

# Polynomial Diffeomorphisms of $\mathbf{C}^2$ :

## VI. Connectivity of $J$

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### 0. Introduction

Polynomial maps  $g : \mathbf{C} \rightarrow \mathbf{C}$  are the simplest holomorphic maps with interesting dynamical behavior. The study of such maps has had an important influence on the field of dynamical systems. On the other hand the traditional focus of the field of dynamical systems has been in a different direction: invertible maps or diffeomorphisms. Thus we are led to study polynomial diffeomorphisms  $f : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ , which are the simplest holomorphic diffeomorphisms with interesting dynamical behavior.

Two features are apparent in much of the contemporary work on polynomial maps of  $\mathbf{C}$  (cf. [DH]). The first is a focus on the connectivity of the Julia set. The second is the use of computer pictures. Computer pictures do not substitute for proofs, but they have provided a tool that has been used to guide research. In this paper we consider these ideas in the context of polynomial diffeomorphisms of  $\mathbf{C}^2$ . This approach to the study of polynomial diffeomorphisms originates with Hubbard.

For a polynomial map of  $\mathbf{C}$  the “filled Julia set”  $K \subset \mathbf{C}$  is the set of points with bounded orbits, and the Julia set  $J$  is defined to be the boundary of  $K$ . The Julia set has several analogs for diffeomorphisms of  $\mathbf{C}^2$ . Since  $f : \mathbf{C}^2 \rightarrow \mathbf{C}^2$  is invertible, we can distinguish properties of points based on both forward and backward iteration. The sets  $K^+$  (resp.  $K^-$ ) consist of points with bounded forward (resp. backward) orbits under  $f$ . We write  $U^\pm$  for the complementary sets  $U^\pm = \mathbf{C}^2 - K^\pm$ . The set of points whose orbits are bounded in both forward and backward time is  $K = K^+ \cap K^-$ . The sets  $J^\pm := \partial K^\pm$  are analogues of the Julia set, as is the set  $J = J^+ \cap J^-$ . We use the notation  $J_+^-$  for  $J^- \cap U^+$ . We will see that in some cases the set  $J_+^-$  plays the role of the Fatou set for polynomial maps of  $\mathbf{C}$ . The focus of this paper is to investigate the  $J$ -connected/ $J$ -disconnected dichotomy in the case

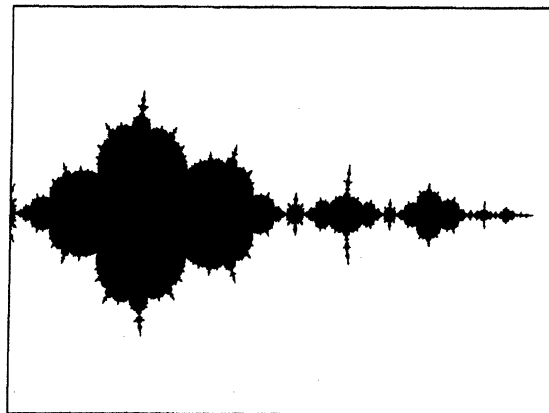
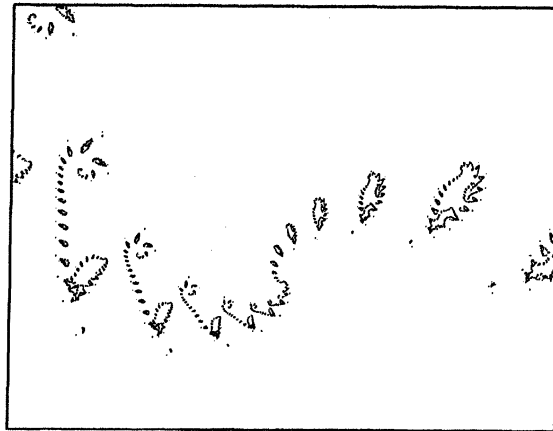
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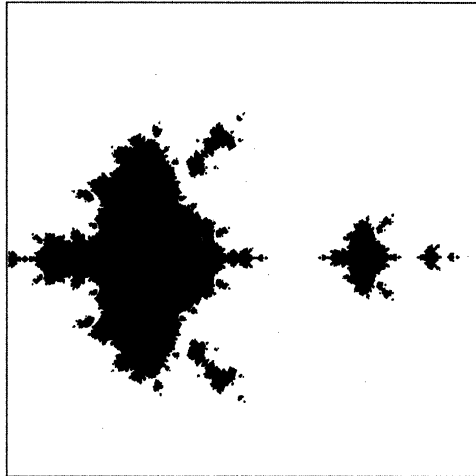
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of polynomial diffeomorphisms of  $\mathbf{C}^2$  and relate it to the structure of the sets  $J^- \cap U^+$  and  $J^+ \cap U^-$ .

One of the attractive features of the study of polynomial maps of  $\mathbf{C}$  is that the Julia sets can be drawn by computer. Thus the connectivity properties of the Julia set can often be demonstrated (visually if not rigorously) by means of computer pictures (see below). One of the daunting features of the study of polynomial diffeomorphisms of  $\mathbf{C}^2$  is that the sets of fundamental importance are complicated subsets of  $\mathbf{C}^2$ . As a consequence of our investigations we will show that it is possible to “see” the connectivity of the Julia set  $J$  for a polynomial diffeomorphism of  $\mathbf{C}^2$ .

Hubbard has suggested the following computer experiment. Let  $f$  be a polynomial diffeomorphism of  $\mathbf{C}^2$ , and let  $p$  be a periodic saddle point of  $f$ . The unstable manifold of  $p$ ,  $W^u(p)$ , is an immersed submanifold conformally equivalent to  $\mathbf{C}$ . We have a partition of  $W^u(p)$  into subsets  $W^u(p) \cap K^+$  and  $W^u(p) \cap U^+$ . The set  $W^u(p) \cap K^+$ , viewed as a subset of  $\mathbf{C}$ , is easily drawn by computer; three such drawings appear below.





In order to make these computer pictures a useful tool for investigating the dynamics of  $f$  it is important to know which features of these pictures reflect dynamically significant properties of the diffeomorphism. In particular it is useful to know which features are independent of the choice of periodic point. We say that  $f$  is *unstably connected with respect to the point  $p$*  if some component of  $W^u(p) \cap U^+$  is simply connected. The following theorem shows that this property is independent of the point  $p$  and furthermore that it is equivalent to the existence of a certain geometric structure of  $J_+^-$ .

**THEOREM 0.1.** *The following are equivalent:*

1. *For some periodic saddle point  $p$ , some component of  $W^u(p) \cap U^+$  is simply connected.*
2. *The set  $J_+^-$  has a lamination by simply connected leaves so that for any periodic saddle point  $p$  each component of  $W^u(p) \cap U^+$  is a leaf of this lamination.*
3. *For any periodic saddle point  $p$ , each component of  $W^u(p) \cap U^+$  is simply connected.*

This result is a consequence of Theorems 2.1 and 4.1.

If the equivalent conditions of the theorem hold, we say that  $f$  is *unstably connected*. When  $f$  is a polynomial diffeomorphism, the inverse function is also a polynomial diffeomorphism. Replacing  $f$  by  $f^{-1}$  interchanges the sets  $J^-$  and  $J^+$ , so any result for, or property of,  $J^-$  has an analog for  $J^+$ . We say that  $f$  is *stably connected* if the equivalent properties of the previous theorem hold for  $f^{-1}$ . In §5 we show that the connectivity of  $J$  is determined by the stable/unstable connectivity of  $f$ .

**THEOREM 5.1.** *The set  $J_f$  is connected if and only if the diffeomorphism  $f$  is either stably connected or unstably connected.*

From Theorem 5.1, it follows that the connectivity of the Julia set  $J$  of a polynomial diffeomorphism of  $\mathbf{C}^2$  can be determined “empirically.” It suffices to pick a periodic saddle point and observe the escape locus inside the stable and unstable manifolds, and according to Theorem 0.1 and its analog for stable connectivity the topology of these sets determines the connectivity of  $J$ .

In one variable, questions involving the connectivity of  $J$  are bound up with properties of the critical points of  $g$ . In particular  $J$  is connected if and only if there are no critical points with unbounded orbits. Now a polynomial diffeomorphism by definition has no critical points, and it seems difficult to define a single analog of critical points which works in all settings. On the other hand, an analog of critical points with unbounded forward orbits was described in [BS5]. This is the set  $\mathcal{C}^u$  of critical points of the Green function  $G^+$  restricted to unstable manifolds.

**THEOREM 7.3.**  *$f$  is unstably connected if and only if for  $\mu$  almost every point  $p$ ,  $W^u(p) \cap U^+$  contains no unstable critical points.*

Recall that the Jacobian determinant of a polynomial diffeomorphism  $\det D(f)$  is constant, and  $\det D(f^{-1}) = \det D(f)^{-1}$ . Replacing  $f$  by  $f^{-1}$  if necessary, we may assume that  $|\det Df| \leq 1$ . We say that  $f$  is dissipative if  $|\det Df| < 1$  and volume preserving if  $|\det Df| = 1$ .

Combining the previous theorem with results from [BS5] we have:

**COROLLARY 7.4.** *If  $f$  is dissipative then  $f$  is not stably connected. If  $f$  is volume preserving, then  $f$  is stably connected if and only if  $f$  is unstably connected.*

Combined with Theorem 5.1 this immediately yields:

**THEOREM 0.2.** *Assume  $|\det Df| \leq 1$ . Then  $J$  is connected if and only if  $f$  is unstably connected.*

Earlier we described a method of determining the connectivity of  $J$  experimentally by considering the stable and unstable manifolds of a periodic point. It follows from Theorem 0.2 that we can determine the connectivity of  $J$  with only half as much data.

When  $f$  is unstably connected we can give further information about the lamination of  $J_+^-$ . Our result can be compared with the fact that for polynomial maps in one variable the connectivity of the Julia set is equivalent to the existence of a canonical model for the dynamics on the complement of the Julia set.



The complex solenoid is defined as the set of bi-infinite sequences  $\Sigma_+ = \{z = (z_j) : |z_j| > 1, j \in \mathbf{Z}, z_j^d = z_{j+1}\}$  with the product topology. The induced mapping  $\sigma : \Sigma_+ \rightarrow \Sigma_+$ , obtained by shifting sequences to the left, is a homeomorphism. Furthermore,  $\Sigma_+$  has a natural lamination by Riemann surfaces.

**THEOREM 3.2.** *If  $f$  is unstably connected, then there is a continuous map  $\Phi : J_+^- \rightarrow \Sigma_+$  which semi-conjugates  $f|_{J_+^- \cap U^+}$  to the mapping  $\sigma|_{\Sigma_+}$ , i.e.  $\sigma \circ \Phi = \Phi \circ f$ . Furthermore,  $\Phi$  preserves the lamination structure and is a holomorphic bijection on each leaf.*

Given this laminar structure on  $J_+^-$  we define external rays to be gradient lines of the function  $G^+$  restricted to leaves of the lamination. As in the case of polynomial maps of  $\mathbf{C}$  it is interesting to know when these rays “land,” i.e., converge to points in  $J$ . Even in one variable it is too much to ask that all rays land, unless we impose additional hypotheses on the map. It is known that the set of rays which fail to land has Lebesgue measure zero. Theorem 3.1 shows that for polynomial diffeomorphisms, there is a natural measure on the set of rays, with respect to which almost all rays land. In [BS7] we show that when  $f$  is unstably connected and hyperbolic, all rays land. We also show in [BS7] that with the same hypotheses the semiconjugacy to the solenoid can be replaced by a conjugacy.

We end this section with a guide to the organization of the paper. As a tool for relating unstable connectivity at a point and the laminar structure on  $J_+^-$ , we introduce in §2 a condition (†), and we show that the existence of a laminar structure is a consequence of (†). In §3 we show that (†) implies the existence of a semiconjugacy to the solenoid. In §4 we show that the topological condition of unstable connectivity implies (†). In §5 we relate stable and unstable connectivity to the connectedness of  $J$ . In §6 we collect several properties equivalent to unstable connectivity. These are summarized in Theorem 6.9. In particular this theorem shows that unstable connectivity can be characterized without making reference to unstable manifolds. In §7 we establish a relationship between unstable connectivity and critical points.

## 1. Notation and preliminaries

In this section we briefly review standard terminology and facts about polynomial diffeomorphisms of  $\mathbf{C}^2$ . For a discussion at greater length see §1 of [BS5].

We consider mappings of the form  $f = f_1 \circ \cdots \circ f_m$ , where each  $f_j$  has the form

$$f_j(x, y) = (y, p_j(y) - a_j x),$$

and  $p_j(y)$  is a monic polynomial of degree  $d_j$ , and  $a_j \in \mathbf{C}$  is nonzero. It follows that  $f$  has degree  $d = d_1 \cdots d_m$ , and the  $n$ -fold iterate  $f^n = f \circ \cdots \circ f$  has degree  $d^n$ . Let  $K^\pm$  denote the points which remain bounded in forward/backward time, and set  $K = K^+ \cap K^-$ ,  $J^\pm = \partial K^\pm$ , and  $J = J^+ \cap J^-$ . Define the projections  $\pi_x(x, y) = x$  and  $\pi_y(x, y) = y$ , and it follows that  $\pi_x(f^n) = y^{d^{n-1}} + \cdots$  and  $\pi_y(f^n) = y^{d^n} + \cdots$ . Set

$$G^\pm(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(x, y)\| = \lim_{n \rightarrow +\infty} \frac{1}{d^n} \log^+ |\pi_y \circ f^{\pm n}(x, y)|.$$

Then  $G^\pm$  is continuous and plurisubharmonic on  $\mathbf{C}^2$ ,  $K^\pm = \{G^\pm = 0\}$ , and  $G^\pm$  is pluriharmonic on the sets  $U^\pm := \mathbf{C}^2 - K^\pm$ . The currents  $\mu^+$  and  $\mu^-$  are defined by  $\mu^\pm = (1/2\pi) dd^c G^\pm$ . The harmonic measure is given by  $\mu = \mu^+ \wedge \mu^-$ .

For  $R > 0$  define  $V^+ = \{|y| > |x|, |y| > R\}$ . It follows that one may choose  $R_0$  sufficiently large that

$$(1.1) \quad J^- \cap \{G^+ > R_0\} \subset V^+.$$

If  $R$  is chosen sufficiently large, then for all  $n \geq 0$ , one may choose a  $d^n$ -th root of  $\pi_y \circ f^n$  on  $V^+$  such that  $(\pi_y(f^n))^{\frac{1}{d^n}} \approx y$ , and with this choice of root, define

$$\varphi^+(x, y) = \lim_{n \rightarrow +\infty} (\pi_y \circ f^n(x, y))^{\frac{1}{d^n}},$$

which is analytic on  $V^+$  and satisfies  $\varphi^+ \circ f = (\varphi^+)^d$  (see [HOV]). Further,

$$(1.2) \quad \varphi^+(x, y) = y + O(1), \text{ and } \frac{\partial \varphi^+(x, y)}{\partial y} = 1 + O(|y|^{-1})$$

on  $V^+$ . The uniformity of the  $O$  terms means that for fixed  $x$ ,  $y \mapsto \varphi^+(x, y)$  is univalent for  $|y| \geq \max(R, |x|)$ . Thus, if we set  $s = \varphi^+$ , then  $(x, s)$  is a global coordinate system on  $V^+$ .

Let  $\nu$  be an ergodic invariant measure supported on  $J$ . Associated to  $\nu$  are two Lyapunov exponents  $\lambda^+(\nu)$  and  $\lambda^-(\nu)$  which describe the growth of tangent vectors under  $Df^n$ . We say that  $\nu$  is an index one hyperbolic measure (or simply a hyperbolic measure when no confusion will result) if  $\lambda^-(\nu) < 0 < \lambda^+(\nu)$ . If  $\nu$  is an index one hyperbolic measure then the following conditions hold:

- (1) for  $\nu$  almost every  $p$  there are complex one-dimensional linear subspaces  $E_p^s$  and  $E_p^u$  of the tangent space  $T_p \mathbf{C}^2$  such that  $E_p^s \oplus E_p^u = T_p \mathbf{C}^2$  where the families  $\{E_p^{s/u}\}$  are invariant:  $Df_p(E_p^{s/u}) = E_{f_p}^{s/u}$ .
- (2) for  $\nu$  almost every point  $p$ , we have

$$(1.3) \quad \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Df_p^n|_{E_p^{s/u}}\| = \lambda^{-/+}(\nu).$$

Conversely if there are numbers  $\lambda^- < 0 < \lambda^+$  such that (1) and (2) hold then  $\nu$  is an index one hyperbolic measure with exponents  $\lambda^\pm$ . The concept of hyperbolic measure will be used frequently throughout this paper; see [P] or [KM] for further information.

Examples of index one hyperbolic measures include the measure given by the average of point masses over the orbit of a periodic saddle point. Such periodic saddle points exist in profusion for every  $f$  (see [BLS2]). Another important example of a hyperbolic measure is the harmonic measure  $\mu$  (see [BS3]).

A consequence of the Pesin theory is that there is a set  $\mathcal{R} \subset J$  which has full  $\nu$  measure for any index one hyperbolic measure  $\nu$  and such that for  $p \in \mathcal{R}$  the stable/unstable manifolds

$$W^{s/u}(p) = \{q \in \mathbf{C}^2 : \lim_{n \rightarrow \pm\infty} \text{dist}(f^n q, f^n p) = 0\}$$

are smooth submanifolds of  $\mathbf{C}^2$ . In fact, (see [BLS], [W]) these are Riemann surfaces which are conformally equivalent to  $\mathbf{C}$ . In general these submanifolds vary only measurably with  $p$ . Loosely speaking, we may *define*  $\mathcal{R}$  to be the set of points  $p \in J$  for which  $W^{s/u}(p)$  are Riemann surfaces conformally equivalent to  $\mathbf{C}$ .

## 2. Extension of $\varphi^+$ and laminar properties of $J_+^-$

In this section we describe a technical condition on complex disks contained in  $J_+^-$  which will play an important role in this paper. We will show that when there is a disk that satisfies this condition,  $J_+^-$  has a lamination by complex leaves. In §4 we will give topological hypotheses that imply the existence of such a disk.

Let  $M$  be a Riemann surface. A positive harmonic function on  $M$  will be called *minimal* if it generates an extreme ray in the cone of positive harmonic functions. In other words, if a minimal harmonic function  $h$  can be written as a sum of positive harmonic functions  $h = h_1 + h_2$  on  $M$ , then  $h, h_1$ , and  $h_2$  are all constant multiples of each other. In case  $M = \Delta$  is the unit disk, the minimal harmonic functions are just the positive constant multiples of the Poisson kernel

$$P(z, e^{i\theta}) = \Re \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right)$$

for some real  $\theta$ . We note that under composition with an automorphism of the disk  $\Delta$ , the Poisson kernel  $P(z, 1)$  is taken to  $cP(z, e^{i\theta})$  for some real numbers  $c > 0$  and  $\theta$ . For any positive harmonic function  $h$  on the unit disk, there

is a unique positive Borel measure  $\lambda$  on  $\partial\Delta$  for which we have the Herglotz representation

$$h(z) = \int P(z, e^{i\theta}) \lambda(\theta).$$

It turns out that minimality of  $G^+$  will enter frequently in our work in a rather specific context. So we formalize this. We call  $\mathcal{O} \subset \mathbf{C}^2$  a *complex disk* if there is a holomorphic injective immersion of the unit disk  $\psi : \Delta \rightarrow \mathbf{C}^2$  with  $\mathcal{O} = \psi(\Delta)$ . We say that a complex disk  $\mathcal{O}$  satisfies condition  $(\dagger)$  if the following three properties hold:

$$(\dagger) \quad \begin{array}{l} \mathcal{O} \subset J_+^-, \quad G^+|\mathcal{O} \text{ is minimal, and} \\ \text{for each } j \in \mathbf{Z} \text{ either } \mathcal{O} \cap f^j\mathcal{O} = \emptyset \text{ or } \mathcal{O} = f^j\mathcal{O}. \end{array}$$

In the most typical situation  $\mathcal{O}$  will be a component of  $W^u(p) - K^+$  where  $W^u(p)$  is an unstable manifold of a point (not necessarily periodic). In this case the minimality of  $G^+$  is the only part of  $(\dagger)$  that is not automatic. We say that a polynomial diffeomorphism satisfies condition  $(\dagger)$  if it possesses a disk which satisfies condition  $(\dagger)$ .

Let us recall the definition of a Riemann surface lamination of a topological space  $X$  (cf. [C]). A chart is a choice of an open set  $U_j \subset X$ , a topological space  $Y_j$ , and a map  $\rho_j : U_j \rightarrow \mathbf{C} \times Y_j$  which is a homeomorphism onto its image. An atlas is a collection of charts such that  $\{U_j\}$  covers  $X$ . The set of points of  $U_j$  for which the second coordinate of  $\rho_j$  assumes a fixed value is called a plaque. For coordinate charts  $(\rho_i, U_i, Y_i)$  and  $(\rho_j, U_j, Y_j)$  with  $U_i \cap U_j \neq \emptyset$ , the transition function is the homeomorphism from  $\rho_j(U_i \cap U_j)$  to  $\rho_i(U_i \cap U_j)$  defined by  $\rho_{ij} = \rho_i \circ \rho_j^{-1}$ . A Riemann surface lamination of a topological space  $X$  is determined by an atlas of charts which satisfy the following consistency condition: the transition functions may be written in the form  $\rho_{ij} = (g(z, y), h(y))$ , where for fixed  $y \in Y_j$  the function  $z \mapsto g(z, y)$  is holomorphic. The condition on the transition functions gives a consistency between the plaques defined in  $U_j$  and those in  $U_i$ . Thus plaques fit together to make global manifolds called leaves of the lamination, and each leaf has the structure of a Riemann surface.

One of the equivalent conditions in Theorem 2.1 below is that  $J_+^-$  carries a unique Riemann surface lamination. After Proposition 2.7, we remark that this lamination in fact carries a special affine structure.

The following theorem collects the basic results of this section.

**THEOREM 2.1.** *If there is a complex disk  $\mathcal{O}$  satisfying  $(\dagger)$ , then*

- (1)  $\varphi^+$  extends to a continuous mapping  $\varphi^+ : J_+^- \rightarrow \{|\zeta| > 1\}$  which satisfies the functional equation

$$\varphi^+(f(p)) = (\varphi^+(p))^d.$$

- (2)  $J_+^-$  has a Riemann surface lamination  $\mathcal{M}^-$  with  $\mathcal{O}$  as one leaf.
- (3) For each leaf  $M$  of  $\mathcal{M}^-$ ,  $G^+|_M$  has no critical points.
- (4) For each leaf  $M$  of the lamination  $\mathcal{M}^-$ , the restriction  $\varphi^+|_M : M \rightarrow \{|\zeta| > 1\}$  is a holomorphic covering map.
- (5) Each leaf  $M$  of  $\mathcal{M}^-$  is a disk satisfying  $(\dagger)$ .
- (6) Each leaf of  $\mathcal{M}^-$  is dense in  $J_+^-$ .
- (7)  $\mathcal{M}^-$  is the unique lamination of  $J_+^-$  by Riemann surfaces.

The remainder of this section is devoted to the proof of Theorem 2.1. In particular we will make the standing assumption that there exists a disk  $\mathcal{O}$  which satisfies  $(\dagger)$ . The general plan of the proof is as follows. Let  $\mathcal{O}^\infty$  be defined as  $\bigcup_{j \in \mathbf{Z}} f^j \mathcal{O}$  in  $J_+^-$ . Let  $\hat{\mathcal{O}}$  denote the closure in  $U^+$  of the set  $\mathcal{O}^\infty$ . We begin by proving analogs of (1–5) with  $J_+^-$  replaced by  $\hat{\mathcal{O}}$ . In Theorem 2.6 we prove an analog of (1). In Proposition 2.7 we prove an analog of (2). In Corollary 2.8 we prove an analog of (3). In Corollary 2.9 we prove an analog of (4). In Corollary 2.10 we prove (5). Corollary 2.14 proves (6) and shows that  $\hat{\mathcal{O}} = J_+^-$ . Thus it follows that items (1–5) hold for  $J_+^-$  as claimed. (7) is proved in Corollary 2.18.

For any set  $X \subset J_+^-$  we will use the notation:

$$X(\rho) := X \cap \{G^+ \geq \log \rho\}.$$

We saw in §1 that  $\varphi^+$  is defined for points in  $J_+^-$  with  $G^+$  sufficiently large. Let  $\rho_0$  be a fixed constant chosen large enough that  $\varphi^+$  is defined on  $J_+^-(\rho_0)$ .

We will also use the notation  $\mathbf{C}(\rho) := \{\zeta \in \mathbf{C} : |\zeta| \geq \rho\}$ . With this notation it follows that for  $\rho \geq \rho_0$  the map  $\varphi^+$  maps the set  $J_+^-(\rho)$  to the set  $\mathbf{C}(\rho)$ .

**PROPOSITION 2.2.** *Let  $H = \{x + iy : x > 0\}$  be the right half-plane, and let  $\mathcal{O}$  be a disk satisfying  $(\dagger)$ . We can choose a conformal coordinate  $\alpha : H \rightarrow \mathcal{O}$  such that for  $x > \log \rho_0$  we have  $\varphi^+(\alpha(x + iy))$  is defined and  $\varphi^+(\alpha(z)) = e^z$ .*

*Proof.* By hypothesis, there is an imbedding  $\psi : \Delta \rightarrow \mathcal{O} \subset \mathbf{C}^2$  such that  $g = G^+ \circ \psi$  is a positive multiple of the Poisson kernel function with pole at  $e^{i\kappa} \in \partial\Delta$ . Choose a conformal map from the right half-plane  $H$  to  $\Delta$  which sends the point at infinity to  $e^{i\kappa}$ . Let  $\alpha : H \rightarrow \mathcal{O}$  be the composition of map from  $H$  to  $\Delta$  and  $\psi$ . On the right half-plane the Poisson kernel function with pole at infinity is  $x + iy \mapsto x$ . Thus  $G^+(\alpha(x + iy)) = cx$ . By composing  $\alpha$  with multiplication by a scalar we may assume that  $c = 1$ . If  $x > \log \rho_0$  then  $\varphi^+(\alpha(x + iy))$  is defined, and  $G^+(\alpha(x + iy)) = \log |\varphi^+(x + iy)|$ . Since  $\log |e^{x+iy}| = x$ , we have  $\log |\varphi^+(x + iy)| = \log |e^{x+iy}|$  so that

$$\begin{aligned} \log |\varphi^+(x + iy)/e^{x+iy}| &= 0 \\ \Re(\varphi^+(x + iy)/e^{x+iy}) &= 1. \end{aligned}$$

A holomorphic function with constant real part is constant so that  $\varphi^+(x+iy) = e^{i\theta_0}e^{x+iy}$ . Now by composing  $\alpha$  with the translation  $z \mapsto z - i\theta_0$  we have the required parametrization. □

Since the map  $\exp : \{x + iy : x > \log \rho\} \rightarrow \{\zeta : |\zeta| > \rho\}$  is a covering when  $\rho \geq \rho_0$ , we have:

**COROLLARY 2.3.** *If  $\mathcal{O}$  satisfies (†) then  $\varphi^+|_{\mathcal{O}(\rho)} : \mathcal{O}(\rho) \rightarrow \{|\zeta| > \rho\}$  is a covering map when  $\rho > \rho_0$ .*

For any subsets  $X \subset J_+^-$  and  $E \subset \{G^+ > \rho\}$  with  $\rho \geq \rho_0$ , we will use the notation  $X_E := X \cap (\varphi^+)^{-1}E$ .

**LEMMA 2.4.** *If  $G \subset \{|s| > \rho_0\}$  is simply connected and  $\zeta \in G$ , then there a homeomorphism  $H : G \times \hat{\mathcal{O}}_{\{\zeta\}} \rightarrow \hat{\mathcal{O}}_G$  satisfying  $\varphi^+(H(s,p)) = s$ . Furthermore  $H(G \times \mathcal{O}_{\{\zeta\}}^\infty) = \mathcal{O}_G^\infty$ .*

*Proof.* Condition (†) implies that  $\mathcal{O}^\infty$  can be written as a disjoint union of disks of the form  $f^n(\mathcal{O})$ , which again satisfy (†). For  $p \in \mathcal{O}^\infty$  let  $\mathcal{O}'$  denote the (unique) disk containing  $p$ , and let  $L_p$  be the component of  $\mathcal{O}'_G$  which contains  $p$ . Now by Corollary 2.3  $\varphi^+|_{L_p}$  is a covering map with connected domain and simply connected range, so in fact  $\varphi^+|_{L_p}$  is a bijection. Let  $g_p : G \rightarrow L_p \subset \mathbf{C}^2$  be the inverse map, so  $g_p$  is a holomorphic. By the remark following (1.2), the functions  $(\pi_x, \varphi^+)$  form a coordinate system on  $V^+$ , so there is a function  $h$  such that  $g_p(z) = (h(z,p), z)$ . Further, if we set  $E := \mathcal{O}_{\{\zeta\}}^\infty$ , then in these coordinates we may identify  $E$  with a subset of  $\mathbf{C}$ .

Now the function  $h : G \times E \rightarrow \mathbf{C}$  is a holomorphic motion, which is to say that:

- (1)  $z \mapsto h(z,p)$  is holomorphic, and
- (2)  $p \mapsto h(z,p)$  is injective.

The second property follows from the fact that  $\mathcal{O}^\infty$  consists of subsets of pairwise disjoint disks in  $\mathbf{C}^2$  which are given as graphs  $\zeta \mapsto (h(\zeta,p), \zeta)$  in the  $(\pi_x, \varphi^+)$ -coordinates. There is a well-developed theory of holomorphic motions which shows that holomorphic motions can be extended to quasiconformal maps  $\mathbf{C} \ni t \mapsto h(z,t)$ . We need only the basic observation from [MSS] that  $h$  has an extension to a function  $\hat{h} : G \times \overline{\mathcal{O}_{\{\zeta\}}^\infty} \rightarrow \mathbf{C}$ , which is also a holomorphic motion, and which is continuous. Now  $\overline{\mathcal{O}_{\{\zeta\}}^\infty} = \hat{\mathcal{O}}_{\{\zeta\}}$ . We define  $H : G \times \hat{\mathcal{O}}_{\{\zeta\}} \rightarrow \hat{X}_G$  by the formula  $H(t,z) = (t, \hat{h}(t,z))$ , where the right-hand side is given in the  $(\pi_x, \varphi^+)$  coordinates on  $\hat{X}_G$ . The function  $H$  is a continuous surjection. The fact that  $\hat{h}$  satisfies (2) shows that  $H$  is injective.  $H$  is proper, so it is a homeomorphism on each compact set. This implies that  $H$  is a homeomorphism. □



Next we show that the function  $\varphi^+$  has a canonical extension to all of  $\hat{\mathcal{O}}$ . This proof will use a purely topological fact which we prove first.

Let  $\pi : \tilde{Y} \rightarrow Y$  be a covering space. Let  $\psi : X \rightarrow Y$  be a map. A lift of  $\psi$  is a map  $\psi' : X \rightarrow \tilde{Y}$  such that  $\pi \circ \psi' = \psi$ . We will reduce the problem of extending  $\varphi^+$  to a problem of finding a lift. On well-behaved spaces lifting problems can be translated into problems in terms of the fundamental group. The spaces we are dealing with here are not locally connected so this translation is not possible. On the other hand standard techniques of topology can be used to solve the problem. Let  $A$  be a closed subset of the topological space  $X$ . To say that  $A$  is a strong deformation retract of  $X$  means that we are given a function  $F : X \times I \rightarrow X$  (the retraction function) such that:

- (1)  $F(x, 0) = x$  for  $x \in X$ ,
- (2)  $F(x, 1) \in A$  for  $x \in X$ ,
- (3)  $F(x, t) = x$  for  $x \in A$ .

LEMMA 2.5. *Let  $A \subset X$  be a strong deformation retract. Let  $\pi : \tilde{Y} \rightarrow Y$  be a covering map. Let  $\psi : X \rightarrow Y$  be a map and assume we are given a map  $\psi_1 : A \rightarrow \tilde{Y}$  which is a lift of  $\psi|_A$ . Then there is a unique lift  $\psi'$  of  $\psi$  which agrees with  $\psi_1$  on  $A$ .*

*Proof.* Let  $F$  denote the strong retraction function as above, and define  $G : X \times I \rightarrow X$  by  $G(x, t) = \psi \circ F(x, 1 - t)$ . The function  $x \mapsto \psi_1 \circ F(x, 1)$  is well defined since  $F(x, 1) \in A$  and it is a lift of the function  $x \mapsto G(x, 0)$ . The homotopy lifting property of the covering map  $\pi$  [S, p. 67, Theorem 3] gives us a unique lift  $G'$  of  $G$  with the property that  $G'(x, 0) = \psi_1 \circ F(x, 1)$ . The restriction of  $G'$  to the set  $A$  is a lift of a constant homotopy (independent of  $t$ ). By uniqueness of lifts of paths [S, p. 68, Theorem 5] the restriction of  $G'$  to  $A$  is itself a constant homotopy. It follows that  $G'(x, t) = \psi_1(x)$  for  $x \in A$ . Now if we set  $\psi'(x) = G(x, 1)$  then  $\psi'$  is a lift of  $\psi \circ G(x, 1) = \psi$  and  $\psi'(x) = x$  for  $x \in A$ .

Now if  $\psi''$  is any other lift of  $\psi$  which agrees with  $\psi_1$  on  $A$  then  $G'' = \psi'' \circ F$  is a lift of  $G$  with the property that  $G''(x, 0) = \psi_1 \circ F(x, 1)$ . By the uniqueness property of lifts of homotopies we have  $G'' = G'$ ; hence  $\psi''(x) = G''(x, 1) = G'(x, 1) = \psi'$ .  $\square$

THEOREM 2.6. *If  $f$  satisfies  $(\dagger)$  then the function  $\varphi^+$  has a continuous extension to all of  $\hat{\mathcal{O}}$  which satisfies the functional equation*

$$(2.1) \quad \varphi^+(f(p)) = (\varphi^+(p))^d.$$

*There is a unique extension satisfying (2.1).*

*Proof.* The function  $\varphi^+$  is defined on  $\hat{\mathcal{O}}(\rho_0)$  and satisfies (2.1) for  $p \in \hat{\mathcal{O}}(\rho_0)$ . To prove the proposition, it suffices to show that  $\varphi^+$  has a unique,



continuous extension to  $\hat{O}(\rho_0^{\frac{1}{d}})$ , which satisfies (2.1). Repeating this argument allows us to extend the function successively to sets  $\hat{O}(\rho_0^{1/d^n})$  and thus to  $\hat{O}$  which is the union of these sets.

Let us define  $\pi : \mathbf{C} \rightarrow \mathbf{C}$  by  $\pi(z) = z^d$  and write  $\psi = \varphi^+ \circ f$ . Finding a  $\psi'$  which satisfies (2.1) is equivalent to finding  $\psi' : \hat{O}(\rho^{1/d}) \rightarrow \mathbf{C} - \bar{\Delta}$  such that  $\pi \circ \psi' = \psi$ . The function  $\pi : \mathbf{C} - \bar{\Delta} \rightarrow \mathbf{C} - \bar{\Delta}$  is a covering map. This is a problem of finding a lift  $\psi'$  with the added constraint that we want  $\psi'$  to have prescribed values on  $\hat{O}(\rho_0)$ .

In order to apply Lemma 2.5, let  $A = \hat{O}(\rho)$ , let  $X = \hat{O}(\rho^{1/d})$ , and let  $\psi_1 = \varphi|_{\hat{O}(\rho)}$ . To verify the hypotheses of the lemma it suffices to show the following. For  $1 < \alpha < \beta$  the set  $\hat{O}(\beta)$  is a deformation retract of  $\hat{O}(\alpha)$ .

Since  $f^n(\hat{O}(\rho)) = \hat{O}(\rho^{d^n})$  by applying a sufficiently high power of  $f$  we may assume that  $\rho_0 \leq \alpha < \beta$ . Applying the homeomorphism  $f^n$  does not change the topological properties of the sets. We will construct a deformation retraction by using  $\varphi^+$  to lift a deformation retraction of  $\{|z| \geq \alpha\}$  to  $\{|z| \geq \beta\}$ .

Let  $F : \{|z| \geq \alpha\} \times I \rightarrow \{|z| \geq \beta\}$  be a strong deformation retraction. Thus  $F(z, 0) = z$ ,  $|F(z, 1)| \geq \beta$  and  $F(z, t) = z$  for  $|z| \geq \beta$ . We may assume that  $F$  preserves radial lines. Define  $G : \hat{O}(\alpha) \times I \rightarrow \mathbf{C}(\alpha)$  by  $G(p, t) = F(\varphi^+(p), t)$ . We wish to find a function  $G' : \hat{O}(\alpha) \times I \rightarrow \hat{O}(\alpha)$  which satisfies  $\varphi^+ \circ G' = G$  and  $G'(p, 0) = p$ . Since  $\varphi^+$  is a covering on each leaf, the homotopy lifting property of covering spaces gives us such a unique such function. *A priori* all we know is that this function is continuous when restricted to each leaf. To see that  $G'$  is continuous on  $\hat{O}(\alpha)$  we use the product structure given by Lemma 2.4. Consider a set  $S = \{|z| \geq \alpha, \theta_0 \leq \arg(z) \leq \theta_1\}$ . Since  $S$  is simply connected the set  $\hat{O}_S$  has a product representation as  $S \times \hat{O}_{\{\zeta\}}$  by Lemma 2.4. Now the restriction of  $F$  to  $S$  is a deformation retraction as is the restriction of  $G$  to  $\hat{O}_S$ . To prove that  $G$  is continuous it suffices to prove that  $G$  restricted to  $\hat{O}_S$  is continuous for any such set  $S$ . Now we can use the product structure to define a deformation retraction on  $\hat{O}_S$  by the formula  $(z, w) \mapsto (F_t(z), w)$  and it is a lift of  $F$ . This function is clearly continuous and by the uniqueness of lifts it must be equal to  $G$  restricted to  $\hat{O}_S$ .  $\square$

We now define the lamination  $\mathcal{M}^-$  on the set  $\hat{O}$ .

**PROPOSITION 2.7.** *There is a Riemann surface lamination of  $\hat{O}$ . The charts have the form  $\rho_G : \hat{O}_G \rightarrow G \times \hat{O}_{\{\zeta\}}$  and satisfy the condition  $\pi_1 \circ \rho_G = \varphi^+$ .*

*Proof.* For each point  $p \in \hat{O}$  we will construct a set  $G$  and a chart as above. If  $G^+(p) > \log \rho_0$  then let  $G \subset \{|z| > \rho_0\}$  be a simply connected open

set that contains  $\varphi^+(p)$ . Let  $\rho_G$  be the inverse of the function  $h$  given by Lemma 2.4.

For a  $p \in \hat{\mathcal{O}}$  not of the above form there is an  $n > 0$  so that  $|\varphi^+(f^n(p))| > \rho_0$ . Let  $\xi = \varphi^+(f^n(p))$ . Let  $\zeta_1, \dots, \zeta_{d^n}$  be the roots of  $z^{d^n} = \xi$  so that  $\zeta_1 = \varphi^+(p)$ . Let  $H$  be a simply connected subset of  $\{|z| > \rho_0\}$  that contains  $\xi$ . Let  $G_1, \dots, G_{d^n}$  be the components of  $\{z : z^{d^n} \in H\}$  numbered so that  $\zeta_k \in G_k$ . Now  $f^{-n}(\hat{\mathcal{O}}_H) = \hat{\mathcal{O}}_{G_1} \cup \dots \cup \hat{\mathcal{O}}_{G_{d^n}}$  where the sets on the right-hand side are disjoint. We have a map  $\rho_G : \hat{\mathcal{O}}_H \rightarrow H \times \hat{\mathcal{O}}_{\{\xi\}}$ . We can write  $\hat{\mathcal{O}}_{\{\xi\}}$  as a disjoint union  $\hat{\mathcal{O}}_{\{\zeta_1\}} \cup \dots \cup \hat{\mathcal{O}}_{\{\zeta_{d^n}\}}$ . In fact  $f^{-n}\rho_G^{-1}(H \times \hat{\mathcal{O}}_{\{\zeta_k\}}) = \hat{\mathcal{O}}_{G_k}$ . This is because for each plaque  $P$  the function  $\varphi^+ \circ f^{-n} \circ (\varphi^+|_P)^{-1}$  is a branch of the  $d^n$ -th root function so its values lie in  $G_k$  if and only if its value at  $\xi$  is  $\zeta_k$ . Thus the function  $\rho_H \circ f^n : \mathcal{O}_{G_1} \rightarrow H \times \hat{\mathcal{O}}_{G_1}$  gives a coordinate chart on  $\mathcal{O}_{G_1}$ .

It remains to check the form of the overlap functions. If  $\hat{\mathcal{O}}'_G$  and  $\hat{\mathcal{O}}''_G$  intersect, then their intersection is  $\hat{\mathcal{O}}_H$  where  $H = G' \cap G''$ . We can analyze the transition function in terms of the functions from  $\hat{\mathcal{O}}_H$  to  $\hat{\mathcal{O}}'_G$  and from  $\hat{\mathcal{O}}_H$  to  $\hat{\mathcal{O}}''_G$ . Thus we begin with the situation of  $H \subset G$ . Let us assume first that  $G \subset \{|z| > \rho_0\}$ . Let us assume that  $\rho_G : \hat{\mathcal{O}}_G \rightarrow G \times \hat{\mathcal{O}}_{\{\zeta\}}$  with  $\zeta$  in  $H$ . The map  $\rho : H \times \hat{\mathcal{O}}_{\{\zeta\}} \rightarrow G \times \hat{\mathcal{O}}_{\{\zeta\}}$  defined as  $\rho = \rho_G \circ \iota \circ \rho_H^{-1}$  is continuous where  $\iota$  is the inclusion. On the set  $H \times \mathcal{O}_{\{\zeta\}}^\infty$  we have  $\rho(z, w) = (z, w)$ . Since  $H \times \mathcal{O}_{\{\zeta\}}^\infty$  is dense  $\rho$  must have this form everywhere. For a set  $G$  such that  $f^n(G) \subset \{|z| > \rho_0\}$  we apply the function  $f^n$  and repeat the previous argument.  $\square$

*Remark.* In Proposition 2.7 we have defined an atlas  $\mathcal{A}$  of charts which gives us a Riemann surface lamination structure, which has the special property that a global function,  $\varphi^+$ , is used as the local holomorphic coordinate. It is sometimes useful to consider a related atlas  $\mathcal{A}'$  which gives us an affine Riemann surface lamination. Recall that the atlas  $\mathcal{A}$  consists of charts  $\rho_G$  where  $G$  is a simply connected subset of  $\mathbf{C}$  and  $\rho_G : \mathcal{O}_G \rightarrow G \times Y$ . Define  $\rho'_G = (\ell, id) \circ \rho_G$  where  $\ell$  is a branch of the logarithm function defined on  $G$ . The overlap functions for  $\mathcal{A}'$  now have the form  $(z, w) \mapsto (z + c, g(w))$  where the constant  $c$  is an integral multiple of  $2\pi$  which arises because of ambiguity in the choice of the branch of the logarithm. The chart  $\mathcal{A}'$  induces an affine structure on each leaf, and with respect to these affine structures the map  $f$  has the form  $z \mapsto d \cdot z + c(w)$  on the local plaque with  $w$  constant. Note further that our transition functions preserve both factors of the product so that not only are leaves well defined but local transversals to the leaves of the form  $\{(z_0, y) : y \in Y_j\}$  are also well defined, independent of the chart (compare Sullivan's comment on the TLC property [S, p. 549]).

The following are consequences of Proposition 2.7.

COROLLARY 2.8. *Let  $\mathcal{O}'$  be a leaf of the lamination  $\mathcal{M}^-$ , then  $\varphi^+|_{\mathcal{O}'}$  has no critical points.*

COROLLARY 2.9. *Let  $\mathcal{O}'$  be a leaf of the lamination  $\mathcal{M}^-$ . Then when  $\mathcal{O}'$  is given the leaf topology the function  $\varphi^+|_{\mathcal{O}'} : \mathcal{O}' \rightarrow \mathbf{C} - \overline{\Delta}$  is a covering map.*

*Proof.* Let  $G$  be any set arising in the previous proposition. Then the proposition implies that  $\varphi^+|_{\mathcal{O}'}^{-1}(G)$  consists of path components mapped bijectively to  $G$ . With respect to the leaf topology each of these path components is an open set.  $\square$

LEMMA 2.10. *Each leaf of  $\mathcal{M}^-$  is a conformal disk, and  $\varphi^+$  is a minimal harmonic function on each leaf.*

*Proof.* Let  $\mathcal{O}'$  be a leaf  $\mathcal{M}^-$ . Assume first that the degree of the covering  $\varphi^+|_{\mathcal{O}'}$  is finite. For  $n$  a natural number let  $\mathcal{O}'_n = f^{-n}(\mathcal{O}')$ . Now  $\varphi^+ \circ f^n = \pi^n \circ \varphi^+$  where we use the letter  $\pi$  for the  $d$ -th power map. The degree of  $f^n|_{\mathcal{O}'_n}$  is one and the degree of  $\pi^n$  is  $d^n$ . Since degrees multiply we have  $\deg(\varphi^+|_{\mathcal{O}'_n}) \cdot d^n = \deg(\varphi^+|_{\mathcal{O}'})$ . Since the right-hand side is divisible by some highest power of  $d$  this equation cannot be valid for all  $n$ . We conclude that the degree of  $\varphi^+|_{\mathcal{O}'}$  must be infinite. It follows from covering space theory that the image of the map  $(\varphi^+)_* : \pi_1(\mathcal{O}') \rightarrow \pi_1(\mathbf{C} - \overline{\Delta})$  is a subgroup of infinite index. But this second group is isomorphic to  $\mathbf{Z}$ ; hence the only subgroup of infinite index is the trivial group. Since by covering space theory  $(\varphi^+)^*$  is injective we conclude that  $\pi_1(\mathcal{O}')$  is trivial. Thus  $\mathcal{O}'$  is a simply connected Riemann surface.

Let  $H = \{x + iy : x > 0\}$ . Consider the map  $\alpha : H \rightarrow \mathbf{C} - \Delta$  given by  $\alpha(z) = e^z$ . Now  $\mathcal{O}'$  and  $H$  are both universal covering spaces of  $\mathbf{C} - \Delta$  so by covering space theory there is a bijection  $\beta : \mathcal{O}' \rightarrow H$  so that  $\varphi^+ = \alpha \circ \beta$ . Since  $\alpha$  and  $\varphi^+$  are holomorphic,  $\beta$  is holomorphic. Thus  $\mathcal{O}'$  is holomorphically equivalent to the right half-plane and  $\log|\varphi^+|$  is holomorphically equivalent to  $\beta(x + iy) \mapsto \log|\alpha(x + iy)| = x$ . The function  $x + iy \mapsto x$  is a minimal function on the right half-plane so  $\log|\varphi^+|$  is minimal on  $U$ .  $\square$

If we fix a simply connected domain  $G$  as above and consider  $\zeta_1, \zeta_2 \in G$ , then the product structure on  $\hat{\mathcal{O}}_G$  gives us a homeomorphism  $\chi_{\zeta_1, \zeta_2} : \hat{\mathcal{O}}_{\{\zeta_1\}} \rightarrow \hat{\mathcal{O}}_{\{\zeta_2\}}$ . If  $\gamma$  is a path in  $\mathbf{C} - \overline{\Delta}$ , then we define a holonomy map  $\chi_\gamma : \hat{\mathcal{O}}_{\{\gamma(0)\}} \rightarrow \hat{\mathcal{O}}_{\{\gamma(1)\}}$  by subdividing the path into small intervals, covering each interval by a disk and composing successive local holonomy maps. The resulting holonomy map depends only on the homotopy type of the path relative to its endpoints.

We now prove that  $\hat{O}$  supports a unique positive, closed current. We will define a family  $\mathcal{L}(\mathcal{M}^-)$  of positive, closed currents which are compatible with the laminar structure of  $\mathcal{M}^-$ . We consider currents  $S$  such that for each  $\zeta \in \mathbf{C} - \bar{\Delta}$  there is a neighborhood  $G$  of  $\zeta$  and a measure  $\lambda_{S,\zeta}$  on  $\hat{O}_{\{\zeta\}}$  such that

$$(2.2) \quad S \llcorner \hat{O}_G = \int_{t \in \hat{O}_{\{\zeta\}}} \lambda_{S,\zeta}(t) [\Gamma_t],$$

where  $\Gamma_t$  are the leaves of the product lamination  $\mathcal{M}^- \cap J_G^-$ . We let  $\mathcal{L}(\mathcal{M}^-)$  denote the set of positive, closed currents  $S$  on  $U^+$  with support in  $\hat{O}$  such that the representation (2.2) holds with  $\lambda_{S,\zeta}$  a probability measure. Thus for each  $S \in \mathcal{L}(\mathcal{M}^-)$  there is a probability measure  $\lambda_\zeta$  on  $\hat{O}_{\{\zeta\}}$  for each  $\zeta \in \mathbf{C} - \bar{\Delta}$ . These measures are connected via the holonomy map: if  $\rho$  is a path connecting the points  $\zeta$  and  $\zeta'$ , then  $(\chi_\rho)_* \lambda_\zeta = \lambda_{\zeta'}$ . Let  $\gamma$  be a loop based at  $\zeta$  which generates the fundamental group of  $\mathbf{C} - \bar{\Delta}$ . If we have a measure  $\lambda$  on  $\hat{O}_{\{\zeta\}}$  which is invariant under the holonomy automorphism  $\chi_\gamma : \hat{O}_{\{\zeta\}} \rightarrow \hat{O}_{\{\zeta\}}$ , we can define a family of measure  $\lambda_{\zeta'}$  on each  $\hat{O}_{\{\zeta'\}}$  by the formula  $\lambda_{\zeta'} = (\chi_\rho)_* \lambda_\zeta$  where  $\rho$  is a path connecting  $\zeta$  and  $\zeta'$ . The resulting measure is independent of the path: any other path between these points is homotopic relative to its endpoints to a path which differs from  $\rho$  by a multiple of  $\gamma$ . This family of measures produces a current in  $\mathcal{L}(\mathcal{M}^-)$  via formula (2.2). Thus we have proved the following lemma.

LEMMA 2.11. *Elements of  $\mathcal{L}(\mathcal{M}^-)$  are in one-to-one correspondence with probability measures on  $\hat{O}_{\{\zeta\}}$  which are invariant under the holonomy map  $\chi_\gamma$ .*

Next we will look at the push-forward of a current under a diffeomorphism. We recall that for a current of integration  $[D]$ , the push-forward is given by  $f_*[D] = [f(D)]$ . Thus the push-forward of a current of the form (2.1) is given by  $f_*(\int \lambda(t) [\Gamma_t]) = \int \lambda(t) [f\Gamma_t]$ .

LEMMA 2.12.  $d^{-1} f_* \mathcal{L}(\mathcal{M}^-) \subset \mathcal{L}(\mathcal{M}^-)$ .

*Proof.* For a point  $\zeta \in \mathbf{C} - \bar{\Delta}$ , we let  $\xi_1, \dots, \xi_d$  denote the solutions to  $\xi^d = \zeta$ , and we let  $D_1, \dots, D_d$  denote the preimages of a neighborhood  $G$  of  $\zeta$ . Let  $S \in \mathcal{L}(\mathcal{M}^-)$  be given, and suppose that over each  $D_j$ ,  $S$  has the form (2.1). It follows that on the neighborhood  $\hat{O}_G$  of the fiber  $\hat{O}_{\{\zeta\}}$ ,

$$f_* S \llcorner (\varphi^+)^{-1} G = \sum_{j=1}^d \int_{t \in \hat{O}_{\xi_j}} \lambda_{\xi_j}(t) [f\Gamma_{j,t}],$$

where  $\Gamma_{j,t}$  denotes a leaf of  $\mathcal{M}^-$  lying over  $D_j$  and containing the point  $t$ . The transversal measure in this case is the push-forward of  $\sum_{j=1}^d \lambda_{\xi_j}$ , which has

total mass  $d$ . After we normalize by multiplying by  $d^{-1}$ ,  $f_*S$  will again belong to  $\mathcal{L}(\mathcal{M}^-)$ .  $\square$

By (1.1) it follows that if  $R$  is large, then for any  $S \in \mathcal{L}(\mathcal{M}^-)$ , the restriction  $S \llcorner \{|y| > R\}$  is closed on  $\{|y| > R\}$ . We will let  $S_\zeta$  denote the slice measure  $S \llcorner \{y = \zeta\}$ . For  $|\zeta| > R$ , the transversal  $\hat{\mathcal{O}} \cap \{y = \zeta\}$  is approximately  $\hat{\mathcal{O}}_{\{\zeta\}}$ , and so the local holonomy map  $\chi : \hat{\mathcal{O}} \cap \{y = \zeta\} \rightarrow \hat{\mathcal{O}}_{\{\zeta\}}$  is well defined and takes the slice measure  $S_\zeta$  to  $\lambda_\zeta$ . Thus the slice measure is a probability measure. We define the potential function

$$P_S(x, y) := \int_{x' \in \mathbb{C}} \log |x - x'| S_y(x').$$

Since  $S$  is a positive, closed current, it follows that on  $\{|y| > R\}$ ,  $P_S$  is plurisubharmonic, and  $\frac{1}{2\pi} dd^c P_S = S$ . Since each slice measure  $S_y$  is supported in the set  $\{|x| \leq |y|\}$ , it follows that

$$(2.3) \quad \log(|x| + |y|) \geq P_S(x, y) \geq \log(|x| - |y|)$$

holds on the set  $\{(x, y) : R < |y| < |x|\}$ .

The following proposition was motivated by the result of Forneaess and Sibony [FS, Theorem 7.12]: *Any positive closed current on  $\mathbb{C}^2$  which has support in  $K^+$  must be a multiple of  $\mu^+$ .*

PROPOSITION 2.13.  $\mathcal{L}(\mathcal{M}^-) = \{\mu^- \llcorner U^+\}$ .

*Proof.* By Lemma 2.11, we know that there is an element  $S \in \mathcal{L}(\mathcal{M}^-)$ . If we write  $S^j := d^j f_*^{-j} S$ , then by Lemma 2.12, we have  $S^j \in \mathcal{L}(\mathcal{M}^-)$ . By our notation,

$$S = d^{-j} (f^{-j})^* d^j f_*^{-j} S = d^{-j} (f^{-j})^* S^j.$$

By the remarks on the potential function, then,

$$S = (d^{-j}) \frac{1}{2\pi} dd^c (P_{S^j} \circ f^{-j}).$$

For any point  $(x, y) \in \{|y| > R\} - K^-$ , we observe that  $f^{-j}(x, y) = (x_{-j}, y_{-j})$  satisfies  $R \ll |y_{-j}| \ll |x_{-j}| \rightarrow \infty$  as  $j \rightarrow \infty$ . We recall that  $d^{-j} \log |x_{-j}| \rightarrow G^-$  uniformly on compact subsets of  $\mathbb{C}^2 - K^-$ . Now if we apply (2.3), we obtain that

$$(2.4) \quad d^{-j} P_{S^j}(f^{-j}(x, y)) \rightarrow G^-(x, y)$$

uniformly on compact subsets of  $\{|y| > R\} - K^-$ . It follows that the sequence on the left-hand side of (2.4) is uniformly bounded above, and any such sequence of plurisubharmonic functions either has a subsequence that converges in  $L_{\text{loc}}^1(|y| > R)$  or the entire sequence converges everywhere to  $-\infty$ . Now, passing to a subsequence, we may assume that it converges in  $L_{\text{loc}}^1(|y| > R)$

to a plurisubharmonic function  $W$ . Since  $W = G^-$  on the complement of  $K^-$ ,  $G^-$  is continuous and  $G^- = 0$  on  $K^-$ , and since  $W$  is upper semicontinuous, we have  $W = 0$  on  $\partial K^-$ . Further, since  $\text{supp}(dd^c P_{S_j}) \subset J^-$ , it follows that  $\text{supp}(dd^c W) \subset J^-$ . Thus  $W$  is pluriharmonic on the interior of  $K^-$ , and so  $W = 0$  on  $K^-$ . Thus  $W = G^-$ , so this sequence converges in  $L_{\text{loc}}^- (|y| > R)$  to  $G^-$ . Applying  $\frac{1}{2\pi} dd^c$  to (2.3), we conclude that  $S = \frac{1}{2\pi} dd^c G^- = \mu^-$  on  $U^+$ . Thus  $S = \mu^- \llcorner U^+$ .  $\square$

COROLLARY 2.14.  $\overline{\mathcal{O}} \cap U^+ = \hat{\mathcal{O}} = J_+^-$ .

*Proof.* Since  $\mathcal{O} \subset J^-$  and  $J^-$  is closed we have  $\overline{\mathcal{O}} \cap U^+ \subset J_+^-$ . For the other containment, let  $\nu^-$  denote the current with  $\text{supp}(\nu^-) \subset \overline{\mathcal{O}}$ , given in Lemma 2.11. By Proposition 2.13, we have  $\mu^- \llcorner U^+ = \nu^-$ , so  $J^- \cap U^+ = \text{supp}(\nu^-) \subset \overline{\mathcal{O}}$ , since  $\text{supp}(\mu^-) = J^-$ .

COROLLARY 2.15. *The holonomy map is uniquely ergodic and the unique invariant measure has full support.*

*Proof.* By Lemma 2.11 a finite probability measure  $\lambda$  on the transversal invariant under the holonomy map gives rise to a current on  $J_+^-$ . There is only one such current namely  $\mu^-$  and its support is all of  $J_+^-$ . It follows that the support of  $\lambda$  is equal to the transversal.

COROLLARY 2.16. *Every leaf of the lamination  $\mathcal{M}^-$  is dense.*

*Proof.* This corresponds to the minimality of the holonomy map. The minimality of the holonomy map is a consequence of the unique ergodicity with a measure of full support (see [Wa, Theorem 6.17]).

The next result establishes the uniqueness of the lamination  $\mathcal{M}^-$ .

THEOREM 2.17. *If  $f$  satisfies condition (†) and  $D$  is a connected Riemann surface contained in  $J_+^-$ , then  $D$  is an open subset of a leaf of the lamination  $\mathcal{M}^-$ .*

*Proof.* Let us suppose that  $D$  is a complex disk such that  $D \cap J_+^-$  contains an open subset of  $D$ . We recall that the Green function  $G^-$ , restricted to the transversals  $\hat{\mathcal{O}}_{\{\zeta\}}$ , is continuous. Thus  $\hat{\mathcal{O}}_{\{\zeta\}}$  cannot have isolated points. Let us consider a neighborhood  $N \subset J_+^-$  such that  $\mathcal{M}^- \cap N$  is the product lamination  $T \times G$ , whose leaves are written  $\Gamma_t$ . Let us suppose, for the sake of contradiction, that  $D$  is not contained in any leaf of  $\mathcal{M}^-$ . It follows, then, that  $D$  must intersect each leaf  $M$  of  $\mathcal{M}^-$  in a zero-dimensional set. If  $D$  intersects  $\Gamma_{t_0}$  tangentially, then by [BLS, Proposition 6.4], the intersection of  $D$  and  $\Gamma_t$  is transverse for all  $t$  close to  $t_0$  with  $t \neq t_0$ .



Thus we may assume that  $D$  and  $\Gamma_t$  intersect transversally for all  $t \in T$ . We may also assume that in local coordinates  $D = \{|x| < 1\} \times \{0\}$ , so that  $D$  is contained in the family of complex disks  $D_s = \{|x| < 1, y = s\}$ . It follows that for  $\varepsilon$  small, there is a holonomy map  $\chi_s : D_s \cap N \rightarrow D \cap N$  for each  $|s| < \varepsilon$ . Since  $D \cap N$  contains an open subset of  $D$ , it follows that  $D_s \cap N$  contains a nonempty open subset of  $D_s$  for each  $s$ , and that  $\bigcup_{|s| < \varepsilon} (D_s \cap N)$  contains a domain in  $\mathbf{C}^2$ , which contradicts the fact that  $J_+^-$  has no interior.  $\square$

**COROLLARY 2.18.** *If  $f$  satisfies condition  $(\dagger)$  then decomposition of  $J_+^-$  into the leaves of  $\mathcal{M}^-$  is the unique way of writing  $J_+^-$  as a union of connected Riemann surfaces.*

**COROLLARY 2.19.** *If  $W \subset J^-$  is a Riemann surface conformally equivalent to  $\mathbf{C}$ , then each component of  $W \cap U^+$  is a disk satisfying  $(\dagger)$ .*

*Remark.* If  $p \in \mathcal{R}$  then  $W = W^u(p)$  satisfies the hypotheses of this corollary.

*Proof.* The set  $U^+ \cap W$  is an open subset of  $W$  so it has a most countably many components. According to Theorem 2.17 each component is contained in a leaf  $L$  of the lamination  $\mathcal{M}^-$ . Let  $L_1, L_2, \dots$  be the leaves that meet  $W$ . Let  $X = W \cup \{L_j\}$ . We claim that  $X$  has the structure of an immersed Riemann surface. Let  $\phi : \mathbf{C} \rightarrow W$  and let  $\phi_j : H \rightarrow L_j$  where  $H$  is the right half-plane. The set of charts  $\mathcal{A} = \{\phi, \phi_1, \dots\}$  forms an atlas for  $X$ . To show that  $\mathcal{A}$  is an atlas we verify that the change of coordinate functions are holomorphic. If  $\phi(\mathbf{C}) \cap \phi_j(H) = W \cap L_j$  is a Riemann surface so  $\phi^{-1}(W \cap L_j)$  is an open subset of  $\mathbf{C}$  and  $\phi_j^{-1}(W \cap L_j)$  is an open subset of  $H$  and  $\phi^{-1} \circ \phi_j$  is holomorphic.

The atlas  $\mathcal{A}$  determines a topology on  $X$  where we define a set  $Y \subset X$  to be open if  $\phi^{-1}(Y)$  is open for each chart  $\phi \in \mathcal{A}$ . With respect to this topology the inclusion from  $X$  into  $\mathbf{C}^2$  is continuous. The existence of an atlas for a topological space does not imply that it is Hausdorff. From the fact that the inclusion is continuous we can deduce that  $X$  is Hausdorff. If  $p$  and  $q$  are distinct points in  $X$  then they are contained in disjoint neighborhoods in  $\mathbf{C}^2$ . The pullbacks of these neighborhoods are disjoint neighborhoods in  $X$ . We also see that  $X$  is path connected. Since  $X$  is holomorphically embedded in  $\mathbf{C}^2$  so it is not compact.

We will finish by showing that  $X = W$ . This will imply that the intersection of  $W$  with  $U^+$  is a union of leaves of the lamination  $\mathcal{M}^-$ . The corollary then follows from Theorem 2.1 (5).

Assume that  $W$  is a proper subset of  $X$ . Replace  $X$  by its universal cover  $\tilde{X}$  and replace  $W$  by some lift  $W_0$  of  $W$ . The inclusion of  $W_0$  in  $\tilde{X}$  is still proper. Now  $\tilde{X}$  is simply connected but not compact thus it is conformally



equivalent to a disk or plane. Now  $\tilde{X}$  contains a copy of  $\mathbf{C}$  so it is not the disk. We conclude that  $\tilde{X}$  is the plane. A proper simply connected subsurface of the plane is equivalent to the disk so  $W_0$  must be equal to  $\tilde{X}$  contradicting our assumption.  $\square$

### 3. External rays and semi-conjugacy to the solenoid

The previous section was devoted to the study of some of the properties of mappings which have a disk satisfying  $(\dagger)$ , and the following sections will be devoted to establishing various conditions which imply this property. Before we proceed with this, however, we describe two ways in which the special features of these maps can be exploited. First we define a family  $\mathcal{E}$  of external rays in  $J_+^-$ . It is expected that, as in the one-dimensional case, a connection between the dynamics of the restriction  $f|_{J_+^-}$  and  $f|_J$  will be obtained by a more careful study of the map of external rays. Our second observation is that  $f|_{J_+^-}$  is semi-conjugate to the shift on the complex solenoid. The paper [BS7] is devoted to further exploration of the relationship between these mappings and the solenoid in the hyperbolic case.

We begin by recalling some properties developed in §2. If  $f$  satisfies  $(\dagger)$ , then by Theorem 2.1 there is a lamination  $\mathcal{M}^-$  of  $J_+^-$  and a holomorphic extension  $\varphi^+ : J_+^- \rightarrow \mathbf{C} - \bar{\Delta}$ , which serves as a canonical projection. For any  $S \subset \mathbf{C} - \bar{\Delta}$  we set  $J_S^- := (\varphi^+)^{-1}(S)$ . The sets  $J_{\{\xi\}}^-$ ,  $\xi \in \mathbf{C} - \bar{\Delta}$  form a canonical family of transversals to  $J_+^-$ . If  $G \subset \mathbf{C} - \bar{\Delta}$  is a simply connected domain, then for any  $\xi \in G$ , Proposition 2.4 shows that the restriction  $\mathcal{M}^- \cap J_G^-$  is equivalent to a product lamination whose leaves are  $\{\Gamma_t : t \in J_{\{\xi\}}^-\}$ , and  $\varphi^+ : \Gamma_t \rightarrow G$  is a canonical biholomorphism. This local product structure also extends to  $\mu^-$  in the sense that there is a measure  $\lambda_\xi$  on  $J_{\{\xi\}}^-$  such that

$$(3.1) \quad \mu^- \llcorner J_G^- = \int_{t \in J_{\{\xi\}}^-} \lambda_\xi(t) [\Gamma_t].$$

For a leaf  $M$  of  $\mathcal{M}^-$ , the 1-form  $d^c G^+$  restricts naturally to  $M$ . The set of integral curves  $\gamma$  of  $d^c G^+|_M$  will be called *external rays*, and the set of all external rays will be denoted by  $\mathcal{E}$ . External rays serve as a substitute for gradient lines of  $G^+$ , and each  $\gamma$  may naturally be parametrized by  $G^+$ . These are called rays because each  $\gamma \in \mathcal{E}$  is contained in some leaf  $M$  of  $\mathcal{M}^-$ , and  $\gamma$  is the lift of some radial ray  $R_\theta = \{re^{i\theta} \in \mathbf{C} : r > 1\}$  under the mapping  $\varphi^+|_M$ . We may define the map  $e_r : \mathcal{E} \rightarrow \{G^+ = r\} \cap J^-$  by letting  $e_r(\gamma)$  be the intersection point  $\gamma \cap \{G^+ = r\}$ . The mapping  $\chi_{r,s} : J^- \cap \{G^+ = r\} \rightarrow J^- \cap \{G^+ = s\}$ , defined by following the external rays, is a homeomorphism

between these two sets, and  $e_s = \chi_{r,s} \circ e_r$ . It is natural to topologize  $\mathcal{E}$  so that  $e_r$  is a homeomorphism.

We will show how external rays give a description of the harmonic measure  $\mu$  when  $(\dagger)$  holds. Let  $\mu_r$  denote the measure on  $J^- \cap \{G^+ = r\}$  defined by

$$(3.2) \quad \mu_r := \frac{1}{(2\pi)^2} (-d^c G^+ \llcorner \partial\{G^+ > r\}) \wedge dd^c G^- = \frac{1}{(2\pi)^2} dd^c(\max(G^+, r)) \wedge dd^c G^-.$$

We note that  $(2\pi)^{-1}(-d^c G^+|_M \llcorner \{G^+ > r\})$  may be interpreted as the harmonic measure of  $M \cap \{G^+ > r\}$ . (This is also the pullback under  $\varphi^+|_M$  of the planar harmonic measure  $(2\pi)^{-1}d^c \log |\zeta|$  of  $\{\zeta \in \mathbf{C} : |\zeta| > r\}$ , which is normalized arclength measure on the circle orthogonal to the rays.) For a simply connected domain  $G \subset \mathbf{C} - \overline{\Delta}$ , it follows from (3.1) that on  $J_G^-$  the middle expression in (3.2) may be interpreted as

$$(3.3) \quad \int_{t \in J_{\{\xi\}}^-} \lambda_\xi(t) \omega_t$$

if we set  $\omega_t := (2\pi)^{-1}(-d^c G^+|_{\Gamma_t} \llcorner \{G^+ > r\})$ . Flowing along the gradient lines of  $G^+|_M$  preserves harmonic measure in the sense that  $\chi_{r,s_*} \omega_r = \omega_s$ , wherever  $\chi_{r,s}$  is correctly defined, so  $\chi_{r,s}$  also preserves  $\mu_r$  in the sense that  $(\chi_{r,s})_* \mu_r = \mu_s$ . Thus we may define a measure  $\nu$  on  $\mathcal{E}$  by the condition that  $(e_r)_* \nu = \mu_r$ , and this definition is independent of  $r$ .

For a leaf  $M$  of  $\mathcal{M}^-$ , let  $\alpha_M : H \rightarrow M$  denote the uniformization given in Lemma 2.2. All external rays  $\gamma \subset M$  are of the form  $\gamma(y) = \{\alpha_M(x + iy) : 0 < x < \infty\}$ . Since  $\alpha_M$  is a bounded analytic function on the set  $\{0 < x < R < \infty\}$  the limit  $\alpha_M^*(y) := \lim_{x \rightarrow 0^+} \alpha_M(x + iy)$  exists for almost every  $y$ . Thus  $\alpha_M^*(y) = \lim_{r \rightarrow 0^+} e_r(\gamma(y))$  exists for almost every  $\gamma(y) \subset M$ . Since harmonic measure  $\omega_t$  corresponds to  $dy$  under  $\alpha_M$ , it follows that within each leaf  $M$ ,  $e$  is defined almost everywhere with respect to  $\omega_t$ . We define the endpoint map

$$e(\gamma) = \lim_{r \rightarrow 0} e_r(\gamma)$$

for all  $\gamma \in \mathcal{E}$  for which this limit exists, and we observe that by (3.3),  $e(\gamma)$  is defined for  $\nu$  almost every  $\gamma$ .

Thus  $e : \mathcal{E} \rightarrow \mathbf{C}^2$  is a Borel measurable map, and we may push the measure  $\nu$  forward to a measure  $e_* \nu$  on  $\mathbf{C}^2$ , which coincides with the weak-\* limit  $\lim_{r \rightarrow 0} (e_r)_* \nu$ . From the right-hand expression in (3.2) and the fact that  $G^+$  is continuous, we see that  $\lim_{r \rightarrow 0^+} \mu_r = \mu$ . It follows by the uniqueness of weak-\* limits of measures that we have the following:

**THEOREM 3.1.** *If  $f$  satisfies the hypotheses of Theorem 2.1, then the endpoint mapping  $e$  is defined  $\nu$  almost everywhere on  $\mathcal{E}$ , and  $e_* \nu = \mu$ .*

We now begin the construction of the semi-conjugacy to the standard model. Let  $\sigma : \mathbf{C}^* \rightarrow \mathbf{C}^*$  denote the map  $\sigma(z) = z^d$ . Let  $\Sigma$  denote the projective limit of the map  $\sigma$ , and call  $\Sigma$  the  $d$ -fold complex solenoid. We may represent the solenoid as a space of bi-infinite sequences

$$\Sigma = \{(\dots z_{-2}z_{-1}z_0z_1z_2\dots) : \text{where } z_j \in \mathbf{C}^* \text{ and } z_{j+1} = \sigma(z_j) \text{ for all } j \in \mathbf{Z}\}.$$

Observe that  $\sigma$  induces a homeomorphism (which we again denote as  $\sigma : \Sigma \rightarrow \Sigma$ ), which is given by left translation:  $\sigma(z) = w$ , where  $z = (z_j)$  and  $w = (w_j)$  has entries  $w_j = z_{j+1}$ .

Define  $\pi : \Sigma \rightarrow \mathbf{C}^*$  by  $\pi(z) = \pi(\dots z_{-1}z_0z_1\dots) = z_0$ . Thus  $\pi\sigma = \pi^d$ . We give  $\Sigma$  the product topology, so the fiber  $\pi^{-1}(\zeta)$  is a Cantor set (i.e. totally disconnected and perfect).

The standard (real) solenoid is given by

$$\Sigma_0 = \{s \in \Sigma : |\pi(s)| = 1\}.$$

This may also be identified with a set of sequences of points of the circle

$$\Sigma_0 = \{\theta = (\dots \theta_{-1}\theta_0\theta_1\dots) : \theta_j \in \mathbf{R}/2\pi\mathbf{Z}, d \cdot \theta_n = \theta_{n+1}\}.$$

We let  $\eta : \Sigma \rightarrow \Sigma_0$  be defined by  $\eta(s) = \tilde{s}$ , where  $\tilde{s}_j = s_j/|s_j|$ . Thus  $\eta$  commutes with  $\sigma$ . We define

$$\Sigma_+ = \{s \in \Sigma : |\pi(s)| > 1\}$$

to be the portion of the complex solenoid lying above the complement of the closed unit disk. We observe that the fibers of the mapping  $\eta : \Sigma_+ \rightarrow \Sigma_0$  are rays in the complex solenoid, so the complex solenoid has a natural structure of a family of external rays, parametrized by the real solenoid  $\Sigma_0$ .

Given a space  $X$ , a mapping  $\Phi : X \rightarrow \Sigma$  is given by a family of mappings  $\phi_j : X \rightarrow \mathbf{C}^*$ ,  $j \in \mathbf{Z}$  such that  $\phi_{j+1}(x) = \phi_j^d(x)$  for all  $x \in X$  and  $j \in \mathbf{Z}$ . If  $\Phi$  is to give a semi-conjugacy between a bijection  $f : X \rightarrow X$  and  $\sigma$ , then we must have  $\phi_j \circ f = \phi_{j+1} = \phi_j^d$ . In this case  $\phi_0 : X \rightarrow \mathbf{C}^*$  determines all of the coordinate maps  $\phi_j$  via the relation  $\phi_j(x) = \phi_0(f^j x)$  for all  $j \in \mathbf{Z}$ . The consistency condition for a given map  $\phi_0 : X \rightarrow \mathbf{C}^*$  to induce an equivariant mapping  $\Phi$  in this fashion is that  $\phi_0 \circ f = \phi_0^d$ . According to Theorem 2.1(1) if there is a disk satisfying condition  $(\dagger)$  then  $\varphi^+$  has an extension to  $J_+^-$  which satisfies  $\varphi^+ \circ f = (\varphi^+)^d$ . Thus  $\varphi^+$  serves as a 0-th coordinate map for an equivariant mapping  $\Phi : J_+^- \rightarrow \Sigma_+$  given by

$$(3.4) \quad \Phi(p) := (\phi_j(p)) = (\varphi^+(f^j p)).$$

**THEOREM 3.2.** *If  $f$  satisfies the hypotheses of Theorem 2.1, then there is a continuous mapping  $\Phi : J_+^- \rightarrow \Sigma_+$  which is holomorphic and injective on the leaves of  $\mathcal{M}^-$  and such that*

$$(3.5) \quad \sigma \circ \Phi = \Phi \circ f,$$

and

$$(3.6) \quad \log |\pi \circ \Phi| = G^+.$$

*Proof.* By Theorem 2.1,  $\varphi^+$  extends to  $J_+^-$  and is holomorphic on the leaves of  $\mathcal{M}^-$ . Thus, if we define  $\Phi$  as in (3.4), then (3.5) and (3.6) are easily seen to be satisfied. For  $M \in \mathcal{M}^-$  it is evident that  $\Phi|_M$  is holomorphic. We must show that  $\Phi|_M$  is injective. We let  $\zeta_0 = \varphi^+(p)$  and let  $j\tau$  denote the path in  $\mathbf{C}$  starting at  $\zeta_0$  which traces the circle  $\{|\zeta| = |\zeta_0|\}$   $j$  times in the counter-clockwise direction. We let  $p_j$  be the point obtained by lifting  $j\tau$  via the map  $\varphi^+|_M$  starting at  $p_0$ , so that  $(\varphi^+)^{-1}(\zeta_0) = \{p_j : j \in \mathbf{Z}\}$ . We let  $\gamma_{j_1, j_2}$  denote the curve above  $\tau$  which goes from  $p_{j_1}$  to  $p_{j_2}$ . Since  $(\varphi^+)^d = \varphi^+ \circ f$ , we see that

$$((\varphi^+ \circ f^{-n})|_M)^{d^n} = \varphi^+|_M.$$

It follows that the curve  $(\varphi^+ \circ f^{-n})(\gamma_{j_1, j_2})$  starts at the point  $\varphi^+ \circ f^{-n}(p_{j_1})$  and moves inside the circle  $\{|\zeta| = |\varphi^+ \circ f^{-n}(p_0)|\}$  through an angle of  $2\pi(j_2 - j_1)d^{-n} \in \mathbf{Z}$ . It follows that if  $\phi_{-n}(p_{j_1}) = \phi_{-n}(p_{j_2})$  for  $n = 1, 2, \dots$ , then  $j_2 - j_1$  is divisible by  $d^n$  for all  $n$ . Thus  $j_1 = j_2$ , and so  $\Phi$  is one-to-one on  $M$ .  $\square$

Under the mapping  $\Phi$ , the external rays  $\mathcal{E}$  are mapped to rays of the solenoid, and the lamination  $\mathcal{M}^-$  is taken to the lamination of  $\Sigma_+$ .

#### 4. Unstable connectedness

Let  $p$  be any point in  $J$  so that  $W^u(p)$  exists and is conformally equivalent to  $\mathbf{C}$ . We will say that  $f$  is unstably connected with respect to  $p$  if  $W^u(p) - K^+$  has at least one simply connected component.

Let  $\nu$  be an ergodic hyperbolic measure with index one. We will say that  $f$  is *unstably connected* with respect to  $\nu$  if for  $\nu$  almost every point  $p$ ,  $f$  is unstably connected with respect to  $p$ . Since  $\nu$  is assumed to be ergodic and our condition on  $p$  is invariant under  $f$ , it is equivalent to assume  $f$  is unstably connected with respect to  $p$  for  $p$  in a set of positive  $\nu$  measure.

**THEOREM 4.1.** *Let  $\nu$  be a hyperbolic index one measure then  $f$  is unstably connected with respect to  $\nu$  if and only if there is a disk satisfying  $(\dagger)$ .*

**COROLLARY 4.2.** *If  $f$  is unstably connected with respect to some hyperbolic measure  $\nu$  then it is unstably connected with respect to any hyperbolic measure  $\nu$ .*

We say that  $f$  is *unstably connected* if  $f$  satisfies the hypothesis of the corollary for any hyperbolic measure.

*Remark.* If  $p$  is a periodic saddle point and  $\nu$  is the normalized counting measure on the orbit then the condition of being unstably connected with respect to  $p$  described in the introduction, i.e. that  $W^u(p) \cap U^+$  has a simply connected component, is equivalent to being unstably connected with respect to  $\nu$ .

The “if” part of Theorem 4.1 is already contained in Corollary 2.19. The rest of this section will be devoted to the proof of the “only if” statement. The proof of Theorem 4.1 can be broken into two parts: function theoretic, involving the growth of  $G^+$  on a single unstable manifold, and dynamical, where we use the properties of the measure  $\nu$ . We will start with some properties of hyperbolic measures. The reader who is unfamiliar with hyperbolic measures may find it convenient upon first reading to consider only the simplest hyperbolic measures: the averages of point masses over periodic saddle orbits. This proves Theorem 4.1 in the case of periodic saddle points and makes the proofs of the following three lemmas rather simple.

For the rest of this section,  $\nu$  will denote an index one hyperbolic measure, and  $f$  will be assumed to be unstably connected with respect to  $\nu$ .

Applying the Pesin Stable Manifold Theorem to an index one hyperbolic measure  $\nu$ , it follows that for  $\nu$  almost every point  $p$  there is an unstable manifold  $W^u(p)$  which is conformally equivalent to  $\mathbf{C}$  (cf. [BLS], [W]). Let us fix a conformal equivalence  $\phi : \mathbf{C} \rightarrow W^u(p)$  with  $\phi(0) = p$ . We will use  $\phi$  to translate concepts from  $W^u(p)$  back to  $\mathbf{C}$ . For  $z \in \mathbf{C}$  we will sometimes write  $G^+(z)$  to denote the function  $G^+(\phi(z))$ . We write  $U^+$  for  $\phi^{-1}(U^+)$  and  $K^+$  for  $\phi^{-1}(K^+)$ . Thus  $G^+$  is a continuous subharmonic function on  $\mathbf{C}$  which is equal to zero on  $K^+$  and positive and harmonic on  $U^+$ .

Given one uniformization  $\phi$ , all other uniformizations  $\phi$  are of the form  $\phi(\alpha z)$  for some constant  $\alpha \in \mathbf{C} - \{0\}$ . We define the function

$$M(p, r) = \max_{|z|=r} G^+(\phi(z)).$$

By the maximum principle  $M$  is an increasing function of  $r$ . We may choose the scale  $|\alpha|$  in the uniformizing function  $\phi(\alpha z)$  so that  $M(p, 1) = 1$ , and we let  $\phi_p$  denote the uniformization which is normalized this way. This determines  $M$  uniquely in terms of  $G^+$  and the conformal structure of  $W^u(p)$ . Since  $G^+ \circ \phi$  is subharmonic on  $\mathbf{C}$ ,  $M(p, r)$  is continuous and increasing in  $r$ . We define a Hermitian metric  $\|\cdot\|_G$  on the subspace  $E_p^u$  by the condition

$$\|D\phi_p(0)\mathbf{1}\|_G = 1,$$

where  $\mathbf{1}$  denotes an element of  $T_0\mathbf{C}$  which has unit length with respect to the standard metric. We will also consider the growth rate of  $Df^n$  with respect to  $F$ :

$$\|Df_p^n|_{E_p^u}\|_G = \|Df_p^n(v)\|_G/\|v\|_G,$$

where  $v$  is any nonzero element of  $E_p^u$ . The advantage of  $\|\cdot\|_G$  is that it transforms naturally under  $f$  in connection with  $G^+$  and  $M(p, r)$ .

LEMMA 4.3. *For all  $n \in \mathbf{Z}$  we have  $M(f^n p, \|Df_p^n\|_G) = d^n$ .*

*Proof.* The map  $\phi_{f^n p}^{-1} \circ f^n \circ \phi_p$  is a holomorphic and bijective map that sends  $\mathbf{C}$  to  $\mathbf{C}$  and takes 0 to 0. Any such map is linear. Thus there is  $\xi \in \mathbf{C} - \{0\}$  so that

$$(4.1) \quad \phi_{f^n p}^{-1} \circ f^n \circ \phi_p(z) = \xi z.$$

Thus  $\phi_{f^n p}(\xi z) = f^n \circ \phi_p(z)$ . By rewriting this equation and applying  $G^+$  we obtain:

$$G^+(\phi_{f^n p}(\xi z)) = G^+(f^n \circ \phi_p(z)).$$

Now  $G^+$  multiplies by  $d$  when  $f$  is applied so

$$G^+(\phi_{f^n p}(\xi z)) = d^n G^+(z).$$

We now evaluate

$$\begin{aligned} M(f^n p, |\xi|) &= \max_{|z|=|\xi|} G^+(\phi_{f^n p}(z)) = \max_{|z|=1} G^+(\phi_{f^n p}(\xi z)) \\ &= d^n \max_{|z|=1} G^+(\phi_p(z)) = d^n. \end{aligned}$$

To evaluate  $|\xi|$ , we differentiate equation (4.1) to get

$$D\phi_{f^n p}^{-1} \circ Df^n \circ D\phi_p(z) = \xi;$$

thus

$$\|D\phi_{f^n p}^{-1}\| \cdot \|Df^n\|_G \cdot \|D\phi_p(z)\| = \|\xi\|.$$

Since  $\phi_p$  and  $\phi_{f^n p}$  were normalized so that  $D\phi_p$  and  $D\phi_{f^n p}$  have norm one, we have  $\|Df^n\|_G = |\xi|$ .  $\square$

The next lemma shows that we may also compute the Lyapunov exponent starting with the metric  $\|\cdot\|_G$ .

LEMMA 4.4. *Let  $\nu$  be an index one hyperbolic measure. For  $\nu$  almost every  $p$ , we have the existence of the limit*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D^n f_p\|_G = \lambda^u(\nu).$$

*Proof.* Let  $v \in E_p^u$  be nonzero and let

$$r(p) = \frac{\|v\|_H}{\|v\|_G},$$



so that

$$\frac{\|Df_p\|_H}{\|Df_p\|_G} = \frac{\|Df_p(v)\|_H}{\|Df_p(v)\|_G} \cdot \frac{\|v\|_G}{\|v\|_H} = r(f(p))/r(p).$$

Let  $\alpha(p) = \log \|Df_p\|_H$ ,  $\beta(p) = \log \|Df_p\|_G$ , and  $\rho(p) = \log r(p)$ . Taking logarithms gives the cocycle equation:

$$\alpha(p) - \beta(p) = \rho(f(p)) - \rho(p).$$

If  $\rho$  were in  $L^1$ , then our lemma would be a consequence of the Ergodic Theorem.

Although we have no information on  $\rho$ , we can show that  $\alpha(p) - \beta(p)$  is bounded below. Since  $M(fp, \|Df_p\|_G) = d$  and  $M(fp, 1) = 1$  and  $M$  is strictly monotone in  $r$ , we conclude that  $\|Df_p\|_G > 1$ . Thus  $\alpha(p) > 0$ . Further,  $\|Df_p\|_H$  is bounded above by the supremum of the Euclidean norm of the Jacobian matrix  $Df_p$  over the compact set  $J$ . Thus  $\beta(p) \leq C$  and  $\alpha(p) - \beta(p) \geq -C$ . According to [LS, Proposition 2.2] we have  $\lim_{n \rightarrow \infty} \frac{1}{n}(\alpha(f^n p) - \beta(f^n p)) = 0$  so  $\lim_{n \rightarrow \infty} \frac{1}{n}\alpha(f^n p) = \lim_{n \rightarrow \infty} \frac{1}{n}\beta(f^n p)$  holds for  $\nu$  almost every point  $p$ , and this limit must be equal to  $\lambda^u(\nu)$ .  $\square$

We fix a component  $\mathcal{O}$  of  $U^+ \cap W^u(p)$ . An *end* of  $\mathcal{O}$ , written  $\mathcal{E}_r(\mathcal{O})$  or just  $\mathcal{E}_r$ , is a connected component of  $\mathcal{O} \cap \{|z| > r\}$ . It is evident that  $\mathcal{E}_0(\mathcal{O}) = \mathcal{O}$ . We say that an end  $\mathcal{E}_r$  has no loops if  $\mathcal{E}_r$  contains no closed curves which encircle the origin.

LEMMA 4.5.  $W^u(p) \cap U^+$  has no loops.

*Proof.* If  $U^+$  contains a loop around zero then the component of  $K^+$  containing 0 is compact. But we have assumed that there are no compact components.  $\square$

For an end  $\mathcal{E}_r \subset U_p^+$ , we define

$$M(p, \mathcal{E}_r, s) = \max_{\zeta \in \bar{\mathcal{E}}_r \cap \mathcal{O} \cap \{|\zeta|=s\}} G^+(\zeta),$$

or we write  $M(\mathcal{E}_r, s) = M(p, \mathcal{E}_r, s)$  if the point  $p$  is understood. Let us fix  $r < s$ . For a subset  $X \subset \partial(\mathcal{O} \cap \{r < |z| < s\})$ , we define a function  $\omega(X, z)$  called the harmonic measure of the set  $X$  to be the greatest function  $0 \leq \omega(X, z) < 1$  that is harmonic on  $\mathcal{O} \cap \{r < |z| < s\}$  and which satisfies

$$\lim_{\zeta \rightarrow \zeta_0} \omega(X, \zeta) = 0 \quad \text{for } \zeta_0 \in \partial(\mathcal{O} \cap \{r < |z| < s\}) - X.$$

We will consider the case where  $X$  has the form  $A_t = \mathcal{O} \cap \{|z| = t\}$  for  $t = r, s$ . Let  $t\Theta(t)$  denote the length of the set  $\mathcal{O} \cap \{|z| = t\}$ . When  $\mathcal{O}$  contains



no loops, classical estimates on the harmonic measures of the sets  $A_r$  and  $A_s$  for  $r < |z| < s$  (see Fuchs [F]) are given by

$$\omega(A_r, z) \leq 4 \cdot \exp \left( -\pi \int_r^{|z|} \frac{dt}{t\Theta(t)} \right)$$

and

$$\omega(A_s, z) \leq 4 \cdot \exp \left( -\pi \int_{|z|}^s \frac{dt}{t\Theta(t)} \right).$$

Since  $\Theta(t) \leq 2\pi$  we have

$$\pi \int_a^b \frac{dt}{t\Theta(t)} \geq \pi \int_a^b \frac{dt}{2\pi t} = \frac{1}{2} \int_a^b \frac{dt}{t} = \frac{\log(b/a)}{2}.$$

Thus

$$\omega(A_r, z) \leq 4\sqrt{r/|z|} \quad \text{and} \quad \omega(A_s, z) \leq 4\sqrt{|z|/s}.$$

We use these estimates as follows. For any end  $\mathcal{E}_r$  of  $\mathcal{O}$ , the maximum principle gives

$$(4.2) \quad G^+(z) \leq M(\mathcal{E}_r, r)\omega(A_r, z) + M(\mathcal{E}_r, s)\omega(A_s, z).$$

Thus

$$(4.3) \quad M(\mathcal{E}_r, |z|) \leq 4M(\mathcal{E}_r, r)\sqrt{r/|z|} + 4M(\mathcal{E}_r, s)\sqrt{|z|/s}.$$

PROPOSITION 4.6. *Fix  $r \geq 0$  and an end  $\mathcal{E}_r$ . Then exactly one of the following two alternatives holds. Either*

$$(4.4) \quad M(\mathcal{E}_r, s) \leq 4M(\mathcal{E}_r, r)\sqrt{r/s} \text{ for all } s > r,$$

or for some constant  $c > 0$

$$(4.5) \quad M(\mathcal{E}_r, s) \geq c\sqrt{s} \quad \text{for all } s > r.$$

When  $r = 0$ , the second alternative must hold.

*Proof.* Let us fix an end  $\mathcal{E}_r$  of  $\mathcal{O}$ . If alternative (4.4) does not hold, then there exists  $t_0 > r$  such that

$$(4.6) \quad M(\mathcal{E}_r, t_0) > 4M(\mathcal{E}_r, r)\sqrt{r/t_0}.$$

Set  $\alpha = M(\mathcal{E}_r, t_0) - 4M(\mathcal{E}_r, r)\sqrt{r/t_0}$ . Then by (4.6),  $\alpha > 0$  and using (4.3) with  $|z| = t_0$  we have

$$\alpha \leq 4M(\mathcal{E}_r, s)\sqrt{\frac{t_0}{s}};$$

thus

$$c\sqrt{s} \leq M(\mathcal{E}_r, s)$$

holds with  $c = \alpha/(4\sqrt{t_0})$ .

Now, in case  $r = 0$  we have  $\mathcal{E}_r = \mathcal{O}$ . Case (4.4) cannot hold, for it implies that  $G^+ = 0$  on  $\mathcal{O}$  by the maximum principle.  $\square$

Let  $\mathcal{E}_r$  be an end. If case (4.4) in Proposition 4.6 applies, we will say that  $\mathcal{E}_r$  is a decay end, and if (4.5) applies we say that,  $\mathcal{E}_r$  is a growth end. By Proposition 4.6 each end is either a growth end or a decay end. For  $f > 0$  we let  $\mathcal{C}(r)$  denote the set of connected components of  $U^+ \cap \{|z| > r\}$ . Let  $c(r)$  be the cardinality of the set  $\mathcal{C}(r)$ , and let  $g(r)$  denote the number of growth components in  $\mathcal{C}(r)$ . Thus  $c(r) \geq g(r)$ . Since  $\mathcal{C}(0)$  corresponds to the set of connected components of  $U^+$ , it follows from Proposition 4.6 and the maximum principle that each component of  $U^+$  is a growth component. Thus  $c(0) = g(0)$ .

If  $r < s$ , then there is the containment map

$$\Upsilon : \mathcal{C}(s) \rightarrow \mathcal{C}(r)$$

where for any component  $\mathcal{O} \in \mathcal{C}(s)$ ,  $\Upsilon(\mathcal{O})$  denotes the element of  $\mathcal{C}(r)$  containing  $\mathcal{O}$ .

**PROPOSITION 4.7.** *The functions  $c(r)$  and  $g(r)$  are nondecreasing in  $r$ , and if  $g(r)$  is finite for some value of  $r$ , then  $\lim_{r \rightarrow 0} g(r) = g(0) = c(0)$ .*

*Proof.* The mapping  $\Upsilon$  is surjective since there are no bounded components (this is a consequence of the maximum principle), so  $c(r) \geq c(s)$  for  $r \leq s$ . Further, by Proposition 4.6, an end  $\mathcal{E}_r$  is a growth end if and only if one of the components of  $\mathcal{E}_r \cap \{|z| > s\}$  is a growth end. Thus  $\Upsilon$  is a surjective mapping from growth ends to growth ends, and so  $g(r)$  is nondecreasing.

Finally, we show that  $\lim_{r \rightarrow 0} g(r) = g(0)$  if  $g(r)$  is finite. Let us suppose, to the contrary, that  $g(0) < g(r)$  for  $0 < r < r_0$ . This means that there are two growth ends  $\mathcal{E}_{r_0}^1$  and  $\mathcal{E}_{r_0}^2$  in  $U^+ \cap \{|z| > r\}$  which are contained in different connected components of  $U^+ \cap \{|z| > r\}$  for  $0 < r < r_0$ , but which are contained in the same connected component of  $U^+ - \{0\}$ . Now if  $\gamma$  is a path in  $U^+ - \{0\}$  which connects these two components, then  $\gamma$  avoids some disk  $\{|z| < \varepsilon\}$ . Thus  $\mathcal{E}_{r_0}^1$  and  $\mathcal{E}_{r_0}^2$  are contained in the same connected component of  $U^+ \cap \{|z| > s\}$ , which is a contradiction.  $\square$

**LEMMA 4.8.** *If  $\mathcal{O}_1, \dots, \mathcal{O}_N$  are disjoint, open sets, then there exists  $j$  such that*

$$\pi \int_t^s \frac{dr}{r\Theta_j(r)} \geq \frac{N}{2} \log \left( \frac{s}{t} \right).$$

*Proof.* By the Cauchy inequality,

$$\sum \Theta_j \sum \Theta_j^{-1} \geq \left( \sum \Theta_j^{\frac{1}{2}} \Theta_j^{-\frac{1}{2}} \right)^2 = N^2.$$

Since the sets  $\mathcal{O}_j$  are disjoint,  $\sum \Theta_j \leq 2\pi$ ; thus

$$\sum_{j=1}^N \frac{1}{\Theta_j(r)} \geq \frac{N^2}{2\pi}.$$

This gives

$$\sum_{j=1}^N \int_t^s \frac{dr}{r\Theta_j(r)} \geq \frac{N^2}{2\pi} \log\left(\frac{s}{t}\right)$$

from which the lemma follows. □

PROPOSITION 4.9. *If for some  $r > 0$  there are  $N$  distinct growth ends in  $U^+ \cap \{|z| > r\}$ , then*

$$M(U^+, s) \geq Cs^{N/2}$$

for some  $C > 0$  and all  $s \geq r$ .

*Proof.* The harmonic measure estimates above yield the estimate

$$M(\mathcal{O}, t) \leq 4M(\mathcal{O}, r)\sqrt{r/t} + 4M(\mathcal{O}, s) \cdot \exp\left(-\pi \int_t^s \frac{dr}{r\Theta(r)}\right).$$

Arguing as in Proposition 4.6 we have

$$M(\mathcal{O}, s) \geq C \cdot \exp\left(-\pi \int_t^s \frac{dr}{r\Theta(r)}\right).$$

If we choose  $\mathcal{O} = \mathcal{O}_j$  satisfying the conclusion of Lemma 4.8, then the proposition follows. □

PROPOSITION 4.10. *For  $\nu$  almost every point  $p$  and every  $r > 0$  we have*

$$g(r) \leq \frac{2 \log d}{\lambda(\nu)}.$$

*Proof.* According to Lemma 4.4,  $M(f^n p, \|Df_p^n\|_G) = d^n$ . Thus, after changing variables, we have  $M(p, \|Df_{f^{-n}p}^n\|_G) = d^n$ .

Let  $N$  be a (finite) integer no greater than  $g(r)$ . Then

$$d^N = \gamma(p, \|Df_{f^{-N}p}^N\|_G) \geq c \|Df_{f^{-N}p}^N\|_G^{N/2}$$

and taking logarithms we see that

$$n \log d \geq \log C + \frac{N}{2} \log \|Df_{f^{-N}p}^N\|_G.$$

We divide by  $n$ , take limits, and apply the chain rule  $\|Df_{f^{-n}p}^n\| = \|Df_p^{-n}\|^{-1}$ , so

$$\log d \geq \frac{N}{2} \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_{f^{-n}p}^n\|_G = -\frac{N}{2} \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_p^{-n}\|_G.$$

By Lemma 4.9 we have

$$\log d \geq \frac{N}{2} \lambda(\nu).$$

Since this holds for all finite  $N \leq g(r)$  we have:

$$\log d \geq \frac{g(r)}{2} \lambda(\nu). \quad \square$$

We say that a component  $\mathcal{O}$  has a unique growth end if for each  $r \geq 0$  there is only one growth end  $\mathcal{E}_r(\mathcal{O})$ .

**THEOREM 4.11.** *For  $\nu$  almost every point  $p$  every component of  $W_p^u \cap U^+$  has a unique growth end. Furthermore the number of components of  $W_p^u \cap U^+$  is equal to a constant  $N$  for  $\nu$ -almost every  $p$ , and  $N \leq \frac{2 \log d}{\lambda}$ .*

*Proof.* We denote by  $g(p, r)$  the number of growth ends in  $\phi_p^{-1}(U^+) \cap \{|z| > r\}$ . We observe that by the property of  $\|\cdot\|_G$ , we have  $g(f(p), \|Df_p\|_G \cdot r) = g(p, r)$ . We begin by showing that for almost every  $p$  the function  $g(p, r)$  is independent of  $r$  for  $r > 0$ . Since  $g(f(p), 1) = g(p, \|Df_p^{-1}\|_G) \leq g(p, 1)$ , we have  $g(p, 1) - g(f(p), 1) \geq 0$ . On the other hand

$$\int g(p, 1) \nu(p) = \int g(f(p), 1) \nu(p)$$

because of the invariance of  $\nu$ . Both integrals are finite because  $g(p, 1)$  is uniformly bounded almost everywhere by Proposition 4.10. This gives

$$\int (g(p, 1) - g(f(p), 1)) \nu(p) = 0,$$

and so we conclude that  $g(p, 1) - g(f(p), 1) = 0$  outside a set measure zero. Thus  $g(f^n p, 1)$  is independent of  $n \in \mathbf{Z}$ , except for  $p$  in a set of measure zero. Let  $r_n = \|Df_{f^{-n}p}^n\|_G$ , then  $g(f^n p, 1) = g(p, r_n)$ . Since  $g(p, r)$  is a monotone function of  $r$ , and since  $g(p, r_n) = g(p, r_m)$ , it follows that  $g(p, r)$  is constant for  $r_n \leq r \leq r_m$ . Finally, since  $r_n \rightarrow 0$  as  $n \rightarrow -\infty$  and  $r_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , it follows that  $g(p, r)$  is constant on the interval  $0 < r < \infty$ .

Now we apply Proposition 4.7 to conclude that  $g(p, 0) = \lim_{r \rightarrow 0} g(p, r)$ . As was observed before Proposition 4.7,  $g(0) = c(0)$  is the number of connected components of  $\phi_p^{-1}(U^+)$ , so the theorem follows from Proposition 4.10.  $\square$

*Remark.* In the special case where  $\nu$  is normalized counting measure on the orbit of a periodic saddle, then it is possible to define a combinatorial rotation number in terms the action of  $f$  on the growth ends. The Pommerenke-Levin-Yoccoz inequality gives the estimate above, as well as information on the possible combinatorial rotation number. Indeed, the proof of this inequality, as given in [H], goes through essentially without change in the case of polynomial diffeomorphisms.

**THEOREM 4.12.** *Let  $\mathcal{O}$  be a simply connected component of  $W_p^u \cap U^+$  with a unique growth end. Then  $\mathcal{O}$  satisfies  $(\dagger)$ .*

*Proof.* Since  $p$  is a Pesin regular point, it follows (see [BLS]) that  $W^u(p) \subset J^-$  and that  $W^u(p) \cap W^u(f^n p) = \emptyset$  or  $W^u(p) = W^u(f^n p)$  for  $n \in \mathbf{Z}$ . We must show that  $G^+|_{\mathcal{O}}$  is minimal.

Let  $\psi : \Delta \rightarrow \mathcal{O}$  be a conformal equivalence with  $\psi(0) = z_0 \in \mathcal{O}$ , and let  $h = G^+ \circ \psi$ . We wish to show that  $h$  is a multiple of the Poisson kernel  $P(z, e^{i\kappa})$  for some real  $\kappa$ . Let  $\lambda$  denote the measure in the Herglotz representation. We will show that the support of  $\lambda$  is a single point. For  $0 < r < \infty$ , let  $\mathcal{E}_r$  be the unique growth end. Let  $\hat{\omega}_r = \psi^{-1}(\mathcal{O} - \overline{\mathcal{E}_r})$ , and let  $\omega_r$  be the component of  $\hat{\omega}_r$  containing 0. Thus  $\Delta \cap \partial\hat{\omega}_r$  is a collection of Jordan arcs with their boundaries in  $\partial\Delta$ . Let  $\gamma_r$  denote the Jordan arc which separates  $\psi^{-1}(\mathcal{E}_r)$  from 0.

Let us write  $\Delta \cap \partial\omega_r = \gamma_r \cup \bigcup_j \sigma_r^j$ . Let  $\omega_r^j$  denote the component of  $\Delta - \sigma_r^j$  which does not contain 0. Then  $\psi(\hat{\omega}_r \cap \omega_r^j)$  is a decay end of  $\mathcal{O}$ . Since there is only one growth end, it follows from Proposition 4.6 that

$$\lim_{\zeta \in \omega_r^j, \zeta \rightarrow \partial\Delta} h(\zeta) = 0.$$

Thus  $\lambda$  is zero outside the region of  $\partial\Delta$  cut out by  $\gamma_r$ . Arguing as in Proposition 4.6, we see that the harmonic length of  $\psi(\gamma_r)$  with respect to  $z_0$  inside  $\mathcal{O}$  is bounded above by  $4\sqrt{|z_0|/r}$ . Transferring this result back to  $\Delta$  via  $\psi$ , we see that the endpoints of  $\gamma_r$  in  $\partial\Delta$  are separated by at most  $(2/\pi)\sqrt{|z_0|/r}$ . Since the family of curves  $\{\gamma_r : 0 < r < \infty\}$  is nested, i.e. if  $r < s < t$ , then  $\gamma_s$  separates  $\gamma_r$  from  $\gamma_t$ , it follows that they must decrease down to a single point, which must be the support of  $\lambda$ .  $\square$

*Proof of Theorem 4.1.* This is an immediate consequence of Theorems 4.11 and 4.12.

## 5. Connectivity of $J$

This section is devoted to proving (Theorem 5.1) that the presence of either stable or unstable connectivity is equivalent to the connectivity of  $J$ .

**THEOREM 5.1.**  *$J$  is connected if and only if either  $f$  or  $f^{-1}$  is unstably connected.*

**LEMMA 5.2.** *Let  $\nu$  be a hyperbolic measure. Then for  $\nu$  almost every point  $p$  each component of  $U^+ \cap W^u(p)$  contains  $p$  in its closure.*

*Proof.* Let  $e(p, r)$  be the number of connected components of  $W^u(p) \cap U^+$  that meet the closed disk of radius  $r$  in  $W^u(p)$ . Note that if  $r < s$  then

every component which intersects the disk of radius  $r$  also intersects the disk of radius  $s$ . Thus  $e(p, r)$  is an increasing function of  $r$ . Also, since every component intersects some disk, we have  $\lim_{r \rightarrow \infty} e(p, r) = c(p)$ , where  $c(p)$  is the number of components of  $W^u(p) \cap U^+$ . By ergodicity,  $c(p)$  is constant  $\nu$  almost everywhere.

Now  $e(fp, \|Df_p\|) = e(p, 1)$  so that

$$e(p, 1) = e(fp, \|Df_p\|) \geq e(fp, 1).$$

Thus  $e(p, 1)$  behaves like  $c(p, 1)$ , and arguing as before, we see that for  $\nu$  almost every point  $p$  we have  $e(p, r) = c(p)$  for every  $r$ . In particular, the set of components that meet the disk of radius  $r$  has the same cardinality as the set of all components. Thus every component meets the disk of radius  $r$ .  $\square$

**LEMMA 5.3.** *Let  $\nu$  be a hyperbolic measure. If  $f$  is unstably connected with respect to  $\nu$ , then for  $\nu$  almost every  $p$  the set  $\{G^+ < \varepsilon\} - \text{int}\{G^+ = 0\} \subset W^u(p)$  is connected.*

*Proof.* The set of points  $p$  for which Lemma 5.2 holds has full measure for  $\nu$ . Let  $p$  be such a point. Then for every connected neighborhood  $N$  of  $p$  in  $W^u(p)$ ,  $N \cup (W^u(p) \cap U^+)$  is connected. If  $\mathcal{O}$  is a component of  $W^u(p) \cap U^+$ , then by Theorem 2.1,  $\mathcal{O}$  satisfies  $(\dagger)$ , so there is a conformal equivalence taking  $\mathcal{O}$  conformally to the upper half-plane and taking  $G^+$  to the function  $y$ . Thus the sublevel set  $\mathcal{O} \cap \{G^+ < \varepsilon\}$  is a strip, which is connected. Taking the union over all components, we see that  $N \cup (W^u(p) \cap \{G^+ < \varepsilon\}) = N \cup (\mathcal{O} \cap \{G^+ < \varepsilon\})$  is connected. Since this holds for all  $N$ , the lemma follows.  $\square$

The following general lemma will be used to show that a certain bidisk contains points of  $J$ .

**LEMMA 5.4.** *Suppose that  $(x, y)$  is a local coordinate system such that  $D = \{|x|, |y| < 1\}$  has nonempty intersection with both  $J^+$  and  $J^-$ . If  $G^+ > 0$  on  $\{|x| < 1, |y| = 1\}$  and  $G^- > 0$  on  $\{|x| = 1, |y| < 1\}$ , then it follows that  $\int_D \mu^+ \wedge \mu^- > 0$ .*

*Proof.* Since  $G^\pm \geq 0$  the condition on the supports of  $\mu^\pm$  allow us to choose  $\varepsilon^\pm > 0$  sufficiently small that  $D \cap \{G^+ < \varepsilon^+\} \subset \{|x| < 1, |y| < r\}$  and  $D \cap \{G^- < \varepsilon^-\} \subset \{|x| < r, |y| < 1\}$  for some  $0 < r < 1$ . We may assume that  $\varepsilon^+ > 0$  is a regular value of  $G^+$ , so that  $dG^+ \neq 0$  on  $D \cap \{G^+ = \varepsilon^+\}$ . Thus

we may apply Stokes' Theorem:

$$\begin{aligned} \int_{D \cap \{G^+ < \varepsilon^+, G^- < \varepsilon^-\}} \mu^+ \wedge \mu^- &= \int_{D \cap \{G^+ < \varepsilon^+, G^- < \varepsilon^-\}} d(d^c G^+ \wedge dd^c G^-) \\ &= \int_{D \cap \{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\}} d^c G^+ \wedge dd^c G^-. \end{aligned}$$

Since  $\varepsilon^+$  is a regular value, we may consider  $D \cap \{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\}$  as a domain inside the smooth manifold  $D \cap \{G^+ = \varepsilon^+\}$ . It is relatively compact since  $D \cap \{G^+ < \varepsilon^+, G^- < \varepsilon^-\} \subset \{|x|, |y| < r\}$ . Since  $G^+$  is pluriharmonic in a neighborhood of  $D \cap \{G^+ = \varepsilon^+\}$ , the closure of  $D \cap \{G^+ = \varepsilon^+\}$  is not relatively compact in  $D$ . Thus  $G^-$  is not constant on this set, so we may choose  $\varepsilon^-$  to be a regular value for the restriction of the map  $G^-|_{\{G^+ = \varepsilon^+\}}$ . This means that  $dG^+ \wedge dG^- \neq 0$  at all points of  $D \cap \{G^+ = \varepsilon^+, G^- = \varepsilon^-\}$ , so this is a smooth 2-manifold. Since  $dd^c G^+ = 0$  on  $\{G^+ = \varepsilon^+\}$ , it follows that  $d^c G^+ \wedge dd^c G^- = d(d^c G^- \wedge d^c G^+)$  on  $D \cap \{G^- < \varepsilon^-\} \cap \{G^+ = \varepsilon^+\}$ , so we may apply Stokes' Theorem again to obtain:

$$\begin{aligned} \int_{D \cap \{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\}} d^c G^+ \wedge dd^c G^- \\ &= \int_{D \cap \{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\}} d(d^c G^- \wedge d^c G^+) \\ &= \int_{D \cap \partial(\{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\})} d^c G^- \wedge d^c G^+. \end{aligned}$$

This last integral is taken over the set  $\{G^+ = \varepsilon^+, G^- = \varepsilon^-\}$ , which is an oriented 2-manifold. The tangent space of a 2-manifold is either a complex subspace of  $\mathbf{C}^2$  (which is equivalent to being invariant under the complex structure operator  $J$ ) or totally real (which means that  $T \oplus JT = T\mathbf{C}^2$ , the generic case). Since this 2-manifold is compact, it cannot be a complex submanifold. Since it is real-analytic, there is an open dense subset of points where the tangent space is totally real. The 2-form  $dG^+ \wedge dG^-$  annihilates the tangent space. And since  $d^c G^\pm$  is obtained from  $dG^\pm$  by applying  $J$ , it follows that  $d^c G^+ \wedge d^c G^-$  does not annihilate the tangent space; thus  $dG^+ \wedge dG^- \wedge d^c G^+ \wedge d^c G^- \neq 0$ . It follows that  $dG^+ \wedge d^c G^+ \wedge dG^- \wedge d^c G^-$  is a nonzero, positive multiple of the standard volume form on  $\mathbf{C}^2$ . From this, we conclude that  $d^c G^+ \wedge dG^- \wedge d^c G^- = dG^- \wedge d^c G^- \wedge d^c G^+$  is a nonzero positive multiple of the volume form on  $D \cap \partial\{G^+ < \varepsilon^+\}$ , with the induced (boundary) orientation; and  $d^c G^- \wedge d^c G^+$  is a nonzero, positive multiple of the volume form on  $D \cap \partial(\{G^- < \varepsilon^-\} \cap \partial\{G^+ < \varepsilon^+\})$  with the induced orientation. It follows that the last integral above, and thus  $\int_D \mu^+ \wedge \mu^-$ , is strictly positive.  $\square$

*Proof of Theorem 5.1.* We begin by showing that if  $f$  is either stably or unstably connected, then  $J$  is connected. Replacing  $f$  by  $f^{-1}$ , we may



assume that  $f$  is unstably connected. According to Proposition 2.3 of [BS3],  $J^*$  intersects every connected component of  $J$ . Periodic saddle points are dense in  $J^*$ , by [BLS], so it suffices to show that any two saddle points  $p$  and  $q$  can be connected by a path lying in an arbitrarily small neighborhood  $U$  of  $J$ . Again by [BLS], any two saddle points are heteroclinic, so  $W^u(p) \cap W^s(q)$  is nonempty and in fact contains a transverse intersection. Now  $W^u(p) \cap W^s(q)$  contains points arbitrarily close to  $q$ , so it suffices to show that any transverse intersection point  $r \in W^u(p) \cap W^s(p)$  can be connected to  $p$  by a path lying inside  $U$ .

It is evident that  $r \in W^u(p) \cap \{G^+ = 0\}$ , and we will show that  $r \notin \text{int}\{G^+ = 0\}$ , where the interior is taken relative to  $W^u(p)$ . Let us suppose, to the contrary, that there is a disk in  $\{G^+ = 0\} \cap W^u(p)$  containing  $r$ . Since the iterates of  $f^n$ ,  $n \geq 0$  remain bounded, the derivative  $Df^n$  tangential to  $W^u(p)$  at  $r$  remains bounded. But this contradicts the smooth Lambda Lemma. Thus  $r$  is not in the interior, and  $r$  must belong to the closure of  $\{0 < G^+ < \varepsilon\} \cap W^u(p)$  inside  $W^u(p)$ , and so it will follow from Lemma 5.3 that there is a path in  $\{G^+ < \varepsilon\} \cap W^u(p)$  connecting  $r$  to  $p$ . Since we may choose  $\varepsilon > 0$  sufficiently small that  $\{G^+ < \varepsilon\} \cap W^u(p) \subset U$ , it follows that  $r$  and  $p$  are in the same connected component of  $J$ . This completes the proof that  $J$  is connected.

Now we show that if neither  $f$  nor  $f^{-1}$  is unstably connected, then  $J$  is not connected. Let  $p$  be a periodic saddle point. Replacing  $f$  by  $f^n$  for an appropriate  $n$  lets us assume that  $p$  is a fixed point. We can choose a coordinate system  $\{|x| < 1, |y| < 1\}$  in a neighborhood  $B$  of  $p$  so that  $p$  corresponds to the point  $(0,0)$ , the set  $\{|x| < 1, y = 0\}$  is a local unstable manifold for  $p$ , and the set  $\{x = 0, |y| < 1\}$  is a stable manifold for  $p$ . Furthermore, by taking  $B$  small, we may assume that the restriction of  $f$  to  $B$  is approximately linear, and thus it is uniformly expanding in the  $x$ -direction and uniformly contracting in the  $y$ -direction.

Since  $f$  is not unstably connected,  $W^u(p) \cap K^+$  contains a compact component. Now  $f^{-1}$  decreases distance in  $W^u(p)$ , so by applying  $f^{-m}$  with  $m$  sufficiently large we may assume that the set  $\{|x| < 1, y = 0\} \cap K^+$  has a compact component. We may further assume that this component does not contain  $p = (0,0)$ . Let  $\gamma$  be a curve in  $\{|x| < 1, y = 0\} \cap U^+$  which encloses this compact component but does not enclose  $p$ . Let  $D_0$  be the region of  $\{|x| < 1, y = 0\}$  enclosed by  $\gamma$ , and let  $E^+$  denote the portion of  $W^u(p) \cap K^+$  enclosed by  $\gamma$ . By the uniform expansion/contraction of  $f|_B$  (or equivalently, by the Lambda Lemma) it follows that for  $n$  sufficiently large and  $|x| < \frac{1}{2}$ , the disk

$$M_n^+(x) := f^{-n}(\{x\} \times \{|y| < \varepsilon\}) \cap \bigcap_{j=0}^{n-1} f^{-j} B$$

is vertical, in the sense that the projection of  $M_n^+(x)$  to  $\{x = 0, |y| < \varepsilon\}$  is a homeomorphism. It follows that

$$\Gamma_n^+ := f^{-n}(\gamma \times \{|y| < \varepsilon\}) \cap \bigcap_{j=0}^{n-1} f^{-j}B = \bigcup_{x \in \gamma} M_n^+(x)$$

is a hypersurface in  $B$  made up of vertical disks. Thus  $\Gamma_n^+$  divides  $B$  into two components, and we let  $B^+$  denote the component of  $B - \Gamma_n^+$  which contains  $f^{-n}D_0$ . We note, also, that  $G^+ > 0$  on  $\Gamma_n^+$ .

Similarly, since  $f^{-1}$  is not unstably connected,  $W^s(p) \cap K^-$  contains a compact component. Arguing as above, we have a hypersurface  $\Gamma_n^-$  of unstable disks  $M_n^-(x)$ , and  $G^- > 0$  on  $\Gamma_n^-$ . Furthermore, there is a component  $B^-$  of  $B - \Gamma_n^-$  which contains the compact component of  $W^s(p) \cap K^-$ . Let  $B_0 = B^+ \cap B^-$ . We consider the family of vertical disks  $\{|x| < \frac{1}{2}\} \ni x \mapsto M_n^+(x)$ . We know that  $G^- > 0$  on  $\Gamma_n^-$ , and so  $dd^c G^-$  vanishes there. Thus  $\{|x| < \frac{1}{2}\} \ni x \mapsto \int_{B^- \cap M_n^+(x)} dd^c G^-$  is constant. We know that  $B^- \cap M_n^+(0) = B^- \cap W^s(p)$  contains a compact component, so it follows  $dd^c G^-$  puts positive mass on  $B^- \cap M_n^+(0)$ . Thus  $dd^c G^-$  puts positive mass on  $B^- \cap M_n^+(x)$  for  $x \in D_0 \subset \{|x| < \frac{1}{2}\}$ . This implies that  $B_0$  intersects  $J^-$ . Similarly,  $B_0$  intersects  $J^+$ . Thus we may apply Lemma 5.4 to conclude that  $\int_{B_0} \mu^+ \wedge \mu^- \neq 0$ , from which it follows that  $J \cap B_0 \neq \emptyset$ . Since  $\partial B_0 \subset \Gamma_n^+ \cup \Gamma_n^-$  is disjoint from  $J$ , and since  $B_0$  does not contain  $p$ , and thus all of  $J$ , it follows that  $J$  is disconnected.  $\square$

### 6. Unstable connectivity and extension of $\varphi^+$

In this section we find several characterizations of the condition of unstable connectivity which are summarized in Theorem 6.3. These relate to the existence of extensions of  $\varphi^+$  and topological properties of  $J_+^-$ .

Recall that  $\varphi^+$  is defined and holomorphic on  $V^+$  and satisfies the functional equation

$$(6.1) \quad \varphi^+(f^n(p)) = (\varphi^+(p))^{d^n}.$$

When  $f$  is unstably connected, then according to Theorem 2.1 the function  $\varphi^+$  has a continuous extension to  $J_+^-$  that satisfies equation (6.1). Let  $\mathcal{G}^+$  be the foliation of  $U^+$  defined by the holomorphic one form  $\partial G^+$ . When restricted to  $V^+$  the leaves of  $\mathcal{G}^+$  are just the sets on which  $\varphi^+$  is constant.

**LEMMA 6.1.** *If  $f$  is unstably connected then each path component of  $J_+^-$  is simply connected.*

*Proof.* Let  $\gamma$  be a loop in  $J_+^-$ . The image  $\varphi^+(\gamma)$  is a loop in  $\mathbf{C} - \overline{\Delta}$ . We begin by showing that the image loop is contractible. Let  $[\varphi^+(\gamma)] \in \mathbf{Z}$  denote

the degree of the image loop. To show that the image loop is contractible we show that the degree is zero. Let  $\gamma'$  denote  $f^{-n}(\gamma)$ . Now the functional equation for  $\varphi^+$  gives:

$$[\varphi^+(\gamma)] = [\varphi^+(f^n(\gamma'))] = [(\varphi^+)^{d^n}(\gamma')] = d^n[\varphi^+(\gamma')].$$

This implies that  $[\varphi^+(\gamma)]/d^n$  is an integer for any  $n$ . Thus  $[\varphi^+(\gamma)] = 0$ .

Lemma 2.4 tells us that the map  $\varphi^+ : J_+^- \rightarrow \mathbf{C} - \bar{\Delta}$  is a locally trivial fibration. Thus it has the homotopy lifting property. Since the image of  $\gamma$  is contractible the loop  $\gamma$  is homotopic to a loop in a fiber of the map  $\varphi^+$ , that is to say a set  $\varphi^+ = \text{const}$ . We complete the proof by showing that each component of a fiber is simply connected. Now a fiber of the map is contained in a leaf  $L$  of the foliation  $\mathcal{G}^+$ . In fact it is contained in the intersection of  $L$  with  $J^-$ . The set  $J^-$  is the zero set of the function  $G^-$ . The leaf  $L$  is conformally equivalent to  $\mathbf{C}$  and the restriction of  $G^-$  to  $L$  is a subharmonic function. The maximum principle implies that each component of the zero set of  $G^-$  in  $L$  is simply connected.  $\square$

LEMMA 6.2. *If  $f$  is unstably connected, then  $\varphi^+$  has an analytic continuation to a neighborhood of  $J_+^-$ .*

*Remark.* It follows from Hubbard and Oberste-Vorth [HO] that  $\varphi^+$  cannot be extended to  $U^+$ . For any holomorphic extension of  $\varphi^+$  is locally constant on the leaves of  $\mathcal{G}^+$ . Thus if an extension of  $\varphi^+$  to a set  $U' \supset V^+$  exists, each leaf of  $\mathcal{G}^+|U'$  can intersect only one disk of  $\mathcal{G}^+|V^+$ , since  $\varphi^+$  takes distinct values on distinct disks. Thus there can be no extension to  $U^+ \supset V^+$ , since as shown in [HO] each leaf of  $\mathcal{G}^+$  intersects  $V^+$  in infinitely many disks.

*Proof.* For  $p \in \mathcal{G}^+$  let  $L_p$  denote the leaf of the  $\mathcal{G}^+$  foliation that passes through  $p$ . For  $p$  and  $q$  in the same leaf let  $d_L(p, q)$  denote the distance measured with respect to the induced Riemannian metric in the leaf. For  $p \in U^+$  let  $\nu_p$  be the set consisting of "nearest points in  $J_+^-$ ," i.e., those points  $q$  in  $L_p \cap J_+^-$  which minimize the function  $d_L(p, q)$  among all points in  $L_p \cap J_+^-$ . Let  $N$  consist of those points  $p$  for which the function  $\varphi^+$  is constant on  $\nu_p$ . For  $p \in N$  define the function  $\tilde{\varphi}(p)$  to be the common value of  $\varphi^+$  on the elements of  $\nu_p$ . We will show that any  $p \in J_+^-$  has a neighborhood in  $N$  on which the function  $\tilde{\varphi}$  is holomorphic. Choose local coordinates  $u$  and  $v$  near  $p$  so that the set  $B = \{(u, v) : |u| \leq 1, |v| \leq 1\}$  is a neighborhood of  $p$ , and the sets  $u = \text{const}$  are contained in leaves of  $\mathcal{G}^+$ . We may assume that the set  $v = 0$  is the local leaf of the  $J^-$  lamination containing  $p$ . Choose an  $n$  sufficiently large so that  $f^n(p) \in V^+$ . We may assume that  $B$  is chosen small enough so that  $f^n(B) \subset V^+$ . For  $(u, v) \in J_+^- \cap B$  define the following function

$\alpha(u, v) = \varphi^+(u, v)/\varphi^+(u, 0)$ . Let  $W$  denote the set where  $\alpha = 1$ . Since  $\alpha$  is continuous  $W$  is a closed set. We claim that  $W$  is an open set of  $J_+^- \cap B$ .

Since  $(u, v)$  and  $(0, v)$  have the same second coordinate they lie in a disk inside a leaf of the  $\mathcal{G}^+$  foliation. Since  $f^n(B) \subset V^+$  the points  $f^n(u, v)$  and  $f^n(0, v)$  are on the same leaf of the  $\mathcal{G}^+$  foliation of  $V^+$  and we have  $\varphi^+(f^n(u, v)) = \varphi^+(f^n(0, v))$ . The functional equation (6.1) gives

$$(\varphi^+)^{d^n}(u, v) = (\varphi^+)^{d^n}(u, 0).$$

So  $\alpha(u, v)^{d^n} = 1$  and the values of  $\alpha$  are  $d^n$ -th roots of unity. The function  $\alpha$  is continuous and takes on a finite set of values so the set where  $\alpha = 1$  is open. The set  $\{(v, 0)\}$  is in  $W$ . Since  $W$  is open we can choose  $\varepsilon$  sufficiently small so that  $|u| < \varepsilon$  implies that the  $d_L$  distance from  $(u, v)$  to  $(u, 0)$  is smaller than the  $d_L$  distance from  $(u, v)$  to any point  $(u, v')$  not in  $W$  or any point  $(u, v')$  on the boundary of  $B$ . Since the nearest neighbors of  $(u, v)$  are in  $W$  we have that  $\varphi^+(u, v') = \varphi^+(u, 0)$  for all nearest neighbors of  $(u, v)$ . Thus for  $|v| < \varepsilon$   $(u, v) \in N$  and  $\tilde{\varphi}(u, v) = \varphi^+(u, 0)$ .  $\square$

**THEOREM 6.3.** *The following are equivalent:*

- (1)  $f$  is unstably connected.
- (2)  $\varphi^+$  extends to a continuous function on  $J_+^-$  which satisfies the equation  $\varphi^+ f^n = (\varphi^+)^{d^n}$ .
- (3)  $\varphi^+$  extends to a continuous function on  $J_+^-$  which is holomorphic on leaves.
- (4) The cohomology class represented by the form  $\eta = (1/2\pi)d^c G^+$  is an integral class on each leaf of the lamination  $\mathcal{M}^-$  of  $J_+^-$ .
- (5) Each path component of  $J_+^-$  is simply connected.
- (6)  $H_1(J_+^-; \mathbf{R}) = 0$ .
- (7)  $\varphi^+$  extends holomorphically to a neighborhood of  $J_+^-$ .

*Proof.* The strategy of proof is to show that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1). We then show that (1) $\Rightarrow$ (5), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (4). We conclude by showing that (1) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2). This follows from Theorem 2.1.

(2) $\Rightarrow$ (3) The function  $\varphi^+$  is defined and holomorphic on  $V^+$ . Let  $p \in J_+^-$ . For some  $n$ ,  $f^n(p) \in V^+$ . Now the function  $\varphi^+ f^n$  is holomorphic when restricted the leaf containing  $p$ . The extension is locally a continuous  $d^n$ -th root of a holomorphic function. Hence the extension is holomorphic on each leaf.

(3) $\Rightarrow$ (4). Since  $\varphi^+$  is holomorphic on leaves the function  $\log |\varphi^+|$  is harmonic on leaves. In the set  $V^+$  we have  $\log |\varphi^+| = G^+$ . Since both sides of the equation are analytic functions the equation holds on the entire leaf. Now

$$\eta = (1/2\pi)d^c G^+ = (1/2\pi)d^c \log |\varphi^+| = (\varphi^+)^*((1/2\pi)d^c \log |z|).$$

The form  $d^c \log |z|$  measures the change in argument so on any closed loop its value is in the set  $2\pi\mathbf{Z}$ . Thus  $\eta$  is the pullback of a form which represents an integral class in  $H^1(\mathbf{C} - \overline{\Delta})$  so  $\eta$  itself is an integral class.

(4) $\Rightarrow$ (1). Let  $p$  be a saddle point. Assume  $\eta$  represents an integral class but that  $f$  is unstably disconnected. Let  $\mathcal{O}$  be a component of  $W^u(p) - K^+$ . By Theorem 4.11 there are only finitely many components, so we may assume that  $\mathcal{O}$  is periodic. Passing to a power of  $f$ , we may assume that  $\mathcal{O}$  is fixed. Since  $f$  is unstably disconnected there is a nonempty compact component  $E$  of  $W^u(p) - \mathcal{O}$ . Let  $\gamma \subset \mathcal{O}$  be a simple closed curve that surrounds  $E$ . Let  $D$  be the topological disk surrounded by  $\gamma$ . Define  $\delta_\gamma := \int_\gamma d^c G^+$ . Given an  $n$  let  $\gamma' = f^{-n}(\gamma)$ . Our hypothesis implies that  $\delta_\gamma$  and  $\delta'_\gamma$  are integers. Now the functional equation for  $G^+$  gives:

$$\delta_\gamma = \int_\gamma d^c G^+ = \int_{\gamma'} d^c G^+ \circ f^n = d^n \int_{\gamma'} d^c G^+ = d^n \delta'_\gamma.$$

Since  $\delta_\gamma/d^n$  is an integer for any  $n$ ,  $\delta_\gamma = 0$ .

Since  $G^+$  is subharmonic we have  $\int_E dd^c G^+ \geq 0$ . If  $\int_E dd^c G^+ \geq 0 = 0$  then  $G^+$  would be harmonic on the region enclosed by  $\gamma$ . Since  $G^+$  is positive on  $\gamma$  and zero on  $E$  this would violate the minimum principle. We conclude that  $\int_E dd^c G^+ > 0$ . But  $\int_E dd^c G^+ = \delta_\gamma = 0$ . This contradiction completes the proof.

(1) $\Rightarrow$ (5). This is Theorem 6.1.

(5) $\Rightarrow$ (6). This is clear.

(6) $\Rightarrow$ (4). If  $H_1(J_+^-; \mathbf{R}) = 0$  then the simplicial cohomology group  $H^1(J_+^-; \mathbf{R})$  is zero. Since  $\eta$  represents an element of this group  $\eta = 0$ . In particular  $\eta$  is integral.

(1) $\Rightarrow$ (7). This is Theorem 6.2.

(7) $\Rightarrow$ (3). This is clear. □

### 7. Critical points and harmonic measure

We first give a dichotomy of possible behaviors: either  $f$  is unstably connected, or  $f$  has a strong unstable disconnectedness property with respect to harmonic measure. Then we will show that  $f$  is unstably connected exactly when it has no critical points with respect to  $\mu$ .

Let  $\mathcal{R}$  denote the set of Pesin regular points for the map  $f$ . For  $p \in \mathcal{R}$  we let  $K^{+,u}(p)$  denote the connected component of  $W^u(p) \cap K^+$  which contains  $p$ . We let  $\mathcal{T}^u$  denote the set of points of  $\mathcal{R}$  for which the corresponding component of the unstable slice of  $K^+$  is trivial, i.e.  $\mathcal{T}^u = \{p \in \mathcal{R} : \{p\} = K^{+,u}(p)\}$ .

**THEOREM 7.1.** *The following dichotomy holds. Either:*

- (1)  $f$  is unstably connected, or  
 (2)  $T^u$  has full measure for  $\mu$ . Equivalently, for  $\mu$  almost every  $p$ ,  $T^u \cap W^u(p)$  has full measure for the induced measure  $\mu^+|_{W^u(p)}$ .

In case (2), it follows that for  $\mu$  almost every point  $p$ , the critical points of  $G^+|_{W^u(p) \cap U^+}$  are dense in the boundary of  $W^u(p) \cap K^+$ , with the boundary being taken inside  $W^u(p)$ .

*Remark.* This dichotomy says that, with respect to harmonic measure, either  $f$  is unstably connected, or unstably totally disconnected. We recall that the induced measure  $\mu^+|_{W^u(p)}$  defines the measure class of harmonic measure. We conclude that when case (2) of Theorem 6.2 holds, and there is a nontrivial compact component  $E$ , then  $E$  has zero harmonic measure. In this case the point components of  $W^u(p) \cap K^+$  form a halo which surrounds  $E$  closely enough that  $E$  can have no harmonic measure.

*Proof.* For  $p \in \mathcal{R}$ ,  $W^u(p) \approx \mathbf{C}$  has an affine structure, and we let  $\|\cdot\|_G$  denote the norm defined in §2. Let us define  $R(p)$  to be the radius with respect to  $\|\cdot\|_G$  of the smallest closed disk centered at  $p \in W^u(p)$  which contains  $K^+(p)$ . It follows from the ergodicity of  $\mu$  that either: (1)  $R(p) = \infty$ ,  $\mu$  almost everywhere, (2)  $R(p) = 0$ ,  $\mu$  almost everywhere, or (3)  $0 < R(p) < \infty$ ,  $\mu$  almost everywhere. The possibility (2) corresponds to the case (2) in the dichotomy above.

We show that case (1) here corresponds to case (1) above. Let us find countably many unstable boxes  $B_j^u$  whose union has full measure. If  $f$  is not unstably connected, we may choose one of the unstable boxes  $B_j^u = \{\Gamma(t) : t \in T_j\}$  so that one of the leaves  $\Gamma(t_0)$  intersects  $K^+$  in a compact component. Thus there is a simple, closed curve  $\gamma \subset \Gamma(t_0)$  which encircles a nonempty portion of  $K^+ \cap \Gamma(t_0)$ . It follows that  $\min_\gamma G^+ > 0$ , and by continuity  $\min_{\gamma_t} G^+ > 0$  for  $\gamma_t \subset \Gamma(t_0)$  near  $\gamma$ . Further, it follows that  $t \mapsto \int_{\gamma_t} d^c G^+$  is locally constant for  $t$  near  $t_0$ . Thus  $\gamma_t$  cuts off a compact portion  $E_t$  of  $K^+ \cap \Gamma(t)$  for  $t$  near  $t_0$ . The set of  $p$  such that  $W^u(p) \cap K^+$  has a compact component contains  $\bigcup_{|t-t_0| < \varepsilon} E_t$  which has positive  $\mu$  measure by the local product structure, since the measure of  $E_t$  is  $\mu^+|_{\Gamma(t)}(E_t) = \int_{\gamma_t} d^c G^+ > 0$ , and  $\{t \in T : |t - t_0| < \varepsilon\}$  has positive transversal measure. Thus  $f$  is not unstably connected.

Thus to prove the theorem, we must show that (3) cannot occur. For  $0 < a < b < \infty$  we define  $S = \{p \in \mathcal{R} : a < R(p) < b\}$ . In case (3), we may choose  $a$  and  $b$  so that  $S$  has positive  $\mu$  measure. Now by (4.1)  $f^n$  is linear with respect to the affine structures of  $W^u(p)$  and  $W^u(f^n p)$ . Thus it follows that

$$R(f^n p) = \|Df^n|_{E^u p}\|_G R(p).$$



By Poincaré recurrence, for almost every  $q \in S$ , there is a sequence  $n_j \rightarrow \infty$  such that  $f^{n_j}q \in S$ . In this case we have  $b/a > \|Df^{n_j}|_{E^u_p}\|_G$ . But this contradicts Lemma 4.4 if  $S$  has positive measure. Thus case (3) cannot occur.

The equivalent statement in (2) follows by the local product structure, which says that  $\mu$  is given locally as the product of the slice measures of  $\mu^+$  and  $\mu^-$ . Thus if  $\mathcal{T}^u$  has full measure for  $\mu$ , then it has full measure for almost every slice measure  $\mu^+|_{W^u(p)}$ .

For the statement concerning critical points, we note that if  $q \in \mathcal{T}^u \cap W^u(p)$ , then  $G^+|_{W^u(p) \cap U^+}$  must have critical points arbitrarily close to  $q$ . Otherwise, by Lemma 6.2,  $q$  is an isolated point of  $W^u(p) \cap K^+$ , which is impossible since it would mean that  $q$  is an isolated zero of  $G^+|_{W^u(p)}$ .  $\square$

LEMMA 7.2. *If  $E$  is a compact component of  $W^u(p) \cap K^+$ , and if  $G^+|_{W^u(p)}$  has no critical points in a neighborhood of  $E$ , then  $E$  is an isolated component of  $W^u(p) \cap K^+$ .*

*Proof.* Let  $V$  denote a relatively compact neighborhood of  $E$  inside  $W^u(p)$  such that  $\partial V \cap K^+ = \emptyset$ . If we set  $\delta_0 = \min_{\partial V} G^+$ , then the sublevel sets  $S_\delta := \{G^+ < \delta\} \cap V$  are bounded if  $0 < \delta < \delta_0$ , and  $\partial S_\delta \subset \{G^+ = \delta\}$ . Since  $G^+$  has no critical points, the set  $\{G^+ = \delta\}$  is smooth, and each component of  $\partial S_\delta$  is homeomorphic to a 1-sphere. It follows by Morse theory that  $E$  can be the only component of  $K^+$  inside its component of  $S_\delta$ . Thus  $E$  is isolated.  $\square$

THEOREM 7.3. *The following are equivalent:*

- (1)  $\lambda^+(\mu) = \log d$ .
- (2) For  $\mu$  almost every  $p$ ,  $G^+|_{W^u(p) - K^+}$  has no critical points.
- (3) For a set of  $p$  of positive  $\mu$  measure there is a component  $\mathcal{O}_p$  of  $W^u(p) - K^+$  such that  $G^+|_{\mathcal{O}_p}$  has no critical points.
- (4)  $f$  is unstably connected.

*Proof.* (1)  $\Leftrightarrow$  (2) follows from Corollary 6.7 of [BS5]. (2)  $\Rightarrow$  (3) is obvious. (4)  $\Rightarrow$  (2) follows from Corollary 2.19. It remains to show (3)  $\Rightarrow$  (4). By Theorem 6.1, there are two possibilities: if  $f$  is not unstably connected, then for almost every  $p$ , the unstable manifold  $W^u(p)$  has the property that  $\mu^+|_{W^u(p)}$  is carried by the set  $\mathcal{T}^u$ . But if  $q \in \mathcal{T}^u$ , then the  $\{q\}$  is the component of  $W^u(p) \cap K^+$  containing  $q$ , which must be isolated by Lemma 6.2, since  $G^+|_{W^u(p)}$  has no critical points. Since  $G^+$  is continuous, however,  $\mu^+|_{W^u(p)}$  can put no mass on an isolated point. Thus the slice cannot put any mass at all on  $\mathcal{T}^u$ , so  $f$  must be unstably connected.  $\square$

COROLLARY 7.4. *If  $f$  is dissipative then  $f$  is unstably connected. If  $f$  preserves volume, then it is stably connected if and only if it is unstably connected.*

*Proof.* By Theorem 7.3, if  $f$  is unstably connected, then  $\Lambda = \log d$ . If  $a \in \mathbf{C}$  denotes the (constant) complex jacobian determinant of  $f$ , then by Proposition 7.7 of [BS5],  $\Lambda = \log d$  implies that  $|a| \leq 1$ . Similarly, if  $|a| = 1$ , then  $\Lambda(f) = \log d$  if and only if  $\Lambda(f^{-1}) = \log d$ .  $\square$

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