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# External Rays in the Dynamics of Polynomial Automorphisms of $\mathbf{C}^{2*}$

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## §1 Introduction

In this exposition we will describe how tools of analysis can be used to answer questions about the dynamics of polynomial maps and automorphisms of $\mathbf{C}^m$. This exposition is aimed at analysts, so we will not assume any background in dynamics on the part of the reader.

Our main interest is to explain some recent work in the case when $m = 2$, and $f$ is a polynomial automorphism; but in order to motivate some of the questions and some of the methods we will give a selective discussion of the case $m = 1$ when $f$ is a polynomial map.

We will begin by describing some of the questions of dynamical systems in a general context. The word "dynamics" suggests change with time, and indeed the subject of dynamical systems is motivated by questions involving the long term behavior of physical systems that evolve with time. Let us say that a scientist is observing some system in his laboratory or in the field, and he has identified some interesting property of his system which he can measure. Let us assume that the result of each measurement is a real number. The scientist measures the system at regular time intervals. Let us say that at time $t_n$ he measures the value $x_n$. After collecting his data the scientist must then analyze it. Among the questions that he might ask, some are quantitative, but others are more qualitative, for example whether the emergence of the data is regular or chaotic. In other words, does the sequence of values $x_n$ seem to follow a definite rule or does it look like it represents a sequence of unrelated events.

Since we are mathematicians and not experimental scientists our interest is not the physical systems themselves but rather the behavior of the mathematical models that are used to describe these systems. We will discuss a certain class of mathematical models that can be used to describe a variety of systems like the one our scientist is observing. Let us assume that the state of the system is completely captured by the values of a finite number of real quantities. (If the system involves planets moving in space for example then these quantities would be the position vectors and momentum vectors of these planets.) Since the values of these quantities (say there are $m$ of them) completely describe our system we can identify a state of our system with a point $p \in \mathbf{R}^m$. The set of allowable states for our system will correspond to some set $X \subset \mathbf{R}^m$.

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We also assume that the time evolution of our system is given by an *evolution rule* which is nothing more than a continuous function $f : X \to X$ so that if $p \in X$ is the state of the system at time $t_j$ then $f(p)$ is the state of the system at time $t_{j+1}$. We represent the measured quantity by a function $h : X \to \mathbf{R}$. Thus if $p_0 \in X$ denotes the initial state of our system then the state at time $t_1$ is $f(p_0)$ and at time $t_n$ the state of our system is $p_n = f^n(p_0)$ (where the $n$ represents the composition of $f$ with itself $n$ times) and the value of the measured quantity is $x_n = h(f^n(p_0))$. The assumptions we have made rule out some systems. For example the fact that the evolution function does not change with time rules out systems where the behavior is caused by input from outside the system. On the other hand this class of models includes a wide range of systems of interest in physics, biology and engineering.

From the point of view of the general behavior of the system the particular choice of the observable function does not play much of a role. We can assume that instead of observing a function of the state of the system that we are observing the entire state of the system. Thus instead of considering the data to be the sequence $h(f^n(p_0))$ we will consider the sequence $f^n(p_0)$. We call this sequence of points the *orbit* of the point $p_0$. (The term orbit is also used for the set of values of this sequence.)

We return to our original question about regular versus chaotic behavior. If we interpret the notion of "rule" loosely we see that the orbit of a point is *never* random: the orbit of $p_0$ is completely determined by the initial value $p_0$. The situation changes, however, if we adopt the viewpoint of the scientist rather than the viewpoint of a mathematician. Let us say that the scientist records the value of each measurement only up to a certain fixed precision. We will see in this case that the sequences of finite precision measurements can look very much like random sequences. It is this "finite precision" notion which we will use to define chaotic behavior. From this point of view chaotic behavior is possible at least in theory. For any particular dynamical system, we can ask whether chaotic behavior occurs. And if it does occur, it is useful to know which initial conditions give rise to chaotic behavior and which give rise to regular behavior (and which give rise to behaviors which do not fit completely into either category).

This discussion motivates a revised definition of dynamical systems. Given a space $X$ and a map $f : X \to X$ dynamical systems deals with questions concerning the qualitative properties of orbits. A typical problem is to identify the set of points $p \in X$ for which the orbit of $p$ is chaotic or to identify the set of points for which the orbit is regular and see what tools of mathematical analysis can be brought to bear to study the structure of these sets. This revised definition opens the possibility that we might deal with spaces and maps that do not arise from physical models. It has indeed proved valuable to consider a wide range of dynamical systems.

We might compare the young field of dynamical systems to the much older field of algebraic geometry. Though the initial questions involve specific questions with concrete applications, the techniques of the theory extend to a much wider class of examples. Indeed it is only by looking at this wider class that some of the unity of the field becomes clear. The systems we will discuss in this paper are not models of any particular physical systems though some of them were motivated by physical considerations. They are systems where both chaotic and regular behavior occur and where the study of the sets on which various dynamical behaviors occur leads to interesting mathematical questions.

We will give an example of a dynamical system of the type we wish to study. In this case we take the set $X$ to be the complex line $\mathbf{C}$. Let $f : \mathbf{C} \to \mathbf{C}$ be the squaring map $f(z) = z^2$. This map is unusual in that it is rather easy to analyze its dynamical behavior directly from the formula. Even though the tools that we use are special to this case, we will see that in many ways the behavior of the squaring map reflects the behavior of the general polynomial map of $\mathbf{C}$.

Let us use the notation $O^+(p) = \{p, f(p), \ldots\}$ for the orbit of a point $p$. (For invertible maps we will have other notions of orbit to consider.) The nature of the orbit of $z \in \mathbf{C}$ depends on $|z|$. Note that if $|z| > 1$ then $|f(z)| > 1$; if $|z| = 1$ then $|f(z)| = 1$ and if $|z| < 1$ then $|f(z)| < 1$. We say a set $X \subset \mathbf{C}$ is *(forward) invariant* if $f(X) \subset X$. In particular the sets $U = \{z : |z| > 1\}$, $J = \{z : |z| = 1\}$ and $K = \{z : |z| \leq 1\}$ are examples of invariant sets. (The notation is chosen to be consistent with some standard notation we will introduce later.) Invariant sets play an important role in dynamics. When we have an invariant set we can view the restriction of $f$ to $X$ as a dynamical system in its own right. In particular if $z \in X$ then $O^+(z) \subset X$.

Let us look at the dynamics on each of these invariant sets. When $z \in U$ then $O^+(z)$ is unbounded and $f^n(z)$ tends to infinity. The set $int(K)$ is also an interesting invariant set in our example. The point 0 is a fixed point. If $0 < |z| < 1$ then the orbit of $f^n(z)$ is bounded and converges to 0.

The unit circle, which we have called $J$, is the invariant set with the most complicated behavior. There is a useful technique which allows us to analyze the dynamics on the unit circle. Let $a = 0.a_1a_2a_3\ldots$ be the binary expansion of a number between 0 and 1. Let $\sigma(a) = 2a \bmod 1$. Multiplying a number by two shifts the digits in its binary expansion to the left. Reducing the number modulo one has the effect of dropping the digits to the left of the decimal point Thus $\sigma(a)$ has the binary expansion $a = 0.a_2a_3a_4\ldots$. Let $\phi(a) = e^{2\pi i a}$. It is easy to see that

$$f(\phi(a)) = \phi(\sigma(a)). \tag{1.1}$$

This technique of associating sequences of symbols to points is very useful. It allows us to construct points with prescribed behavior.

We will consider an example of this. An important notion in dynamics is the notion of a periodic point. A point $p$ is *periodic* if $f^n(p) = p$ for some $n \geq 1$. The least $n$ for which this equation holds is the *period* of the periodic point. A periodic point with period one is called a *fixed point*. Consider for example the point $z_0$ which is $\phi(a_0)$ with $a = .010101\ldots$. To find the binary expansion which gives rise to $f(z_0)$ we note that $f(\phi(a_0)) = \phi(\sigma(a))$. Now $\sigma(a) = .101010\ldots$ Similarly $f^2(z_0) = f^2(\phi(a)) = \phi(\sigma^2(a))$ and $\sigma^2(a) = .010101\ldots = a$. We conclude that we have found a periodic point of period 2. It is not hard to see that if we start with a point $a$ with a periodic binary expansion then $\phi(a)$ is a periodic point for the map $f$. This technique allows us to show that periodic points are dense in the circle.

Let us look more carefully at what we have done. The first observation is that we can think of the shift map acting on the set of binary expansions as a dynamical system. The set of binary expansions can be identified with the infinite product

$$\prod_{n=1}^{\infty}\{0,1\}.$$

As such it has a natural product topology. The shift map $\sigma$ is a continuous map of this space to itself. Such a system is commonly referred to as a symbolic dynamical system. Such systems have the convenient feature that it is very easy to construct orbits with prescribed behavior. For example, a point is periodic precisely when its sequence of symbols is periodic.

The map $\phi$ gives us a map from the space of sequences to $J$, and the equation (1.1) can be interpreted as saying that the map $\phi$ "preserves the dynamics". In particular the equation (1.1) implies that

$$f^n(\phi(a)) = \phi(\sigma^n(a)). \tag{1.2}$$

In particular the map $\phi$ takes periodic points (of $\sigma$) to periodic points (of $f$) and it takes orbits of $\sigma$ to orbits of $f$. The map $\phi$ is called a *semiconjugacy*. The map $\phi$ is not injective. This is due to the ambiguity in the representation of certain numbers; for example $\phi(.0000\ldots) = \phi(.1111\ldots)$. A semiconjugacy which is also a homeomorphism is called a conjugacy. The construction of conjugacies and semiconjugacies is a powerful tool in dynamics which we will use repeatedly.

In the example of the squaring map we have identified a set $J$ corresponding to initial conditions with interesting dynamics. Can the behaviors of the orbits of these points be described as chaotic? Recall how a map can give rise to sequence of measurements. Let $h : \mathbf{C} \to \mathbf{R}$ be the function $h(z) = \Im(z)$. Assume that $h$ represents the quantity we are measuring. For a given initial condition our data is $h(f^n(z_0))$. Let us say that our scientist only records the value of $h$ up to a finite precision. Let us assume in fact that he only records the sign of $h$. What sequences of signs can occur? If we assume that the sequence $a$ does not terminate in either all zeros or all ones, then for the initial condition $z_0 = \phi(a)$ the sign of $x_n$ is positive if $a_n$ is zero and negative if $a_n$ is one. Thus we can achieve almost any sequence of signs. This deserves to be called chaotic behavior.

We have seen that on the complement of $J$ the dynamics is rather regular. Rather than define the notions of chaotic and regular let us consider some properties of $f$ that are associated with the various sets. One feature that we see on $J$, the Julia set, is *expansion* in that the size of the derivative of $f^n$ grows with $n$. In fact $|Df^n| = 2^n$. Expansion is associated with two nearby points having very different orbits.

Regular behavior is associated with the dynamical notion of stability. Loosely speaking a point is stable if all nearby orbits have similar behavior. For example in the case of the squaring map for any $z$ near 0 then the sequence $f^n(z)$ converges to 0 so we see the same dynamical behavior for all these points.

Another important dynamical notion is recurrence. A point $p$ is forward recurrent if $p$ is a limit of points $f^n(p)$ as $n \to \infty$. One strong form of recurrence is periodicity. Let us consider the question of recurrence for points in the unit circle. The point $\phi(a)$ is recurrent if and only if every sequence which occurs in its binary expansion occurs infinitely often. For example if $a = .100000\ldots$ then $\phi(a) = -1$. Now $O^+(-1) = \{-1, 1, 1, 1\ldots\}$ so $-1$ is not recurrent. We can see this on the symbolic level by noting that the binary digit 1 occurs only once in its binary expansion. It is not difficult to see that both the set of recurrent points and the set of nonrecurrent points are dense in the unit circle.

In order to distinguish the nonrecurrent behavior that we see for some points on the unit circle from the nonrecurrent behavior that we see for points with $|z| > 1$ it is useful to introduce the idea of wandering points. A point is wandering if there is a neighborhood $U$ of $p$ so that all iterates $f^n(U)$ are disjoint. Note that this is not a property of the orbit of $p$ but rather a property of the structure of nearby orbits.

This is the way in which we have to modify our original notion of a "dynamical" property of a point.

We define the nonwandering set to be the set of points which are not wandering. The nonwandering set is automatically invariant. In our example the nonwandering set consists of the unit circle and the point 0. The behavior of 0 and of the points on the unit circle are quite different.

In the example of the squaring map nonwandering unstable behavior is associated with chaos. This turns out to be a general phenomena. The study of chaotic behavior is quite interesting. On the other hand we will also see that there are interesting analytic questions about points with stable behavior and points with non-recurrent behavior as well.

This discussion has motivated our general program. Our program for a polynomial map $f : \mathbf{C}^m \to \mathbf{C}^m$ is to decompose $\mathbf{C}^m$ into invariant sets with distinct dynamical behavior and to try to describe these sets up to isomorphism where the natural notion of isomorphism for dynamical systems is the notion of conjugacy.

This paper is organized as follows. In §2 we describe some results from the dynamics of a polynomial mapping $p : \mathbf{C} \to \mathbf{C}$. We treat a small part of the theory, just the things that seem to us to be most likely to be useful in the study of complex dynamics in two dimensions. In §3 we give a treatment of the complex solenoid. In §4, we describe the approach given in [H] and [HO1] to the study of the analytic conjugacy type of the set of points with unbounded forward orbits. In §5 we describe some of the [FM] work on polynomial automorphisms. We also give the normal family arguments which allow us to describe the dynamics on the Fatou set (see [BS2] and [FS2]). In §6 we describe the approach to the theory of mappings with connected Julia set $J$, which has been developed in [BS5–7].

We wish to thank Ricardo Oliva for providing the computer pictures that we have used.

## §2 Dynamics of Polynomials in one complex variable

Before attempting this program for automorphisms of $\mathbf{C}^m$ we will consider the case of general polynomial maps of $\mathbf{C}$. We will give a brief summary of some well understood facts about the dynamics of polynomial mappings in $\mathbf{C}$. References are [CG, Chapter VIII], [S, Chapter 6] and [M, §18]. Our presentation is organized so as to motivate our development of the analogous model for an automorphism of $\mathbf{C}^2$.

We consider a polynomial of degree $d$ of the form

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + a_{d-2} z^{d-2} + a_{d-3} z^{d-3} + \ldots + a_0.$$

As we have seen before conjugate maps have the same dynamics. We can begin by asking when two polynomials are conjugate by polynomial automorphisms. Since all polynomial automorphisms of $\mathbf{C}$ are linear this is a relatively easy question to answer. (The corresponding question in $\mathbf{C}^2$ is much harder to answer. We will discuss this in §5.) It turns out that every polynomial map is conjugate to one of the form:

$$f(z) = z^d + a_{d-2} z^{d-2} + a_{d-3} z^{d-3} + \ldots + a_0$$

since we may conjugate by an affine map $z \mapsto bz + c$ to obtain $a_d = 1$ and $a_{d-1} = 0$.

In considering the squaring map we saw that there was a decomposition of $\mathbf{C}$ into three invariant sets. We can find a corresponding decomposition in general.

Let $K$ be the set of points in $\mathbf{C}$ with bounded orbits. Let $U$ be the set of points with unbounded orbits. We can further decompose $K$ into $int(K)$ and $J = \partial K$.

Note that this decomposition yields the decomposition of $\mathbf{C}$ described for the squaring map. We want to study the question of the extent to which the properties of these sets for general polynomials mirror the properties of the sets in the case of the squaring map.

Let us start by considering the set of points with bounded orbits. A point $x$ is *Lyapunov stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $d(f^n x, f^n y) < \epsilon$ for all $n \geq 0$. In other words, if points start sufficiently close together, they will remain near to each other for all forward time. The notion of Lyapunov stability is quite similar to a notion which is familiar to complex analysts: the condition that $\{f^n : n \geq 0\}$ is a normal family in a neighborhood of $x$. Since $\{f^n : n \geq 0\}$ is a normal family on $int(K) \cup (\mathbf{C} - K) = \mathbf{C} - J$, we conclude that all points are Lyapunov stable there. Conversely, points in $J$ are not Lyapunov stable, for any neighborhood of a point contains points which remain bounded and points which tend to $\infty$.

Now let us consider the set of points with unbounded orbits. We will see that these points are all wandering. We let

$$U = \{z \in \mathbf{C} : \lim_{n \to \infty} f^n(z) = \infty\}. \tag{2.1}$$

The iterates have the asymptotic form $p^n(z) = z^{d^n} + O(z^{d^n - 2})$ for $z$ large. We may choose $R$ sufficiently large that the limit

$$\phi(z) = \lim_{n \to \infty} (p^n(z))^{\frac{1}{d^n}} = z + o(1) \tag{2.2}$$

converges uniformly for $|z| > R$, since there is a unique choice of root which is close to $z$ at infinity. It follows that $\phi \circ p = \sigma \circ \phi$, where we define $\sigma(z) = z^d$. Thus $\phi$ serves to give a conjugacy between $f$ and $\sigma$ in a neighborhood of $\infty$. In particular we see that every point in $U$ is wandering.

The next question is when this conjugacy can be extended to all of $U$.

We also define

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|. \tag{2.3}$$

It follows formally from the definition that

$$G \circ f = d \cdot G, \text{ and } \log |\phi| = G, \tag{2.4}$$

with the second property valid only where $\phi$ is defined. It may be shown that $G$ is the Green function with pole at infinity for the set $\mathbf{C} - K$. This means that $G$ is harmonic and $G > 0$ on $\mathbf{C} - K$, $G = 0$ on $K$, and $G = \log |z| + O(1)$ near $z = \infty$.

For the squaring map we have $G(z) = \log^+(|z|)$.

Equation (2.4) allows us to make an analytic continuation of $\phi$ along any path in $\mathbf{C} - K$ which starts in the region $\{|z| > R\}$. If $K$ is connected, then $U$ is simply connected. Thus the analytic continuation of $\phi$ to $U$ is single-valued. It follows that $\phi : U \to \mathbf{C} - \overline{\Delta}$ is a conformal equivalence. To summarize, *If $K$ (or equivalently, $J$) is connected, then $\phi$ extends to a conjugacy between $p|_U$ and $\sigma|_{\mathbf{C} - \overline{\Delta}}$.*

Recall that $z_0$ is a *critical point* if $p'(z_0) = 0$. Critical points allow us to characterize the class of mappings for which $K$ is connected as follows: *If all the* *critical points of p are contained in $K$, then $K$ is connected.* This is seen as follows. If the critical points of $p$ lie inside $K$, then for each $n \geq 0$, $p^n : p^{-n}\{|z| > R\} \to \{|z| > R\}$ is an unbranched covering map. It follows that $p^{-n}\{|z| > R\}$ is an annulus, and the complement $\mathbf{C} - p^{-n}\{|z| > R\}$ is connected for each $n \geq 0$. Thus $K$ is connected. Conversely, if there is a critical point outside $K$, then there is a critical point in $U$. Thus for $n$ sufficiently large $p^n$ is branched on $p^{-n}\{|z| > R\}$, and so $p^{-n}\{|z| > R\}$ is not an annulus, and the complement is not connected.

Now we analyze $J$ in the connected case.

Now we discuss how, in case $K$ is connected, we can pass to the object of primary interest, which is $p|_{\partial U} = p|_J$. Let us consider the inverse mapping

$$\psi = \phi^{-1} : \mathbf{C} - \overline{\Delta} \to U.$$

For fixed $\theta$, we set $R_\theta = \{re^{i\theta} : r > 1\}$, and we define the *external ray* $\gamma_\theta := \psi(R_\theta)$. We let $\mathcal{R} = \{\gamma_\theta\}$ denote the set of external rays, which is isomorphic to $\partial\Delta$. It follows that the action of $p$ on $\mathcal{R}$ is equivalent to $\sigma$ on $\partial\Delta$, since $p(\gamma_\theta) = \gamma_{d\cdot\theta}$. *If the mapping $\psi$ extends continuously to $\partial\Delta$, then $\psi : \partial\Delta \to J$ gives a representation of $J$ as a quotient of $\partial\Delta$, and $\psi$ gives a semiconjugacy from $\sigma|_{\partial\Delta}$ to $p|_J$.*

The question arises when we can expect $\psi$ to extend continuously to $\partial\Delta$. The simplest criterion is that $p$ is hyperbolic (i.e., uniformly expanding) on $J$. There is a nice criterion for this: *p is hyperbolic iff all the critical points of p lie in basins of attraction.* This criterion is, in principle, easy to check, since it is just a matter of iterating the critical points and seeing whether they are attracted to sinks.

The approach now is to represent the dynamical system $p|_J$ as a quotient of the model system $\sigma|_{\partial\Delta}$. If we use $\mathbf{R}/\mathbf{Z}$ to represent $\partial\Delta$ via the mapping $x \mapsto 2\pi x = \theta \mapsto e^{i\theta}$ then $\sigma$ is given by the mapping $x \mapsto d\cdot x \,(\text{mod } \mathbf{Z})$. In the case $d = 2$ this is just the restriction of the squaring map to the circle.

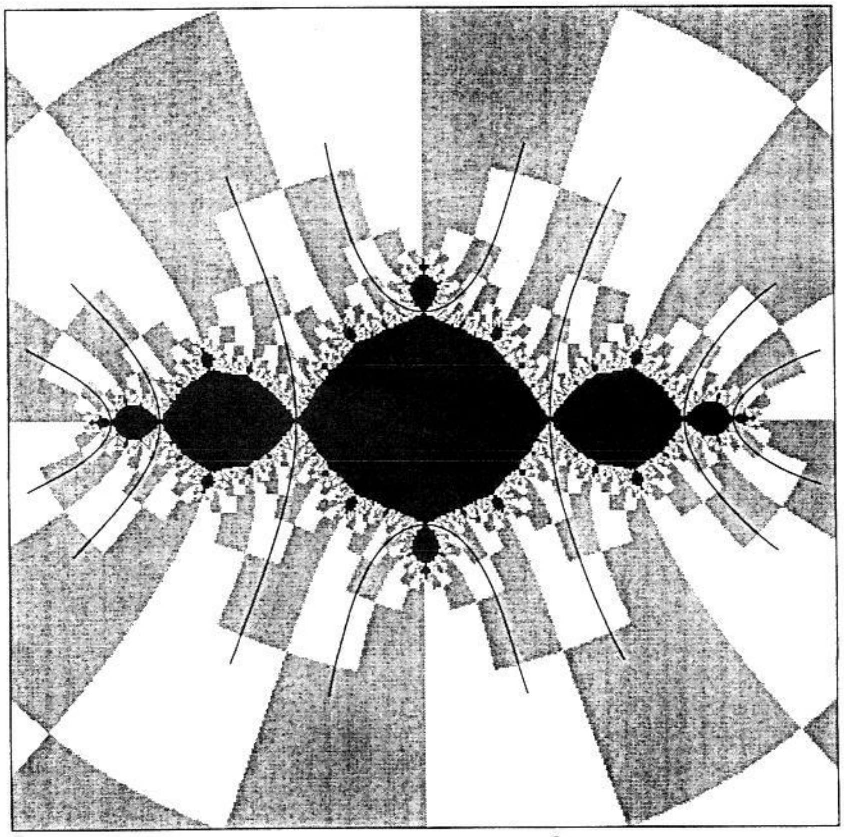
A feature that arises when we use the symbolic representation is the nonuniqueness of the base $d$ representation:

$$.x_1x_2\ldots x_k(j-1)(d-1)^\infty{}_{(d)} = .x_1x_2\ldots x_kj0^\infty{}_{(d)}, \tag{2.5}$$

where $0 \leq j \leq d-1$. For any finite word in the symbols $\{0, 1, \ldots, d-1\}$, we use the notation $w^k$ to indicate the $k$-fold concatenation of $w$. The only numbers with nonunique representation base $d$ are those of the form $nd^{-k}$.

There is a canonical computer picture which illustrates the external rays very well. For a point $z \in \{0 < G < 1\}$, there exist $n \geq 1$ and $0 \leq j < d^{n+1}$ such that $d^{-n} \leq G(z) < d^{-n+1}$, and $jd^{-n}2\pi \leq \operatorname{Arg}(\phi(z)) < (j+1)d^{-n}2\pi$. A palate of $d$ colors is selected, and the point $z$ is then assigned the color $j \,(\text{mod } d)$. This picture indicates both the level sets (equipotential lines) $\{G = d^{-n}\}$ and external rays (flux lines) $\gamma_{jd^{-n}}$. Further, for a general external ray $\gamma_\theta$, it is possible to determine the external angle $\theta$ as follows. If $a_n$ is the color of the region where $\gamma_\theta$ passes through $\{d^{-n} \leq G < d^{-n+1}\}$, then the base $d$ representation of $\theta$ is given by $\frac{\theta}{2\pi} = .a_1a_2a_3\ldots_{(d)}$.

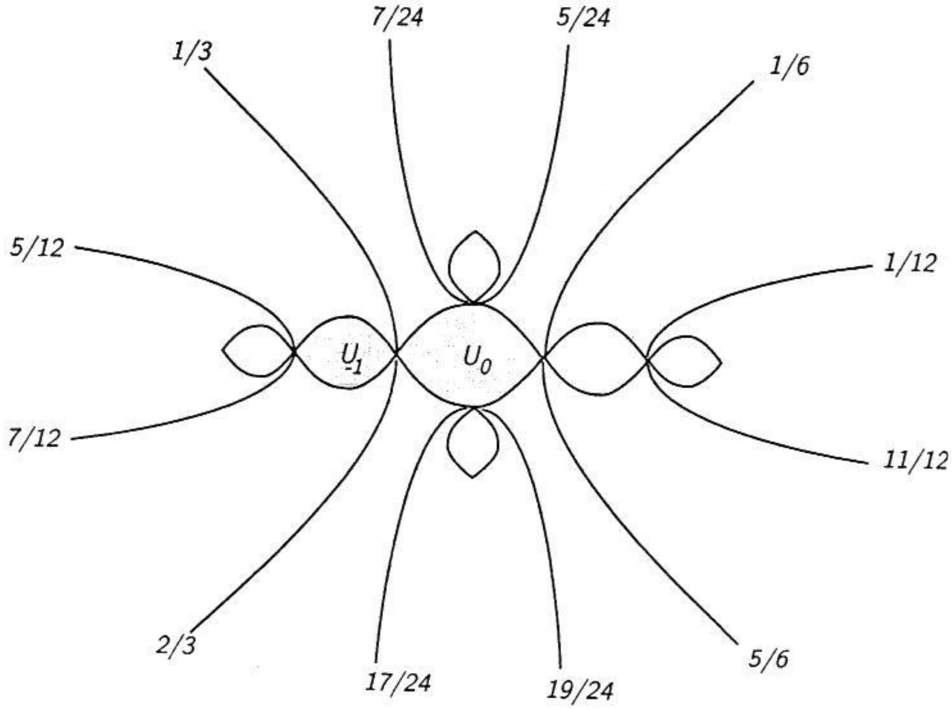
An example of how this partition of the plane looks is given in the computer picture of $z^2 - 1$. If we look at the large oval figure in this picture, we see that there are $32 = 2^5 = 2^{n+1}$ black and white regions around its perimeter, so $n = 4$, and this oval corresponds to the set $\{G = 2^{-4}\}$. In addition, this picture contains 16 arcs of external rays. Inspection shows that each of these rays alternates between black and white regions, even moving back (where the drawing of the ray is not continued) to the region $\{2^{-4} < G < 2^{-3}\}$. Thus each of these rays has the form $.xxx(01)^{\infty}{}_{(2)}$ or $.xxx(10)^{\infty}{}_{(2)}$. There are 16 such rays altogether. These are the binary expansions of the numbers of the form $j/24$, with $j$ relatively prime to 3.



Computer picture of $z^2 - 1$

Other external rays that appear implicitly in this picture are the rays that are defined by the boundaries between black and white regions which are also gradient lines of $G$. These correspond to the identifications (2.5). The rays that have pairs of codings of the forms $.xxx10^{\infty}{}_{(2)}$ and $.xxx01^{\infty}{}_{(2)}$ originate on the curve $\{G = 2^{-4}\}$. (The values of $x$ will not be the same in the identified codings.) The ones that originate in the set $\{G > 2^{-4}\}$ have pairs codings the forms $.xxx0^{\infty}{}_{(2)}$ and $.xxx1^{\infty}{}_{(2)}$. To see how these rays compare to the ones drawn in the computer picture, we note that if $j$ is not relatively prime to 3, then 3 divides $j$. The numbers $j/24$ with $j$ divisible by 3 are then the numbers of the form $k/8$, which have base 2 expansions $.xxx0^{\infty}{}_{(2)}$.

Within the amount of picture shown, we are a priori only given the terms $a_n$, $n \geq 4$, in the expansion $\theta = .a_1a_2a_3\ldots$ However, the point on the extreme right of the picture is the fixed point, corresponding to $.0^{\infty}{}_{(2)} = .1^{\infty}{}_{(2)}$. With this extra knowledge, we may determine $a_n$, $n \leq 3$ by counting our way around the level set $\{G = 2^{-4}\}$.




*Identification Relations Among External Rays*

Douady and Hubbard have developed the theory of external rays into a powerful tool for understanding $J$ by understanding the sort of quotient that can occur. We consider the equivalence relation $\theta_1 \sim \theta_2$ which is defined by the condition $\psi(\theta_1) = \psi(\theta_2)$. We recall that the orientation on the circle allows to define an ordered interval $(a, c)$, starting at $a$ and running to $c$. Thus for three points we may define $a < b < c$ to mean that $b$ is in the interval $(a, c)$. A basic observation is that the equivalence relation satisfies a *planarity condition.* This says that if $\theta_1 \sim \theta_2$ and $\theta_3 \sim \theta_4$, then $\theta_3$ and $\theta_4$ both lie in the same component of the complement (in the circle) of $\{\theta_1, \theta_2\}$. In other words, the intervals $(\theta_1, \theta_2)$ and $(\theta_3, \theta_4)$ are either disjoint or nested. In particular, we cannot have $\theta_1 < \theta_3 < \theta_2 < \theta_4$. The reason for this condition is that two distinct rays are disjoint subsets of $\mathbf{C} - K$, and they cannot cross. The following example will make this more clear.

Let us illustrate how the theory of external rays works out in the simplest case where $J$ is connected but the equivalence $\sim$ is nontrivial. This is the case $p(z) = z^2 - 1$. The defining feature of this mapping is that there is one attracting 2-cycle, $0 \leftrightarrow -1$. Since the critical point is part of an attracting cycle, the mapping is hyperbolic, and the external rays will give $J$ as a finite quotient of the circle. Let $U_0$ and $U_{-1}$ denote the connected components of $int(K)$ containing 0 and $-1$, respectively. By a normal families argument, we see that $U_0$ is the largest connected open set containing 0 such that $p^{2n}(z) \to 0$ on $U_0$ as $n \to \infty$. An analogous statement holds for $U_{-1}$. A special property of the parameter $c = -1$ is that $\partial U_0 \cap \partial U_{-1} = \{\beta\}$ is a point. Since $U_0 \cup U_{-1}$ is mapped to itself, $\beta$ must be a fixed point.

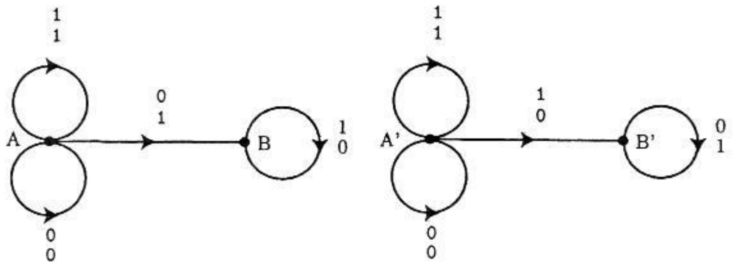
We start with the elementary observation that $\psi : \partial\Delta \to J$ takes a periodic point $\zeta$ of $\sigma|_{\partial\Delta}$ to a periodic point $\psi(\zeta)$ of $p|_J$. Further, if $\sigma^n(\zeta) = \zeta$, then $p^n(\psi(\zeta)) = \psi(\zeta)$. Thus the period of $\psi(\zeta)$ must divide the period of $\zeta$. By inspection, we see that $\#\{z \in \mathbf{C} : p^n(z) = z\} = 2^n$ holds for $n = 1, 2$. Thus $p$ has 2 fixed points and a cycle of length 2. The unique fixed point $1 \in \partial\Delta$ will be mapped onto one of the fixed points. The 2-cycle for $p$, $\{0, -1\} \subset \text{int}\, K$, however, is disjoint from $J$. Thus the unique 2-cycle $\{1/3, 2/3\}$ in $\partial\Delta$ will be taken to the other fixed point. We are led to the identification relations among the external rays as pictured, where the external rays $\gamma_{1/3}$ and $\gamma_{2/3}$ land at the same point $\beta \in \partial U_0 \cap \partial U_{-1}$. It follows that the preimage of this pair, which is $\gamma_{1/6}, \gamma_{5/6}$ must land at a common point. The next preimage is

$$p^{-1}(\gamma_{1/6} \cup \gamma_{5/6}) = \gamma_{1/12} \cup \gamma_{5/12} \cup \gamma_{7/12} \cup \gamma_{11/12}.$$

To apply the planarity condition, we note that $\gamma_{1/3} \cup \gamma_{2/3}$ and $\gamma_{1/6} \cup \gamma_{5/6}$ separate the plane into three components, which we may describe as being on the left and right with a vertical strip separating them. Since $\gamma_{5/12}, \gamma_{7/12}$ lie in the left-hand component, and $\gamma_{1/12}, \gamma_{11/12}$ lie in the right-hand component, the external rays must pair off as in the illustration. The next preimage is

$$p^{-1}(\gamma_{5/12} \cup \gamma_{7/12}) = \gamma_{5/24} \cup \gamma_{7/24} \cup \gamma_{17/24} \cup \gamma_{19/24}.$$

These external rays pair off as $5/24 \sim 7/24$ and $17/24 \sim 19/24$. To see this, we note that the other pairings, such as $5/24 \sim 19/24$, are not possible since they would have to cross the basins $U_{-1} \cup U_0$.




*Graphs Generating Binary Equiavlences*

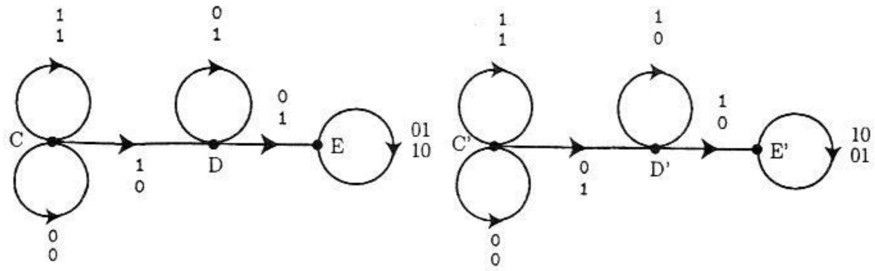
We have described the equivalence relation in terms of geometric properties of the external rays. We may also describe the symbolic equivalence relation from a combinatorial point of view. In particular, suppose that two sequences of symbols $.\theta_1'\theta_2'\theta_3' \ldots_{(2)} \sim .\theta_1''\theta_2''\theta_3'' \ldots_{(2)}$. We now consider the pair of sequences as a sequence of pairs $\begin{smallmatrix}\theta_1' & \theta_2' & \theta_3' \\ \theta_1'' & \theta_2'' & \theta_3''\end{smallmatrix} \ldots$ and discuss the combinatorics of finding pairs.

We show how the identifications among the external rays that were obtained geometrically in the preceding paragraphs may also be generated by a graph.

Let us start by considering the (relatively simple) equivalence relation arising from the ambiguity of base $d = 2$ representations, given in (2.5), i.e. $.\theta_1'\theta_2'\theta_3' \ldots_{(2)} \sim .\theta_1''\theta_2''\theta_3'' \ldots_{(2)}$ means that $\theta' = \theta''$. This is generated by the graph given below. This is interpreted as follows: an identification arises as a sequence of vertical pairs $\begin{smallmatrix}\theta_j' \\ \theta_j''\end{smallmatrix}$ obtained by following a path in the graph. For each edge traversed in the directed path, the corresponding paired symbols are read off of the label of that edge. For instance, the identification $\begin{smallmatrix}0 & 1 & 1 & 0 & 1^\infty \\ 0 & 1 & 1 & 1 & 0^\infty\end{smallmatrix}$, corresponding to $.01101^\infty{}_{(2)} = .01110^\infty{}_{(2)}$

is obtained by following 3 edges from, and returning to, vertex A, then the edge leading to vertex B, and after that (infinitely often), the edge that leaves and returns to vertex B.

Thus we see the purpose of using this graph. Any infinite path through the graph generates a sequence $\genfrac{}{}{0pt}{}{\theta'_1}{\theta''_1}\genfrac{}{}{0pt}{}{\theta'_2}{\theta''_2}\genfrac{}{}{0pt}{}{\theta'_3}{\theta''_3}\ldots$ such that $.\theta'_1\theta'_2\theta'_3\ldots_{(2)} = .\theta''_1\theta''_2\theta''_3\ldots_{(2)}$ is an identification of the form (2.5). Conversely, all such identifications arise in this way, and we may generate the identifications by considering arbitrary (infinite) paths through the graph.




*Graphs Generating Identifications of External Rays*

There is also a graph which generates the equivalence relation corresponding to the identification of the external rays for the Julia set of $p(z) = z^2 - 1$. In this graph, the edge starting and ending at E has a label with two symbols. The identification $1/3 = .(01)^\infty{}_{(2)} \sim 2/3 = .(10)^\infty{}_{(2)}$ is given by taking the loop at E an infinite number of times. The identification $1/12 = .00(01)^\infty{}_{(2)} \sim 11/12 = .11(10)^\infty{}_{(2)}$, corresponding to $\genfrac{}{}{0pt}{}{00(01)^\infty}{11(10)^\infty}$ is obtained as follows. We start at D, traverse the loop at D one time, which generates $\genfrac{}{}{0pt}{}{0}{1}$, then we pass to E, generating another $\genfrac{}{}{0pt}{}{0}{1}$, and finally we traverse the loop at E an infinite number of times, generating an infinite sequence of $\genfrac{}{}{0pt}{}{01}{10}$'s. To generate the identification $5/24 = .001(10)^\infty{}_{(2)} \sim 7/24 = .010(01)^\infty{}_{(2)}$, we need to generate the sequence $\genfrac{}{}{0pt}{}{001(10)^\infty}{010(01)^\infty}$. To do this, we make a $\genfrac{}{}{0pt}{}{0}{0}$ loop at C', then the edge from D' to E' (to give $\genfrac{}{}{0pt}{}{1}{0}$), and finally an infinite repetition of the loop at E'.

Understanding the generation of these symbol sequences yields insight into the topology of $J$. We give two examples of this in our example. A point $z \in J$ is called a *cut point* if $K - \{z\}$ is disconnected. We note that by the maximum principle, the set $K - \{z\}$ is disconnected if and only if it is locally disconnected at $z$. In the case of the mapping $z \mapsto z^2 - 1$, these are the points of attachment between the various components of the interior of $K$. If $J$ is represented as a quotient of the circle via the landing of external rays, then a point $z \in J$ may be seen to be a cut point if and only if $z$ is the landing point of at least two external rays. Thus the points $\theta$ which correspond to identifications as above are the external angles of the cut points of $J$. We say that a point $z \in J$ is a *tip point* if there is an infinite sequence of cut points $z_j$ converging to $z$ with the property: the diameter of the component of $J - \{z_j\}$ containing $z$ goes to zero. The ray $\gamma_0$ lands at a fixed point $\alpha \in J$. There are identifications $\genfrac{}{}{0pt}{}{01^k(01)^\infty}{10^k(10)^\infty}$ which cut off an interval containing 0. These identifications correspond to a sequence of cut points $z_k$, which converge to $\alpha$ as $k \to \infty$. The intervals containing 0 correspond to the components of $J - \{z_k\}$ containing $\alpha$. Since the length of these intervals vanishes as $k \to \infty$, the diameters of these components also tends to zero, and $\alpha$ is seen to be a tip point.

## §3 The Complex Solenoid

The dynamical systems which we wish to consider are invertible. Let us begin with some general comments on the relation between invertible and non-invertible dynamical systems.

Let $f : X \to X$ be a continuous map. When $f$ is invertible we define the orbit of a point $p$ to be

$$O(p) = \{\ldots, f^{-1}(p), p, f(p), \ldots\}.$$

In some cases it is useful to break the orbit into two pieces. We can speak of the *forward orbit*

$$O^+(p) = \{p, f(p), \ldots\}$$

or the *backward orbit*

$$O^-(p) = \{p, f^{-1}(p), \ldots\}.$$

separately. For invertible systems we have defined bi-infinite orbits while for non-invertible systems we have only defined forward orbits. It turns out that there is a very natural definition of bi-infinite orbit for a not necessarily invertible dynamical system $f : X \to X$. We say that a sequence of points $\{\ldots, p_{-1}, p_0, p_1, \ldots\}$ is an orbit if $f(p_j) = p_{j+1}$. In the invertible case this definition gives us the same notion of orbit as we had before. Indeed the map which takes an orbit $\hat{p}$ to $p_0$ gives a one-to-one correspondence between orbits and points in $X$. In the non-invertible case the set of orbits is not the same as the set of points. The observation is that in some cases it is preferable to deal with the set of orbits rather than the set of points.

The set of orbits is a topological space in a natural way if we think of it as a subset of the infinite product. The set of orbits also has a self-map which makes it a dynamical system. This dynamical system is called the natural extension of $f$ and we denote it by $\hat{f}$. This is the map given by shifting an orbit to the left. The evaluation map is a semi-conjugacy from the set of orbits to the original dynamical system. The shift map on the space of orbits has the especially nice feature that it is invertible.

The disadvantage of the set of orbits is that its definition makes it seem hard to deal with. Let us consider two examples. The first is the space of one sided binary sequences. The second is the squaring from $\mathbf{C}$ to $\mathbf{C}$.

If $X$ is a compact space, and if $f : X \to X$ is a continuous mapping, then we may define the *natural extension* $\hat{f} : \hat{X} \to \hat{X}$, which is a canonical homeomorphism associated to $f$. This is the projective limit, which may be described as follows. We set

$$\hat{X} = \{\hat{x} = (x_j)_{j\in\mathbf{Z}} : f(x_j) = x_{j+1}\}$$

as the set of all sequences in $X$ such that the $j$th entry is mapped under $f$ to the $(j+1)$st entry. We recall that the infinite product $\prod_{j\in\mathbf{Z}} X$ is compact in the product topology. We give $\hat{X}$ the topology of a subset of the infinite product. Thus a sequence of elements $(x_j^{(\nu)})$ converges to $(x_j)$ if and only if for fixed $j$ we have $x_j^{(\nu)} \to x_j$ as $\nu \to \infty$. We define the mapping $\hat{f}(x)$ on $\hat{X}$ by setting

$$\hat{f}(\hat{x}) = (f(x_j)) = \hat{y} = (y_j), \text{ where } y_j = x_{j+1}.$$

That is, shifting the elements of the sequence $(x_j)$ one to the left is the same as applying $f$ to each element of the sequence. Note that $\hat{f}$ is a homeomorphism, since the inverse is given by the shift in the other direction.

We let $\pi : \hat{X} \to X$ be the projection defined by $\pi(x_j) = x_0$. Thus

$$\pi \circ \hat{f} = f \circ \pi.$$

Thus we may think of the original mapping $f$ as being given by the one-sided infinite sequences. Thus the relation between $f$ and $\hat{f}$ may be seen as the relation between the unilateral and the bilateral shift operators.

We set $\mathbf{C}^* = \mathbf{C} - \{0\}$, and we let $\sigma : \mathbf{C}^* \to \mathbf{C}^*$ be given by $\sigma(\zeta) = \zeta^d$. We define the complex solenoid as a set of bi-infinite sequences

$$\Sigma = \{\zeta = (\cdots \zeta_{-1}\zeta_0\zeta_1\zeta_2 \cdots) : \zeta_j \in \mathbf{C}^*, \zeta_j^d = \zeta_{j+1}\}.$$

It follows that $\sigma$ extends to a mapping $\sigma : \Sigma \to \Sigma$, which acts on bi-infinite sequences as

$$\sigma(\zeta_j) = (\zeta_j^d) = (\zeta_{j+1}).$$

Thus this mapping is the natural extension of the $d$-to-1 mapping $\sigma$ of the punctured plane.

$\Sigma$ is a multiplicative group, under componentwise multiplication $(z_n)(w_n) = (z_n w_n)$. The group identity element is given by $\mathbf{1}$, the sequence all of whose components are equal to 1.

We define

$$\pi : \Sigma \to \mathbf{C}^*, \quad \pi(\zeta) = \zeta_0.$$

Let us set

$$\Sigma_+ = \Sigma \cap \pi^{-1}\{|\zeta_0| > 1\}, \quad \Sigma_0 = \Sigma \cap \pi^{-1}\{|\zeta_0| = 1\}.$$

We will refer to $\Sigma_+$ as the *complex solenoid* in the sequel and to $\Sigma_0$ as the *real solenoid*. Note that $\sigma$ acts on both $\Sigma_+$ and $\Sigma_0$. The mapping

$$\rho : \Sigma_+ \to \Sigma_0 \times (1, \infty), \quad \rho(\zeta) = (\zeta/|\pi(\zeta)|, |\pi(\zeta)|)$$

is a homeomorphism. Thus we may think of $\Sigma_0$ as giving the "angle" part of a polar representation of $\Sigma_+$. The real solenoid may also be written as

$$\Sigma_0 = \{x = (x_j) : x_j \in \mathbf{R}/\mathbf{Z}, d \cdot x_j \equiv x_{j+1}\}, \tag{3.1}$$

which is equivalent to the previous definition via the mapping $(x_j) \mapsto (e^{2\pi i x_j})$. From this representation, we see that $\Sigma_0$ is a compact subgroup of the compact group $(\mathbf{R}/\mathbf{Z})^\infty$.

It will be useful to rewrite these concepts in terms of sequences of symbols. Here we define the relevant spaces of sequences, but we postpone defining the isomorphism with the real solenoid until later in this section. We let $S_d = \{0, 1, \ldots, d-1\}$ be the set of $d$ symbols, which we will also treat as the additive group modulo $d$. The infinite products

$$S_d^{-\infty} := \{(s_0 s_{-1} s_{-2} \cdots) : s_j \in S_d\}, \quad S_d^{+\infty} := \{(s_1 s_2 \cdots) : s_j \in S_d\}$$

$$S_d^\infty := \{s = (s_j)_{j\in\mathbf{Z}} : s_j \in S_d,\}, \quad S_d^\infty \cong S_d^{-\infty} \times S_d^{+\infty}$$

are compact topological groups and all equivalent (topologically homeomorphic and algebraically isomorphic) to each other. If $\mathcal{S}$ denotes any of these infinite product spaces, the topology may be described by the cylinder sets

$$C_n(a) := \{b = (b_j) \in \mathcal{S} : b_j = a_j \text{ for } |j| \le n\}$$

which give a fundamental system of neighborhoods of a point $a$ in $\mathcal{S}$. Since $S_d$ is finite, it follows that each cylinder set is both open and closed. It follows that each point $a \in S$ is a connected component of $\mathcal{S}$. We also conclude that the cylinder sets in $S_d^{+\infty}$ and $S_d^{-\infty}$ have a "generational" structure, where the specification of the $n$th generation corresponds to fitting $C_n(a)$ inside $C_{n-1}(a)$. A consequence of this is that $\mathcal{S}$ is homeomorphic to the $d$-ary Cantor set of the unit interval $[0, 1]$, obtained by proceeding in "generations," removing open sets to divide into $d$ pieces each component from the previous generation. Furthermore, $\mathcal{S}$ is homeomorphic to the standard Cantor set so as to respect the "generational" structure.

Let us observe that the fiber of the mapping $\pi : \Sigma \to \mathbf{C}^*$ may be identified with the Cantor set $S_d^{-\infty}$. In fact, *If $U \subset \mathbf{C}^*$ is a connected, simply connected domain, it follows that $\pi^{-1}U$ is homeomorphic to $S_d^{-\infty} \times U$.* To see this, we define the $n$th truncation $\tau_n : \Sigma \to \mathbf{C}^{2n+1}$ by $\tau_n(\zeta) = (\zeta_{-n} \cdots \zeta_0 \cdots \zeta_n)$. We claim that if $U \subset \mathbf{C}^*$ is connected and simply connected, $\tau_n(\pi^{-1}U)$ is homeomorphic to $S_d^n \times U$. The reason for this is that if $\zeta \in U$, and if $(\zeta_j) \in \pi^{-1}\zeta$, then $\zeta_j = (\zeta)^{d^j}$ for $j \ge 0$. The assignment of $\zeta_k$ for $k < 0$ allows for choices. There are $d$ choices of $\zeta_{-1}$, i.e., the solutions of $\zeta_1^d = \zeta_0$ for any fixed $\zeta_0$. These may be identified in an arbitrary way with $S_d$, $\zeta_{-1} \mapsto s_{-1}$. Such a choice determines a unique branch of $z \mapsto z^{\frac{1}{d}}$ on $U$, and this branch of the $d$th root allows us to extend the choice continuously to $U$. To continue to the next entry of $(\zeta_j)$, i.e., $\zeta_{-2}$, we observe that the $d^2$ solutions of $(\zeta_{-2})^{d^2} = \zeta$ are naturally divided into $d$ groups of $d$ elements of the form $\{\eta : \eta^d = \zeta'_{-1}\}$ for a fixed choice of $\zeta'_{-1}$. The $d$ elements of this set may be assigned arbitrarily to the $d$ elements of $S_d$. If $\eta'$ is one of the roots of $\eta^d = \zeta'_{-1}$, then this defines a unique branch of $z \mapsto z^{\frac{1}{d^2}}$ on $U$. If $s' \in S_d$ is the symbol assigned to $\eta'$, then $s'$ will be assigned to the choice of $\zeta_{-2}$ determined by this branch of $d^2$ root of $z$ over $U$. Continuing this way, we obtain the homeomorphism between $\tau_n(\pi^{-1}U)$ and $S_d^n \times U$. By the definition of the product topology it follows that $\pi^{-1}U$ is homeomorphic to $S_d^{-\infty} \times U$.

Let us consider the exponential mapping

$$\mathbf{C} \ni t \mapsto \mathbf{exp}(t) := (e^{d^n t}) \in \Sigma.$$

It is evident that $\mathbf{exp}$ is injective, and $\mathbf{exp}(0) = \mathbf{1}$. Since $\Sigma$ is a group, we may view $\mathbf{exp}(t)$ either as an element of $\Sigma$ or as a mapping from $\Sigma$ to itself by multiplication. Further, $\mathbf{exp}$ is a group homomorphism from the additive group $\mathbf{C}$ to $\Sigma$, so $\mathbf{exp}(\mathbf{C})$ is a subgroup of $\Sigma$, isomorphic to $\mathbf{C}$. Now let us observe that the path components of $\Sigma$ all have the form $\mathbf{exp}(\mathbf{C})\zeta$, and thus are translates of the subgroup $\mathbf{exp}(\mathbf{C})$. In fact, the path components all have the form $\mathbf{exp}(\mathbf{C})\zeta$ for some $\zeta \in \Sigma$. To see this, note that if $\gamma$ is a path in $\Sigma$, then $\pi(\gamma)$ is a path in $\mathbf{C}^*$. By the previous paragraph, the fiber of $\pi$ is totally disconnected, so that any local lifting of a path $\pi(\gamma)$ must lie inside $\{s\} \times U$ for some $s \in S_d^{-\infty}$. Thus the lifting of a path is locally unique. Further, $\pi(\mathbf{exp}(t)) = e^t$, $\mathbf{exp}$ provides a local lifting of the mapping $\log \circ \pi : \Sigma \to \mathbf{C}^*$, and it follows that any path component is contained in $\mathbf{exp}(\mathbf{C})\zeta^{(0)}$ for some $\zeta^{(0)} \in \Sigma$.

A similar argument shows that $\mathbf{exp}(i\mathbf{R})$ is a subgroup of the real solenoid $\Sigma_0$, and the path components of $\Sigma_0$ are given by $s \cdot \mathbf{exp}(i\mathbf{R})$ and are thus homeomorphic to $\mathbf{R}$. If $H = \{z \in \mathbf{C} : \Re(z) > 0\}$ denotes the right half plane, then the path components of $\Sigma_+$ are given by $s \cdot \mathbf{exp}(H)$ for some $s \in \Sigma_0$.

Let $\gamma$ be a path in $\mathbf{C}^*$ starting at $z_1$ and ending at $z_2$. Then for each $z_1' \in \pi^{-1}z_1$ there is a unique lifting $\gamma'$ of $\gamma$ starting at $z_1'$ and ending at a point $z_2'$. The *holonomy map* $\chi_\gamma : \pi^{-1}z_1 \to \pi^{-1}z_2$ is given by $\chi_\gamma(z_1') = z_2'$. This is a homeomorphism, depending on $\gamma$, between fibers of $\pi$ in $\Sigma$. The holonomy map is also given in an explicit way by $\mathbf{exp}$. In the case where $z = z_1 = z_2$, and $\gamma \subset \mathbf{C}^*$ winds once around 0 in the clockwise direction, the holonomy map of $\pi^{-1}z_1$ to itself is given by

$$\chi_\gamma : \pi^{-1}z \ni z' \mapsto \mathbf{exp}(2\pi i)z' \in \pi^{-1}z.$$

Note that $\mathbf{exp}(2\pi i)$ is a fixed point free mapping of the fiber to itself. Writing $\mathbf{exp}(2\pi i) = (\epsilon_n)$, we have that $\epsilon_{-1} = e^{2\pi i/d}$, so $\mathbf{exp}(2\pi i)z$ differs from $z$ in the $-1$ coordinate. In general, the holonomy mapping has the property that $\chi^n = (\mathbf{exp}(2\pi i))^n$ does not change the entries above $-n$ and multiplies the $-n$ entry of a point by $e^{2\pi i/d}$.

Let us continue our discussion of how to represent the real solenoid by sequences of symbols. We recall the base $d$ representation of the circle. A point $x \in \mathbf{R}/\mathbf{Z}$ may be written $x = .b_1b_2b_3\cdots_{(d)}$ as a number in base $d$. Let $\delta : S_d^{+\infty} \to \mathbf{R}/\mathbf{Z}$ be given by

$$\delta(b_1b_2b_3\ldots) = .b_1b_2b_3\ldots_{(d)}\,.$$

The base $d$ representation of $x$ is not unique, with the nonuniqueness as described in (2.5), and we define an equivalence relation $s_1 \sim s_2$ by $\delta(s_1) = \delta(s_2)$. Thus we have a mapping

$$\delta : (S_d^{+\infty}/\sim) \to \mathbf{R}/\mathbf{Z}$$

which is both a homeomorphism and an isomorphism. Let $\tau_+ : S_d^\infty \to S_d^{+\infty}$ denote the truncation operator

$$\tau^+(s_n : n \in \mathbf{Z}) = (s_n : n \geq 1) = (s_1, s_2, \ldots).$$

Thus we define

$$\eta : S_d^\infty \to \Sigma_0, \quad \eta(b) = (\theta_n), \quad \theta_n = \delta\tau^+\sigma^n(b) = .b_{n+1}b_{n+2}b_{n+3}\cdots_{(d)} \in \mathbf{R}/\mathbf{Z}.$$

Now we show that, modulo the identification $\sim$ of base $d$ expansions, this will become an isomorphism. For $b', b'' \in S_\infty^d$, we say that $b' \sim b''$ if there exists $n \in \mathbf{Z}$ such that $b_j' = b_j''$ for $j \leq n$, and $\delta\tau^+\sigma^n(b') = \delta\tau^+\sigma^n(b'')$. Now $\eta$, defined in this way, induces an isomorphism

$$\eta : (S_d^\infty/\sim) \to \Sigma_0, \quad \eta(b) = \eta(b_j) = \theta = (\theta_j).$$

It is useful, too, to be able to determine $\eta^{-1}(\theta)$. To do this, we let $\theta = (\theta_j)$ be represented by a sequence of real numbers satisfying $0 \leq \theta_j < 1$. Then for $j \in \mathbf{Z}$ we define $b_j \in S_d = \{0, 1, \ldots, d-1\}$ such that

$$\theta_{j-1} = b_j/d + \theta_j/d. \tag{3.2}$$

This mapping $\eta$ is both a homeomorphism and an isomorphism. In particular, it involves the additive structure of $S_d$, so it involves more structure of the group $S_d^{-\infty}$ than was used in discussing the topological structure of $\pi^{-1}U$.

We observed above that the path components of $\Sigma_0$ are the orbits under the expontential map. The point $\mathbf{1} \in \Sigma_0$ has two representations $\eta^{-1}\mathbf{1} = ({}^\infty 0.0^\infty) \sim ({}^\infty 1.1^\infty)$. We will use the notation $t = b_{-n} \ldots b_0.b_1b_2b_3 \cdots_{(d)}$ for the base $d$ representation of $t$. Thus by (3.2) we see that for $t \in \mathbf{R}$, $\eta^{-1}\mathbf{exp}(t) = (b_j)$, where $b_j = 0$ for $j < -n$, and otherwise $b_j$ is the $j$th entry in the base $d$ representation of $t$. Thus the symbolic representation of any point on the connected component of the identity $\mathbf{1}$ agrees with the usual base $d$ representation. Thus we see that

$$\mathbf{exp}(i\mathbf{R}) = \{{}^\infty 0c_{-n} \cdots c_0 \cdot c_1c_2 \ldots : n \in \mathbf{Z}, c_j \in \{0, \ldots, d-1\}\} \tag{3.3}$$

For any other point $\zeta_0 \in \sum_0$, the connected component through $\zeta_0$ may be parametrized as a translate of a subgroup, i.e. as $\mathbf{exp}(t)\zeta_0$. The symbolic representation on this component is

$$\eta^{-1}(\mathbf{exp}(t)\zeta_0) = \eta^{-1}(\mathbf{exp}(t)) + \eta^{-1}\zeta_0. \tag{3.4}$$

Thus the symbolic representation is the base $d$ representation of $t$, translated by the fixed symbolic code for the point $\zeta_0$, i.e. $\eta^{-1}\zeta_0$.

A general concept concerning a dynamical system $f : X \to X$ is that of the *stable manifold* of a point $a$:

$$W^s(a) = \{x \in X : \lim_{n\to+\infty} \operatorname{dist}(f^n x, f^n a) = 0\}. \tag{3.5}$$

The appearance of the word "manifold" is explained by the fact that in many cases, such as when $a$ is a saddle point (a case which will be discussed in §5), $W^s(a)$ is an immersed submanifold. For the moment, however, $W^s(a)$ will be just considered as the stable set of $a$. In the case where $\sigma$ is the mapping of $\{\zeta \in \mathbf{C} : |\zeta| = 1\}$ defined by $\sigma(\zeta) = \zeta^2$, which was discussed in §2, we have

$$W^s(a, \sigma) = \bigcup_{n\geq 0} \{b \in \mathbf{C} : |b| = 1, \sigma^n(b) = \sigma^n(a)\} = \bigcup_{n\geq 0} \sigma^{-n}\sigma^n(a).$$

In other words, since $\sigma$ is expanding on the unit circle, we cannot have $\lim_{n\to\infty} dist(\sigma^n a, \sigma^n b) = 0$ unless we actually have $\sigma^N a = \sigma^N b$ for some $N$.

The stable manifold of a point $a$ of the solenoid $\Sigma_0$ is given by

$$W^s(a) = \{b \in \Sigma_0 : \lim_{n\to+\infty} \operatorname{dist}(\sigma^n a, \sigma^n b) = 0\}.$$

In terms of sequences, we see that $W^s(a)$ consists of all points $b = (b_n)$ such that there is a number $N$ such that $b_j = a_j$ for all $j > N$. This is the same as

$$W^s(a) = \bigcup_{n\geq 0} \pi^{-1}\sigma^{-n}\sigma^n\pi(a) = \pi^{-1}W^s(\pi a)$$

where the $W^s(\pi a)$ is the stable set of $\sigma$ on the unit circle.

The unstable manifold of a mapping is the same as the stable manifold for its inverse. Thus

$$W^u(a) = \{b \in \Sigma_0 : \lim_{n\to+\infty} \operatorname{dist}(\sigma^{-n} a, \sigma^{-n} b) = 0\}. \tag{3.6}$$

This coincides with the set of all sequences $b$ such that $b_{-n} = a_{-n}$ for $-n < -N$ for some $N$. We note that: *$W^u(a)$ coincides with the path component of $\Sigma$ containing $a$.* This is perhaps most easily seen by observing that $\mathbf{exp}(i\mathbf{R}) = W^u(\mathbf{1})$, as in (3.3).

The intersection $W^u(a) \cap W^s(b)$ in the real solenoid is best seen by looking at symbol sequences. It consists, loosely speaking, of points of the form $a * . * b$, i.e. $x = (x_j) \in W^u(a) \cap W^s(b)$ if there exists $N$ such that $a_{-j} = x_{-j}$ and $b_j = x_j$ for $j \geq N$. It is evident that $W^u(a) \cap W^s(b)$ is dense in $\Sigma_0$.

We are also interested in unstable manifolds of $\sigma$ on $\Sigma$ and $\Sigma_+$. A similar argument shows that $W^u(a, \Sigma)$ (resp. $W^u(a, \Sigma_+)$) is the path component in $\Sigma$ (resp. $\Sigma_+$) containing $a$.

The periodic points of $\sigma : \Sigma \to \Sigma$ are all contained in $\Sigma_0$. In fact, there is a one-to-one correspondence

$$\mathrm{Per}_n(\sigma : \partial\Delta \to \partial\Delta) \leftrightarrow \mathrm{Per}_n(\sigma : \Sigma_0 \to \Sigma_0)$$

given by the representation as symbol sequences:

$$.w^{\infty}{}_{(d)} \leftrightarrow {}^{\infty}w.w^{\infty} \in \Sigma_0.$$

It is evident that the periodic points are dense in $\Sigma_0$.

## §4. Polynomial Automorphisms: the Basin of Infinity

We consider mappings of the form $f(x, y) = (y, p(y) - ax)$, where

$$p(y) = y^d + a_{d-2}y^{d-2} + a_{d-3}y^{d-3} + \cdots + a_0. \tag{4.1}$$

This mapping has a polynomial inverse

$$f^{-1}(x, y) = (a^{-1}(p(x) - y), x). \tag{4.2}$$

We define

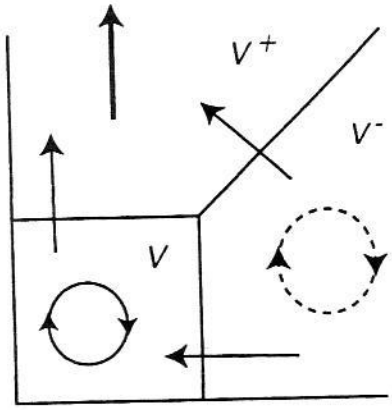
$$K^{\pm} = \{(x, y) \in \mathbf{C}^2 : \{f^{\pm n}(x, y) : n \geq 0\} \text{ is bounded}\}, \quad U^{\pm} = \mathbf{C}^2 - K^{\pm}$$

$$J^{\pm} = \partial K^{\pm}, \quad J = J^+ \cap J^-, \quad K = K^+ \cap K^-.$$

The notation suggests an analogy with the case of dimension 1. For $R$ large, we set

$$V^+ = \{|y| \geq |x|, |y| \geq R\}, \quad V^- = \{|x| \geq |y|, |x| \geq R\}, \quad V = \{|x|, |y| \leq R\}. \tag{4.3}$$

Filtration




These sets have the filtration properties:

1. If $(x,y) \in V^+$, then $f(x,y) \in V^+$, and $f^n(x,y) \to \infty$ as $n \to +\infty$.
2. If $(x,y) \in V$, then $f(x,y) \notin V^-$
3. If $(x,y) \in V^-$, then the orbit of $(x,y)$ can remain in $V^-$ for at most finite forward time. (More generally, any compact set must leave $V^-$ in finite time.)

We note that an additional feature of property 1 is that we have $f(x,y) \approx y^d$ for $(x,y) \in V^+$ with $\|(x,y)\|$ large. Thus for $(x,y) \in V^+$ the forward orbit is tangent to the $y$-axis at infinity.

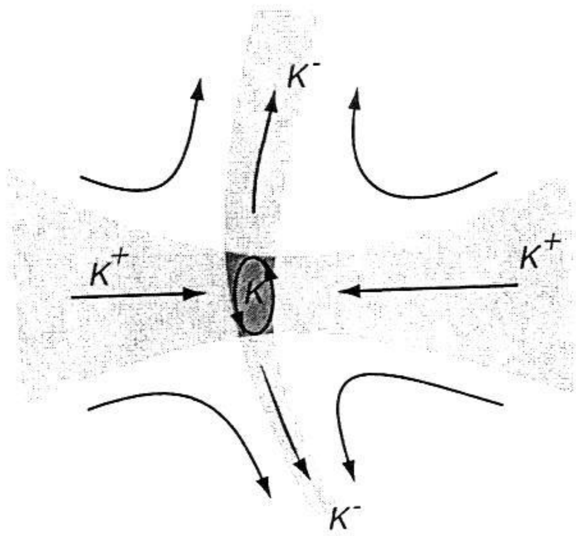
We note, too, that interchanging $V^+ \leftrightarrow V^-$, and possibly after further increasing $R$, we have the corresponding filtration properties for the function $f^{-1}$. This gives us a useful saddle-type picture of the "dynamics in the large" of $f$. Although little is known about mappings in higher dimension, there is a class of polynomial automorphisms of $\mathbf{C}^N$, $N \geq 2$ which posses filtrations (see [BP]).

A consequence of the filtration is that $f^{\pm 1}V^{\pm} \subset V^{\pm}$ and we have

$$U^{\pm} = \bigcup_{n \geq 0} f^{\mp n} V^{\pm}, \quad K^{\pm} = \bigcap_{n \geq 0} f^{\pm n}(V \cup V^{\mp}).$$

If $\sigma^{\pm}$ denotes a generator of $\pi_1(V^{\pm})$, then the curves $f_*^n \sigma^{\pm}$, $n \in \mathbf{Z}$ generate $H_1(U^{\pm})$.

Another consequence of the filtration is that any compact subset of $K^+$ is mapped, in finite time, inside of $V$. If we define the forward Fatou set as the set of normality for the forward iterates $\{f^n, n \geq 0\}$, then by the existence of the filtration, the forward Fatou set is seen to be $\mathbf{C}^2 - J^+$. Similarly, the set of normality for the backward iterates is seen to be $\mathbf{C}^2 - J^-$. We will discuss the Fatou sets in this section. We start with the components $U^{\pm}$.




*Dynamics in the Large*

The limit defining

$$G^{\pm}(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(x,y)\| \tag{4.4}$$

is locally uniform on $\mathbf{C}^2$, and $G^{\pm}$ is pluriharmonic on $U^{\pm}$. Then it follows that

$$G^+ \circ f = d \cdot G^+.$$

In the sequel we will focus on $G^+$ and $U^+$, but it is clear that an analogous discussion could be carried out for $G^-$ and $U^-$. The function $G^+$ is pluri-harmonic on $U^+$, and $U^+ = \{G^+ > 0\}$, so $U^+$ is pseudo-concave. One approach is to study $G^+ : U^+ \to \mathbf{R}^+$ as a fibration, i.e. in terms of the level sets $\{G^+ = c\}$. The level sets of $G^+$ in $U^+$ (a pluri-harmonic function) are foliated by Riemann surfaces, which are given by level sets of $G^+ + i(G^+)^*$, where $(G^+)^*$ is a (locally defined) harmonic conjugate function. This foliation of $U^+$, which we denote by $\mathcal{G}^+$, is generated by the holomorphic 1-form $\partial G^+$. It is shown in [HO1, §5] that the leaves of $\mathcal{G}^+$ are conformally equivalent to $\mathbf{C}$.

Let $\pi_y(x,y) = y$ denote the projection onto the second coordinate, so

$$\pi_y \circ f^n(x,y) = y^{d^n} + O(y^{d^n-2}) \tag{4.5}$$

for $(x,y) \in V^+$. It follows that the sequence

$$\varphi^+(x,y) := \lim_{n\to\infty} (\pi_y \circ f^n(x,y))^{\frac{1}{d^n}} \sim y + O(y^{-1}) \tag{4.6}$$

converges uniformly on $V^+$ and defines $\varphi^+$ as an analytic function there. Thus $\varphi^+$ satisfies $\varphi^+ \circ f = \sigma \circ \varphi^+$, so that $\varphi^+$ gives a semi-conjugacy from $f|_{V^+}$ to $\sigma|_{\mathbf{C}-\overline{\Delta}}$.

By (4.6) we see that $G^+ = \log|\varphi^+|$ on $V^+$. Thus the restriction of the foliation $\mathcal{G}^+$ to $V^+$ is given by the level sets of $\varphi^+$. Another consequence is that $\varphi^+ = e^{G^+ + iG^*}$ is determined by a choice of (pluri)harmonic conjugate $G^*$ of $G^+$, and this allows us to continue $\varphi^+$ analytically along any path in $U^+$ which starts in $V^+$. In view of the fundamental importance of $\varphi^+$ in the case of dimension 1, it is worthwhile to explore the analytic continuation of $\varphi^+$.

Let us consider the closed 1-form $\eta = \frac{1}{2\pi} d^c G^+$ on $U^+$. The integral $\int_\gamma \eta$ represents the obstruction to a single-valued harmonic conjugate $G^*$ of $G^+$ on $\gamma$. Thus $\int_\gamma \eta \,(\mathrm{mod}\ \mathbf{Z})$ is the obstruction to a single-valued analytic continuation of $\varphi^+$ along $\gamma$. If $\sigma \in \pi_1(U^+)$ satisfies $\int_\sigma \eta = 1$, then $\sigma$ corresponds to a generator of $\pi_1(V^+)$. If $\gamma \subset U^+$ is any closed path, then there exists $n \in \mathbf{Z}$ such that $f^n \gamma \subset V^+$. Thus $f^n_* \gamma \sim k\sigma$ for some $k \in \mathbf{Z}$. It follows that

$$d^n \int_\gamma \eta = d^n \int_\gamma \frac{1}{2\pi} d^c G^+ = \int_\gamma \frac{1}{2\pi} d^c (G^+ \circ f^n) = \int_\gamma f^{n*} \eta = k \in \mathbf{Z}.$$

If $d$ and $k$ are relatively prime, it follows that $(\varphi^+)^{d^n}$ has a single-valued analytic continuation around $\gamma$, but $(\varphi^+)^{d^{n-1}}$ does not. We conclude, also, that

$$\eta : \pi_1(U^+) \to \mathbf{Z}[\frac{1}{d}], \quad \eta(\gamma) := \int_\gamma \eta$$

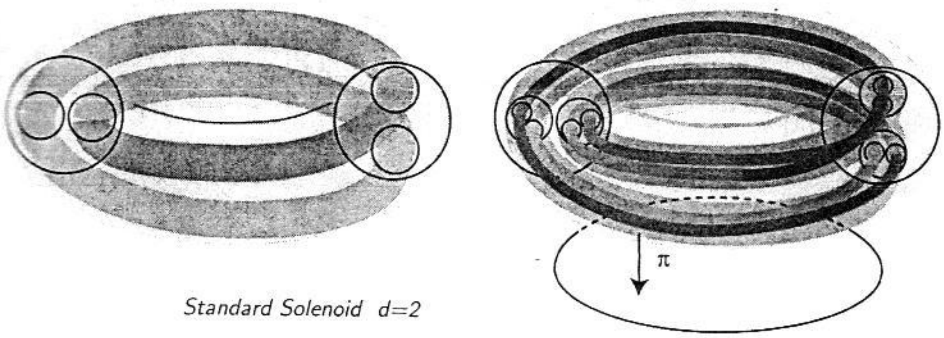
is an isomorphism.

Here we summarize some results from §8 of [HO1]. We let $\tilde{U}^+$ denote the Riemann domain over $U^+$ such that the multiple-valued function $\varphi^+ : U^+ \to \mathbf{C} - \overline{\Delta}$ lifts to a single-valued map $\tilde{\varphi}^+ : \tilde{U}^+ \to \mathbf{C} - \overline{\Delta}$. There is a biholomorphic map $\tau : \mathbf{C} \times (\mathbf{C} - \overline{\Delta}) \to \tilde{U}^+$ such that $\tilde{\varphi}^+ \circ \tau(t, \zeta) = \zeta$. The foliation $\mathcal{G}^+$ lifts to a foliation $\tilde{\mathcal{G}}^+$, whose leaves are level sets of $\tilde{\varphi}^+$, and under the uniformization $\tau$, they correspond to $\{(t, \zeta) : \zeta = const\}$. The mapping $f : U^+ \to U^+$ lifts to a mapping $\tilde{f} : \tilde{U}^+ \to \tilde{U}^+$.

[HO1] gives an explicit representation for $\tilde{f}$ in the case of degree 2. (The case of higher degree seems to introduce computational complications.) In this case, we write the mapping as $f(x, y) = (y, y^2 + c - ax)$. (Or via the conjugacy $(x, y) \mapsto (y, x)$, this is equivalent to $F(z, w) = (z^2 + c - aw, z)$.) The uniformizing coordinates $(t, \zeta)$ on $\mathbf{C} \times (\mathbf{C} - \overline{\Delta}) \cong \tilde{U}^+$ may be chosen so that the induced mapping of the covering space is given by

$$\tilde{f} : (t, \zeta) \mapsto (\frac{a}{2} t + \zeta^3 - \frac{c}{2} \zeta, \zeta^2).$$

We note that $\tilde{f}$ is not an automorphism of $\tilde{U}^+$.




*Standard Solenoid d=2*

By the discussion above, the only elements of $\pi_1(U^+)$ over which $\varphi^+$ has a single-valued analytic continuation are those in the range of $i_* : \pi_1(V^+) \to \pi_1(U^+)$. It follows that $\pi_1(\tilde{U}^+) \cong \mathbf{Z}$. Let $\Gamma \subset \mathrm{Aut}(\tilde{U}^+)$ denote the group of covering transformations for the covering $\tilde{U}^+ \to U^+$. It follows that

$$1 \to \pi_1(\tilde{U}^+) \to \pi_1(U^+) \to \Gamma \to 0$$

is exact. Thus $\Gamma \cong \mathbf{Z}[\frac{1}{d}]/\mathbf{Z}$. Another approach to understanding $U^+$ would be to describe the quotient $\tilde{U}^+/\Gamma$. A concrete description of this group $\Gamma$ is given in [HO1, Theorem 8.9]. This description of $\Gamma$ makes it possible to show that the complex structure of $U^+_{a,c} \cong \tilde{U}^+_{a,c}/\Gamma_{a,c}$ varies with $a$ and $c$.

We recommend that the reader also look at the work of Bousch [B] for futher material that we have not covered here.

It is interesting to note that, in contrast to the case of dimension 1, the topology of the set $U^+$ depends only on the degree $d$ of the mapping $f$ and is independent of the mapping itself. In fact, $U^+$ is homeomorphic to the complement of a cone over a real solenoid imbedded inside the 4-sphere $S^4$. A topological realization of the "standard" imbedding of a solenoid in 3-space is pictured in the case $d = 2$. This is obtained as follows. Let $\tau$ denote an injective mapping of the solid torus $S^1 \times D^2$ into itself, so that the image winds around twice inside the solid torus. The solenoid is then obtained as the infinite intersection $\bigcap_{n\geq 0} \tau^n(S^1 \times D^2)$. The generational structure is shown by the nested sequence of circles in the fiber. The projection to $\mathbf{R}/\mathbf{Z}$ is given by the vertical projection shown. The reader is referred to [HO1] for a more thorough discussion of different topological possibilities for imbeddings of the solenoid.

Since there is a symmetry $f \leftrightarrow f^{-1}$, $U^-_f$ is just the set $U^+_{f^{-1}}$, corresponding to another degree $d$ mapping, namely $f^{-1}$. Thus $U^-$ is homeomorphic to $U^+$.

We note a variation on the definition of the filtration $V, V^{\pm}$. Given $\epsilon > 0$, we may choose $R$ sufficiently large that $V^+ = \{|y| \geq \epsilon|x|, |y| \geq R\}$ satisfies $fV^+ \subset V^+$. Further, the definition of $\varphi^+$ converges uniformly on this larger set $V^+$. There is a small modification needed to define $\varphi^-$, since we may not, in general, assume that the polynomials in the definitions of $f$ and $f^{-1}$ are both monic. By the form (4.2), we have

$$\pi_x \circ f^{-n}(x,y) \sim a^{-(1+d+\cdots+d^{n-1})} x^{d^n} = a^{-\frac{d^n-1}{d-1}} x^{d^n}.$$

Let $c$ be any root of the equation $c^{d-1} = a^{-1}$. Thus we may define $\varphi^-$ on $V^-$ by setting

$$\varphi^-(x,y) = (\pi_x \circ f^{-n}(x,y))^{\frac{1}{d^n}} \sim cx,$$

since for $n$ large $a^{-(d^n-1)/(d-1)}$ is approximately $a^{-d^n/(d-1)}$, so there is a unique $d^n$-th root closest to $c$. It follows that the functions $\varphi^+$ and $\varphi^-$ are both defined and holomorphic on the set $\{\frac{1}{\epsilon}|x| \ge |y| \ge \epsilon|x|, |y| \ge R\} \subset U^+ \cap U^-$.

Nothing much seems to be known about the set $U^+ \cap U^-$, which carries both foliations $\mathcal{G}^\pm$ and the invariant 2-form $\partial G^+ \wedge \partial G^-$, as well as the invariant function $G^+G^-$.

## §5. Polynomial Automorphisms: General Properties

In the previous Section, we studied an automorphism with the special form (4.1). Friedland and Milnor [FM] addressed the general problem of describing the dynamics of a polynomial automorphism in two variables. To clarify the issue of what automorphisms occur, they classified the family of polynomial automorphisms modulo conjugacy, since two mappings which are conjugate necessarily have the same dynamics. For the conjugacy classes corresponding to mappings which they call "elementary," they give a rather complete description of the dynamics (see [FM, §6–8]). The other possibility in the [FM] classification consists of mappings which are finite compositions

$$f = f_m \circ \cdots \circ f_1, \tag{5.1}$$

where each $f_j$ is of the form (4.1). The entropy of this map is computed in the combined work of [FM, §4] and [Sm]; it is equal to the logarithm of the degree of $f$. Thus these mappings have complicated dynamical behavior. In [FM, §5] the $d$-fold horseshoe is discussed as a model for these mappings. Many of the results of §4 were based primarily on the use of the filtration for $f$ and will carry forward to mappings of the form (5.1).

The degree of a polynomial automorphism is not a conjugacy invariant. For instance, if $L$ is an invertible linear map, then the degree of the automorphism $f^{-1} \circ L \circ f$ is in general somewhere between 1 and the degree of $f$. A mapping of the form (5.1) has minimal degree within its conjugacy class. If the degree of $f_j$ is $d_j$, then it is evident from the form (5.1) that $f$ has degree $d = d_m \cdots d_1$, and the degree of the $n$-th iterate is $d^n$. As [FM] show, this degree $d$ is equal to the quantity

$$\delta(g) := \lim_{n\to\infty} (\deg(g^n))^{1/n}$$

for any $g$ which is conjugate to $f$. An elementary map $h$ is characterized by the property that $\delta(h) = 1$.

In [FM, §3] it is shown that if $f$ is of the form (5.1), then $\mathrm{Per}_n(f) = \{z \in \mathbf{C}^2 : f^n z = z\}$ contains $d^n$ points, counted with multiplicity. In particular, the fixed point set is discrete.

The question also arises whether the "typical" polynomial automorphism is trivial or nontrivial. [HOV1, §2] shows that for polynomial automorphisms of degree 2, there is a 2 (complex) parameter family of mappings of the form (4.1), which are dynamically nontrivial. On the other hand, the family of conjugacy classes of elementary mappings form several 1 and 2 parameter families. Examples of mappings which are elementary from the dynamical point of view are the "shears"

$$(z,w) \mapsto (z + g(w), w), \quad \text{and } (z,w) \mapsto (z, w + h(z)).$$

The $n$th iterates of these mappings are

$$(z,w) \mapsto (z+ng(w),w), \quad \text{and } (z,w) \mapsto (z, w+nh(z)),$$

which have the same fixed points and the same degrees as the original mappings. E. Andersen [A] showed that the group of finite compositions of shear transformations is dense in the set of volume-preserving automorphisms of $\mathbf{C}^2$. It is evident that a holomorphic shear may be approximated by a polynomial shear. In fact, from Theorem D of [A], it follows easily that the polynomial automorphisms are numerous: *If $F : \mathbf{C}^2 \to \mathbf{C}^2$ denotes a holomorphic mapping with constant Jacobian, and if $k$ is prescribed, then there is a polynomial automorphism $f$ of $\mathbf{C}^2$ such that $f - F$ vanishes to order $k$ at the origin.*

Let us continue our discussion of the Fatou components by considering the interior of $K^+$. We will derive as many properties of the components of the interior of $K^+$ as we can obtain from normal families arguments. More details concerning this kind of argument are found in [B], [C], and [FS2]. The Jacobian determinant of a mapping of the form (4.1) is equal to the constant $a$, so it follows that the Jacobian of a composition of such mappings is a constant, which we again write as $a$. If $|a| > 1$ then $K^+$ has no interior (see [FM, Lemma 3.7]), so if the interior of $K^+$ is nonempty, we necessarily have $|a| \le 1$. The case $|a| = 1$, in which $f$ preserves volume, is especially interesting, but we do not discuss it here. We refer the reader instead to the discussion in [BS2, Appendix]. Here we will assume that $|a| < 1$.

We say that a component $\Omega$ is *periodic* if $f^N\Omega = \Omega$ for some $N \in \mathbf{Z}$. It is not known whether all components of the interior of $K^+$ are periodic. We let $\Omega$ be a periodic component, and after replacing $f$ by $f^N$ if necessary, we may assume that $\Omega$ is fixed, i.e. $f\Omega = \Omega$. Since the restriction $\{f^n|_\Omega : n \ge 0\}$ is a normal family, we consider the set

$$\mathcal{H} = \{h : \Omega \to \mathbf{C}^2 : h = \lim_{j\to\infty} f^{n_j} \ \text{ for some subsequence } n_j \to +\infty\}$$

of all uniform limits on compact subsets of $\Omega$. Passing to a limit of a subsequence of $f \circ f^{n_j} = f^{n_j} \circ f$, we conclude that

$$f \circ h = h \circ f \tag{5.2}$$

holds for all $z \in \Omega$ and for all $h \in \mathcal{H}$. It follows that if $h \in \mathcal{H}$, then

$$f(\Sigma) = \Sigma \quad \text{for } \ \Sigma := h(\Omega), \tag{5.3}$$

so the elements of $\mathcal{H}$ may be viewed as a source of $f$-invariant sets. We observe that since the Jacobian determinant of $f^{n_j}$ is $a^{n_j}$, the Jacobian determinant of $h$ must be identically zero. Thus each $h \in \mathcal{H}$ has rank 1 on a dense open set, or the rank is identically 0.

We define a periodic component $\Omega$ to be *recurrent* if there is a point $z_0 \in \Omega$ such that the forward iterates of $z_0$ do not all tend to $\partial\Omega$. By definition, then, $\Omega$ is not recurrent if and only if $h(\Omega) \subset \partial\Omega$ for all $h \in \mathcal{H}$. To see the reason for working with recurrent domains, suppose that $\{p_j\}$ and $\{q_j := p_{j+1} - p_j\}$ are sequences with $p_j \to \infty$, $q_j \to \infty$, and $F := \lim_{j\to\infty} f^{p_j} \in \mathcal{H}$, $\rho := \lim_{j\to\infty} f^{q_j} \in \mathcal{H}$. Formally, we have the convergence of identities

$$f^{p_{j+1}} = f^{p_j} \circ f^{q_j} = f^{q_j} \circ f^{p_j} \to F = F \circ \rho = \rho \circ F \tag{5.4}$$

as $j \to \infty$, but the composition makes no sense if $\rho(\Omega) \subset \partial\Omega$ or $F(\Omega) \subset \partial\Omega$.

**Proposition.** *If $\Omega$ is not recurrent, and all the mappings of $\mathcal{H}$ have rank 0, then there is a fixed point $z_0 \in \partial\Omega$ such that $\lim_{n\to\infty} f^n z = z_0$ for all $z \in \Omega$.*

**Proof.** If $h_0 \in \mathcal{H}$ has rank 0, then $h_0(\Omega) = z_0$ is a point, and by (5.3), $z_0$ is a fixed point. An equivalent formulation of what remains to be proved is: The constant function $z_0$ is the only function in $\mathcal{H}$. It will suffice to show that any other function $h_1 \in \mathcal{H}$ coincides with $h_0$. By the previous discussion, we know that there $h_1 = z_1$ for some fixed point $z_1 \in \partial\Omega$. The number of fixed points is finite. Let us write them as $\{z_0, z_1, \dots, z_N\}$. We may find neighborhoods $V_j$ of $z_j$ such that $(V_j \cup f(V_j)) \cap V_k$ for $k \neq j$. Since $h_0, h_1 \in \mathcal{H}$, there are sequences $\{n_j\}$ and $\{m_j\}$, both tending to infinity, and with $n_j < m_j < n_{j+1}$, such that for some fixed $w' \in \Omega$ we have $f^{n_j}(w') \in V_0$, and $f^{m_j}(w') \in V_1$. Since $f^{n_j}w' \in V_0$ and $f^{m_j}w' \in V_1$, we must have $f^p w' \notin V_0$ for some $n_j < p < m_j$. Choosing $p_j$ to be the first value of $p$ for which this occurs, we have $f^{p_j}w' \notin \bigcup_{j=0}^N V_j$. Since $\{f^{p_j}\}$ is a normal family, we may extract a subsequence which converges to a mapping $\hat{h} \in \mathcal{H}$ with $\hat{h}(w') \in \overline{\Omega} - \bigcup_{j=0}^N V_j$. Thus $\hat{h}(\Omega)$ is not a fixed point, which is a contradiction. □

Let $\Omega$ and $z_0$ be as in the Proposition, and let $\lambda_1$ and $\lambda_2$, $|\lambda_1| \le |\lambda_2|$, denote the eigenvalues of the differential $Df(z_0)$. We cannot have $|\lambda_1|, |\lambda_2| < 1$, for in this case $z_0$ would be an attracting fixed point and thus part of the interior of $K^+$, which is not possible since $z_0$ is on the boundary of the component $\Omega$. Neither can we have $|\lambda_1| < 1 < |\lambda_2|$, for in this case $z_0$ is a saddle point, and (see (5.7)) there is a complex manifold $W^s(z_0)$ such that the only points that can approach $z_0$ as $n \to +\infty$ lie within the stable manifold $W^s(z_0)$. But $W^s(z_0)$ cannot contain $\Omega$, an open set. Thus the only possibility is $|\lambda_1| < |\lambda_2| = 1$. The question is open as to whether $\lambda_2$ is necessarily a root of unity.

In this case, replacing $f$ by some iterate $f^N$, we may assume that $\lambda_2 = 1$. Thus there are local coordinates such that $z_0 = 0$, and the map is given by

$$f : (x, y) \mapsto (x + x^{p+1} + O(|x|^{p+2} + |xy| + |y|^2), \lambda_1 y + O(|y|^2 + |x|^2)).$$

Such a fixed point is called *semi-attracting* if $|\lambda_1| < 1$. (Since the other eigenvalue is 1, it could also be called semi-parabolic.) In this case, let $\mathcal{D}$ denote the set of points where the iterates $\{f^n, n \ge 0\}$ converge locally uniformly to $z_0$. It is shown in [U1,2] in the case $p = 1$ and in [H] and [Kw] for $p \ge 1$ that $\mathcal{D}$ consists of $p$ components of the interior of $K^+$, each of which is non-recurrent. Further for each component $\mathcal{D}'$ there is a biholomorphism $T : \mathcal{D}' \to \mathbf{C}^2$ such that $T \circ f \circ T^{-1}$ is the translation $(z, w) \mapsto (z + 1, w)$ on $\mathbf{C}^2$. Thus $\mathcal{D}'$ gives an example of a rank zero, non-recurrent component of the interior of $K^+$.

It is not known whether there can be a non-recurrent periodic component such that $\mathcal{H}$ contains a mapping with rank one. In this case, the boundary has nontrivial complex structure given by $\Sigma := h(\Omega) \subset \partial\Omega$. In contrast to the semi-attracting case, $\Omega$ cannot be biholomorphically equivalent to $\mathbf{C}^2$, since it carries a non-constant, bounded function. Further, $\Omega$ would have a partition into varieties $V_\sigma := \{z \in \Omega : h(z) = \sigma\}$ for $\sigma \in \Sigma$, and $f(V_\sigma) = V_{f\sigma}$.

Let us discuss recurrent components. We start by defining a rotation domain. Let $\hat{\mathcal{R}} \subset \mathbf{C}$ denote a connected circular domain in the plane, i.e. the disk or an annulus. Let $\chi : \hat{\mathcal{R}} \to \mathbf{C}^2$ denote a holomorphic mapping such that

$$\mathcal{R} : \zeta \mapsto \chi^{-1} \circ f \circ \chi(\zeta) = e^{2\pi i \kappa}\zeta \tag{5.5}$$

with $\kappa \in \mathbf{R}$ irrational. The image $\mathcal{R} := \chi(\hat{\mathcal{R}})$ will be called a *rotation domain* for $f$. Under the dynamics of an irrational rotation, the orbit of a point is dense in the concentric circle containing that point. If $\mathcal{R}'$ is a circular domain which is relatively compact in $\mathcal{R}$, then the image $\chi(\mathcal{R}')$ is locally a variety and can have only a discrete singular set. Since the singular set is invariant under the irrational rotation, it must be empty. Thus we conclude that: *A rotation domain $\mathcal{R}$ is nonsingular, except possibly at $\chi(0)$.* Since $f$ is conjugate to a rotation on a rotation domain, it is an isometry of any intrinsic metric of $\mathcal{R}$. If we consider $f$ as a mapping of $\mathbf{C}^2$ leaving $\mathcal{R}$ invariant, then $f$ contracts volume, so $f$ must be strictly contracting in the direction normal to $\mathcal{R}$. It follows (proofs are given in [BS2, Proposition 2] and [FS1, §2]) that $\mathcal{R}$ is in the interior of $K^+$, so there is a component of the interior of $K^+$ associated with each rotation domain.

If $\Omega$ is recurrent, then there exists $z_0 \in \Omega$ and a sequence $p_j \to \infty$ such that $f^{p_j}(z_0)$ remains inside a compact subset of $\Omega$. Let us define the functions $F$ and $\rho$ as in (5.4). We set

$$W := \{z \in \Omega : \rho(z) = z\},$$

which is a subvariety of $\Omega$. Our analysis of recurrent domains begins with:

**Lemma.** *Either $W$ is an attracting fixed point for $f$, or $W$ is a rotation domain.*

**Proof.** If we set $\Sigma := \rho(\Omega)$, then we have

$$F(z_0) \in F(\Omega) \cap \Omega \subset W = \Sigma \cap \Omega \subset \Sigma, \tag{5.6}$$

where the left hand containment follows from (5.4). If $W = \{Q\}$ is a single point, then $F$ has rank zero. As we have seen above, $Q$ is a fixed point for $f$. If $F$ has rank zero, then

$$DF(Q) = \lim_{j\to\infty} Df^{p_j}(Q) = \lim_{j\to\infty} (Df(Q))^{p_j} = 0,$$

so we conclude that the eigenvalues of $Df(Q)$ are both less than one, which means that $Q$ is an attracting fixed point.

Otherwise, $F$, and thus $\rho$ has rank one. It is evident that $\rho : \rho^{-1}W \to W$ is a retraction, i.e. $\rho \circ \rho = \rho$, and it is an elementary property of retractions that the range $W$ is nonsingular. If $w \in \Sigma = \rho(\Omega)$, then $w = \rho(z) = \lim_{j\to\infty} f^{q_j} z$ for some $z \in \Omega \subset K^+$, so by the filtration properties, we conclude that $w \in V$. Thus $W \subset \Sigma \subset V$ is bounded.

Now we claim that there is a Riemann surface $\hat{\Sigma}$ and a finite, holomorphic mapping $\phi : \hat{\Sigma} \to \Sigma$ which is one-to-one outside of a finite set. To construct $\hat{\Sigma}$, we take an exhaustion $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ by relatively compact open sets. Each set $\Sigma_j := \rho(\Omega_j)$ is locally a complex variety, so there is a Riemann surface $\hat{\Sigma}_j$ and a mapping $\phi_j : \hat{\Sigma}_j \to \Sigma_j$, which resolves the nodes and crossings of $\Sigma_j$. The Riemann surface $\hat{\Sigma}$ is the union of the $\hat{\Sigma}_j$. Since $\Sigma$ is bounded, it carries nonconstant, bounded, holomorphic functions, and thus $\hat{\Sigma}$ is a hyperbolic Riemann surface. Thus $Aut(\hat{\Sigma})$ is a Lie group. We may remove a discrete set from $\hat{\Sigma}$, and $\phi$ becomes a biholomorphism to its image. Thus $f : \Sigma \to \Sigma$ lifts to an injective, holomorphic mapping $\hat{f}$ of the complement of a discrete subset. By the Riemann Removable Singularity Theorem, $\hat{f}$ extends to a holomorphic mapping of $\hat{\Sigma}$. Lifting $f^{-1}$, we have that $\hat{f} \in Aut(\hat{\Sigma})$.

Let $\mathcal{G}$ denote the Lie subgroup of $Aut(\hat{\Sigma})$ generated by the transformations $f^n$, $n \in \mathbf{Z}$. Since the fixed points of $f$ are isolated, it follows that the restrictions $f^n|_{\hat{\Sigma}}$ are distinct from the identity transformation on $\hat{\Sigma}$. Thus for distinct $n_1, n_2 \in \mathbf{Z}$, $\hat{f}^{n_1}$ is a distinct element from $\hat{f}^{n_2}$ in $Aut(\hat{\Sigma})$. Thus $\mathcal{G}$ is an infinite group, and since it has one generator, it is abelian. The abelian Lie groups are of the form $\mathbf{R}^m \times \mathbf{T}^n \times D$, where $D$ is a discrete abelian group.

We will show that $\mathcal{G}$ is not discrete. For this we note that $f^{q_j} \to \rho$ as $j \to \infty$. Since $\rho(z) = z$ on $W \subset \Sigma$, and $F(z_0) \in W$, it follows that $f^{q_j}$ converges to the identity transformation in a neighborhood of $F(z_0)$ inside $W$. We wish to conclude that $\hat{f}^{q_j}$ converges in $Aut(\hat{\Sigma})$ to the identity as $j \to \infty$. The topology of $\hat{\Sigma}$ is a priori different from the topology of its image $\phi(\hat{\Sigma}) \subset \mathbf{C}^2$, as a subset of $\mathbf{C}^2$. However, for the portion $\phi^{-1}W \subset \hat{\Sigma}$, the two topologies coincide, since $W$ is a (closed) subvariety of $\rho^{-1}W \subset \Omega$. A sequence of elements of $Aut(\hat{\Sigma})$ either converges to an element of $Aut(\hat{\Sigma})$, or it diverges to infinity uniformly on compact subsets of $\hat{\Sigma}$. Since $\hat{f}^{q_j}$ converges to the identity in a neighborhood of $\phi^{-1}(F(z_0))$, it cannot diverge to infinity. And since it converges to a limit in $Aut(\hat{\Sigma})$, the limit must be the identity.

Since $\mathcal{G}$ has a sequence converging to the identity, the connected component of the identity, written $\mathcal{G}_0$, must be $\mathbf{R}^m \times \mathbf{T}^n$, with $m$ and $n$ not both zero. Let us consider $\{n \in \mathbf{Z} : f^n \in \mathcal{G}_0\}$, which is a cyclic subgroup of $\mathbf{Z}$, generated by an element $N \in \mathbf{Z}$. Thus $\mathcal{G}_0$ must be generated by a single element, $f^N$. But $\mathbf{R}^m \times \mathbf{T}^n$ cannot be generated by a single element unless $m = 0$.

A hyperbolic Riemann surface with a circle action is necessarily a disk or an annulus. Thus we have a conformal equivalence $\hat{\chi} : \hat{\mathcal{R}} \to \hat{\Sigma}$, where $\hat{\mathcal{R}}$ is either the disk or an annulus. If $\hat{\Sigma}$ is simply connected, $f$ has a fixed point, and we may choose $\chi$ to take the origin to the fixed point. It is evident, then, that $\chi := \phi \circ \hat{\chi} : \hat{\mathcal{R}} \to \Sigma$ is a rotation domain, satisfying (5.5). □

Thus we have shown that if $h \in \mathcal{H}$ has rank one, then its image is a rotation domain. By analyzing the attracting behavior in the direction normal to the rotation basin, we get a global picture of the behavior of $f$ on the recurrent component $\Omega$:

**Theorem.** *A recurrent component must be one of the following:*

1. *a basin of attraction:* $\Omega = W^s(q)$, *where* $q$ *is an attracting periodic point.*
2. *a Siegel cylinder:* $\Omega \cong \mathbf{C} \times \Delta$, *and* $f^N|\Omega$ *is biholomorphically conjugate to the self-mapping* $(z, w) \mapsto (\alpha z, \beta w)$ *of* $\mathbf{C} \times \Delta$, *and the coefficients satisfy* $|\alpha| < 1$, $|\beta| = 1$, $\beta$ *not a root of unity.*
3. *a Herman cylinder: There exist* $|\alpha| < 1$, $|\beta| = 1$, $\beta$ *not a root of unity, and* $0 < r < 1$ *such that* $\Omega \cong \mathbf{C} \times \{r < |w| < r^{-1}\}$, *and* $f^N|\Omega$ *is biholomorphically conjugate to the self-mapping* $(z, w) \mapsto (\alpha z, \beta w)$ *of* $\mathbf{C} \times \{r < |w| < r^{-1}\}$.

This is proved in [BS2, §5]. An interesting question is whether a Herman cylinder can actually exist. A component of the interior of $K^+$ is Runge in $\mathbf{C}^2$ (see [BS2, Proposition 8], or [FS1]), so a Herman cylinder would be a Runge domain which is biholomorphically equivalent to the product of $\mathbf{C}$ and an annulus.

By a celebrated result of S. Newhouse and C. Robinson, there are real, quadratic mappings $f(x, y) = (y, y^2 + c - ax)$ with infinitely many real (and thus complex) sink orbits. We refer the reader to the work Buzzard [Bz] for a treatment of diffeomorphisms with infinitely many sinks and related questions.

One general result that we may bring to bear on this situation is [BS2, Theorem 4]: If $\Omega$ is a recurrent component, then $\partial\Omega = J^+$. In case there are more than one recurrent component, e.g. if there are two (or infinitely many) sinks, this forces a certain amount of topological complication on the way the sinks may be imbedded in $\mathbf{C}^2$.

The sets $J$, $J^-$ and $J^+$ carry the interesting dynamics of $f$. There seem to be few properties of $J^\pm$ that hold for all mappings $f$. One of them is that $K^+$, $K^-$, $J^+$ and $J^-$ are always connected (Theorem 7.2 of [BS1]). Another property concerns saddle points. A periodic point $P$, $f^N P = P$ is a *saddle point* if the eigenvalues of the differential $Df^N(P)$ of $f^N$ at the point $P$ do not have modulus 1. Thus if $P$ is a saddle point, the eigenvalues may be written as $\lambda^+, \lambda^-$, with $|\lambda^+| > 1 > |\lambda^-|$. The (global) stable and unstable manifolds are defined in the direct analogy of (3.5). By the Stable Manifold Theorem, these are imbedded Riemann surfaces. Since the Riemann surface $W^s(P)$ (resp. $W^u(P)$) has an invertible self-mapping with a contracting (resp. expanding) fixed point, it follows that it must be conformally equivalent to $\mathbf{C}$. In the case of the unstable manifold, for instance, there is an entire mapping

$$\psi : \mathbf{C} \to W^u(P) \subset \mathbf{C}^2, \quad \psi(0) = P, \quad \psi(\lambda^+\zeta) = f(\psi(\zeta)) \tag{5.7}$$

for $\zeta \in \mathbf{C}$. The set $J^-$ is also characterized as the closure of an arbitrary unstable manifold in [BS2, Theorem1]: *For any saddle point $P$, $W^u(P)$ is a dense subset of $J^-$.*

## §6. Mappings with $J$ connected

Our goal here is to try to carry through meaningful analogies with the conditions and techniques presented in §2: the extension of $\varphi$, the connectivity of $J$, the simple connectivity of $U$ (i.e. $K$ has no compact components), the concept of "no critical points" (i.e. all critical points of $p$ are contained in $K$), the canonical model $\sigma$ on $\mathbf{C} - \overline{\Delta}$, and the external rays.

The plan is to use the map $\sigma$ on the complex solenoid to replace the canonical model of $w \mapsto w^d$ on the complement of the unit disk. The set $U$, which was used in dimension one, will be replaced by the set $J^- \cap U^+ = J^- - K$. As in §4,5, we will assume that $|a| \le 1$, i.e. $f$ does not increase volume, but there is no longer a symmetry between $f$ and $f^{-1}$: the corresponding results do not hold with $f$ replaced by $f^{-1}$. The function $\varphi^+$ is analytic and single-valued on $V^+$. We consider the restriction $\varphi^+|_{V^+\cap J^-}$ and ask when $\varphi^+$ has a continuation to $J^+ \cap U^+$. As we have observed, $\varphi^+$ has a multiple-valued continuation to $U^+$, so this question is equivalent to asking whether $\varphi^+$ has a continuous extension to $J^+ \cap U^+$.

This problem is addressed in [BS6], which gives several conditions equivalent to the existence of this continuation. We summarize some of them in the following:

**Theorem.** *([BS6]) The following are equivalent for a mapping with $|a| \le 1$:*
1. *$\varphi^+$ extends continuously to $J^- \cap U^+$.*
2. *$H_1(J^- \cap U^+; \mathbf{R}) = 0$.*
3. *For every saddle point $P$, $K^+ \cap W^u(P)$ has no compact components.*
4. *There is a saddle point $P$ such that $K^+ \cap W^u(P)$ has no compact components.*
5. *$J$ is connected.*

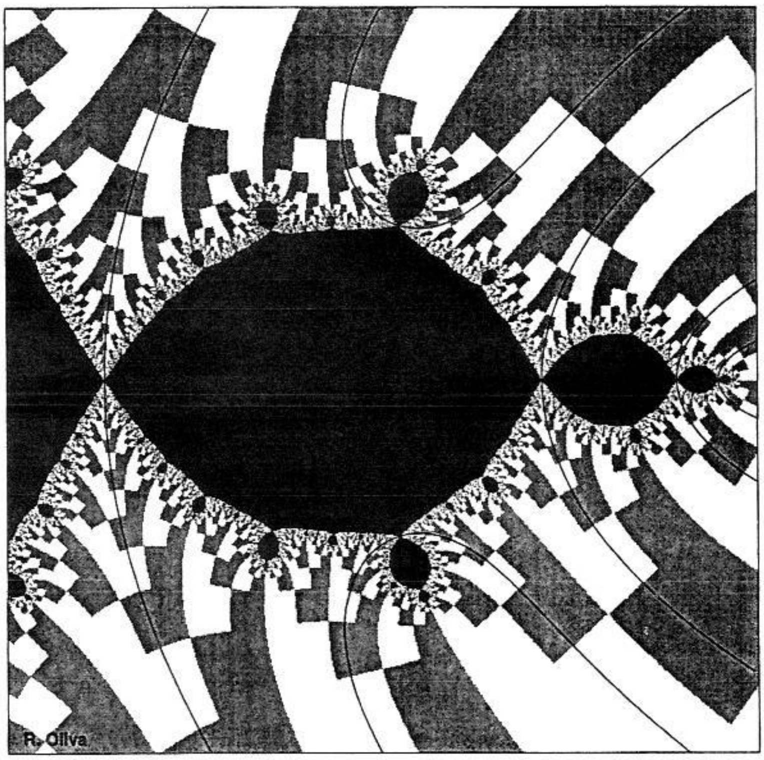
This Theorem may be modified to hold for mappings with $|a| \ge 1$ if in conditions 3 and 4 we replace $K^+ \cap W^s(P)$ by $K^- \cap W^s(P)$.

Before entering further into the general theory, it is perhaps interesting to consider an example. One case where the sets $J$, as well as $J^\pm$ can be described rather completely is when $p(y)$ is a hyperbolic polynomial of one variable and $a \approx 0$. The constructions for $J$ and $J^{\pm}$, are given by projective and inductive limits in terms of $p$. Let $J_p \subset \mathbf{C}$ denote the 1-dimensional Julia set of $p$. Then (see [FS1] and [HO2]) for $|a| \ll 1$, $f|J$ is conjugate to the natural dynamical system $\tilde{p} : \tilde{J}_p \to \tilde{J}_p$, associated with $p|_{J_p} : J_p \to J_p$. It is known by the one-variable theory that if $J_p$ is connected, it is a quotient of circle. Thus in constructing the natural extension mapping $\tilde{p}$ and the projective limit $\tilde{J}_p$, we are also constructing a quotient of the real solenoid.

In the special case $f(x, y) = (y, y^2 + c - ax)$ with $a \approx 0$ and $c \approx 0$, the "quotient" involves no identifications, i.e. $J_p$ is homeomorphic to $\partial\Delta$, and the $(\tilde{p}, \tilde{J}_p)$ is conjugate to $(\sigma, \Sigma_0)$. In fact, a global topological model for the mapping $(f, \mathbf{C}^2)$ has been given in [HO3].

Let us discuss the problem of describing the identifications that arise in the quotient of the solenoid, represented as in (3.2), and how we would represent them in terms of bi-infinite sequences of symbols, 0's and 1's. In this case, we may start with the corresponding problem, representing $J_p$ as the quotient of $\partial\Delta$ in terms of identifying base $d$ expansions. It is known that these may be represented by finite graphs with labeled edges. The graphs corresponding to $p(z) = z^2 - 1$ were drawn in §2. The passage to the natural extension is automatic: we use the same graphs, but we use them instead to generate bi-infinite sequences. This means that we follow the arrows forward to generate the part of the sequence for $n \geq 0$ and follow the arrows backward to generate the part of the sequence for $n \leq 0$.

a=0.01 , c=-1




An Unstable Slice: $f(x, y) = (y, y^2 + c - ax)$

An intrinsic object which is useful for understanding $f$ is $W^u(P) \cap K^+ = W^u(P) \cap K$, for a periodic point $P$, which we call the "unstable slice of $K^+$." If $\varphi^+$ can be extended to $J^-$ (one of the conditions equivalent to the connectivity of $J$), then we may draw the computer picture in 2 dimensions, which is directly analogous to the 1-dimensional picture. Namely, we choose a saddle point $P$, let $\psi_P$ be as in (5.7), and define the composition $\varphi_P^+ := \varphi^+ \circ \psi_P$. The computer picture is drawn in $\mathbf{C}$ in terms of the level sets of the modulus and argument of $\varphi_P^+$. This was introduced by J.H. Hubbard and has been very useful. A short explanation of how to read the solenoidal data from one of these pictures is given in Appendix B of [BS7]; see also the Thesis [O] of Oliva for an extensive treatment.

To illustrate this algorithm, we give here the computer picture for the mapping $f(x, y) = (y, y^2 + c - ax)$ in the case $c = -1$, $a = .01$. This mapping may be thought of as a perturbation of $z^2 - 1$, and it is visually evident that this picture looks like a small neighborhood of the fixed point $.0^\infty = .1^\infty$ in "Computer Picture of $z^2 - 1$." In fact, it follows from the construction of the natural extension that a neighborhood of $.0^\infty$ in the Julia set of $z^2 - 1$ is homeomorphic to a neighborhood of $P$ inside $W^u(P) \cap K^+$. The point on the extreme right of the picture is the saddle point corresponding to ${}^\infty 0.0^\infty = {}^\infty 1.1^\infty$. By (5.7), the slice by an unstable manifold, with the natural parametrization by $\mathbf{C}$, is invariant under the mapping $\zeta \mapsto \lambda^u \zeta$, the picture is self-similar. Thus all of the information of the picture is contained already in a small neighborhood of the fixed point. Since this picture represents a slice by the unstable manifold $W^u({}^\infty 0.0^\infty) = W^u({}^\infty 1.1^\infty)$, it follows that all rays in this picture have solenoidal codings of the form ${}^\infty 0 * .*$ or ${}^\infty 1 * .*$. By inspection of the black and white regions these rays pass through, the external rays that are drawn in the figure all have the form $*.*(01)^\infty$. The landing points of these rays are then in the stable manifold of either ${}^\infty(01).(01)^\infty$ or ${}^\infty(10).(10)^\infty$. Let $Q \in \mathbf{C}^2$ denote the corresponding fixed point. The points in the solenoid corresponding to these rays are of the form ${}^\infty 0 * . * (01)^\infty$ or ${}^\infty 1 * . * (01)^\infty$.

Let us recall the concept of a *Riemann surface lamination* of a topological space $X$. We start by giving a heuristic definition, and in the subsequent paragraph we give a more precise definition. If $\Delta$ is a complex disk, and if $Y$ is a closed set, then the *product lamination* is the partition of $\Delta \times Y$ given by $\{\Delta \times \{y\} : y \in Y\}$. More generally, let $\mathcal{L}$ denote a partition of $X$ by connected manifolds which have the structure of Riemann surfaces. For an open set $U$, we let $\mathcal{L}|U$ denote the restriction lamination, whose leaves are the connected components of intersections of the leaves of $\mathcal{L}$ with $U$. We will say that the partition $\mathcal{L}$ is a lamination if each point $P \in X$ has a neighborhood $U$ such that $\mathcal{L}|U$ is a product lamination. We observe that if $X$ is a manifold, then a lamination of $X$ is just a foliation. Thus a lamination may be thought of as being a foliation of a closed set that is not a manifold.

To give the definition of a lamination, we make several preliminary definitions. A chart is a choice of an open set $U_j \subset X$, a topological space $Y_j$ and a map $\rho_j : U_j \to \mathbf{C} \times Y_j$ which is a homeomorphism onto its image. An atlas is a collection of charts such that $\{U_j\}$ covers $X$. The set of points of $U_j$ for which the second coordinate of $\rho_j$ assumes a fixed value is called a plaque. For coordinate charts $(\rho_i, U_i, Y_i)$ and $(\rho_j, U_j, Y_j)$ with $U_i \cap U_j \neq \emptyset$, the transition function is the homeomorphism from $\rho_j(U_i \cap U_j)$ to $\rho_i(U_i \cap U_j)$ determined by $\rho_{ij} = \rho_i \circ \rho_j^{-1}$. A Riemann surface lamination of a topological space $X$ is determined by an atlas of charts which satisfy the following consistency condition: the transition functions may be written in the form $\rho_{ij} = (g(x,y), h(y))$, where for fixed $y \in Y_j$, the function $z \mapsto g(z,y)$ is holomorphic. The condition on the transition functions gives a consistency between the plaques defined in $U_j$ and those in $U_i$. Thus plaques fit together to make global manifolds called leaves of the lamination, and each leaf has the structure of a Riemann surface.

In [BS6, Theorem 2.1] we show that: *If $J$ is connected, then there is a Riemann surface lamination $\mathcal{M}^-$ of the set $J^- \cap U^+$.* This lamination is unique in the sense that if $M \subset \mathbf{C}^2$ is any Riemann surface which is also contained in $J^-$, then $M$ is a piece of the lamination $\mathcal{M}^-$. Further, $\varphi^+$ is holomorphic and locally injective on the leaves of $\mathcal{M}^-$.

The connection with the complex solenoid is as follows. If $\varphi^+$ extends to $J^- \cap U^+$, then we have a mapping

$$\Phi : J^- \cap U^+ \to \Sigma_+, \quad \Phi(p) = (\phi_n(p)), \quad \phi_n(p) = \varphi^+(f^n p). \tag{6.1}$$

A consequence of the identity $(\varphi^+)^d = \varphi^+ \circ f$ is the fact that $\Phi \circ f = \sigma \circ \Phi$. Thus $\Phi$ gives a semi-conjugacy from $(f, J^- \cap U^+)$ to $(\sigma, \Sigma_+)$. In fact (see [BS6, Theorem 3.2]) the connection with the lamination $\mathcal{M}^-$ is as follows: *For any leaf $M$ of $\mathcal{M}^-$, the restriction $\Phi|_M : M \to \Sigma_+$ is injective, and $\pi \circ \Phi|_M : M \to H$ is a covering of the right half plane $H$.*

A further consequence is that $\Phi$ is locally injective on the leaves of the lamination $\mathcal{M}^-$. This may be seen as the generalization of the condition of "no critical points." We do not discuss critical points here as they are an issue somewhat different from solenoids. The reader is referred to [BS5] for a development of the concept of "dynamical critical point" for the diffeomorphism $f$.

For $s \in \Sigma_0$, we will define a ray in $\Sigma_+$ as $R_s = \{\mathbf{exp}(t)s \in \Sigma_+ : t > 0\}$. Another way to define this ray is to set $\pi(s) = e^{i\theta}$ and let $R_s$ be the lift under $\pi$, passing through $s$, of the ray $R_\theta \subset \mathbf{C} - \overline{\Delta}$. The set of rays in the complex solenoid $\Sigma_+$ is parametrized by the real solenoid $\Sigma_0$.

We define the set of *external rays* $\mathcal{E}$ as follows: for any leaf $M$ of the lamination $\mathcal{M}^-$, and any point $m \in M$, we let the external ray $\gamma_m$ passing through $m$ be defined as the lift of the corresponding ray in $\Sigma_+$, i.e. $(\Phi|_M)^{-1} R_s$, where $R_s$ is the ray passing through $\Phi(m)$. An alternative definition of the family $\mathcal{E}$ involves the Green function. If $J$ is connected, then for any leaf $M$ of $\mathcal{M}^-$, the restriction $G^+|_M$ has no critical points. Thus we could also define the external rays as the set of gradient lines of $G^+|_M$.

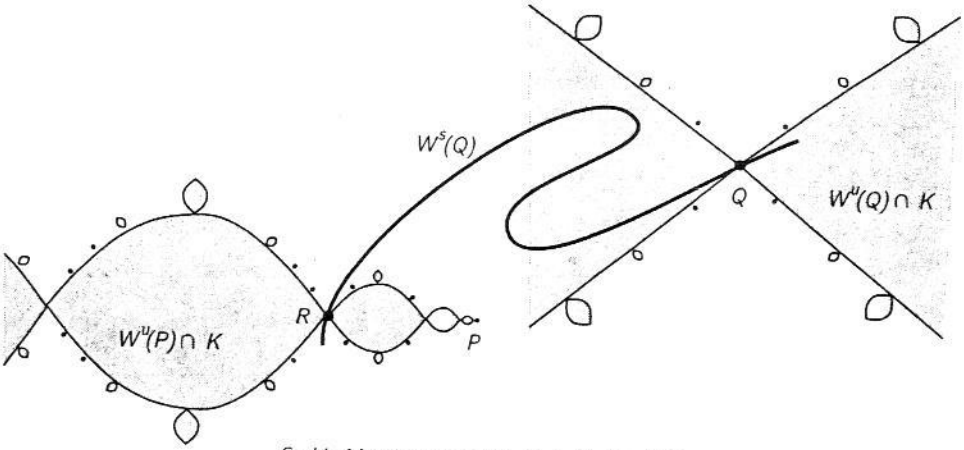
It follows that $f$ maps $\mathcal{E}$ to itself, and it would be good to represent the mappings $(f, J^-)$ and $f|J$ in terms of the solenoid.

In the sequel, we assume that $J$ is a *hyperbolic set* for $f$. In this case we will also say that $f$ is hyperbolic. That means that there is a splitting of the tangent bundle over $J$, i.e. $T\mathbf{C}^2 = E^s + E^u$, and there are constants $C < \infty$ and $\lambda < 1$ such that

$$\|Df_p^n v\| \le C\lambda^n, \quad n \ge 0, v \in E^s,$$

$$\|Df_p^{-n} v\| \le C\lambda^n, \quad n \ge 0, v \in E^u,$$

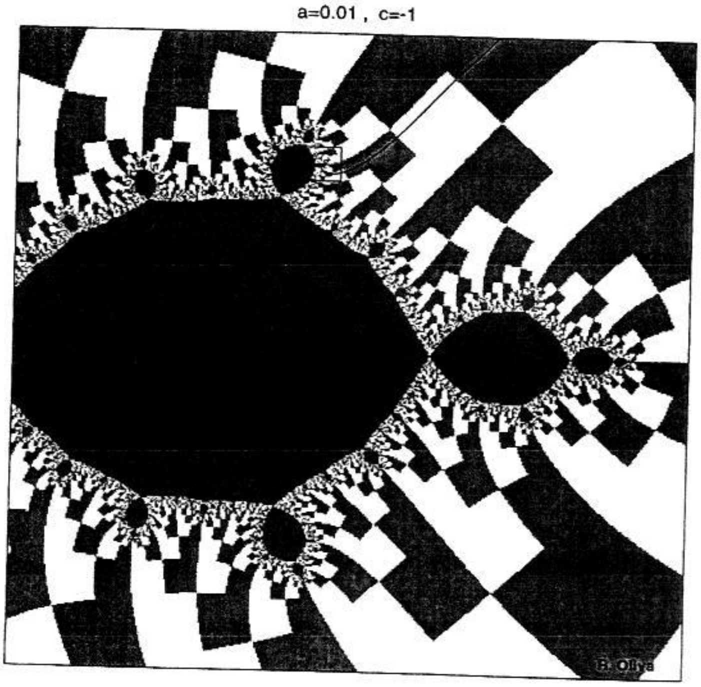
i.e. there are directions in which $Df_p$ is uniformly expanding/contracting for all $p \in J$.




*Stable Manifold Connects Unstable Manifolds*

If $f$ is hyperbolic, then by the Stable Manifold Theorem (see [S] for a treatment of this basic result), the stable and unstable manifolds form Riemann surface laminations $\mathcal{W}^s = \{W^s(P) : P \in J\}$ and $\mathcal{W}^u = \{W^u(P) : P \in J\}$. By [BS1], $\mathcal{W}^s$ is a lamination of $J^+$, and $\mathcal{W}^u$ is a lamination of $J^- - \mathcal{S}$, where $\mathcal{S}$ denotes the set of sinks (i.e. attracting periodic points). If $J$ is also connected, the leaves of $\mathcal{M}^-$ are contained in leaves of $\mathcal{W}^u$. We have already observed that $U^\pm$ have holomorphic foliations $\mathcal{G}^\pm$. Each component $M$ of $W^u(P) \cap U^+$ is a component of $\mathcal{M}^-$.

a=0.01 , c=-1



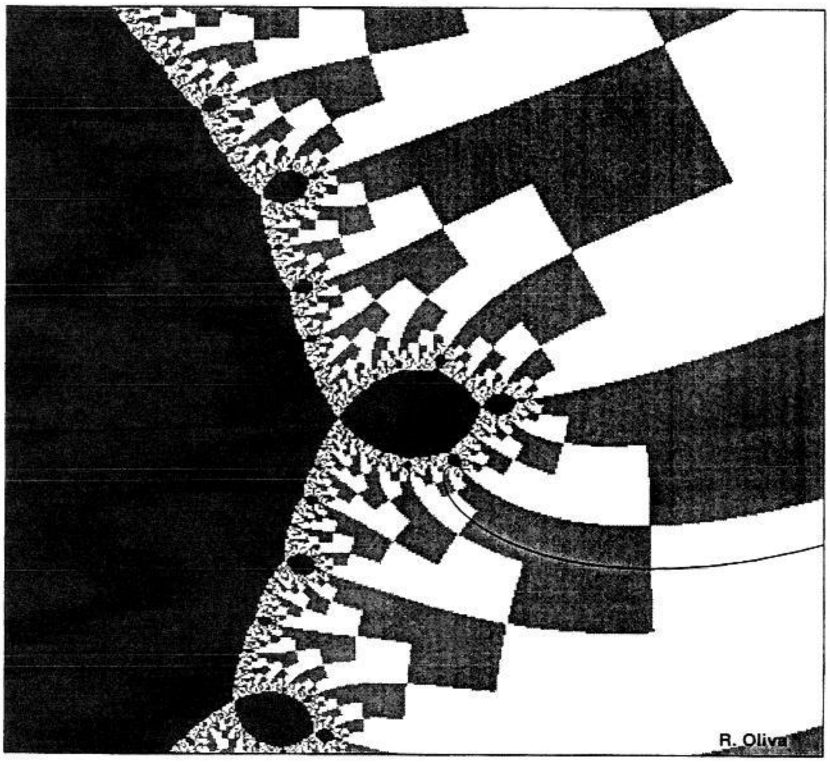

Ray $\gamma$ Landing at point $T$

A general property of the stable and unstable laminations is the *local product structure*. Let $f$ be a hyperbolic mapping, and let $P \in J$ be any point. Then there is a neighborhood $N^+$ of $P$ inside $J^+$ such that $\mathcal{W}^s|N^+$ is a product lamination $\Delta \times X$. Similarly, there is a neighborhood $N^-$ inside $J^-$ such that $\mathcal{W}^u|N^-$ is a product lamination $\Delta \times Y$. The local product structure enters from the mapping

$$X \times Y \ni (x, y) \mapsto [x, y] := W^s_x \cap W^y_y$$

where $W^s_x$ denotes the piece of stable manifold corresponding to the piece of lamination $\Delta \times \{x\} \subset N^+$, and similarly for $W^u_y$. (For further discussion of the local product structure, see §5 of [BS7].) A consequence of the local product structure is that for $y \in Y$, the slice $W^u_y \cap J$ is homeomorphic to $X$. Thus all these local slices are homeomorphic to each other.

a=0.01 , c=-1




Ray $\gamma$ Landing at point $T$ (detail)

The local product structure leads to a further understanding of the picture $c = -1$, $a = .01$. We have observed that the external rays that are pictured land at points belonging to $W^s(Q)$. Thus the stable manifold $W^s(Q)$ serves to connect the local picture of $W^u(P)$ at the point $R$ with the local picture of $W^u(Q)$ at $Q$. The reason for this is that $W^u(P) = W^u(R)$. If $R$ were sufficiently close to $Q$, then by the local product structure property of $J$ the slice $W^u(R) \cap J$ would be locally homeomorphic to $W^u(Q) \cap J$ at $Q$. However, everything is invariant under $f$, so there is no loss of generality if we first apply $f^n$. Since $R$ is in the stable manifold, $f^n R$ approaches $Q$ as $n \to \infty$. Thus we may assume that $R$ is sufficiently close to $Q$ that it lies in a local product neighborhood. Our conclusion from this is that $W^u(Q) \cap J$, seen within $W^u(Q)$ parametrized by $\mathbf{C}$ is a self-similar set which looks just like $W^u(P)$ in a neighborhood of $R$. In the illustration "Stable Manifold Connects Unstable Manifolds" we draw a heuristic picture of $W^u(Q) \cap K$, based on what we would obtain if we scaled out the picture of $W^u(P) \cap K$ to make it self-similar at the point $R$. In other words, the computer picture for $c = -1$, $a = .01$, which was based at the fixed point $P$, also contains all the information of the corresponding computer picture based at the other fixed point $Q$.

A similar procedure lets us describe the picture $W^u(S) \cap K$ for a saddle point $S$ of period 3. To do this, we find an external ray, which we may call $\gamma$, and which has a trajectory that approaches $K$ through a sequence which keeps repeating "black-white-black." This takes some practice. Perhaps the best place to start is in the picture for the map $a = .01$, $c = -1$. By $\gamma_1$ we denote the ray which exits just above the upper right hand corner of the picture. $\gamma_1$ enters the picture with "black" and has an infinite repetition of "black-white." The ray $\gamma$ will be just above and to the left of $\gamma_1$ in the starting black region. Then, as $\gamma$ and $\gamma_1$ move in toward $K$, they enter the same white region. But $\gamma$ curls to the left to enter "black-black" whereas $\gamma_1$ enters "black-white." The coding of $\gamma$ is ${}^{\infty}0 * . * (101)^{\infty}$, which lands at a point $T \in W^u(P) \cap K$. By the coding of $\gamma$ (equivalent to the coding of $T$), it follows that $T$ is contained in the stable manifold of $S$, where $S$ is in the 3-cycle generated by ${}^{\infty}(101).(101)^{\infty}$.

The picture of the unstable slice $W^u(S) \cap K$ at $S$ may be found by blowing up the picture $W^u(P) \cap K$ at the pont $T$ so that it is self-similar with respect to the multiplier $\lambda_S^+$. In fact, $W^s(S) \cap K$ has a spiral behavior at $S$. We can get an impression of this from the picture of ray landing at the point $T$ and the more detailed blow-up picture at $T$.

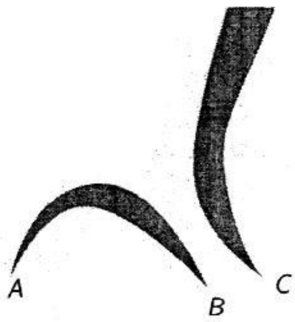
In [BS7, Proposition 2.7] it is shown that *If $J$ is connected and hyperbolic, then $\mathcal{G}^+ \cup \mathcal{W}^s$ forms a Riemann surface lamination of $U^+ \cup J^-$.* It is clear that $\mathcal{G}^+ \cup \mathcal{W}^s$ is a partition of $U^+ \cup J^+$ by Riemann surfaces. What needs to be shown is that $\mathcal{G}^+ \cup \mathcal{W}^s$ is locally trivial at every point $P \in J^-$. We note, that on the other hand, the partition $\mathcal{G}^- \cup \mathcal{W}^u$ of $U^- \cup J^-$ is not locally trivial if $J$ is connected. (See Appendix A of [BS7].)

If $J$ is hyperbolic, then for each $P \in J$, the stable and unstable manifolds $W^{s/u}(P)$ are conformally uniformized by $\mathbf{C}$. The only conformal self-equivalences of $\mathbf{C}$ are the complex affine maps $z \mapsto \alpha z + \beta$ for $\alpha, \beta \in \mathbf{C}$, $\alpha \neq 0$. Thus the conformal structure of $\mathbf{C}$ naturally admits a complex affine structure. We may assign a complex affine coordinate to $W^u(P)$, which is unique modulo affine self-equivalences of $\mathbf{C}$. (In the discussion of affine structure, all results remain valid with $W^u(P)$ replaced by $W^s(P)$; for convenience, we will only formulate results for $\mathcal{W}^u$.) If $P \in J$, if $\zeta$ is a complex affine coordinate on $W^u(P)$, and $\zeta'$ is a complex affine coordinate on $W^u(fP)$, then the mapping $f : W^u(P) \to W^u(fP)$ is given by $\zeta \mapsto \zeta' = \alpha'\zeta + \beta'$. Any metric that is compatible with this affine structure is a positive multiple of the Euclidean metric on $\mathbf{C}$. Although length is not an affine invariant, a geodesic for an affine metric is a geodesic for the Euclidean metric, and angles in this metric coincide with Euclidean angles.

Having an affine structure on each $W^u(P)$ gives us an affine structure on the lamination $\mathcal{W}^u$. Let us consider a small neighborhood $U$ of $P$ such that the restriction lamination $\mathcal{W}^u|_U$ is locally trivial, i.e. there is a compact space $Y$ such that $\mathcal{W}^u|_U \cong \{\Delta \times \{y\} : y \in Y\}$, where $\Delta$ denotes a topological disk. The affine structure on $W^u(P)$ induces an affine structure on each topological disk $\Delta \times \{y\}$. If $T_A$, $T_B$, and $T_C$ are disks in $\mathbf{C}^2$ which intersect the lamination $\mathcal{W}^u|_U$ transversally, then for $y \in Y$, these transversals intersect the plaque $\Delta \times \{y\}$ in points $A_y$, $B_y$, and $C_y$. The affine structure induced on $\Delta \times \{y\}$ allows us to define an angle $\angle A_y B_y C_y$. We say that the affine structure on $\mathcal{W}^u$ is *continuous* if $y \mapsto \angle A_y B_y C_y$ is continuous. We have: *If $f$ is hyperbolic, then the affine structure induced on $\mathcal{W}^u$ is continuous.* This is [BS7, Proposition 5.1], which is an adaptation of [G] to our case.

The question naturally arises whether the complex affine coordinate on $W^u(P)$ is somehow consistent with its imbedding into $\mathbf{C}^2$. As we have observed already, the imbedding of the unstable manifold $W^u(P)$ in $\mathbf{C}^2$ is not closed. Thus the topology of $W^u(P)$ as a complex manifold differs from (is inconsistent with) its topology as a subspace of $\mathbf{C}^2$. For the affine structure, let $U$ be a neighborhood such that $\mathcal{W}^u|_U$ is the product lamination $\Delta \times Y$. Let $y_0$ be such that $\Delta \times \{y_0\}$ is the plaque of $W^u(P) \cap U$ passing through $P$. Since $W^u(P)$ is dense in $J^-$, it follows that there is a sequence of plaques $\Delta \times \{y_j\}$ converging to $\Delta \times \{y_0\}$. Under the mapping $\psi : \mathbf{C} \to W^u(P)$, each plaque $\Delta \times \{y\}$ with $y = y_j$ has a preimage under $\psi$ which is a simply connected domain $\hat{U}_j \subset \mathbf{C}$, and the angle $\angle A_y B_y C_y$ is the same as the angle $\angle(\psi^{-1}A_y, \psi^{-1}B_y, \psi^{-1}B_y)$, formed in $\mathbf{C}$ by the preimages. Since $\hat{U}_j$ travels off to infinity as $j \to \infty$, it is difficult to know a priori whether the angle $\angle A_{y_j} B_{y_j} C_{y_j}$ converges as $j \to \infty$. This convergence, however, is a consequence of the continuity of the affine structure of $\mathcal{W}^u$.

Carrots and Cigars




If $J$ is connected, then we have the mapping $\Phi : J^- \cap U^+ \to \Sigma_+$ given in (6.1). We do not know whether $\Phi$ is a homeomorphism. However (see [BS7, Theorem 4.2]): *If $f$ is hyperbolic, then there is a homeomorphism $\Psi : \Sigma_+ \to J^- \cap U^+$ such that $f \circ \Psi = \Psi \circ \sigma$.* Note that for $k \in \mathbf{Z}$, $k \geq 0$, $(k,d) = 1$, the $k$th power mapping $s \mapsto s^k$ (defined by raising each coordinate in $s = (s_j)$ to the $k$th power) defines a $k$-fold covering of $\Sigma_+$. Of course, if $k = d$, then $s \mapsto s^k = s^d = \sigma(s)$ is a homeomorphism. $\Psi^{-1}$ is not far from being $\Phi$ itself, for in the Theorem above, it is shown that there exist $k \in \mathbf{Z}$, $k \geq 0$, $(k,d) = 1$, and $s_0 \in \Sigma_0$ such that

$$\Phi = m_{s_0} \circ (\Psi^{-1})^k,$$

where we define $m_{s_0}$ as the multiplication operator $m_{s_0}(t) = s_0 t$ for all $t \in \Sigma_0$.

We let $E \subset \mathbf{C}$ denote a path from $a$ to $b$. For $c > 0$ we let

$$car(E,c) = \{z \in \mathbf{C} : |z - x| < c|x - a| \text{ for some } x \in E\}$$

$$cig(E, c) = \{z \in \mathbf{C} : |z - x| < c \min(|x - a|, |x - b|) \text{ for some } x \in E\}.$$

We call $car(E, c)$ the *carrot* about $E$ with opening $c$, or *c-carrot*, and $cig(E, c)$ is the *c-cigar* about $E$. Although the definition of carrots and cigars is given in terms of the Euclidean metric, any metric compatible with the affine structure of $\mathbf{C}$ (i.e. any constant multiple of the Euclidean metric) defines the same $c$-cigar and $c$-carrot. Thus $c$-cigars and $c$-carrots are properties of the complex affine structure. A domain $D \subset \mathbf{C}$ is said to satisfy the *c-cigar condition* if each pair of points in $D$ may be joined by a path $E$ such that $cig(E, c)$ is contained in $D$. We note that the $c$-cigar property is preserved under complex affine transformations.

The mapping $f : \mathbf{C}^2 \to \mathbf{C}^2$ is affine with respect to the affine structure of $\mathcal{W}^u$, since if $\psi_P : \mathbf{C} \to W^u(P)$ and $\psi_{fP} : \mathbf{C} \to W^u(fP)$ are uniformizations, $\psi_{fP}^{-1} \circ f \circ \psi_P : \mathbf{C} \to \mathbf{C}$ is a conformal equivalence, and thus affine. It follows that a $c$-cigar or $c$-carrot in $W^u(P)$ is mapped to a $c$-cigar or $c$-carrot in $W^u(fP)$. Using this invariance and the continuity of the affine structure we get ([BS7, Theorem 5.2]): *If $f$ is hyperbolic, and if $J$ is connected, then there exists $c > 0$ such that for $P \in J$, each component of $W^u(P) - K^+ = W^u(P) \cap U^+$ satisfies the c-cigar condition, and for each $Q \in J$, there is a carrot $car(E, c) \subset W^u(P) \cap U^+ \cup \{Q\}$ connecting $Q$ to infinity.* The illustration "Carrots and Cigars" shows a domain with the properties of this Theorem. The crescent connecting $A$ to $B$ is a cigar, which shows that the opening of a fjord must be proportional to its length. The carrot connecting $C$ to infinity shows that there must be good access to a boundary point from the interior of the domain, and there is a wide exit to infinity. The consequence of this Theorem is similar to saying that $W^u(P) \cap U^+$ satisfies the John condition (see [NV]). A difference, however, is that $W^u(P) \cap K^+$ is not compact in the topology of $W^u(P)$, and so infinity can not be treated as an interior point of $W^u(P) - K$.

One useful property is that the conformal uniformization of a John domain by the unit disk (or upper half plane) extends continously to the boundary. Something analogous holds for the mapping $\Psi$ defined in (6.1): *If $f$ is hyperbolic, and if $J$ is connected, then the mapping $\Psi : \Sigma_+ \to J^- \cap U^+$ extends to a continuous mapping $\Psi : \Sigma_+ \cup \Sigma_0 \to (J^- \cap U^+) \cup J$. Further, $\Psi|_{\Sigma_0} : \Sigma_0 \to J$ is finite-to-one.* The mapping $\Psi : \Sigma_0 \to J$ is the map we have been looking for: it represents $J$ as a finite quotient of the real solenoid $\Sigma_0$.

It will be interesting to explore what possible quotients can arise from a polynomial automorphism $f : \mathbf{C}^2 \to \mathbf{C}^2$ for which $J$ is connected. There are restrictions on the possible identifications given in [BS7].

Let us say that the slices by unstable manifolds at points $p, p' \in J$ are *locally homeomorphic* if there is a homeomorphism of a neighborhood $N$ of $p$ in $W^u(p)$ to a neighborhood $N'$ of $p'$ in $W^u(p')$ which takes $K \cap N$ to $K \cap N'$. We will want to try to explain one of the local homeomorphism properties that is most visible to the eye in computer pictures. For $p \in J$, we define the number $v(p)$, the *valence* at $p$ to be the number of local components of $W^u \cap K - \{p\}$ at $p$. Since $W^u(p) - K^+$ has the cigar condition, it follows that the number of local components is bounded. Further, this is equal to the number of local components $W^u - K^+$ at $p$. We defined a cut point in §2. Here we see that $p$ is a cut point for $W^u(p) \cap K^+$ if and only if $v(p) \geq 2$.

Now we define $a(p)$ to be the number of components of $W^u(p) \cap \text{int}(K^+)$ which contain $p$ in their boundaries. Clearly $a(p) \leq v(p)$. We define a *pinch point* to be a point $p$ such that either $v(p) \geq 3$ or $a(p) \geq 2$. If $p$ is a cut point, then it is also a pinch point unless $a(p) = 0$, i.e. if there is no component of $W^u(p) \cap \text{int}(K^+)$

whose closure contains $p$. For instance, an inspection of the picture of the unstable slice in the case $c = -1$, $a = .01$ shows that all cut points for this set are in fact pinch points.

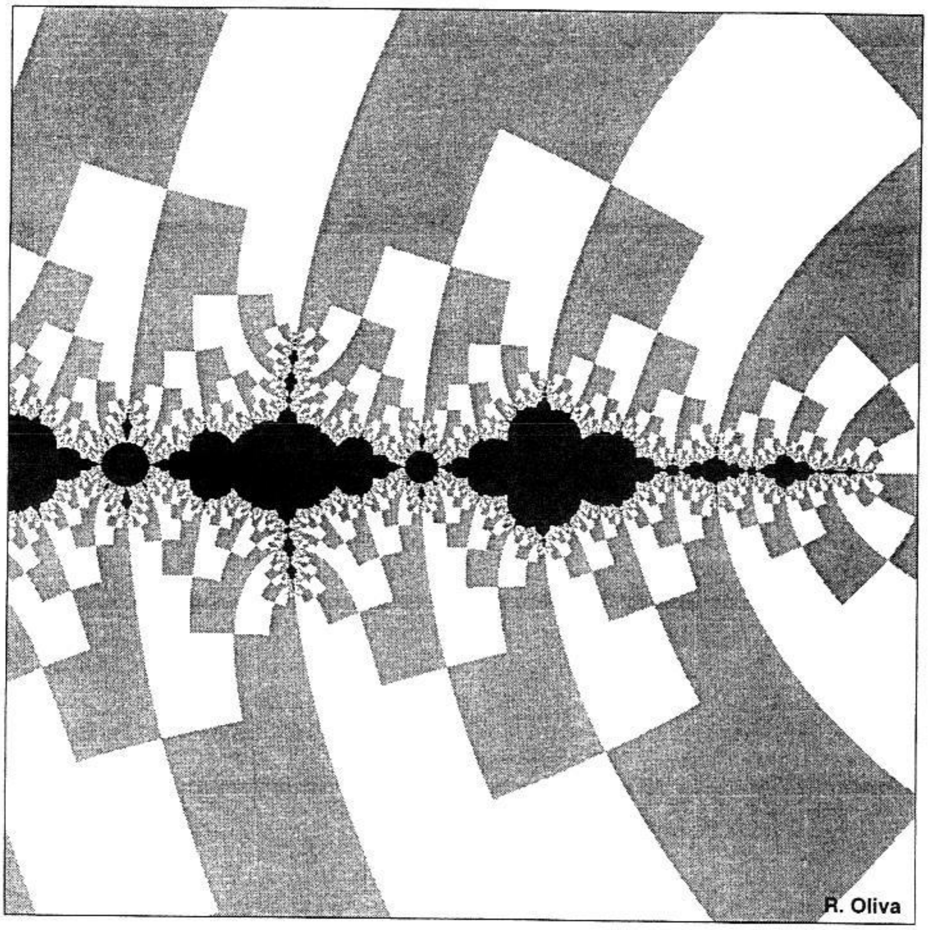
We get a different notion of valence if we only count the unbounded components of $W^u(p) \cap K - \{p\}$. This quantity is no longer a local homeomorphism invariant. The number of unbounded components of $W^u \cap K - \{p\}$ is the same as the number of path components of $W^u - K$ which have $p$ in their closure. We define $v'(p)$ to be this number. Let us define $a'(p)$ to be the number of unbounded components of the interior of $W^u \cap K - \{p\}$ which contain $p$ in their closure. We say that $p$ is a *primary pinch point* if $v'(p) \geq 3$ or $v'(p) = 1$ and $a'(p) \geq 1$.

**Theorem.** *The set of primary pinch points is finite. If $P \in J$ is any pinch point, then there exists a primary pinch point $Q \in J$ such that $P \in W^s(Q)$.*

We conclude with the computer picture of another unstable slice of $K^+$. This slice is taken through a saddle (fixed) point, which is at the right hand side of the picture. If the portion of the picture to the right of the fixed point were visible, it could be seen that the horizontal axis is an external ray which lands at the fixed point. Further, this (solenoidal) external ray has coding ${}^\infty 0.0^\infty = {}^\infty 1.1^\infty$, which corresponds to a special coloring: the picture is white just above the horizontal axis and black just below. Since all rays in this picture are contained in $W^u({}^\infty 0.0^\infty) = W^u({}^\infty 1.1^\infty)$, they have codings ${}^\infty 0 * .*$ (if the ray is in the upper half plane) or ${}^\infty 1 * .*$ (if the ray is in the lower half plane.)

The parameters of this are $a = .3$ and $c = -1.17$, which were chosen so that this mapping has one attracting fixed point and one attracting 3-cycle. The basin $\Omega_1$ of the attracting fixed point is connected and biholomorphically equivalent to $\mathbf{C}^2$. The basin of the attracting 3-cycle has the form $\Omega_3' \cup \Omega_3'' \cup \Omega_3'''$, where each of the three components is biholomorphically equivalent to $\mathbf{C}^2$. The mapping $f$ permutes these components. In the picture we can perceive that some components of $W^u(P) \cap int(K)$ are arranged along the $x$-axis like beads on a necklace: some of these components are smaller and look relatively circular (these correspond to components of the intersection with the attracting 3-cycle), while others are larger and far from circular (these correspond to components of the intersection $W^u(P) \cap \Omega_1$). The mapping $f$ acts on each domain intersecting the $x$-axis by moving it four to the left.

We may count that there are 8 solutions to $f^3(p) = p$, of which 2 are fixed points. Since one 3-cycle is attracting, there can be only one 3-cycle of saddle type. Let us denote this 3-cycle as $\{Q', Q'', Q'''\}$. Since $f$ is a real mapping, the complex conjugate $\{\overline{Q}', \overline{Q}'', \overline{Q}'''\}$ is also a 3-cycle, and must coincide with the original 3-cycle. We claim that $\{Q', Q'', Q'''\} \subset \mathbf{R}^2$. (For otherwise, if $\overline{Q}' \neq Q'$, and, say, $\overline{Q}' = Q''$, we have $f^2 Q' = fQ'' = f\overline{Q}' = \overline{fQ'} = \overline{Q}'' = Q'$, which means that $Q'$ generates a 2-cycle.) Let us observe that in the picture, the cut points along the horizontal axis appear to be pinch points. Thus they lie in the stable manifolds of primary points. Careful inspection of the external rays landing at these points reveals repeated patterns of white-black-black or black-white-white, and so their addresses are of the form $*.*(011)^\infty$ or $*.*(100)^\infty$. Thus they are on the stable manifolds of the 3-cycle of saddle type, which is a 3-cycle of primary pinch points. The picture of $W^u(Q') \cap K$ will look like a self-similar blow-up of one of these pinch points. In particular, there will be rays of the form $*.*(011)^\infty$ and $*.*(100)^\infty$ approaching $Q'$ from opposite directions.




An Unstable Slice: $f(x,y) = (y, y^2 + c - ax)$

The reader is referred to the Thesis [O] of R. Oliva for the combinatorial analysis of pictures of this sort. In particular, it is explained how, for certain mappings of this type, it is possible to construct automata which generate the identifications, in analogy with what was done in §2.

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