Julia Sets for Polynomial Diffeomorphisms of $\mathbb{C}^2$ are not Semianalytic

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Abstract. For any polynomial diffeomorphism f of $\mathbb{C}^2$ with positive entropy, neither the Julia set of f nor of its inverse f^{-1} is semianalytic.

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Introduction

If X is a complex manifold, and $f : X \to X$ is a holomorphic mapping, then the Fatou set is the largest open set where the iterates $f^n := f \circ \cdots \circ f$ are locally equicontinuous. Equivalently, these are the points where f is Lyapunov stable. The complement of the Fatou set is the Julia set. While we refer to this as *the* Julia set, it is sometimes possible to define several Julia sets, (see [9, 20]). In dimension 1, the principal case is where $X = \mathbb{P}^1$ is the Riemann sphere, and f is a rational function. In this case, Fatou showed that if J has a tangent at some point, then J is either a circle or a circular arc. In the case of the circle, f is conjugate to z^d for $d \in \mathbb{Z}$, $|d| \geq 2$; and in the case of an arc, f is conjugate to a Chebyshev polynomial. In higher dimension, there are of course product maps, and in this case the Julia set is a union of product sets. There are also nontrivial examples of polynomial maps for which the Julia set is (real) algebraic; examples were given in $\mathbb{C}^2$ by Nakane [13] and in $\mathbb{C}^3$ by Uchimura [16, 17, 18].
These maps discussed above are non-invertible; in the sequel we consider invertible maps. In this case, we have both a forward Julia set $J^+ := J(f)$ and a backward Julia set $J^- := J(f^{-1})$. The invertible polynomial maps of $\mathbb{C}^2$ have been classified by Friedland and Milnor [10]. The polynomial diffeomorphisms

with nontrivial dynamical behavior are conjugate to compositions of generalized Hénon maps, and each such composition has a degree d. (See [4, 8, 12] for the basic dynamical properties of these maps.) By [10, 15], it follows that the topological entropy of f is $\log(d)$. Hubbard [11] defined the escape locus U^+ for such a map f, and it is easily seen that $J^+ = \partial U^+$. By [3], J^+ cannot contain an algebraic curve, so it follows (see Proposition 1.2) that: *Neither J^+ nor J^- can be (real) algebraic.* Our main result is:

Theorem. *Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ with positive entropy. Then neither J^+ nor J^- is a semianalytic subset of $\mathbb{C}^2$.*

Fornæss and Sibony [8] showed that, for a generic Hénon map, the Julia set is neither smooth nor semianalytic. In [1] we showed that J^+ can never be C^1 smooth. However, the Julia sets in [13] and [16, 17, 18] have singular points and thus are not C^1, and this non-smoothness was our motivation for the present Theorem.

1 Levi flat hypersurfaces

Let $U \subset \mathbb{C}^2$ be an open subset. A function ρ on U is said to be *real analytic* if for every $q \in U$, ρ can be written as a real power series which converges in a neighborhood of q. Let us suppose that $q = 0$ and write

$$\rho(z, \bar{z}) = \sum_{I,J} c_{I,J} z^I \bar{z}^J$$

where $I = (i_1, i_2)$ is a pair of nonnegative integers, and $z^I = z_1^{i_1} \cdot z_2^{i_2}$, and similarly for J and $\bar{z}^J$. We may treat z and $\bar{z}$ as independent variables and write

$$\rho(z, \bar{w}) = \sum_{I,J} c_{I,J} z^I \bar{w}^J$$

The reality condition on ρ is that $c_{I,J} = \overline{c_{J,I}}$, which means that $\rho(z, \bar{w}) = \overline{\rho(w, \bar{z})}$. A set X is *real analytic* if it can be written locally as $X \cap U = \{\rho = 0\}$. A point $x_0 \in X$ is said to be *regular* if X is a smooth manifold in a neighborhood of x_0. We write $Reg(X)$ for the set of regular points, and $Reg(X)$ is dense in X (see [7]), although the dimension may be different at different points.

A smooth real hypersurface is said to be *Levi flat* if it is (locally) pseudoconvex from both sides. We recall that the Green function is given by the superexponential rate of escape to infinity: $G^+ = \lim_{n\to\infty} d^{-n} \log^+ ||f^n||$, and G^+ is pluriharmonic on the set $U^+ = \{G^+ > 0\}$. It follows that: *If the set $J^+ = \partial\{G^+ > 0\}$ is C^1 smooth on some open set, then it is Levi flat there.* A real analytic set is said to be Levi flat if it is Levi flat at each regular point. If X is a real analytic, Levi flat hypersurface, then at each regular point, there is a local holomorphic coordinate system such that X is locally given as $\{z_1 + \bar{z}_1 = 0\}$. At singular points, the situation is more complicated.

The following allows us to replace the semianalytic J^+ by a real analytic Levi flat hypersurface.

Lemma 1.1. *Suppose that J^+ is semianalytic, and $p \in J^+$. Then there is a neighborhood U of p such that*

$$J^+ \cap U \subset X := X_1 \cup \cdots \cup X_N \subset U$$

where X_j is analytic and locally irreducible at p, the real dimension of $X_j = 3$, X_j is Levi flat, and for each j, p is contained in the closure of $Reg(X_j) \cap J^+ - \bigcup_{i \neq j} X_i$. Further, if p is a fixed point, then X is invariant in the sense that $f(X) \cap U \subset X$.

Proof. The semianalytic sets are generated locally by taking finite unions, intersections and complements of sets of the form $\{r_j = 0, s_j > 0\}$. (See Bierstone and Milman [7] for further information on semianalyticity.) Thus, if J^+ is contained in a semianalytic set, it is contained in an analytic set X. Now X will have a finite number of irreducible components $X_1, \ldots, X_N$ at p, and we can take the minimal number of components necessary to contain $J^+ \cap U$. Now for each of the components X_j, minimality means that we must have $X_j \cap J^+ - \bigcup_{i \neq j} X_i \neq \emptyset$. Now we know that for any saddle point q, the stable manifold $W^s(q)$ is dense in J^+, (see [5]). Thus for any neighborhood V which intersects $X_j \cap J^+ - \bigcup_{i \neq j} X_i$, we have that $W^s(q) \cap V$ is a nonempty subset of $X_j \cap J^+ - \bigcup_{i \neq j} X_i$. Since $W^s(q) \cap V$ is a 2-dimensional set which is not locally equal to $V \cap X_j \cap J^+ - \bigcup_{i \neq j} X_i$, we conclude that X_j must have dimension 3. The statement that p is contained in the closure of $Reg(X_j) \cap J^+ - \bigcup_{i \neq j} X_i$ is a consequence of the minimality of the set of varieties X_j.
Finally, if p is a fixed point, then $f(U)$ is a neighborhood of p, and $f(X)$ is a real analytic set which contains $J^+ \cap f(U)$. By the minimality of X, $f(X)$ must coincide with X in a neighborhood of p. □

Let us discuss the hypersurface $X = \{\rho = 0\}$, where $\rho(z, \bar{w})$ converges for $z, w \in U$. If for fixed $w \in U$, $\rho(z, \bar{w}) = 0$ for all z, we say that X is *Segre degenerate* at w. If X is not degenerate at $w \in U$, then the *Segre variety*, which is defined as

$$Q_w := \{z \in U : \rho(z, \bar{w}) = 0\},$$

is a proper subvariety of U. (In other words, the condition that w is Segre degenerate means that the Segre variety is the whole open set U.) We may choose the defining function ρ to be minimal, which means that if ρ' is any other defining function, then $\rho' = h\rho$. The family of Segre varieties is independent of the choice of minimal defining function.
A basic property of analytic varieties is that if p is not Segre degenerate, then for q near p, the dependence $q \mapsto Q_q$ is continuous in the Hausdorff topology in a neighborhood of p. Another basic property is that if M is a complex analytic curve (possibly singular), and if $M \subset X$, then $M \subset Q_\zeta$ for all $\zeta \in M$.
At this stage, we can conclude that $J^\pm$ cannot be algebraic.

Proposition 1.2. *Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ with positive entropy. Then neither J^+ nor J^- is a real algebraic set.*

Proof. Let us suppose that $J^+ = \{\rho(z, \bar{z}) = 0\}$ is defined by a real polynomial. At a regular point, $w \in J^+$ must be Levi flat, since every stable manifold is a complex and dense in J^+. Since J^+ is Segre nondegenerate at w, Q_w is a proper subvariety of $\mathbb{C}^2$ which is contained in J^+. On the other hand, this is not possible, since by [3] there is no complex algebraic subvariety of $\mathbb{C}^2$ which is contained in K^+. □

The set of Segre degenerate points is a complex subvariety of codimension at least 2 (see [14, Section 3]). Thus in $\mathbb{C}^2$, the Segre degenerate points are isolated, so we may assume that U is sufficiently small that all points of $X \cap U - \{p\}$ are Segre nondegenerate.
A basic result (see Pinchuk, Shafikov and Sukhov [14]) is that if X is Levi flat, then for each regular point $w \in X$, the Segre variety Q_w is contained in X. We say that p is *dicritical* if there are infinitely many distinct varieties Q_q passing through p. If X is locally irreducible at p, it follows that if infinitely many varieties Q_q contain p, then all varieties Q_q contain p. We will make use of the following result:

Theorem 1.3 ([14, Theorem 3.1]). *A point is Segre degenerate if and only if it is dicritical.*

Lemma 1.4. *If p and $X = X_1 \cup \cdots \cup X_N$ are as in Lemma 1.1, then p is not dicritical for any X_j.*

Proof. If r_0 is a saddle point, then by [5], the stable manifold $W^s(r_0)$ is dense in J^+. Since there are infinitely many saddle points, we may suppose that $r_0 \neq p$. Let $q \in W^s(r_0) \cap X - \{p\}$ be a regular point of X. We may assume that q is Segre nondegenerate, so that Q_q is a complex subvariety of X. Further, since the leaves of the complex foliation of a Levi flat hypersurface are unique, it follows that $W^s(r_0)$ intersects Q_q in an open set. If p is dicritical, then $p \in Q_q$. On the other hand, since p is fixed, it cannot belong to $W^s(r_0)$. Thus $\hat{W}^s(r_0) := W^s(r_0) \cup Q_q$ is a complex manifold which is strictly larger than $W^s(r_0)$. (Note that we may desingularize $\hat{W}^s(r_0)$ if p is a singular point of Q_q.) Now recall that $W^s(r_0)$ is uniformized by $\mathbb{C}$, and the only Riemann surface which strictly contains $\mathbb{C}$ is the Riemann sphere, which is compact. Since $\mathbb{C}^2$ can contain no compact, Riemann surfaces, we have a contradiction, by which we conclude that Q_q cannot contain p. Thus p is not dicritical. □

Lemma 1.5. *Let p and $X_1 \cup \cdots \cup X_N$ be as in Lemma 1.1. Then for each j , the Segre variety $Q_p^{(j)}$ corresponding to X_j, satisfies $Q_p^{(j)} \subset J^+$.*

Proof. Let r_0 be a saddle point, and let $W^s(r_0)$ be its stable manifold. Then the set $W^s(r_0) \cap X_j$ is dense in $Reg(X_j) \cap J^+ - \bigcup_{i \neq j} X_i$. Let $\zeta \in W^s(r_0) \cap Reg(X_j) \cap J^+ - \bigcup_{i \neq j} X_i$. It follows that $W^s(r_0)$ coincides with Q_ζ in a neighborhood of

ζ. Thus $Q_\zeta \subset J^+$. Now as we have observed, Q_ζ depends continuously on ζ, so letting $\zeta \to p$, we conclude that $Q_p \subset J^+$. □

2 Multipliers at a fixed point

In the following Lemmas, we will assume that f is a composition of generalized Hénon mappings, J^+ is semianalytic, $p \in J^+$ is a fixed point of f, and the multipliers of Df at p are α and β with $|\alpha| \le |\beta|$. Let $X_1, \ldots, X_N$ be the Levi flat hypersurfaces given by Lemma 1.1. By Lemma 1.5, the germs of varieties $Q_p^{(j)}$ at p are invariant under some iterate of f. There is an injective holomorphic map $\varphi : \Delta \to Q_p^{(j)}$ such that $\varphi(0) = p$, and $\varphi(\Delta) = Q_p^{(j)}$. The map $f|_{Q_p^{(j)}}$ induces a locally biholomorphic map g of Δ, fixing 0. Since $Q_p^{(j)} \subset J^+$ the forward iterates of g are a normal family, so we have $|g'(0)| \le 1$. However, it is evident that if the eigenvalues of Df are both greater than 1, then we must have $|g'(0)| > 1$. Thus we conclude:

Lemma 2.1. *We cannot have* $1 < |\alpha| \le |\beta|$.

The next observation is less immediate.

Lemma 2.2. *We cannot have* $1 = |\alpha| \le |\beta|$.

Proof. If Q_p is as in Lemma 1.5, then there is an invariant germ $Q \subset Q_p$ and an injective holomorphic map $\varphi : \Delta \to Q$ such that $\varphi(0) = p$ and $\varphi(\Delta) = Q$. We let g denote the selfmap of Δ induced by $f|_Q$. If $|\alpha| = 1$, then we must have $|g'(0)| = 1$. If $g'(0)$ is a root of unity then an iterate of g is the identity and therefore Q consists of periodic points, but this is not the case for Hénon maps (see [10]). Let $\hat{Q}$ denote the maximal analytic continuation of Q. Since $g'(0)$ is not a root of unity, the iterates of f on $\hat{Q}$ generate $\mathbb{T}^1$ of rotations and are bounded in both forward and backward time. Thus it follows that $\hat{Q} \subset K$. Thus there is an injective holomorphic map $\varphi : M \to \hat{Q}$ with $\varphi(0) = p$ and $\varphi(M) = \hat{Q}$. M must be equivalent to the disk Δ or to $\mathbb{C}$. Since $\hat{Q} \subset K$ is bounded, it follows that we must have $M = \Delta$.
Now φ is a bounded holomorphic function on Δ, so if follows that for almost every θ there is a radial limit $\lim_{r\to 1} \varphi(re^{i\theta})$. Let θ have this property. Let $\gamma := \{\varphi(re^{i\theta}) : 0 \le r < 1\}$, and let $\hat{p} = \lim_{r\to 1} \varphi(re^{i\theta})$ be the endpoint of γ. As in Lemma 1.1, let $\hat{X}$ be a real analytic hypersurface defined in a neighborhood U of $\hat{p}$ such that $U \cap J^+ \subset \hat{X}$. Thus $\gamma \cap U \subset \hat{Q} \cap U \subset \hat{X}$. A basic property of Segre varieties is that $Q_\zeta \subset \hat{X}$ for every $\zeta \in \hat{Q} \cap U$. In particular, if $\zeta \in \gamma$, there is an irreducible component Q'_ζ of Q_ζ that contains $\gamma \cap U$. Thus $Q'_\zeta \cap U$ is independent of $\zeta \in \gamma \cap U$. If we choose $\zeta \in \gamma$, $\zeta \to \hat{p}$, then we see by the continuity of varieties that $Q'_\zeta \subset Q_{\hat{p}}$. Thus there is an irreducible component Q'' of $Q_{\hat{p}}$ such that $\gamma \cap U \subset Q''$. We conclude that Q'' gives an analytic continuation of $\hat{Q}$ along γ, which contradicts the maximality of $\hat{Q}$. This contradiction shows that we cannot have $|\alpha| \ge 1$. □

If the multipliers at p satisfy $|\alpha|<1$ and $|\alpha|<|\beta|$, then the *strong stable set* of p is defined as

$$W^{ss}(p)=\{p\}\cup\{q\in\mathbb{C}^2:\lim_{n\to\infty}\frac{1}{n}\log(\,\mathrm{dist}(f^n(p),f^n(q)))=\log|\alpha|\}$$

By the Strong Stable Manifold Theorem, $W^{ss}(p)$ is a complex submanifold of $\mathbb{C}^2$ which is uniformized by $\mathbb{C}$. The *local strong stable manifold* is defined as

$$W^{ss}_{\mathrm{loc}}(p):=\{q\in W^{ss}(p):f^n(q)\in U\text{ for all }n\geq 0\}.$$

Let us choose coordinates (x,y) near $p=(0,0)$ so that the coordinate axes are the eigenspaces for $Df(p)$. Then if we take $U=\{|x|<r_1,|y|<r_2\}$ to be a small bidisk, then $W^{ss}_{\mathrm{loc}}(p)$ is the connected component of $W^{ss}(p)\cap U$ which contains p.

Lemma 2.3. *We have $|\alpha|<1\leq|\beta|$, and $Q_p=W^{ss}_{loc}(p)$.*

Proof. By Lemmas 2.1 and 2.2 we know that $|\alpha|<1$. If $|\beta|<1$, then p is an attracting fixed point, which means that p belongs to the interior of K^+. Since $p\in J^+=\partial K^+$, we must have $|\beta|\geq 1$. Thus the eigenvalues are distinct, and we may diagonalize $Df(p)$. We may suppose that $p=(0,0)$, and $f(x,y)=(x_1,y_1)=(\beta x+\cdots,\alpha y+\cdots)$. Further, we may choose local coordinates such that $W^{ss}_{\mathrm{loc}}(p)=\{x=0\}$.
If V be an irreducible component of Q_p, and V is not the same as $W^{ss}_{\mathrm{loc}}(p)$, then we may choose U sufficiently small that $Q_p\cap W^{ss}_{\mathrm{loc}}(p)=Q_p\cap\{x=0\}=\{(0,0)\}=\{p\}$. Thus for some positive integer μ we may choose a root $x^{1/\mu}$ and represent V locally as a Puiseux expansion $V=\{y=\sum_{j=1}^{\infty}a_jx^{j/\mu}\}$. The local invariance of V at $p=(0,0)$ means that we will have $y_1=\sum_{j=1}^{\infty}a_jx_1^{j/\mu}$. If a_{j_0} is the first nonvanishing coefficient, we must have $\alpha=\beta^{j_0/\mu}$. But this is impossible since $j_0/\mu>0$, and $|\alpha|<1\leq|\beta|$. It follows, then that the only irreducible component of Q_p is $\{x=0\}$. ∎

Lemma 2.4. *$|\alpha|<1<|\beta|$, and thus p is a saddle point.*

Proof. By Lemma 2.3, we know that the multipliers of Df at p are $|\alpha|<1$ and $|\beta|\geq 1$. We must show that $|\beta|>1$. If not, then $|\beta|=1$. First, we observe that β cannot be a root of unity. For in that case, p is a semi-attracting, semi-parabolic fixed point. Such a fixed point has a semi-parabolic basin $\mathcal{B}$, which has been studied by Ueda [19] and Hakim [11], and more recently in [6]. By [5], we know that $\partial\mathcal{B}=J^+$. However, the boundary of $\mathcal{B}$ has a fractal "cusp" at p (reminiscent of the cauliflower Julia set) and is not contained in a semianalytic set. To see this, we consider the strong stable manifold $W^{ss}(p)$ (called the "Poincaré curve" in [19]). The local structure of a semi-parabolic map means that $\partial\mathcal{B}$ cannot be smooth at points of $W^{ss}(p)$. Thus, if $\partial\mathcal{B}$ is contained in a semianalytic set X, then $W^{ss}(p)$ must be contained in the singular locus of X. Ueda [19] shows that $W^{ss}(p)$ is dense in $\partial\mathcal{B}$ (this also follows from [5]).

But the singular locus of a semianalytic set is again semianalytic and cannot be dense; so $\partial\mathcal{B}$ cannot be contained in a semianalytic set. We conclude that $\beta^k \neq 1$ for all nonzero integers k.

Now let us use coordinates from the proof of Lemma 1.5. Since $Q_{(0,0)} = \{x = 0\}$, we may write $\rho(x, y, 0, 0) = x^k u(x, y)$, where $u(x, y)$ is a holomorphic function with $u(0, 0) = 1$. This means that

$$\rho(x, y, \bar{x}, \bar{y}) = x^k u(x, y) + \bar{x}^k \overline{u(x, y)} + \Psi(x, y, \bar{x}, \bar{y})$$

where in the expansion of ρ, all of the purely holomorphic terms are contained in $x^k u(x, y)$, and x^k is the purely holomorphic part of lowest order. Now there is a real analytic unit $h(x, y, \bar{x}, \bar{y})$ such that $\rho \circ f = h\,\rho$, and $h(0, 0) = c \neq 0$ is real. Thus the purely holomorphic part of lowest order are cx^k. On the other hand, as in the proof of Lemma 1.5, we have

$$\begin{aligned}\rho(f(x,y)) &= \rho(x_1, y_1, \bar{x}_1, \bar{y}_1) = \\ &= \rho(\beta x + \cdots, \alpha y + \cdots, \bar{\beta}\bar{x} + \cdots, \bar{\alpha}\bar{y} + \cdots) = \beta^k x^k + \bar{\beta}^k \bar{x}^k + \Psi_1\end{aligned}$$

Thus we see that the purely holomorphic terms of lowest order are $\beta^k x^k$, from which we conclude that β^k is real, which is a contradiction. ☐

Now p is a saddle point, and the multipliers are $|\alpha| < 1 < |\beta|$. Let $W^u(p)$ be the unstable manifold at p. There is a holomorphic uniformization $\psi_p : \mathbb{C} \to W^u(p) \subset \mathbb{C}^2$ such that $\psi_p(0) = 0$, and $\psi_p(\beta\zeta) = f(\psi_p(\zeta))$. We set $J_p := \psi_p^{-1}(J^+ \cap W^u(p))$ and $g_p^+ := G^+ \circ \psi_p$. By the invariance of J^+ it follows that J_p is invariant under $\zeta \mapsto \beta\zeta$.

Lemma 2.5. *If $p \in J^+$ is fixed, then $\beta \in \mathbb{R}$, and J_p is a straight line in $\mathbb{C}$ passing through the origin.*

Proof. Let X be as in Lemma 1.1. With ψ_p as above, it follows that $J_p := \psi_p^{-1}(J^+ \cap W^u(p) \subset \mathbb{C}$ is semianalytic. Since it is invariant under $\zeta \mapsto \beta\zeta$, we conclude that $\beta \in \mathbb{R}$, and J_p consists of a finite number of rays passing through the origin. We know that $\partial\{g_p^+ > 0\} \subset J_p$, so it follows that g_p^+ is harmonic on $\mathbb{C} - J_p$. Further, g_p^+ cannot be identically zero on $\mathbb{C}$, so there must be a component of $\mathbb{C} - J_p$ where $g_p^+ > 0$. Such components are sectors with vertex at the origin, and let L denote a line which forms part of the boundary of a sector with $g_p^+ > 0$. If J^+ is semianalytic, then so is J_p, and it follows that J_p must contain at least a half-line inside L. We will show that $J_p = L$.

We consider the points $r_0 \in J_p$ which correspond to transverse intersections between $W^s(p)$ and $W^u(p)$. By [2] this set is dense in the set $\partial\{g_p^+ > 0\}$ and thus it is dense in the interval $J_p \cap L$. Let $\Delta_0 \subset \mathbb{C}$ denote a small disk about the origin, and let $\Delta \subset \mathbb{C}$ denote a disk about such a point r_0, small enough that it is disjoint from the other lines in J_p. $\Delta \cap L$ is a segment I which divides Δ into halves Δ' and Δ''. g_p^+ is harmonic on $\Delta - I = \Delta' \cup \Delta''$, and we may assume it is strictly positive on at least one of the half disks Δ' or Δ''. Similarly, it will be strictly positive on (at least) one of the half disks of $\Delta_0 - L$.

Consider the complex disks in $\mathbb{C}^2$ given by $\mathcal{D}_0 := \psi_p(\Delta_0)$ and $\mathcal{D} := \psi_p(\Delta)$. Since $\mathcal{D}$ is transverse to $W^s(p)$ at $\psi_p(r_0)$, we may apply the Lambda Lemma to conclude that there are disks $\mathcal{D}_j \subset f^j(\mathcal{D})$ containing $f^j(\psi_p(r_0))$ which may be written as graphs over $\mathcal{D}_0$, and $\mathcal{D}_j \to \mathcal{D}_0$ in the C^1 topology. Let $\gamma_j := f^j(\psi_p(I)) \cap \mathcal{D}_j$. This is a smooth curve which divides $\mathcal{D}_j$ into halves $\mathcal{D}'_j$ and $\mathcal{D}''_j$, corresponding to the partition $\Delta = \Delta' \cup I \cup \Delta''$. It follows that the γ_j converge to a smooth curve $\gamma_0 \subset \mathcal{D}_0$. Further, the half disks $\mathcal{D}'_j$ and $\mathcal{D}''_j$ converge in C^1 to two half disks $\mathcal{D}'_0$ and $\mathcal{D}''_0$ with $\mathcal{D}_0 - \gamma_0 = \mathcal{D}'_0 \cup \mathcal{D}''_0$. Now $G^+ > 0$ is harmonic on $\mathcal{D}'_j$, so either $G^+ > 0$ on $\mathcal{D}'_0$ or G^+ vanishes everywhere there. However, G^+ does not vanish identically on $\mathcal{D}_0$, so we have $G^+ > 0$ on at least one of the components of $\mathcal{D}_0 - \gamma_0$, which means that $J^+ \cap \mathcal{D}_0 = \gamma_0$. Since γ_0 is f-invariant, it follows that $\psi_p^{-1}(\gamma_0)$ is a straight line in $\mathbb{C}$, which completes the proof. □

Lemma 2.6. *There is a dense set of complex lines $L \subset \mathbb{C}^2$ such that $K^+ \cap L$ contains interior.*

Proof. If $L \subset \mathbb{C}^2$ is a complex line, then by [10], $L \cap J^+$ is compact. Since J^+ is semianalytic of dimension 3, it follows that for generic L, $X \cap L$ has real dimension ≤ 1. Recall that $\partial\{G^+|_L > 0\} \subset L \cap J^+$. Thus any component γ of $J^+ \cap L$ with $\gamma \cap J^+ \neq \emptyset$ cannot be a point, and thus must have dimension 1. If J^+ is semianalytic, then $J^+ \cap L$ consists of a finite union of semianalytic arcs. Given a complex line L_0, we will show that there exists a line L arbitrarily close to L_0 such that $K^+ \cap L$ contains interior. If $J^+ \cap L$ is not simply connected, then it divides L into (at least) two connected components. Only one of these components can be unbounded, so we let $\omega \subset L$ denote a bounded component of the complement of $J^+ \cap L$. On the other hand, $G^+ \geq 0$ vanishes on J^+, so by the maximum principle, $G^+ = 0$ on ω, so $\omega \subset K^+ = \{G^+ = 0\}$.
Thus if $K^+ \cap L$ does not contain interior, $J^+ \cap L$ must be a simply connected union of arcs, and thus it must be a tree. Let p be an endpoint of this tree, and let $X_1 \cup \cdots \cup X_N$ be as in Lemma 1.1. It follows that for some j, $L \cap X_j$ contains a real analytic curve γ which contains p. Since γ is real analytic, it cannot have p as its endpoint. Thus, γ cannot be contained in J^+ and p is in the boundary of $J^+ \cap X_j$ as a subset of X_j, in the sense that every neighborhood of p intersects both $J^+ \cap X_j$ and $X_j - J^+$. By Lemma 1.5, the Segre variety $Q_p^{(j)} \subset J^+$. Due to the continuous dependence of $\eta \mapsto Q_\eta$, we see that $Q_p^{(j)}$ is in the X_j-boundary of J^+, and this boundary of J^+ is given by the union of Segre varieties. It follows that the boundary of J^+ is a complex subvariety of $\mathbb{C}^2$. However, there is no complex subvariety contained in K^+ (see [3, 9]), which is a contradiction. □

Lemma 2.7. *Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ with positive entropy, and let d be the degree of f. If $p \in J^+$ is a fixed point, then d is one of the eigenvalues of Df at p.*

Proof. We continue with the notation $\psi_p : \mathbb{C} \to W^u(p)$ and $g_p^+(\zeta) := G^+(\psi_p(\zeta))$. Thus g_p^+ is subharmonic on $\mathbb{C}$ and satisfies the functional equation $g_p^+(\beta\zeta) = d \cdot g_p^+(\zeta)$. By Lemma 2.5, we may assume that J_p is the real axis. Thus on the upper/lower half plane, $g_p^+(\zeta) = c^\pm \Im(\zeta)$ for some constants $c^+ \geq 0$ and $c^- \leq 0$, which are not both zero. By the functional equation, we have $c^+\Im(\beta\zeta) = d \cdot c^+\Im(\zeta)$ if $\beta > 0$, so $\beta = d$ in this case. If $\beta < 0$, then we have $c^+ = -c^-$, and $\beta = -d$.
Now we will show that one of the $c^\pm$ is zero, so we must have $\beta = d$. By Lemma 2.6, we may choose a $L \subset \mathbb{C}^2$ such that $K^+ \cap L$ contains an interior component ω. We may choose a point $r \in W^s(p) \cap \partial\omega$ which is a regular point of $\partial\omega$. Further, we may suppose that L is transverse to $W^s(p)$ at r. Now we let $\Delta \subset L$ denote a small disk containing r, so that $\Delta \cap \partial\omega$ is a smooth arc which divides Δ into two open components. We have $G^+ = 0$ on $\omega \cap \Delta$ and $G^+ > 0$ on the complementary component. Now we apply the Lambda Lemma as we did in Lemma 2.5, and we conclude that $G^+ = 0$ on one of the components of the complement of $\mathcal{D}_0 \cap J^+ \subset W^u(p)$. Thus we have $c^+ = 0$ or $c^- = 0$. □

Our Theorem is now a consequence of Lemma 2.7:

Proof of Theorem. We claim that there can be at most one fixed point $p \in \text{int}(K^+)$. We observe first that f contracts volume. Otherwise by [10] the interior of K^+ is bounded, then it is disjoint from an open set of complex lines, which contradicts Lemma 2.6. Now if there are two fixed points inside $\text{int}(K^+)$, by [5] there must be two basins $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\partial\mathcal{B}_1 = \partial\mathcal{B}_2 = \partial U^+ = J^+$. This is not possible if J^+ is semianalytic. Thus every fixed point, with at most one exception, is contained in J^+. By Lemma 2.7, d is a multiplier for Df at each fixed point, except possibly one. However, by Proposition 5.1 of [1], this is not possible, so J^+ cannot be contained in a semianalytic set. □

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