposition 2.4.

Now we consider the case $b > 0$; the argument for the case $b < 0$ is similar and is omitted. Let $q \in W^s(p_0) \cap W^u(p_0)$ be any point of intersection. Replacing q by $f^{-m}q$ if necessary, we may assume that $q \in B_0$. Let I denote the admissible sequence given by Proposition 2.4. For n sufficiently large, we have $f^n q \in W^s_0$, which is a neighborhood of p_0 inside $W^s(p_0)$. Thus, writing $I(n) := i_0 i_1 \cdots i_n$, we have $f^n q \in W^u_{I(n)}$. By Proposition 3.10, it follows that $W^s_0 \cap W^u_{I(n)} \subset \mathbb{R}^2$. Since $f^n q \in W^s_0 \cap W^u_{I(n)}$, it follows that $q \in \mathbb{R}^2$. Thus, $W^s(p_0) \cap W^u(p_0) \subset \mathbb{R}^2$, so that $f_{a,b}$ has entropy equal to $\log 2$. □

Remark. There is an alternative approach to the 'conversely' part of this theorem. Namely, we could use the arguments of this section to show that W^s_{02} and W^u_I have certain trellis properties, and then we can apply the work of Collins [**C**] to conclude that the real map f has entropy $\log 2$.

4. *The quadratic horseshoe locus*

In this section we analyze the real, maximal entropy bifurcations in a neighborhood of $(2, 0)$.

LEMMA 4.1. *Suppose that* $(a, b) \in \mathcal{A}$*, and suppose that* $\Delta/3 \le \delta \le e^2 - 4 - 2\Delta$*. If we define* B'_0 *and* B'_2 *by*

$$B'_0 := \{|x + 2| < \delta, |y| < e\}, \quad B'_2 := \{|x - 2| < \delta, |y| < e\}$$

then f induces crossed mappings from B_0' to itself and from B_0' to B_2'. In particular, the sets W_0^s and W_{02}^s (as in (2.10)) are given by

$$W_0^s = \bigcap_{n\geq 0} f^{-n} B_0' \quad \textit{and} \quad W_{02}^s = \bigcap_{n\geq 1} B_2' \cap f^{-n} B_0'.$$

Proof. Let us fix δ such that $\Delta/3 \leq \delta \leq e^2 - 4 - 2\Delta$ and set $B_0' := \{|x+2| < \delta, |y| < e\}$. By the upper bound on δ, we have $\{|4 - x^2| < \Delta + \delta\} \subset \{|x| < e\}$. We compute

$$\begin{aligned} f^{-1}B_0' \cap \{|y| < e\} &= \{|2 + \pi_v f(x, y)| < \delta, |\pi_h f(x, y)| < e, |y| < e\} \\ &\subset \{|2 + a - x^2 - by| < \delta, |x| < e, |y| < e\} \\ &\subset \{|4 - x^2| < |a - 2| + |by| + \delta, |x| < e\} \\ &\subset \{|4 - x^2| < \Delta + \delta, |x| < e\} \\ &\subset \{|2 - x| < \sqrt{4 + \Delta + \delta} - 2\} \cup \{|2 + x| < \sqrt{4 + \Delta + \delta} - 2\} \\ &\subset \left\{|2 - x| < \frac{\Delta + \delta}{4}\right\} \cup \left\{|2 + x| < \frac{\Delta + \delta}{4}\right\}. \end{aligned}$$

In the next to last line we have removed the condition $|x| < e$ by the upper bound condition on δ. The last line uses the concavity of the square root. By the lower bound on δ, we have $(\Delta + \delta)/4 < \delta$, so it follows that $f^{-1}\bar{B}_0' \cap \partial_v B_0' = \emptyset$.

Next we consider a point $(x', y') \in f^{-1}(\partial_h B_0')$. By (2.4),

$$\begin{aligned} |y'| = \left|\frac{1}{b}(a - y^2 - x)\right| &> \frac{e}{\Delta}(|y|^2 - 4 - |a - 2| - |x + 2|) \\ &> \frac{e}{\Delta}(e^2 - 4 - \Delta - \delta). \end{aligned}$$

This last quantity is greater than e by the upper bound on δ, so $(x', y') \notin \bar{B}_0'$. Thus, f induces a crossed mapping from B_0' to itself. The proof that f induces a crossed mapping from B_2' to B_0' is the same □

COROLLARY 4.2. *If $(a, b) \in \mathcal{A}$, then $(*)$ holds.*

PROPOSITION 4.3. *If $(a, b) \in \mathcal{A}$, then the horizontal disk W_1^u has degree one.*

Proof. Let $\Gamma \in \mathcal{D}^1_{h,r}(B_1)$ be any real, horizontal disk passing through the saddle point $p_1 \in B_1$. By Proposition 3.6, $\tilde{f}_{2,1}\tilde{f}_{1,2}\Gamma$ consists of two horizontal disks of degree one. One of these disks, which we will denote $(\tilde{f}_{2,1}\tilde{f}_{1,2})^{\#}\Gamma$, will contain p_1. It follows that $(\tilde{f}_{2,1}\tilde{f}_{1,2})^{\#n}\Gamma$ is a sequence of horizontal disks of degree one, passing through p_1, which converge to W_1^u as $n \to \infty$. □

Now let us examine the case $b = 0$. The image of $f_{a,0}$ is the parabola

$$\Gamma := f_{a,0}(\mathbb{C}^2) = \{x = a - y^2\} = \{(p(t), t) : t \in \mathbb{C}\},$$

where $p(z) = a - z^2$. Throughout our discussion, we assume that $|a - 2| < \Delta$. Thus, $a \notin D_0 \cup D_1$, and so there are two holomorphic branches of $p^{-1}(z) = \pm\sqrt{a - z}$ over $D_0 \cup D_1$. For $j = 0, 1$, $\Gamma \cap B_j$ consists of two components Γ_j' and Γ_j'', each of which is a smooth graph of a branch of p^{-1}. We note that $f_{a,0}$ is injective on each component Γ'_j and Γ''_j, $j = 0, 1$. On the other hand, $\Gamma \cap B_2$ is connected, and $f_{a,0}$ is two-to-one on $\Gamma \cap B_2$.

Let $p_0 = (t_0, t_0)$ denote the fixed point which belongs to B_0. (The following discussion can be adapted to work with the other fixed point $p_1 \in B_1 \cap B_2$, as well.) Let $\varphi_a : \mathbb{C} \to \mathbb{C}$ denote the linearizing coordinate such that $\varphi_a(0) = t_0$, $\varphi'_a(0) = 1$, and $p(\varphi_a(\zeta)) = \varphi_a(\lambda\zeta)$, where $\lambda := p'(t_0)$. If we write $\varphi = \varphi_a$, it follows that

$$\psi_{a,0}(\zeta) := (\varphi(\zeta), \varphi(\lambda^{-1}\zeta))$$

defines a mapping $\psi_{a,0} : \mathbb{C} \to \Gamma$ which satisfies $f_{a,0} \circ \psi(\zeta) = \psi(\lambda\zeta)$.

We wish to define the sets W^u_I in the case $b = 0$. We let W^u_0 be the connected component of $\Gamma \cap B_0$ containing p_0; W^u_{01} is the connected component of $\Gamma \cap B_1$ which intersects W^u_0; and $W^u_{012} = \Gamma \cap B_2$. As we try to consider longer I, we run into the difficulty that the mappings $\tilde{f}_{i,j}$ are not invertible. To deal with this, we identify W^u_I in terms of the parameterization $\psi_{a,0}$ of Γ. To do this, let $\Omega_0 \subset \mathbb{C}$ be the connected component of $\psi^{-1}_{a,0}(W^u_0) = \varphi^{-1}(D_0)$ that contains the origin. In general, we set

$$\Omega_I := \lambda^n\Omega_0 \cap \varphi^{-n} D_{i_1} \cap \cdots \cap \varphi^{-n}(D_{i_n}) = \lambda^n\Omega_0 \cap \psi^{-n}_{a,0} B_{i_1} \cap \cdots \cap \psi^{-n}_{a,0}(B_{i_n}), \quad (4.1)$$

where $I = 0i_1 \cdots i_n$ is an admissible sequence. We then identify W^u_I in terms of the map $\psi_{a,0} : \Omega_I \to W^u_I$.

The usefulness of the case $b = 0$ is that it is the limit of the case $b \neq 0$. When $b \neq 0$, we let $\psi_{a,b} : \mathbb{C} \to W^u_{p_0}$ be the uniformization of $W^u(p_0)$, normalized by the condition $(\pi_v \circ \psi_{a,b})'(0) = 1$. In this case, $(a, b, \zeta) \mapsto \psi_{a,b}(\zeta)$ is holomorphic, and we have

$$\lim_{b\to 0} \psi_{a,b} = \psi_{a,0}, \quad (4.2)$$

with uniform convergence on compact subsets. Restricting this to the image of Ω_I, we have

$$\lim_{b\to 0} W^u_I(f_{a,b}) = W^u_I(f_{a,0}), \quad (4.3)$$

where the convergence is in the sense of the Hausdorff topology. Taking multiplicities of $W^{s/u}_I(f_{a,0})$ into account, the convergence also holds in the sense of currents.

LEMMA 4.4. *If $|a - 2| < \Delta$, then for $I = 01212$ and $I = 12012$, Ω_I consists of two connected components with disjoint closures. If $b \neq 0$ is sufficiently small, then W^u_I consists of two components.*

Proof. Since $p : D_0 \to p(D_0)$ is a conformal equivalence, and $p^{-1}D_0 \subset D_0$, we may define a holomorphic map $\lim_{n\to\infty} \lambda^n p^{-n} : p(D_0) \to \mathbb{C}$. This is the inverse of φ, and so $\varphi : \lambda\Omega_0 \to p(D_0)$ is univalent. Thus, $\Omega_{01} = \Omega_0 \cap \varphi^{-1}(D_1)$ is connected and relatively compact in Ω_0. Let c_{01} be the unique point of $\lambda\Omega_0$ such that $\varphi(c_{01}) = 0$. It follows that $\varphi'(\lambda c_{01}) = (p \circ \varphi(c_{01}))' = p'(0)\varphi'(c_0) = 0$. Conversely, if $\zeta \in \lambda^2\Omega_0$, and if $0 = \varphi'(\zeta) = p'(\varphi(\lambda^{-1}\zeta))\varphi'(\lambda^{-1}\zeta)\lambda^{-1}$, then we must have $p'(\varphi(\lambda^{-1}\zeta)) = 0$ since $\varphi' \neq 0$ on $\lambda\Omega_0$. It follows that $\zeta = \lambda c_{01}$, so λc_{01} is the unique critical point in $\lambda^2\Omega_0$. It follows that $\psi_{a,0}(\zeta) = (\varphi(\zeta), \varphi(\lambda^{-1}\zeta))$ has no critical point on $\lambda^2\Omega_0$. Since $\psi_{a,0}(\Omega_{012}) = \Gamma \cap B_2$ is simply connected, it follows that $\psi_{a,0} : \Omega_{012} \to \Gamma \cap B_2$ is univalent. By the argument above, $\lambda^2 c_{01}$ is the unique critical point for $\psi_{a,0}$ in $\lambda^3\Omega_0$.

Now $\psi_{a,0}(\lambda^2 c_{01}) = f_{a,0}(a,0) = (a - a^2, a)$, which does not belong to B_1, since $\Re(a - a^2) < -c$. It follows that $\psi_{a,0}$ is unbranched on the closure of $\lambda^3\Omega_0 \cap \psi_{a,0}^{-1}(B_1)$. Recall that $f_{a,0} : W^u_{012} = \Gamma \cap B_2 \to f_{a,0}(W^u_{012})$ is a mapping of degree two. Thus, W^u_{0121} is the component of $\Gamma \cap B_1$ which is disjoint from W^u_{01}, and W^u_{0121} has multiplicity two. It follows that $\psi_{a,0} : \Omega_{0121} \to W^u_{0121}$ is a covering of degree two. Since $\psi_{a,0}$ is unbranched on the closure of $\Omega_{0121} \subset \lambda^3 \cap \psi_{a,0}^{-1}(B_1)$, it follows that Ω_{0121} consists of two components with disjoint closures.

Let us move forward one more step: since $f_{a,0}$ is injective on W^u_{0121}, it follows that $\psi_{a,0}$ gives a conformal equivalence between each component of $\lambda\Omega_{0121}$ and $f_{a,0}W^u_{0121}$. Intersecting $\lambda\Omega_{0121}$ with $\psi_{a,0}^{-1}(B_2 \cap f_{a,0}(W^u_{0121})) = \psi_{a,0}^{-1}(B_2 \cap \Gamma) = \varphi^{-1}(D_2)$, then Ω_I consists of two components Ω_I' and Ω_I'' that have disjoint closures. If $b \neq 0$ is sufficiently small, then $\psi_{a,b}^{-1}(W^u_I)$ will be close to Ω_I. Thus, it (as well as W^u_I) has two components. □

Now we pass from unstable manifolds to stable manifolds. The vertical complex line through the fixed point p_0 is mapped to p_0 under $f_{a,0}$. If we write $p_0 = (t_0, t_0)$, then

$$W^s_0 = \{(x,y) : x = t_0, |y| < e\} \quad \text{and} \quad W^s_{02} = \{(x,y) : x = t_0', |y| < e\},$$

where $t_0' \in \mathbb{C}$ is the solution to $p(t_0') = t_0$ such that $t_0' \neq t_0$. If (‡) holds, then

$$W^s_{02} \cap \Gamma = \{(\zeta, \pm\sqrt{a - \zeta}) : \zeta = t_0'\}. \tag{4.4}$$

This intersection consists of two distinct points unless $t_0' = a$, which happens exactly when $a = 2$. We can work our way backwards, taking successive preimages, to define $W^s_J(f_{a,0})$ for an admissible sequence J. As in the case of unstable manifolds, we have

$$\lim_{b \to 0} W^s_J(f_{a,b}) = W^s_J(f_{a,0}). \tag{4.5}$$

PROPOSITION 4.5. *Suppose that $(a,b) \in \mathcal{A}$ and $|a-2| \geq (e+\Delta)|b|$. Then for $I = 01212$ and $I = 12012$, W^s_{02} intersects W^u_I in four distinct points, and thus the intersection is not tangential.*

Proof. We begin by noting

$$W^u_I \subset B_2 \cap f B_0 \subset \{|b^{-1}(a - x - y^2)| < \delta, |y| < e\}.$$

If we set $\delta = \Delta/3$, then by Lemma 3.5 we have

$$W^s_{02} \subset B_2' \subset \left\{|x - 2| < \frac{\Delta + \delta}{4}, |y| < e\right\}.$$

Thus,

$$\begin{aligned} W^s_{02} \cap W^u_I &\subset \left\{|x-2| < \frac{\Delta+\delta}{4}, |a - x - y^2| < \delta|b|\right\} \\ &\subset \left\{|x-2| < \frac{\Delta+\delta}{4}, |a - 2 - y^2| < |b|\delta + |x - 2|\right\} \\ &\subset \left\{|a - 2 - y^2| < |b|\delta + \frac{\Delta+\delta}{4}\right\} \\ &=: U_{a,b}. \end{aligned}$$

The set $U_{a,b}$ is symmetric with respect to $y \mapsto -y$ and is seen to be disconnected if (and only if) it does not contain $y = 0$. This occurs exactly when $|a-2| \geq |b|\delta + (\Delta+\delta)/4$. Now we recall that $\delta = \Delta/3$ and substitute the condition (‡), which gives $|a-2| \geq |b|\Delta/3 + \Delta/3 \geq |b|\Delta/3 + (|a-2| + e|b|)/3$, and this is equivalent to $|a-2| \geq (\Delta+e)|b|$.

Now consider the case $b = 0$. By Lemma 4.4, Ω_I consists of components Ω'_I and Ω''_I. Since $a \neq 2$, the intersection (4.4) contains two points, which lie in different components of $U_{a,b}$. Thus, $\psi_{a,0}(\Omega'_I)$ and $\psi_{a,0}(\Omega''_I)$ each intersect W^s_{02} in two points, which lie in different components of $U_{a,0}$. If $b \neq 0$, $|a-2| \geq (e+\Delta)|b|$, then W^u_I consists of two components $(W^u_I)' = \psi_{a,b}(\Omega'_I(a,b))$ and $(W^u_I)'' = \psi_{a,b}(\Omega''_I(a,b))$. Further, the set $U_{a,b}$ continues to be disconnected, and by (4.3) each component of $U_{a,b}$ will continue to contain a point of $W^s_{02} \cap W^{u\prime}_I$. Since $\delta(W^u_I)' = 2$ and $W^s_{02} \cap (W^u_I)'$ contains two distinct points, the intersection is not tangential. A similar argument for $(W^u_I)'' \cap W^s_{02}$ shows that $W^s_{02} \cap W^u_I$ has no tangency. □

Let us define

$$\mathcal{D} := \{(a,b) \in \mathbb{C}^2 : |a-2| < 0.2, |b| < 0.06\}$$
$$T_I := \{(a,b) \in \mathcal{D} : W^s_{02} \text{ intersects } W^u_I \text{ tangentially}\}.$$

In the definition of T_I, we interpret the case $b = 0$ as follows. By §1, we know that $T_I \cap \{b \neq 0\}$ is a complex subvariety of $\mathcal{D} - \{b \neq 0\}$. By (4.5) and (4.2), we have that $(T_I \cap \{b \neq 0\}) \cup (2,0)$ is the closure of $T_I \cap \{b \neq 0\}$ in $\mathcal{D}$. With this interpretation, T_I is a complex subvariety of $\mathcal{D}$.

PROPOSITION 4.6. *For $I = 01212$ and 12012, T_I is a complex subvariety of $\mathcal{D}$ with the following properties.*
(i) *The projection $\pi_h : T_I \to \{|b| < 0.06\}$ is a proper mapping of degree two.*
(ii) *T_I is locally reducible at $(2,0)$.*
(iii) *There are real analytic functions $\kappa^{\pm}_I : [-0.06, 0.06] \to \mathbb{R}$ with $\kappa^-_I(t) < \kappa^+_I(t)$ for $t > 0$ such that $T_I \cap \mathbb{R}^2$ is the union of the graphs of κ^+_I and κ^-_I.*

Proof. Note that with our values of e and Δ, (‡) holds for $(a,b) \in \mathcal{D}$ whenever $b \neq 0$. Further, the condition $|a-2| \geq (e+\Delta)|b|$ holds for $(a,b) \in \partial_v\mathcal{D}$. By Proposition 4.5, then $\bar{T}_I \cap \partial_v\mathcal{D} = \emptyset$. Thus, π_h is a proper mapping. To determine the multiplicity of π_h, it suffices to determine the multiplicity at $b = 0$. If $b = 0$, then the only tangency occurs at $a = 2$. Now $W^u_I = \Gamma \cap B_2$, with multiplicity two, so in case $a = 0$, W^s_{02} makes a tangential intersection with each component of W^u_I. It follows that $T_I \cap \{b = 0\} = \{(2,0)\}$, with multiplicity two. Thus, π_h has multiplicity two.

For (ii), let $b = 0$. By Lemma 4.4, Ω_I consists of components Ω'_I and Ω''_I that have disjoint closures. Thus, for $b \neq 0$ small, there are domains $\Omega'_I(a,b)$ and $\Omega''_I(a,b)$ that are mapped under $\psi_{a,b}$ to the two components of W^u_I. Thus, for $|b| < r_0$ small, we may split T_I into $T'_I = \{(a,b) \in \mathcal{D} : |b| < r_0, W^s_{02} \text{ intersects} W^u_I(a,b)' \text{ tangentially}\}$, and a similar set T''_I for $W^u_I(a,b)''$.

Now we consider the projection $\pi_h : T_I \cap \mathbb{R}^2 \to (-0.06, 0.06)$. This is a proper mapping of degree two. Consider a point $(a,b) \in T_I \cap \mathbb{R}^2$ with $b < 0$ and suppose that $I = 12012$. We may repeat the argument of Proposition 3.5 to conclude that W^u_{12012}

consists of two curves in $B_{2,r}$ which open to the left. By γ' and γ'' we denote the components of W^u_{12012} such that $\gamma' \cap B_{2,r}$ forms the inner curve, and $\gamma'' \cap B_{2,r}$ forms the outer curve.

Let us note at the outset that $\delta(\gamma') = \delta(\gamma'') = 2$, and so $\#(W^s_{02} \cap \gamma') = \#(W^s_{02} \cap \gamma'') = 2$. If there is a tangency between γ' and W^s_{02}, then the tangency must be real. Otherwise, if there were a point of tangency $q \in B_2 - B_{2,r}$, the complex conjugate $\bar{q}$ would also be a point of tangency, so the total intersection of γ' and W^s_{02} in B_2 would be at least four.

Now suppose that the outer curve γ'' is tangential to W^s_{02}. Then this point of tangency must have order two, and can be the only intersection with W^s_{02} since the total intersection satisfies $\#(W^s_{02} \cap W^u_{12012}) = 2$. Since $\gamma'' \cap B_{2,r}$ opens to the left, it follows that $\gamma'' \cap B_{2,r}$ must lie to the left of W^s_{02}. Thus, γ' cannot intersect $W^s_{02} \cap B_{2,r}$. Thus, there can be no tangency between the complex disks W^s_{02} and γ'.

Thus, in the case $b \neq 0$, with a and b both real, there cannot be tangencies (necessarily real) between both components of W^u_I and W^s_{02}. In other words, if $(a, b) \in T'_I \cap \mathbb{R}^2$, $b \neq 0$, then $(a, b) \notin T''_I \cap \mathbb{R}^2$. This gives a splitting of T_I into two components in a neighborhood of $\pi_h^{-1}(-0.06, 0.06)$. Since π_h has degree one on $T'_I \cap \mathbb{R}^2$ and $T''_I \cap \mathbb{R}^2$ these sets are given as the graphs of real analytic functions. □

Let us set
$$\kappa(t) := \max(\kappa^+_{01212}(t), \kappa^-_{12012}(t)).$$

COROLLARY 4.7. *We have* $\{(a, b) \in \mathcal{D} \cap \mathbb{R}^2 : b \neq 0, (**)\ \textit{holds}\} = \{(a, b) \in \mathcal{D} \cap \mathbb{R}^2 : b \neq 0, a \geq \kappa(b)\}$.

Proof. We consider only the case $b > 0$; the other case is similar. For $I = 01212$, set $T^\pm_I := \{a = \kappa^\pm_{01212}(b)\}$. Thus, $T_I \cap \mathbb{R}^2 = T^+_I \cup T^-_I$. As was noted in the proof of Proposition 4.6, T^-_I is the set of parameters for which one component of W^u_I is tangent to W^s_{02}, and the other component is disjoint from W^s_{02}. T^+_I is the set of parameters for which one component of W^u_I is tangential to W^s_{02}, and the other component intersects W^s_{02} in two points. □

Let us write
$$\mathcal{E} := \{(a, b) \in \mathbb{R}^2 : f_{a,b} \text{ has entropy } < \log 2\}$$
$$\mathcal{H} := \{(a, b) \in \mathbb{R}^2 : f_{a,b} \text{ is a real horseshoe}\}.$$

THEOREM 4.8. *We have*
$$\mathcal{H} \cap \mathcal{D} = \{(a, b) \in \mathcal{D} \cap \mathbb{R}^2 : a > \kappa(b), b \neq 0\},$$
$$\mathcal{E} \cap \mathcal{D} = \{(a, b) \in \mathcal{D} \cap \mathbb{R}^2 : a < \kappa(b), b \neq 0\}.$$

Proof. By Theorem 3.11 and Corollary 4.7, the set of parameters $(a, b) \in \mathcal{D} \cap \mathbb{R}^2$ for which the entropy is $\log 2$ is exactly the set $\{a \geq \kappa(b)\}$. On the other hand, if $a > \kappa(b)$, then by Proposition 3.9 there is no tangency. Since f has maximal entropy, it follows from [**BS2**] that f is hyperbolic. Now $\mathcal{D} \cap \mathbb{R}^2 \cap \{a > \kappa(b)\}$ is a connected set of parameters for which $f_{a,b}$ is hyperbolic. By Theorem 1.1, this set contains parameters for which $f_{a,b}$ is a real horseshoe. It follows, then, from the structural stability of hyperbolic maps that all of these maps are horseshoes. □

5. *Generic unfolding*

In Theorem 5.2 we establish the 'generic unfolding' statement in Theorem 2. Let us fix $I = 01212$ or $I = 12012$. In §4 we saw that for $(a, b) \in \mathcal{D}$, $b \neq 0$, the set W_I^u is disconnected and may be split into

$$W_I^u(a, b) = W_I^u(a, b)' \cup W_I^u(a, b)''. \tag{5.1}$$

Further, we saw that if $(a_0, b_0) \in \mathcal{D} \cap \mathbb{R}^2 \cap \partial\mathcal{H}$, then one of these components, say $W_I^u(a_0, b_0)'$, has a quadratic tangency with $W_{02}^s(a_0, b_0)$. This splitting may be done for all $(a, b) \in \mathcal{D} \cap \mathbb{R}^2$ in such a way that we obtain a continuous family

$$\mathcal{D} \cap \mathbb{R}^2 \ni (a, b) \mapsto W_I^u(a, b)'.$$

The horizontal projection $\pi_h(x, y) = y$, establishes a conformal equivalence

$$\pi_h : W_{02}^s(a, b) \to \{|y| < e\}.$$

For $(a, b) \in \mathcal{D}$, $b \neq 0$, we define the function

$$h(a, b) = \prod_{i \neq j}(\pi_h(p_i) - \pi_h(p_j))$$

where the p_i and p_j in the product range over the four points of intersection $W_{02}^s(a, b) \cap W_I^u(a, b)$. Since $\pi_h|_{W_{02}^s(a,b)}$ is invertible, we see that $h(a, b) \neq 0$ if and only if there are four distinct points of intersection. Thus, $h(a, b) \neq 0$ means that the multiplicities of all four intersections are 1, and thus all four intersections are transverse. As in §4 we may extend the definition of h to the case $b = 0$, and we see that h is analytic in $\mathcal{D}$.

THEOREM 5.1. *For $(a, b) \in \mathcal{D} \cap \mathbb{R}^2 \cap T_I$ with $b \neq 0$, we have $\partial h/\partial a \neq 0$.*

Proof. If $b = 0$, then by the discussion in §4, we see that $a \mapsto h(a, 0)$ has a zero of order 2 at $a = 2$, and $h(a, 0) \neq 0$ for $\{0 < |a - 2| < 0.2\}$.

By Theorem 4.5, none of the tangencies T_I occur on the vertical boundary of $\mathcal{D}$. Thus, $h \neq 0$ there. It follows that for each fixed value $|b_0| \leq 0.06$, the function

$$\{|a - 2| < 0.2\} \ni a \mapsto h(a, b_0)$$

is analytic and has exactly two zeros (counted with multiplicity). One zero corresponds to a point $(a', b_0) \in T_I'$ and the other corresponds to $(a'', b_0) \in T_I''$. We have seen that $T_I' \cap \mathcal{D} \cap \mathbb{R}^2 \cap \{b \neq 0\}$ is disjoint from $T_I'' \cap \mathcal{D} \cap \mathbb{R}^2 \cap \{b \neq 0\}$. Since the total multiplicity is 4, each of these zeros must be a simple zero. In particular, we conclude that $(\partial h/\partial z)(a, b) \neq 0$ for $(a, b) \in T_I \cap \mathcal{D} \cap \mathbb{R}^2 \cap \{b \neq 0\}$. □

Let us discuss this situation further. We will consider a sequence of holomorphic coordinate changes $(x', y') = (x'(x, y), y'(x, y))$, which, in addition, depend holomorphically on the parameter (a, b). First, we may change coordinates so that $W_{02}^s(a, b) = \{x = 0\}$ since $W_{02}^s(a, b)$ has degree one in B_2. Now let us split $W_I^u(a, b)$ as in (5.1). We will show that we may introduce coordinates such that we have

$$W_{02}^s = \{x = 0\} \quad \text{and} \quad W_I^u(a, b) = \{x = c_0(a, b) + y^2\}. \tag{5.2}$$

The *generic unfolding* condition is that $\partial c_0(a,b)/\partial a \neq 0$ for $a = a_b$ (see [**PT**, p. 35]).

Now let us fix $b_0 \in (-0.06, 0.06)$, $b_0 \neq 0$, and set $a_0 = a_{b_0}$. Thus, we have

$$W^u_I(a_0, b_0)' = \left\{x = \sum_{j=2}^{\infty} c_j (y - y_0)^j\right\},$$

where $(0, y_0)$ is the point of tangential intersection, and so $c_0 = c_1 = 0$. The coefficient c_2 is non-zero because the intersection is quadratic (see [**BS2**]). Without loss of generality we may assume that $y_0 = 0$. Now for (a, b) near (a_0, b_0), we have

$$W^u_I(a, b)' = \{x = c_0(a, b) + c_1(a, b)y + c_2(a, b)y^2 + \cdots\}.$$

Now since $c_2(a, b) \neq 0$ and $c_0(a_0, b_0) = c_1(a_0, b_0) = 0$, we may solve $\tilde{y} = \tilde{y}(a, b) \sim -c_1/(2c_2)$ such that

$$\frac{\partial x}{\partial y} = c_1(a, b) + 2c_2(a, b)\tilde{y} + \cdots = 0.$$

Replacing y by $y - \tilde{y}$, we have

$$W^u_I(a, b)' = \{x = \tilde{c}_0(a, b) + \tilde{c}_2(a, b)y^2 + \cdots\}.$$

Finally, since $\tilde{c}_2 \neq 0$, we may change coordinates $y' = \sigma(a, b)y$ to obtain (5.2).

Now we consider the function $h(a, b)$ in the coordinates (x, y). We have $W^s_{02}(a, b) \cap W^u_I(a, b)' = \{(0, \pm\sqrt{-\tilde{c}_0(a, b)})\}$. Since $W^u_I(a, b)' \cap W^u_I(a, b)'' = \emptyset$, and $W^u_I(a, b)''$ has no tangency for (a, b) near (a_0, b_0), we have

$$h(a, b) = -(\sqrt{-\tilde{c}_0(a, b)} + \sqrt{-\tilde{c}_0(a, b)})^2\alpha(a, b) = 4\tilde{c}_0(a, b)\alpha(a, b)$$

where α is a non-vanishing analytic function. Since $\tilde{c}_0(a_0, b_0) = 0$, we have

$$\frac{\partial h}{\partial a}(a, b_0) = \frac{\partial \tilde{c}_0}{\partial a}(a, b_0) \cdot \alpha(a, b_0)$$

for $a = a_0$. By Theorem 5.1, then, $\partial\tilde{c}_0(a_0, b_0)/\partial a \neq 0$. Thus, we have the following.

THEOREM 5.2. *We have that $(a, b) \mapsto (W^s_{02}(a, b), W^u_I(a, b))$ is a generic unfolding of a tangency at the parameter value (a_0, b_0).*

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