WEIGHTED PLURIPOTENTIAL THEORY

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ABSTRACT. These are notes from a course at Indiana University in fall 2017. It is an expanded version of notes from a ten lecture course given to a general audience of PhD students at the University of Padova October 17-28, 2011. The goal is to present some basic notions in potential theory and weighted potential theory in the complex plane $\mathbb{C}$ (chapters 1-6) with an eye towards developing pluripotential theory and weighted pluripotential theory in $\mathbb{C}^N,\ N > 1$ (chapters 7-10). The final chapter 11 offers some very recent results in the subject.

CONTENTS

1. Subharmonic functions and potential theory in $\mathbb{C}$.

To motivate the definition of subharmonic functions on domains in the complex plane, we begin with their analogue on the real line $\mathbb{R}$. A twice-differentiable function $h : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$ is linear if and only if $h''(x) = 0$ on I. A twice-differentiable function $g : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$ is convex if and only if $g''(x) \geq 0$ on I. The relation between these classes of functions is as follows: if $g \leq h$ at the endpoints of any subinterval $I' \subset I$, then $g \leq h$ on I'. Of course, the notion of convexity does not require any differentiability.

In $\mathbb{C} = \mathbb{R}^2$ with variables $z = x + iy$, let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian operator. Recall that a twice-differentiable function $h : D \to \mathbb{R}$ on a domain $D \subset \mathbb{C}$ is harmonic in D if $\Delta h = 0$ there. Here is our first definition of subharmonic:

Definition 1.1. A function $u : D \to \mathbb{R}$ is *subharmonic* (shm) in a domain $D \subset \mathbb{C}$ if u is uppersemicontinuous (usc) in D and for any subdomain $D' \subset\subset D$ and any h harmonic on a neighborhood of $\overline{D}'$, if $u \leq h$ on $\partial D'$ then $u \leq h$ on D'.

Recall u is usc on D means that for each $a \in \mathbb{R}$, the set $\{z \in D : u(z) < a\}$ is open; for such a function and a compact subset K of D one can find a decreasing sequence of continuous functions $\{u_j\}$ with $u_j \downarrow u$ on K (cf., Theorem 2.1.3 of [30]). Also, u is bounded above on K and attains its maximum value there (exercise 3 (b)). There is an analogous notion of lowersemicontinuous (lsc): v is lsc on D means that for each $a \in \mathbb{R}$, the set $\{z \in D : v(z) > a\}$ is open; equivalently, $u = -v$ is usc. Thus a function is continuous on D if and only if u is usc and lsc on D. If $D = \mathbb{C}$ and $u(z) = -1$ for $|z| < 1$ while $u(z) = 0$ for $|z| \geq 1$, then u is usc. For completeness, we say a function $v : D \to \mathbb{R}$ is *superharmonic* in D if $u = -v$ is shm there.

A second, equivalent definition of shm is the following:

Definition 1.2. A function $u : D \to \mathbb{R}$ is *subharmonic* in a domain $D \subset \mathbb{C}$ if u is usc in D and u satisfies a subaveraging property in D: for each $z_0 \in D$ and $r > 0$ with $B(z_0, r) := \{z : |z - z_0| < r\} \subset D$,

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta. \tag{1.1}$$

A *harmonic* function h on D satisfies a *mean-value* property: for each $z_0 \in D$ and $r > 0$ with $B(z_0, r) \subset D$,

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) d\theta. \tag{1.2}$$

Moreover, $\Delta h = 0$ in D. We recall that if h is harmonic in a domain D and continuous in $\overline{D}$, if $h \leq M$ on ∂D then $h \leq M$ in D (maximum principle); also, since $-h$ is harmonic, harmonic functions satisfy a minimum principle as well. From our second definition, we will see that shm functions satisfy a maximum principle.

Proposition 1.3. *Let u be usc in a domain $D \subset \mathbb{C}$ and satisfy (1.1). Then*

1. *if $u(z_0) = \sup_{z \in D} u(z)$ for some $z_0 \in D$, then $u(z) \equiv u(z_0)$;*
2. *if D is bounded and $\limsup_{z \to \zeta} u(z) \leq M$ for all $\zeta \in \partial D$, then $u \leq M$ in D.*

Proof. For (1), let $U = \{z \in D : u(z) = u(z_0)\}$. Then $U \neq \emptyset$ and $D \setminus U = \{z \in D : u(z) < u(z_0)\}$ is open by usc of u. Hence U is closed. Using property (1.1), we show U is open. If $w \in U$ then for any $r > 0$ with $B(w, r) \subset D$,

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(w) d\theta = u(w)$$

hence equality holds. Since $u(w + re^{i\theta}) \leq u(w)$, we must have that $u(w + re^{i\theta}) = u(w)$ for almost all θ for all $r > 0$ with $B(w, r) \subset D$. To complete the proof that U is open, we observe that, again by usc, if $u(w + r_0 e^{i\theta_0}) < u(w)$ for some point $w + r_0 e^{i\theta_0}$ in D, then the inequality $u(w') < u(w)$ persists for all points w' in an open neighborhood of $w + r_0 e^{i\theta_0}$. This contradicts the equality $u(w + re^{i\theta}) = u(w)$ for almost all θ for all $r > 0$ with $B(w, r) \subset D$.

For (2), the extension of u to ∂D via $u(\zeta) := \limsup_{z \to \zeta} u(z)$ if $\zeta \in \partial D$ gives an usc function on the compact set $\overline{D}$. From the exercises u attains its maximum value in $\overline{D}$ at some point w. If $w \in \partial D$, by hypothesis $u \leq u(w) \leq M$ in D. If $w \in D$, by (1) u is constant on D and hence on $\overline{D}$ so $u \leq M$ in D. $\square$

We prove the equivalence of Definitions 1.1 and 1.2: To show that the second definition implies the first, it clearly suffices to check the domination property in the first definition on disks $B(z_0, r) \subset D$. If

h is harmonic on a neighborhood of $\overline{B}(z_0, r)$ and $u \leq h$ on $\partial B(z_0, r)$, then by (1.1) and (1.2) $u - h$ satisfies (1.1). Furthermore, $u - h$ is usc (why?); hence, by Proposition 1.3, $u - h \leq 0$ on $\partial B(z_0, r)$ implies $u - h \leq 0$ on $B(z_0, r)$.

For the converse, we recall the solution of the Dirichlet problem in the unit disk $B := B(0, 1)$. Let f be a continuous, real-valued function on ∂B. We seek a harmonic function h in B, $h \in C(\overline{B})$, with $h = f$ on ∂B. This is achieved by writing down the Poisson integral formula:

$$P_{f,B}(z) := h(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta}) d\theta.$$

Note that

$$P_{f,B}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta$$

is the mean value of f over ∂B. A formula can easily be given for the solution of the Dirichlet problem with boundary data f in any disk $B(z_0, r)$ and we will use the notation $P_{f,B(z_0,r)}$ for such a function.

Given u usc satisfying (1.1), since u is usc, on $\partial B(z_0, r)$ we can find a decreasing sequence of continuous functions f_j with $f_j \downarrow u$ there. The functions $h_j(z) := P_{f_j,B(z_0,r)}(z)$ then form a decreasing sequence of harmonic functions in $B(z_0, r)$. Then $u \leq f_j$ on $\partial B(z_0, r)$ implies that $u \leq h_j$ on $B(z_0, r)$. Hence

$$u(z_0) \leq \lim_{j\to\infty} h_j(z_0) = \lim_{j\to\infty} \big(\frac{1}{2\pi} \int_0^{2\pi} h_j(z_0 + re^{i\theta}) d\theta\big)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [\lim_{j\to\infty} h_j(z_0 + re^{i\theta})] d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

by monotone convergence.

Corollary 1.4. *A function $u : D \to \mathbb{R}$ with $u \in C^2(D)$ is shm in D if and only if $\Delta u \geq 0$ in D.*

Proof. If $\Delta u \geq 0$ in D, take $D' \subset\subset D$ and h harmonic on $\overline{D}'$ with $u \leq h$ on $\partial D'$. Then for $\epsilon > 0$, the function

$$u_\epsilon(z) := u(z) - h(z) + \epsilon |z|^2$$

is usc in $\overline{D}'$ and hence attains a maximum there. This maximum value cannot be attained in D' since $\Delta u_\epsilon = \Delta u + 4\epsilon > 0$ there (hence u_ϵ

cannot attain a *local* maximum in D'. Thus it occurs on $\partial D'$ and hence for $z \in \overline{D}'$,

$$u(z) - h(z) + \epsilon|z|^2 \leq \max_{z \in \partial D'}[u(z) - h(z) + \epsilon|z|^2] \leq \epsilon \cdot \max_{z \in \partial D'} |z|^2.$$

Now let $\epsilon \downarrow 0$.

The converse is trivial: if $u \in C^2(D)$ is shm in D and $\Delta u(w) < 0$ for some $w \in D$, by continuity we have $\Delta u(z) < 0$ for $z \in B(w,r)$ for $r > 0$ sufficiently small. From the first part, $-u$ is thus subharmonic in $B(w,r)$ and hence u is harmonic there. In particular, $\Delta u(w) = 0$, a contradiction; thus $\Delta u \geq 0$ in D.

□

The canonical examples of shm functions are those of the form $u = \log|f|$ where $f \in \mathcal{O}(D)$ (the holomorphic functions on D). Recall that a function $u : D \to \mathbb{R}$ is *locally integrable* on D if for each compact set $K \subset D$, $\int_K |u(z)|dm(z) < +\infty$. The function $\log|z|$ is locally integrable which shows that $\log|f|$ is locally integrable; indeed, exercise 7 shows that all subharmonic functions are locally integrable. The class of shm functions on a domain D, denoted $SH(D)$, forms a convex cone; i.e., if $u, v \in SH(D)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in SH(D)$. The maximum $\max(u,v)$ of two shm functions in D is shm in D, and one can "glue" shm functions (see exercise 6). Thus shm functions are very flexible to work with as opposed to holomorphic or harmonic functions. The limit function $u(z) := \lim_{n\to\infty} u_n(z)$ of a decreasing sequence $\{u_n\} \subset SH(D)$ is shm in D (we may have $u \equiv -\infty$); while for any family $\{v_\alpha\} \subset SH(D)$ (resp., sequence $\{v_n\} \subset SH(D)$) which is uniformly bounded above on any compact subset of D, the functions

$$v(z) := \sup_\alpha v_\alpha(z) \text{ and } w(z) := \limsup_{n\to\infty} v_n(z)$$

are "nearly" shm: the usc regularizations

$$v^*(z) := \limsup_{\zeta \to z} v(\zeta) \text{ and } w^*(z) := \limsup_{\zeta \to z} w(\zeta)$$

are shm in D. Finally, if ϕ is a real-valued, convex increasing function of a real variable, and u is shm in D, then so is $\phi \circ u$ – to see this, use *Jensen's inequality* to show subaveraging: if $\phi : (a,b) \to \mathbb{R}$ is convex, μ is a probability measure on U and $f : U \to (a,b)$ is $\mu-$integrable, then

$$\phi(\int_U f d\mu) \leq \int_U (\phi \circ f) d\mu.$$

Then if $B(z,r) \subset D$,

$$\phi(u(z)) \leq \phi\big(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\big) \leq \frac{1}{2\pi}\int_0^{2\pi} (\phi \circ u)(z+re^{i\theta})d\theta.$$

We will use the complex differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\big) \text{ and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big).$$

For a function u, $\partial u := \frac{\partial u}{\partial z}dz$ and $\overline{\partial} u := \frac{\partial}{\partial \overline{z}}d\overline{z}$ where $dz = dx + idy$ and $d\overline{z} = dx - idy$. We let

$$d = \partial + \overline{\partial},\ d^c = i(\overline{\partial} - \partial),\ \text{so } dd^c = 2i\partial\overline{\partial}.$$

Thus for $u \in C^2(D)$, $dd^c u = \Delta u dx \wedge dy$ and u is shm if and only if the Laplacian Δu is a nonnegative function on D. In this notation, a complex-valued function $f : D \to \mathbb{C}$ is holomorphic in D if $f \in C^1(D)$ and $\frac{\partial f}{\partial \overline{z}} = 0$ in D; this is easily seen to be equivalent, writing $f = u + iv$, to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We can smooth a shm function u by convolving with a regularizing kernel $\chi(z) = \chi(|z|) \geq 0$ with $\chi \in C_0^\infty(\mathbb{C})$ ($C^\infty-$functions with compact support) and $\int_{\mathbb{C}} \chi dm = 1$ (here dm is Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$); i.e., if $\text{supp}\chi \subset B(0,r)$,

$$(u * \chi)(z) := \int_{\mathbb{C}} u(z-\zeta)\chi(\zeta)dm(\zeta)$$

is shm and C^∞ on $\{z \in D : \text{dist}(z, \partial D) > r\}$. (See exercise 12 for more on regularizing kernels). The regularity follows via a change of variables:

$$(u * \chi)(z) = \int_{\mathbb{C}} u(\zeta)\chi(z-\zeta)dm(\zeta);$$

differentiating under the integral sign, we see that $u * \chi$ is as differentiable as χ. The subharmonicity follows from Fubini's theorem:

$$\frac{1}{2\pi}\int_0^{2\pi} (u * \chi)(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} u(z_0 + re^{i\theta} - \zeta)\chi(\zeta)dm(\zeta)d\theta$$

$$= \int_{\mathbb{C}} \chi(\zeta)\big(\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta} - \zeta)d\theta\big)dm(\zeta)$$

$$\geq \int_{\mathbb{C}} \chi(\zeta)u(z_0-\zeta)dm(\zeta) = (u*\chi)(z_0).$$

Note that if u is *harmonic* in D, then $u*\chi$ is harmonic on $\{z \in D : \mathrm{dist}(z,\partial D) > r\}$ (why?).

We claim that given u shm in a domain D, we can find a decreasing sequence $\{u_j\}$ of smooth shm functions with $\Delta u_j \geq 0$ defined on $\{z \in D : \mathrm{dist}(z,\partial D) > 1/j\}$ and $\lim_j u_j = u$ in D. For example, if $\mathrm{supp}\chi \subset B(0,1)$, we can take $u_j = u * \chi_{1/j}$ where $\chi_{1/j}(z) := j^2\chi(jz)$. This will allow us to first verify properties of smooth shm functions and then pass to the limit. It remains to show that $u_j = u*\chi_{1/j}$ decrease to u on D as $j \to \infty$. We proceed in several steps, each one being interesting in itself.

(1) *A radial function $u(z) = u(|z|) = u(r)$ on a disk $B(0,R)$ is shm if and only if $r \to u(r)$ is a convex, increasing function of $\log r$.*

Note since $v(z) = \log|z|$ is shm in $\mathbb{C}$ and $f \circ v$ is shm for f convex and increasing, the "if" direction is proved. For the converse, if $u = u(r)$ is shm, then u is increasing by the maximum principle Proposition 1.3. The convexity is less obvious; a relatively painless way to verify it goes as follows: given r_1, r_2 between 0 and R, choose constants a, b so that

$$a + b\log r_1 = u(r_1) \text{ and } a + b\log r_2 = u(r_2).$$

Note that $r \to \log r$ is harmonic for $r > 0$. Thus $u(r) - [a + b\log r]$ is shm on the annulus $B(0,r_2) - \overline{B}(0,r_1)$. Applying the maximum principle, we see that

$$u(r) \leq a + b\log r \text{ on } B(0,r_2) - \overline{B}(0,r_1).$$

Thus for $r_1 \leq r \leq r_2$, writing $\log r = (1-t)\log r_1 + t\log r_2$ for some $0 \leq t \leq 1$, we have

$$u(r) \leq a + b\log r = (1-t)[a + b\log r_1] + t[a + b\log r_2]$$
$$= (1-t)u(r_1) + tu(r_2).$$

(2) *For $u(z)$ shm on a disk $B(0,R)$, the function*

$$M_u(r) := \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta$$

is a convex, increasing function of $\log r$ and $\lim_{r\to 0^-} M_u(r) = u(0)$.

This is left as an exercise for the reader. *Hint: Show* $v(r) := M_u(r)$ *is subharmonic on* $B(0,R)$ *– use Fubini to verify the subaveraging property. Now use (1). To show* $\lim_{r\to 0^-} M_u(r) = u(0)$*, one inequality uses usc of* u*.*

(3) $u_j = u * \chi_{1/j}$ *decrease to* u *on* D *as* $j \to \infty$*.*

We have

$$u_j(\zeta) = \int_{\mathbb{C}} u(\zeta - z)\chi_{1/j}(z)dm(z)$$

$$= \int_0^{2\pi}\int_0^{1/j} u(\zeta - re^{it})\chi_{1/j}(re^{it})rdrdt \text{ (why?)}$$

$$= \int_0^{2\pi}\int_0^{1} v(\frac{s}{j}e^{it})\chi(s)sdsdt$$

$$= \int_0^{1}\big(\int_0^{2\pi} v(\frac{s}{j}e^{it})dt\big)\chi(s)sds$$

where we let $s = rj$ and $v(z) := u(\zeta - z)$. By (2), $\int_0^{2\pi} v(\frac{s}{j}e^{it})dt$ decreases to $2\pi v(0) = 2\pi u(\zeta)$ as $j \uparrow \infty$; thus by monotone convergence, $u_j(\zeta)$ decreases to $2\pi\int_0^1 u(\zeta)\chi(s)sds = u(\zeta)$ (why?).

We remark that the occurrence of the combination $a + b\log r$ in step (1) is very natural: see also exercise 10 and Proposition 1.8 below.

Corollary 1.5. *If* u, v *are shm on* D *and* $u = v$ *a.e. then* $u \equiv v$*.*

Proof. Since $u = v$ a.e., $u_j = u * \chi_{1/j} \equiv v * \chi_{1/j} = v_j$. The result follows since $u_j \downarrow u$ and $v_j \downarrow v$. □

Corollary 1.6. *If* u *is harmonic in* D *then* $u_j = u$ *on* $D_{1/j}$*.*

Proof. We can apply the result to u and $-u$; in particular, $u \leq u_j$ on $D_{1/j}$ and $-u \leq -u_j$ on $D_{1/j}$. □

Remark 1.7. Many properties of shm functions can now be verified by first proving the property for smooth shm functions and then showing this remains valid under decreasing limits.

We can solve the Dirichlet problem on more general bounded domains $D \subset \mathbb{C}$ with reasonable boundaries; i.e., we can construct h satisfying $\Delta h = 0$ in D and $h = f$ on ∂D, one forms the envelope

$$U(0; f)(z) := \sup\{v(z) : v \in SH(D) : \limsup_{z\to\zeta} v(z) \leq f(\zeta)$$

$$\text{for all } \zeta \in \partial D\}.$$

This family of all $v \in SH(D)$ satisfying $\limsup_{z\to\zeta} v(z) \leq f(\zeta)$ for all $\zeta \in \partial D$ is a *Perron* family: for any such v and any disk $\tilde{B} \subset D$, the function $\tilde{v}$ defined as v in $D \setminus \tilde{B}$ and as $P_{v|_{\partial \tilde{B}}, \tilde{B}}$ in $\tilde{B}$ is in the family (and is harmonic in $\tilde{B}$). This follows from the Gluing lemma – see exercise 6. To show $U(0; f)$ is harmonic in D, it suffices to show harmonicity on any disk $\tilde{B} \subset D$. We return to this issue in the next section.

Subharmonic functions need not be twice-differentiable, let alone continuous. Thus we need a way of interpreting derivatives, in particular, the Laplacian, in a generalized sense. A *distribution* $\mathcal{L}$ in one real variable is a continuous linear functional on the vector space $C_0^\infty(\mathbb{R})$ of test functions, i.e., C^∞ functions on $\mathbb{R}$ with compact support. To define continuity of $\mathcal{L}$ one needs to define a topology on $C_0^\infty(\mathbb{R})$. We say $\{\phi_n\} \subset C_0^\infty(\mathbb{R})$ converges to $\phi \in C_0^\infty(\mathbb{R})$ if the supports of each ϕ_n and ϕ are contained in a compact set K and $\phi_n^{(j)} \to \phi^{(j)}$ uniformly on K for $j = 0, 1, 2, ...$; then $\mathcal{L}$ is continuous if $\mathcal{L}(\phi_n) \to \mathcal{L}(\phi)$ whenever ϕ_n converges to ϕ in $C_0^\infty(\mathbb{R})$. Standard examples include, for any $\psi \in C(\mathbb{R})$, the distribution $\mathcal{L}_\psi$ of integration with respect to ψ:

$$\mathcal{L}_\psi(f) := \int f(x)\psi(x)dx;$$

and the distribution $\mathcal{L}(f) := f(0)$, known as the *delta function*: we often write $\delta_0(f) = f(0)$. More generally, for any $x \in \mathbb{R}$, $\delta_x(f) := f(x)$ is the *delta function at* x. These delta functions are examples of *positive* distributions: $\mathcal{L}$ is positive if $f \geq 0$ implies $\mathcal{L}(f) \geq 0$ for $f \in C_0^\infty(\mathbb{R})$. It turns out that *a positive distribution is a positive measure*; in particular, δ_x is represented by a point mass at the point x. If $\psi \in C(\mathbb{R})$ is a nonnegative function, then $\mathcal{L}_\psi$ is a positive distribution (and $\psi(x)dx$ is a positive measure).

We define the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ by $\mathcal{L}'(f) := -\mathcal{L}(f')$. The reader may check that if $\mathcal{L} = \mathcal{L}_g$ for a C^1 function g, then $\mathcal{L}_g' = \mathcal{L}_{g'}$. We can also multiply a distribution by a smooth (C^∞) function: since, clearly, for $g, h \in C(\mathbb{R})$ and $f \in C_0(\mathbb{R})$ (continuous functions on $\mathbb{R}$ with compact support) we have

$$\int f(x)[g(x)h(x)]dx = \int [f(x)g(x)]h(x)dx,$$

we then define, for a distribution $\mathcal{L}$ and a smooth function g, the new distribution $g \cdot \mathcal{L}$ via

$$(g \cdot \mathcal{L})(f) := \mathcal{L}(gf).$$

Convergence of a sequence $\{\mathcal{L}^{(n)}\}$ of distributions is akin to, but easier than, weak-* convergence of a sequence of measures: $\mathcal{L}^{(n)} \to \mathcal{L}$ as distributions if $\mathcal{L}^{(n)}(\phi) \to \mathcal{L}(\phi)$ for all $\phi \in C_0^\infty(\mathbb{R})$. We include some optional exercises on distributions in Appendix B at the end of these notes. For example, exercise 1 shows that $g \in L^1_{loc}(\mathbb{R})$ defines a distribution $\mathcal{L}_g$ via

$$\mathcal{L}_g(\phi) := \int_{\mathbb{R}} \phi(x)g(x)dx$$

for $\phi \in C_0^\infty(\mathbb{R})$; and if $\{g_n\} \subset L^1_{loc}(\mathbb{R})$ with $g_n \to g$ in $L^1_{loc}(\mathbb{R})$, then $\mathcal{L}_{g_n} \to \mathcal{L}_g$ as distributions.

All these notions are easily extended to higher (real) dimensions; of particular interest to us is the case of $\mathbb{R}^2 = \mathbb{C}$. In this setting, if $u \in L^1_{loc}(\mathbb{C})$ we will simply write $u = \mathcal{L}_u$ for the associated distribution. Then note that Δu is well-defined as a distribution: for $\phi \in C_0^\infty(\mathbb{C})$, $\Delta u(\phi) = u(\Delta\phi)$. It follows trivially that:

(1) if $\{u_n\} \subset L^1_{loc}(\mathbb{C})$ and $u_n \to u$ in $L^1_{loc}(\mathbb{C})$, then $\Delta u_n \to \Delta u$ as distributions; and, in particular,
(2) if $\{u_n\}, u$ are subharmonic and either $u_n \downarrow u$ or $u_n \uparrow u$ a.e. (with respect to Lebesgue measure on $\mathbb{C}$), then $u_n \to u$ in $L^1_{loc}(\mathbb{C})$ so that $\Delta u_n \to \Delta u$ as *positive* distributions.

As with distributions on $\mathbb{R}$, a positive distribution $\mathcal{L}$ on $\mathbb{C}$ is a positive *measure*: $\mathcal{L}(\phi) \geq 0$ for $\phi \in C_0^\infty(\mathbb{C})$ with $\phi \geq 0$ implies $\mathcal{L}(\phi)$ is well-defined and nonnegative for $\phi \in C_0(\mathbb{C})$ with $\phi \geq 0$. Thus: *if u is subharmonic, then Δu is a positive measure.* In this case, $dd^c u = \Delta u dx \wedge dy$ is a differential form with distribution coefficients; it is an example of a *positive, closed current of bidegree* $(1,1)$ (see Appendix A). Currents will be more relevant when we work in $\mathbb{C}^n$, $n > 1$.

Using some standard multivariate calculus, we prove a fundamental result on the Laplace operator in $\mathbb{R}^2 = \mathbb{C}$.

Proposition 1.8. *$E(z) := \frac{1}{2\pi}\log|z|$ is a fundamental solution for Δ: we have $\Delta(\frac{1}{2\pi}\log|z|) = \delta_0$, the unit point mass at the origin, in the sense of distributions.*

Proof. To this end, fix $\phi \in C_0^\infty(D)$ where D is a neighborhood of the origin. We want to show that

$$\int_D \Delta\phi(z) \cdot E(z) dm(z) = \phi(0).$$

We make use of a standard multivariate calculus result, sometimes known as a *Green's identity*: let u, v be twice-differentiable functions defined in a neighborhood of the closure $\overline{\Omega}$ of a bounded, open set Ω with C^1-boundary. Then

$$\int_\Omega (u\Delta v - v\Delta u) dm = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) ds \tag{1.3}$$

where ds denotes arclength measure on $\partial\Omega$. Apply (1.3) to the functions ϕ, E in $D_\epsilon := \{z \in D : |z| > \epsilon\}$ to obtain

$$\int_{D_\epsilon} [\Delta\phi(z) \cdot E(z) - \Delta E(z) \cdot \phi(z)] dm(z) = \int_{D_\epsilon} \Delta\phi(z) \cdot E(z) dm(z)$$

$$= \int_{\partial D_\epsilon} [E\frac{\partial \phi}{\partial n} - \phi\frac{\partial E}{\partial n}] ds = -\int_{\partial B(0,\epsilon)} [E\frac{\partial \phi}{\partial n} - \phi\frac{\partial E}{\partial n}] ds.$$

The area integral tends to $\int_D \Delta\phi(z) \cdot E(z) dm(z)$ as $\epsilon \to 0$ since E is locally integrable. Since $\epsilon \log \epsilon \to 0$ as $\epsilon \to 0$,

$$\int_{\partial B(0,\epsilon)} E\frac{\partial \phi}{\partial n} ds \to 0$$

and

$$-\int_{\partial B(0,\epsilon)} \phi\frac{\partial E}{\partial n} ds = \frac{1}{2\pi} \int_0^{2\pi} \phi(\epsilon e^{i\theta}) \frac{1}{\epsilon} \epsilon d\theta \to \phi(0).$$

□

We remark that (1.3) is the same as

$$\int_\Omega (udd^c v - vdd^c u) = \int_{\partial\Omega} (ud^c v - vd^c u)$$

which follows from Stokes theorem. Note that $d^c u = \frac{\partial u}{\partial n} ds$ (see also exercise 1).

Since the function $u(z) = \log|z|$ is locally integrable, it follows that given a positive measure μ of finite total mass and, say, compact support, one can form the convolution

$$V_\mu(z) := -p_\mu(z) := (u * \mu)(z) := \int_{\mathbb{C}} \log|z - \zeta| d\mu(\zeta).$$

This yields a shm function V_μ on $\mathbb{C}$; and since δ_0 acts as the identity under convolution (why?)

$$\Delta V_\mu = \Delta(u * \mu) = \Delta u * \mu = 2\pi\delta_0 * \mu = 2\pi\mu$$

(the equality $\Delta V_\mu = 2\pi\mu$ as distributions can also be directly verified by an argument similar to the proof of Proposition 1.8 together with Fubini's theorem). Note that V_μ is *harmonic* on $\mathbb{C} \setminus \mathrm{supp}\mu$ and we have

$$V_\mu(z) = \mu(\mathbb{C}) \log|z| + 0(1/|z|) \text{ (why?).}$$

We call p_μ the *logarithmic potential function* of μ. The notation for the superharmonic function p_μ is standard; to emphasize the difference with the subharmonic function $-p_\mu$, we have introduced the notation V_μ. If μ is a probability measure, i.e., $\mu(\mathbb{C}) = 1$, then V_μ is in the class

$$L(\mathbb{C}) := \{u \text{ shm on } \mathbb{C},\ u(z) - \log|z| = 0(1),\ |z| \to \infty\}. \tag{1.4}$$

of global shm functions of at most logarithmic growth. A typical example of $u \in L(\mathbb{C})$ is

$$u(z) = \frac{1}{n} \log|p_n(z)| \text{ where } p_n(z) = a_n z^n + \cdots + a_1 z + a_0$$

is a polynomial of degree $n \geq 1$. We will see the importance of this collection of shm functions, and its *plurisubharmonic* generalization, throughout this course. Indeed, the following *Riesz decomposition theorem* shows that, locally, *all* subharmonic functions are logarithmic potential functions modulo a harmonic function.

Theorem 1.9. *Let $u \not\equiv -\infty$ be subharmonic on D. For any open $U \subset D$ with $\overline{U} \subset D$, we can write $u = -p_\mu + h$ on U where $2\pi\mu = \Delta u|_U$ and h is harmonic on U.*

Proof. It suffices to prove a version of *Weyl's lemma*: if $u, v \not\equiv -\infty$ are subharmonic on U and $\Delta u = \Delta v$ then $u - v$ is harmonic on U. Given this, the theorem follows from the previously noted fact that $\Delta(-p_\mu) = 2\pi\mu = \Delta u$ on (components of) U.

For $r > 0$ let $U_r := \{z \in U : \mathrm{dist}(z, \partial U) > r\}$ and let $\chi_r(z) := \frac{1}{r^2}\chi(z/r)$ be a normalized regularizing kernel. Then $\Delta u = \Delta v$ in U implies $\Delta(u * \chi_r) = \Delta(v * \chi_r)$ in U_r. Since $u * \chi_r, v * \chi_r$ are smooth functions in U_r, $u * \chi_r - v * \chi_r = h_r$ for some harmonic function h_r in U_r. Then for any $s > 0$, applying Corollary 1.6, $h_r = h_r * \chi_s = [(u - v) * \chi_r] * \chi_s = [(u - v) * \chi_s] * \chi_r = h_s * \chi_r = h_s$ on U_{r+s} (the second equality is justified since two of the terms have compact support). Hence there is a single harmonic h in U with $(u-v)*\chi_r = h$ on U_r for each r; now let $r \downarrow 0$.

□

We next give an important continuity property of logarithmc potentials.

Proposition 1.10. *Let μ be a positive measure of finite total mass and compact support K and let*

$$V_\mu(z) := \int_{\mathbb{C}} \log|z-\zeta| d\mu(\zeta).$$

For $z_0 \in K$,

$$\liminf_{z \to z_0} V_\mu(z) = \liminf_{z \to z_0,\ z \in K} V_\mu(z).$$

In particular, if $V_\mu|_K$ is continuous, then V_μ is continuous on $\mathbb{C}$; and if $V_\mu \geq M$ on K, then $V_\mu \geq M$ on $\mathbb{C}$.

Proof. First, if $V_\mu(z_0) = -\infty$ the result is clear by usc of V_μ. If $V_\mu(z_0) > -\infty$ then μ puts no mass on the point $\{z_0\}$ (why?); hence, given $\epsilon > 0$ we can find $r > 0$ with $\mu(B(z_0, r)) < \epsilon$. Now given $z \in \mathbb{C} \setminus K$, take a point $z' \in K$ such that $|z - z'| = \min_{w \in K} |z - w|$. Then for any $w \in K$ we have

$$\frac{|z'-w|}{|z-w|} \leq \frac{|z'-z| + |z-w|}{|z-w|} \leq 2$$

and

$$V_\mu(z) = V_\mu(z') - \int_K \log \frac{|z'-w|}{|z-w|} d\mu(w)$$

$$\geq V_\mu(z') - \epsilon \log 2 - \int_{K \setminus B(z_0,r)} \log \frac{|z'-w|}{|z-w|} d\mu(w).$$

Now as $z \to z_0$ clearly $z' \to z_0$ so that

$$\liminf_{z \to z_0} V_\mu(z) \geq \liminf_{z' \to z_0,\ z' \in K} V_\mu(z') - \epsilon \log 2.$$

The last statement is left for the exercises. □

An interesting example of a measure μ with V_μ discontinuous at a point of $K = \mathrm{supp}(\mu)$ is $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{2^n}}$. Then

$$V_\mu(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \log|z - \frac{1}{2^n}|.$$

Clearly $0 \in K$ but

$$-\infty = \limsup_{z \in K \setminus \{0\}} V_\mu(z) < V(0) = \sum_{n=1}^{\infty} \frac{1}{2^n} \log \frac{1}{2^n}.$$

Here, the set K is *thin* at 0; see section 3 for the definition.

Corollary 1.11. *Let $u \not\equiv -\infty$ be subharmonic in D with Δu having compact support K in D. If $u|_K$ is continuous on K then u is continuous in D.*

Corollary 1.12. *Let μ be a positive measure of finite total mass and compact support K and suppose $V_\mu > -\infty$ $\mu-$a.e. Given $\epsilon > 0$, there exists $K' \subset K$ compact such that $\mu(K \setminus K') < \epsilon$ and, setting $\mu' := \mu|_K$, we have $V_{\mu'} \in C(\mathbb{C})$.*

Proof. Lusin's theorem gives the existence of $K' \subset K$ with $\mu(K \setminus K') < \epsilon$ such that V_μ is continuous on K'. We show that $V_{\mu'}$ is continuous on K' and the result follows from Proposition 1.10. Clearly we have

$$V_{\mu'} = V_\mu - V_{\mu-\mu'}$$

and, on K', the left-hand-side is usc; on the right-hand-side, V_μ is continuous on K' while $V_{\mu-\mu'}$ is usc (note $\mu - \mu'$ is a *positive* measure). Thus the right-hand-side is lsc on K'.

□

Two standard examples of functions V_μ are the following:

(1) If $\mu = \delta_0$, then $V_\mu(z) = \log|z|$. More generally, if $\mu = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$, then

$$V_\mu(z) = \frac{1}{n}\log|p_n(z)| \text{ where } p_n(z) = \prod_{j=1}^{n}(z - z_j).$$

(2) If $\mu = \frac{1}{2\pi}d\theta$ on $|z| = 1$, then $V_\mu(z) = \log^+|z| := \max[\log|z|, 0]$.

A useful result, which generalizes to $\mathbb{C}^N$ for $N > 1$, is the *comparison principle*. We do not state it in its most general form; instead, we state a version which generalizes to $\mathbb{C}^n,\ n > 1$.

Proposition 1.13. *Let u, v be shm and bounded in a bounded, open set $D \subset \mathbb{C}$. Suppose $\liminf_{z\to\zeta}[u(z) - v(z)] \geq 0$ for all $\zeta \in \partial D$. Then*

$$\int_{\{u<v\}} dd^c v \leq \int_{\{u<v\}} dd^c u. \tag{1.5}$$

Remark 1.14. A "geometric" interpretation in the one-real dimensional case of convex functions is clear: u is more "curved" (convex) than v. Also, some sort of boundedness or regularity hypothesis is necessary for finiteness of the integral(s): check, e.g., that $u(z) = \log \frac{1}{1-|z|}$ is shm in $D = \{z : |z| < 1\}$ but $\int_D dd^c u = +\infty$.

Proof. The case where ∂D is C^1, $u, v \in C^2(D) \cap C^1(\overline{D})$ and $u = v$ on ∂D is straightforward. In this case, $\lim_{z\to\zeta}[u(z) - v(z)] \geq 0$ for all $\zeta \in \partial D$ and we may assume $u = v$ on ∂D so that $D = \{u < v\}$. Then $d^c(u - v) = \frac{\partial(u-v)}{\partial n} ds$ and $\frac{\partial(u-v)}{\partial n} \geq 0$ on ∂D (see exercise 1 below). Stokes' theorem gives

$$\int_{\{u<v\}} dd^c(u - v) = \int_D dd^c(u - v) = \int_{\partial D} d^c(u - v) \geq 0.$$

Note in this setting subharmonicity of u, v is not needed.

Even if u, v are only assumed continuous on $\overline{D}$ we need some care. Again, in this case, we may assume $u = v$ on ∂D and $D = \{u < v\}$. Given $\epsilon > 0$, let $v_\epsilon := \max[v - \epsilon, u]$ and note that $v_\epsilon = u$ near ∂D. We use this to show

$$\int_D dd^c v_\epsilon = \int_D dd^c u. \tag{1.6}$$

To this end, let $\phi \in C_0^\infty(D)$ with $\phi \equiv 1$ on a neighborhood of the closure of $\{z \in D : v_\epsilon(z) > u(z)\}$. Then

$$\int_D \phi dd^c v_\epsilon = \int_D v_\epsilon dd^c \phi \text{ (why?)}$$

$$= \int_D u dd^c \phi = \int_D \phi dd^c u$$

(note $u = v_\epsilon$ on the support of $dd^c\phi$). This proves (1.6).

Now we use the fact that v_ϵ increase to v so that $dd^c v_\epsilon \to dd^c v$ as positive measures in D. Take $\psi \in C_0(D)$ with $0 \leq \psi \leq 1$ and observe that

$$\int_D \psi dd^c v = \lim_{\epsilon\to 0} \int_D \psi dd^c v_\epsilon \leq \lim_{\epsilon\to 0} \int_D dd^c v_\epsilon = \int_D dd^c u,$$

the last equality by (1.6). This holds for any such ψ; hence

$$\int_D dd^c v \leq \int_D dd^c u.$$

The general case requires more work. We will need this to prove Proposition 2.16 in the next chapter but there we will give an independent proof.

□

Note this result says that *harmonic* functions have "minimal" Laplacian (indeed, 0!) among shm functions. In section 7, we discuss an analogue of this in $\mathbb{C}^N,\ N>1$ where "dd^c" is replaced by the *complex Monge-Ampère operator*, "$(dd^c\ \cdot)^N$." Using Proposition 1.13 we can prove a type of domination principle for subharmonic functions, which will also have an analogue in $\mathbb{C}^N,\ N>1$.

Proposition 1.15. *Let u,v be shm and bounded in a bounded domain $D\subset\mathbb{C}$. Suppose $\liminf_{z\to\zeta}[v(z)-u(z)]\geq 0$ for all $\zeta\in\partial D$ and assume that*

$$dd^c u \geq dd^c v \ \textit{in}\ D.$$

Then $v\geq u$ in D.

Remark 1.16. The hypothesis $dd^cu\geq dd^cv$ means simply that $\Delta(u-v)$ is a positive distribution (and hence a positive measure) in D. Note if v were continuous this simply says that $u-v$ is subharmonic in D; in this case, the result is simply a consequence of Proposition 1.3 (2) (maximum principle for shm functions).

Proof. Assume not, i.e., suppose $\{z\in D: u(z)>v(z)\}\neq\emptyset$. We can choose $\epsilon,\delta>0$ small so that we have

$$u(z)+\epsilon|z|^2-\delta<u(z) \text{ in } D,$$

hence

$$\liminf_{z\to\zeta}[v(z)-(u(z)+\epsilon|z|^2-\delta)]\geq 0 \text{ for all } \zeta\in\partial D,$$

and so that

$$S:=\{z\in D: u(z)+\epsilon|z|^2-\delta>v(z)\}\neq\emptyset.$$

If u were continuous, S is open; in the general case, S still has positive Lebesgue measure by Corollary 1.5. By Proposition 1.13

$$\int_S dd^c(u+\epsilon|z|^2-\delta)\leq\int_S dd^cv.$$

By hypothesis, $\int_S dd^c v \leq \int_S dd^c u$. On the other hand, since S has positive Lebesgue measure, $\int_S dd^c|z|^2 > 0$ and

$$\int_S dd^c(u + \epsilon|z|^2 - \delta) = \int_S dd^c u + \epsilon \int_S dd^c|z|^2 > \int_S dd^c u,$$

a contradiction. $\square$

Exercises.

(1) Let $\rho(z) = |z|^2 - 1$. Show that, on the unit circle $T = \{z = e^{i\theta} : \theta \in [0, 2\pi]\}$, $d^c\rho = 2d\theta$ and, writing $d^c\rho = adx + bdy$, show that $a = -2y$ and $b = 2x$. In particular, the coefficients $< a, b >=< -2y, 2x >$ give a tangent vector to T at each point. More generally, if $D = \{z \in \mathbb{C} : \rho(z) < 0\}$ is a bounded domain with C^1 boundary where ρ is a C^1 function on a neighborhood of $\overline{D}$ and $\nabla\rho \neq 0$ on ∂D, then the coefficient functions of $d^c\rho$ at $p \in \partial D$ define a tangent vector to ∂D at p and $d^c\rho = \frac{\partial \rho}{\partial n} ds$ with $\frac{\partial \rho}{\partial n} \geq 0$ on ∂D.

(2) Verify that if $u \in C^2(D)$ then $dd^c u = \Delta u dx \wedge dy$ in D.

(3) Suppose $u : D \to \mathbb{R}$ is usc; i.e., for each $a \in \mathbb{R}$, the set $\{z \in D : u(z) < a\}$ is open. Show that
   (a) For each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) \leq u(z)$ (this is equivalent to usc of u in D).
   (b) For each $K \subset D$ compact, $M := \sup_{z \in K} u(z) < \infty$ and there exists $z_0 \in K$ with $u(z_0) = M$.

(4) Use part (a) of the previous exercise and the subaveraging property to show that if u is shm in D, then for each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) = u(z)$.

(5) An exercise on convolutions on $\mathbb{R}$:
   (a) Let $f(x) = e^{-x^2}$ and $g(x) = e^{-2x^2}$, Compute $f * g$. (*Hint: You may use the fact that* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)
   (b) More generally, let $f_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ for $t > 0$, Prove that this family of functions acts as a *one-parameter subgroup* in the sense that, for $s, t > 0$

$$f_t * f_s = f_{t+s}.$$

(6) *Gluing shm functions.* Let u, v be shm in open sets U, V where $U \subset V$ and assume that $\limsup_{\zeta \to z} u(\zeta) \leq v(z)$ for $z \in V \cap \partial U$.

Show that the function w defined to be $w = \max(u, v)$ in U and $w = v$ in $V \setminus U$ is shm in V.

(7) In this exercise, you will show that a shm function $u \not\equiv -\infty$ on a domain D is locally integrable on D.
   (a) Verify that it suffices to show for all $z \in D$ there exists $r = r(z) > 0$ with $\int_{B(z,r)} |u(\zeta)|dm(\zeta) < +\infty$.
   (b) Let P denote the set of points $z \in D$ with this property. Show P is both open and closed.
   (c) Show that $u = -\infty$ on $D \setminus P$ to conclude the proof (why?).

(8) Prove that if $u : D \to \mathbb{R}$ is shm on the domain D, then

$$P := \{z \in D : u(z) = -\infty\}$$

is a G_δ−set, i.e., a countable intersection of open sets.

(9) Use the fact mentioned that for a function $u \not\equiv c$ shm in a ball $B(0, R)$, the mean value over circles,

$$r \to M_u(r) := \frac{1}{2\pi} \int_o^{2\pi} u(r(e^{i\theta})d\theta$$

is a convex increasing function of $\log r$, to show that if u is shm in $\mathbb{C}$ and $u(z) = o(\log |z|)$ as $|z| \to \infty$, then u must be constant. Thus functions in the class $L(\mathbb{C})$ are of "minimal" growth.

(10) Show that if $u(z) = u(|z|)$ is a radial function which is harmonic in an annulus $A = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ where $r_1 > 0$ and $r_2 \leq +\infty$, then u is of the form

$$u(z) = a + b \log |z|$$

for some $a, b \in \mathbb{R}$. (*Hint: Write Δu in polar coordinates*).

(11) Verify the claims in the last sentence of Proposition 1.10.

(12) (*Optional*). For those unfamiliar with regularizing kernels, we mention and leave as exercises the following general results for $u : D \to \mathbb{R}$.
   (a) If $u \in C(D)$, on any compact subset $K \subset D$ we have $u_j = u * \chi_{1/j} \to u$ uniformly on K as $j \to \infty$.
   (b) If $u \in L^p_{loc}(D)$ with $1 \leq p < \infty$, we have $u_j \to u$ in $L^p_{loc}(D)$.

2. Logarithmic energy, transfinite diameter and applications.

Now let $K \subset \mathbb{C}$ be compact and let $\mathcal{M}(K)$ denote the convex set of probability measures on K. For $\mu \in \mathcal{M}(K)$ define the *logarithmic energy*

$$I(\mu) := \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta) = \int_K p_\mu(z) d\mu(z).$$

Consider the *energy minimization problem*: minimize $I(\mu)$ over all $\mu \in \mathcal{M}(K)$. We remark that you've likely seen a (real) three-dimensional version of an analogous problem in *Newtonian* potential theory: thinking in terms of electrostatics, given a compact set K (conductor) in $\mathbb{R}^3$, we want to minimize the Newtonian potential energy

$$N(\mu) := \int_K \int_K \frac{1}{|\mathbf{x}-\mathbf{y}|} d\mu(\mathbf{x}) d\mu(\mathbf{y})$$

over all probability measures (positive charges of total charge one) on K. The difference between the formulas for $I(\mu)$ in $\mathbb{C} = \mathbb{R}^2$ and $N(\mu)$ in $\mathbb{R}^3$ is explained by the fact that whereas $\frac{1}{2\pi} \log |z|$ is a fundamental solution of the Laplacian Δ in two (real) dimensions as we saw in Proposition 1.8, up to a dimensional constant, $E(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$ is a fundamental solution of the Laplacian Δ in three (real) dimensions.

Returning to our problem: let $K \subset \mathbb{C}$ be compact and minimize $I(\mu)$ over all $\mu \in \mathcal{M}(K)$.

Proposition 2.1. *Either* $\inf_{\mu \in \mathcal{M}(K)} I(\mu) =: I(\mu_K) < +\infty$ *for a unique* $\mu_K \in \mathcal{M}(K)$ *or else* $I(\mu) = +\infty$ *for all* $\mu \in \mathcal{M}(K)$.

Remark 2.2. In the case where $I(\mu) = +\infty$ for all $\mu \in \mathcal{M}(K)$, we say K is *polar*. We will define polarity for general (not necessarily compact) subsets of $\mathbb{C}$ later.

Proof. The *existence* of an energy-minimizing measure $\mu_K \in \mathcal{M}(K)$ is standard: let $M := \inf_{\mu \in \mathcal{M}(K)} I(\mu)$ and take a sequence $\{\mu_n\} \in \mathcal{M}(K)$ with $\lim_{n\to\infty} I(\mu_n) = M$. There exists a subsequence, which we still label as $\{\mu_n\}$ for simplicity, which converges weak-* to a measure $\mu \in \mathcal{M}(K)$ (why?) and thus by definition, $I(\mu) \geq M$. We claim that

$$\liminf_{n\to\infty} I(\mu_n) \geq I(\mu). \tag{2.1}$$

Given (2.1), we have $I(\mu) \leq \liminf_{n\to\infty} I(\mu_n) = M$ and hence $I(\mu) = M$. The proof of (2.1), which is left to the exercises, follows from weak-*

convergence of $\mu_n \times \mu_n$ to $\mu \times \mu$ and lowersemicontinuity of $z \to \log \frac{1}{|z-\zeta|}$. Uniqueness will follow from Corollary 2.4. □

We verify *uniqueness* by proving a convexity property of the function $\mu \to I(\mu)$. We state the key element (cf., [32] Lemma I.1.8).

Proposition 2.3. *For μ a signed measure with compact support and total mass 0; i.e., $\int_{\mathbb{C}} d\mu = 0$, $I(\mu) \geq 0$ with equality if and only if μ is the zero measure.*

Proof. We only provide a sketch. The key idea is to show

$$I(\mu) = \frac{1}{2\pi} \int_{\mathbb{C}} \Big(\int \frac{1}{|z-t|} d\mu(z) \Big)^2 dm(t) \tag{2.2}$$

which shows $I(\mu) \geq 0$. For the second part, if $I(\mu) = 0$ we then have $F(t) := \int \frac{1}{|z-t|} d\mu(z) = 0$ for a.e. t; since $F(t)$ is continuous for $|t|$ large (μ has compact support) we have $F(t) = 0$ for such t. Expanding $\frac{1}{|z-t|}$ as a power series in $z, \overline{z}$ for $t = Re^{\imath\phi}$ fixed, R large, using $F(t) = 0$ one can show that $\int z^m \overline{z}^{m+k} d\mu(z) = 0$ for all $m, k \geq 0$. Conjugating, we see that, indeed, $\int z^m \overline{z}^j d\mu(z) = 0$ for all $m, j \geq 0$ which shows $\mu = 0$.

To verify (2.2), given z_1, z_2 distinct points in $\text{supp}(\mu)$, we let

$$J_R(z_1, z_2) := \frac{1}{2\pi} \int_{|t| \leq R} \frac{1}{|t-z_1||t-z_2|} dm(t)$$

and we show

$$J_R(z_1, z_2) = \log R - \log|z_1 - z_2| + C + 0(1/R). \tag{2.3}$$

Assuming (2.3), we integrate $J_R(z_1, z_2)$ with respect to $d\mu(z_1)d\mu(z_2)$, and, using $\mu(\mathbb{C}) = 0$, we obtain

$$I(\mu) = \int \int \log \frac{1}{|z_1 - z_2|} d\mu(z_1) d\mu(z_2)$$

$$= \int \int \Big(\frac{1}{2\pi} \int_{|t| \leq R} \frac{1}{|t-z_1||t-z_2|} dm(t) \Big) d\mu(z_1) d\mu(z_2) + 0(1/R)$$

$$= \frac{1}{2\pi} \int_{|t| \leq R} \Big(\int \frac{1}{|z-t|} d\mu(z) \Big) dm(t) + 0(1/R).$$

Letting $R \to \infty$ gives (2.2).

Finally, we verify (2.3). Replacing t by $t + z_2$ yields

$$J_R(z_1, z_2) := \frac{1}{2\pi} \int_{|t| \leq R} \frac{1}{|t - z_1 + z_2||t|} dm(t) + 0(1/R)$$

$$= \frac{1}{2\pi}\int_0^R \int_{-\pi}^{\pi} \frac{1}{|re^{i\theta} - z_1 + z_2|} d\theta dr + 0(1/R)$$

$$= \frac{1}{2\pi}\int_0^R \int_{-\pi}^{\pi} \frac{1}{\big|re^{i\theta} - |z_1 - z_2|\big|} d\theta dr + 0(1/R) \text{ (by symmetry)}$$

$$= \frac{1}{2\pi}\int_0^{R/|z_1-z_2|} \int_{-\pi}^{\pi} \frac{1}{|1 - ue^{i\theta}|} d\theta du + 0(1/R)$$

$$= c + \int_2^{R/|z_1-z_2|} \big(1/u + 0(1/u^2)\big) du + 0(1/R)$$

$$= \log R - \log|z_1 - z_2| + C + 0(1/R).$$

□

Corollary 2.4. *For a compact set K, the functional $\mu \to I(\mu)$ is convex on $\mathcal{M}(K)$. Hence if $\inf_{\mu\in\mathcal{M}(K)} I(\mu) := M < +\infty$ and if $\mu_1, \mu_2 \in \mathcal{M}(K)$ satisfy $I(\mu_1) = I(\mu_2) = M$, then $\mu_1 = \mu_2$.*

Proof. To verify the convexity, it suffices to show that for any $\mu_1, \mu_2 \in \mathcal{M}(K)$,

$$I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) \leq \frac{1}{2}I(\mu_1) + \frac{1}{2}I(\mu_2) \tag{2.4}$$

(midpoint convexity) since $\mu \to I(\mu)$ is uppersemicontinuous (exercise). In any case, we only utilize (2.4). Clearly we need only consider the case where $I(\mu_1), I(\mu_2) < \infty$. We introduce the temporary notation

$$< \mu, \nu >= \int_{\mathbb{C}} p_\mu d\nu = \int_{\mathbb{C}} p_\nu d\mu.$$

Note that for any $c \in \mathbb{R}$,

$$I(c\mu) = c^2 I(\mu). \tag{2.5}$$

Now

$$I(\mu_1 + \mu_2) = I(\mu_1) + I(\mu_2) + 2 < \mu_1, \mu_2 > \tag{2.6}$$

and

$$I(\mu_1 - \mu_2) = I(\mu_1) + I(\mu_2) - 2 < \mu_1, \mu_2 >\geq 0 \tag{2.7}$$

by the Proposition 2.3. Thus

$$2 < \mu_1, \mu_2 >\leq I(\mu_1) + I(\mu_2);$$

plugging this into (2.6) gives

$$I(\mu_1 + \mu_2) \leq 2[I(\mu_1) + I(\mu_2)].$$

Replacing μ_1, μ_2 by $\mu_1/2, \mu_2/2$ and using (2.5) gives (2.4).

For the uniqueness of the energy minimizing measure, if $I(\mu_1) = I(\mu_2) = M$, by (2.4), $I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) \leq M$ and hence, since $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 \in \mathcal{M}(K)$, we have $I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) = M$. From (2.6), (2.5) and (2.7) we have

$$I(\mu_1 - \mu_2) = 2[I(\mu_1) + I(\mu_2)] - I(\mu_1 + \mu_2) = 0.$$

But $I(\mu_1 - \mu_2) \geq 0$ from (2.7) and the result follows from Proposition 2.3. □

We will give a characterization of the energy-minimizing measure μ_K for compact sets K with $\inf_{\mu \in \mathcal{M}(K)} I(\mu) < +\infty$ in Theorem 2.15. First, we show that the energy minimization problem is related to the following discretized version: for each $n = 1, 2, ...$

$$\delta_n(K) := \max_{z_0,\ldots,z_n \in K} \prod_{j<k} |z_j - z_k|^{1/\binom{n+1}{2}}$$

is called the $n-th$ order diameter of K. With this notation, $\delta_1(K) = \max_{z_0,z_1 \in K} |z_0 - z_1|$ is the "ordinary" diameter of K. Note that

$$VDM(z_0, ..., z_n) = \det[z_i^j]_{i,j=0,1,\ldots,n} = \prod_{j<k}(z_j - z_k)$$

$$= \det \begin{bmatrix} 1 & z_0 & \ldots & z_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \ldots & z_n^n \end{bmatrix}$$

is a classical Vandermonde determinant; the basis monomials $1, z, ..., z^n$ for the space of polynomials of degree at most n are evaluated at the points $z_0, ..., z_n$.

If, for example, $\lambda_0, \lambda_1, \lambda_2 \in K$ are points which achieve $\delta_2(K)$, we have

$$[\delta_2(K)]^3 = |\lambda_0 - \lambda_1| \cdot |\lambda_1 - \lambda_2| \cdot |\lambda_0 - \lambda_2| \leq \delta_1(K)^3$$

so that $\delta_2(K) \leq \delta_1(K)$. More generally, the sequence of numbers $\{\delta_n(K)\}$ is decreasing (exercise 7) and hence the limit

$$\lim_{n\to\infty} \big[\max_{\lambda_i \in K} |VDM(\lambda_0, ..., \lambda_n)\big]^{1/\binom{n+1}{2}} := \delta(K) \tag{2.8}$$

exists and is called the transfinite diameter of K.

Points $\lambda_0, ..., \lambda_n \in K$ for which

$$|VDM(\lambda_0, ..., \lambda_n)| = |\det \begin{bmatrix} 1 & \lambda_0 & \dots & \lambda_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^n \end{bmatrix}|$$

is maximal are called *Fekete points of order* n. We call

$$F_{n+1}(z) := \prod_{j=0}^{n}(z - \lambda_j)$$

a *Fekete polynomial of order* n (but degree $n+1$!) for K. We make the following observation for future use.

Lemma 2.5. $||F_{n+1}||_K^{1/n+1} \leq \delta_n(K)$.

Proof. For any $z \in K$, $z, \lambda_0, ..., \lambda_n$ are a set of $n+2$ points in K, hence

$$\prod_{j=0}^{n}|z - \lambda_j| \cdot \prod_{j<k}|\lambda_j - \lambda_k| = |F_{n+1}(z)| \cdot \delta_n(K)^{\binom{n+1}{2}}$$

$$\leq \delta_{n+1}(K)^{\binom{n+2}{2}} \leq \delta_n(K)^{\binom{n+2}{2}}$$

and the result follows by dividing.

□

The quantity $\delta(K)$ in (2.8) coincides with $e^{-I(\mu_K)}$ when $\delta(K) > 0$.

Proposition 2.6. *For $K \subset \mathbb{C}$ compact with $\delta(K) > 0$,*

$$e^{-I(\mu_K)} = \delta(K).$$

Proof. To show

$$e^{-I(\mu_K)} \leq \delta(K), \tag{2.9}$$

we begin by forming the function

$$F_n(z_0, ..., z_n) := \sum_{0 \leq i < j \leq n} \log \frac{1}{|z_i - z_j|}$$

on K^{n+1} and we observe that for Fekete points $\lambda_0, ..., \lambda_n$ of order n for K,

$$F_n(\lambda_0, ..., \lambda_n) = \binom{n+1}{2} \log \frac{1}{\delta_n(K)} = \min_{z_0, ..., z_n \in K} F_n(z_0, ..., z_n).$$

Thus we have

$$\binom{n+1}{2} I(\mu_K) = \int_K \cdots \int_K F_n(z_0, ..., z_n) d\mu_K(z_0) \cdots d\mu_K(z_n)$$
$$\geq \binom{n+1}{2} \log \frac{1}{\delta_n(K)}$$

since μ_K is a probability measure. This gives (2.9).

For the reverse inequality, let μ be any weak-* limit of the sequence of Fekete measures

$$\mu_n := \frac{1}{n+1} \sum_{j=0}^{n} \delta_{\lambda_j}$$

(question: why does such a limit exist?). Then $\mu \in \mathcal{M}(K)$ (why?) and

$$I(\mu) = \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta)$$
$$= \lim_{M\to\infty} \int_K \int_K \min[M, \log \frac{1}{|z-\zeta|}] d\mu(z) d\mu(\zeta)$$
$$= \lim_{M\to\infty} \lim_{n\to\infty} \int_K \int_K \min[M, \log \frac{1}{|z-\zeta|}] d\mu_n(z) d\mu_n(\zeta)$$
$$\leq \lim_{M\to\infty} \lim_{n\to\infty} \Big(\frac{2}{(n+1)^2} \binom{n+1}{2} \log \frac{1}{\delta_n(K)} + \frac{M}{n+1} \Big) = \log \frac{1}{\delta(K)}.$$

Thus from (2.9) we have shown that

$$I(\mu) \leq \log \frac{1}{\delta(K)} \leq I(\mu_K).$$

But $I(\mu_K) = \inf_{\nu \in \mathcal{M}(K)} I(\nu)$ and the proposition is proved. $\square$

As an example, for the unit circle $T = \{z : |z| = 1\}$, clearly the $(n+1)-$st roots of unity $1, \omega := e^{2\pi i/(n+1)}, \omega^2, ..., \omega^n$ or any rotation of these points forms a set of Fekete points of order n; and the weak-* limit of these Fekete measures is normalized arclength $d\mu_T := \frac{1}{2\pi} d\theta$. Note that the same conclusions hold for the closed unit disk $\overline{D} := \{z : |z| \leq 1\}$. Indeed, Fekete points for a compact set K always lie on the *outer boundary* $\partial_e K$ of K; i.e., on the boundary of the unbounded component of $\mathbb{C} \setminus K$ (why?). Note $\partial_e K = \partial \widehat{K}$ where

$$\widehat{K} \equiv \{z \in \mathbb{C} : |p(z)| \leq ||p||_K, \ p \text{ polynomial}\}$$

is the *polynomial hull* of K (why?).

Note as a consequence of the uniqueness of the energy minimizing measure μ_K, we have proved that if $\delta(K) > 0$, any sequence of Fekete measures $\{\mu_n\}$ converges weak-* to μ_K (see also Proposition 4.9). Thus the support of μ_K is in the outer boundary of K. Indeed, examination of the proof that any weak-* limit of a sequence $\{\mu_n\}$ of Fekete measures is μ_K shows that the same conclusion holds if $\{\mu_n = \frac{1}{n+1}\sum_{j=0}^n \delta_{z_j^{(n)}}\}$ is a sequence of measures associated to an *asymptotically Fekete array* $\{z_j^{(n)},\ j = 0, ..., n\}_{n=1,...} \subset K$: this means

$$\lim_{n\to\infty} |VDM(z_0^{(n)}, ..., z_n^{(n)})|^{1/\binom{n+1}{2}} = \delta(K).$$

Now Proposition 2.6 implies that for $K \subset \mathbb{C}$ compact, if $\delta(K) = 0$, then K is polar. It turns out the converse is true; we take this opportunity to make some more remarks on polar sets.

Definition 2.7. Given a set $E \subset \mathbb{C}$, we say the set E is a *polar* set if $I(\mu) = +\infty$ for every finite Borel measure μ with compact support in E.

It turns out this is equivalent to the following:

Definition 2.8. Given a set $E \subset \mathbb{C}$, we say the set E is polar set if there exists a function u shm, $u \not\equiv -\infty$, with $E \subset \{u(z) = -\infty\}$.

Indeed, this latter notion is equivalent to a *local* notion of polarity: $E \subset \mathbb{C}$ is polar if for all $z \in E$, there exists a neighborhood U of z and $u \in SH(U),\ u \not\equiv -\infty$, with $E \cap U \subset \{z \in U : u(z) = -\infty\}$.

Using the second (equivalent) definition of polar set, from the fact that $u(z) = \log|f(z)|$ is shm if f is holomorphic it follows that any discrete set in $\mathbb{C}$ is polar. We can give a direct proof that any bounded countable set is polar, as follows: let $S = \{a_j\} \subset D$ where D is a disk. Let $M_j := \max_{z\in\overline{D}} \log|z - a_j|$. Fix any point $p \in D \setminus S$, and choose $\epsilon_j > 0$ and sufficiently small so that $\sum_j \epsilon_j < +\infty$ and

$$\sum_j \epsilon_j[\log|p - a_j| - M_j] > -\infty.$$

Then

$$u(z) := \sum_j \epsilon_j[\log|z - a_j| - M_j]$$

is shm in D (why?), $u(a_j) = -\infty$ for all j, and $u \not\equiv -\infty$ since $u(p) > -\infty$.

For a *compact* polar set, we can use Lemma 2.5 to construct a logarithmic potential function which is $-\infty$ on K.

Proposition 2.9. *Let $K \subset \mathbb{C}$ be compact and polar. Then there exists $\mu \in \mathcal{M}(K)$ with $V_\mu = -\infty$ on K.*

Proof. Let $\lambda_0, ..., \lambda_n \in K$ be $n-$Fekete points with associate measure $\mu_n := \frac{1}{n+1}\sum_{j=0}^n \delta_{\lambda_j}$ and Fekete polynomial $F_{n+1}(z) = \prod_{j=0}^n (z-\lambda_j)$. Note that

$$V_{\mu_n}(z) = \frac{1}{n+1}\log|F_{n+1}(z)|.$$

By Lemma 2.5, for $z \in K$,

$$V_{\mu_n}(z) \leq \log \delta_n(K).$$

Since $\delta(K) = 0$, $\log \delta_n(K) \to -\infty$. Take a subsequence $\{\mu_n\}$ such that $V_{\mu_n} \leq -2^n$ on K. Then $\mu := \sum \frac{1}{2^n}\mu_n \in \mathcal{M}(K)$ and for $z \in K$,

$$V_\mu(z) = \sum \frac{1}{2^n} V_{\mu_n}(z) = -\infty.$$

□

Note since V_μ is harmonic outside of K, the above proposition shows that $K = \{z \in \mathbb{C} : V_\mu(z) = -\infty\}$.

Definition 2.10. A polar set $E \subset \mathbb{C}$ is *complete polar* if there exists u subharmonic in a neighborhood of E with $E = \{z : u(z) = -\infty\}$.

You showed in exercise 8 of section 1 that the (polar) set of points where a shm function takes the value $-\infty$ is a G_δ set; a theorem of Deny shows a type of converse: given a polar set P which is a $G_\delta-$set, there exists a shm function u in $\mathbb{C}$ with $P = \{z \in \mathbb{C} : u(z) = -\infty\}$; i.e., the only obstruction to a polar set being complete polar is the obvious topological one.

Our next goal is Frostman's theorem (Theorem 2.15). For this, we will need the following notion.

Definition 2.11. If a property **P** holds on a set S except perhaps for a polar subset of S, we say **P** holds q.e. (quasi-everywhere) on S.

For K compact and not polar, how can we find μ_K (or, equivalently, V_{μ_K} or p_{μ_K} since $\Delta V_{\mu_K} = \frac{1}{2\pi}\mu_K$)? Frostman's theorem will show that $p_{\mu_K} = I(\mu_K)$ q.e. on K. Thus

$$V_{\mu_K} + I(\mu_K) = 0 \text{ q.e. on } K.$$

This function $V_{\mu_K}(z)+I(\mu_K)$ belongs to the class

$$L(\mathbb{C})=\{u \text{ shm on } \mathbb{C},\ u(z)-\log|z|=0(1),\ |z|\to\infty\};$$

indeed, it belongs to the restricted subclass

$$L^+(\mathbb{C}):=\{u\in L(\mathbb{C}): u(z)\geq \log^+|z|+C\}$$

where $C=C(u)$ (clearly we can replace $\log^+|z|$ by $\frac{1}{2}\log(1+|z|^2)$ in this definition). It turns out that $V_{\mu_K}(z)+I(\mu_K)=V_K^*(z)$ where

$$V_K(z):=\sup\{u(z): u\in L(\mathbb{C}),\ u\leq 0 \text{ on } K\}. \tag{2.10}$$

and $V_K^*(z):=\limsup_{\zeta\to z}V_K(\zeta)\in L^+(\mathbb{C})$. Hence

$$\mu_K=\frac{1}{2\pi}\Delta V_K^*=\frac{1}{2\pi}dd^cV_K^*. \tag{2.11}$$

We discuss this "upper envelope" in the next section; and we will see in Section 4 that

$$V_K(z)=\sup\{\frac{1}{deg(p)}\log|p(z)|: ||p||_K:=\sup_K|p|\leq 1\} \tag{2.12}$$

where the supremum is taken over all nonconstant holomorphic polynomials. For the unit circle T we have

$$V_T(z)=\max[\log|z|,0] \text{ and } \mu_T=\frac{1}{2\pi}d\theta.$$

Note that $V_T=V_T^*$ (why?).

How "small" are polar sets? If $u\in SH(D)$ with $u\not\equiv-\infty$, then $u\in L^1_{loc}(D)$. Hence polar sets have (two-dimensional) Lebesgue measure zero. Indeed, if $E\subset\mathbb{R}$ is (Borel and) polar, then E has one-dimensional Lebesgue measure zero. These facts follow from the the next result, together with a direct calculation that if μ is two-dimensional Lebesgue measure on a disk $B(0,r),\ r>0$ or one-dimensional Lebesgue measure on an interval $[-a,a],\ a>0$, then $I(\mu)<\infty$.

Proposition 2.12. *If μ is a finite Borel measure with compact support and $I(\mu)<\infty$, then $\mu(E)=0$ for each Borel polar set E.*

Proof. If E is a Borel set with $\mu(E)>0$, we show E is not polar. To this end, take $K\subset E$ compact with $\mu(K)>0$.The measure $\tilde{\mu}:=\mu|_K$ is a finite Borel measure with compact support. Setting $d:=$diam(suppμ), we have

$$I(\tilde{\mu})=\int_K\int_K\log\frac{d}{|z-\zeta|}d\mu(z)d\mu(\zeta)-\mu(K)^2\log d$$

$$\leq \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{d}{|z-\zeta|} d\mu(z) d\mu(\zeta) - \mu(K)^2 \log d$$
$$= I(\mu) + \mu(\mathbb{C})^2 \log d - \mu(K)^2 \log d < \infty.$$

□

Remark 2.13. The middle-thirds Cantor set is not polar so one-dimensional Lebesgue measure zero does not characterize polar subsets of $\mathbb{R}$.

Corollary 2.14. *A countable union of Borel polar sets is polar.*

Proof. Let $E = \cup_j E_j$ where E_j are Borel polar sets and let μ be a finite measure with compact support in E. If $I(\mu) < \infty$, by Proposition 2.12 $\mu(E_j) = 0$ for all j which implies $\mu(E) = 0$.

□

We come to the characterization of the equilibrium measure μ_K for a nonpolar compact set K; Frostman's theorem.

Theorem 2.15. **[Frostman]** *Let $K \subset \mathbb{C}$ with $I(\mu_K) < +\infty$. Then*
(1) *$p_{\mu_K}(z) \leq I(\mu_K)$ for all $z \in \mathbb{C}$; and*
(2) *$p_{\mu_K}(z) = I(\mu_K)$ q.e. on K.*

Proof. For each $n = 1, 2, \ldots$ let

$$K_n := \{z \in K : p_{\mu_K}(z) \leq I(\mu_K) - 1/n\} \text{ and}$$

$$L_n := \{z \in \text{supp}\mu_K : p_{\mu_K}(z) > I(\mu_K) + 1/n\}.$$

We will verify two items:
(1) K_n is polar for each $n = 1, 2, \ldots$ and
(2) $L_n = \emptyset$ for each $n = 1, 2, \ldots$

Given these two items, the second one implies that $p_{\mu_K}(z) \leq I(\mu_K)$ on suppμ_K and hence on $\mathbb{C}$ by Proposition 1.10. This is (1) of the theorem. Next, setting $E := \cup_{n=1}^{\infty} K_n$, the first item and Corollary 2.14 imply that E is a polar set; moreover we have $p_{\mu_K}(z) = I(\mu_K)$ on $K \setminus E$.

We prove item (1) by contradiction. Thus we suppose K_n is not polar for some n so we can find $\mu \in \mathcal{M}(K_n)$ with $I(\mu) < +\infty$. We have $I(\mu_K) = \int_K p_{\mu_K} d\mu_K$ so that we can find $z_0 \in$ supp(μ_K) with $p_{\mu_K}(z_0) \geq I(\mu_K)$; by lsc of p_{μ_K}, there exists $r > 0$ with $p_{\mu_K} > I(\mu_K) - \frac{1}{2n}$ on $\overline{B}(z_0, r)$. Thus $K_n \cap \overline{B}(z_0, r) = \emptyset$; also, we note that $a := \mu_K(\overline{B}(z_0, r)) >$

0 since $z_0 \in$ supp(μ_K). We next define a *signed* measure σ on K by setting

$$\sigma = \mu \text{ on } K_n; \ \sigma = -\mu_K/a \text{ on } \overline{B}(z_0, r).$$

Since $I(\mu)$, $I(\mu_K) < +\infty$, clearly $I(|\sigma|) < +\infty$. For each $t \in (0, a)$, the measure $\mu_t := \mu_K + t\sigma$ is *positive* and, indeed, $\mu_t \in \mathcal{M}(K)$ for such t (why?). We estimate the difference $I(\mu_t) - I(\mu_K)$:

$$I(\mu_t) - I(\mu_K) = I(\mu_K + t\sigma) - I(\mu_K)$$

$$= 2t \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu_K(\zeta) d\sigma(z) + t^2 I(\sigma)$$

$$= 2t \int_K p_{\mu_K}(z) d\sigma(z) + 0(t^2)$$

$$= 2t\Big(\int_{K_n} p_{\mu_K}(z) d\mu(z) - \frac{1}{a} \int_{\overline{B}(z_0,r)} p_{\mu_K}(z) d\mu_K(z) + 0(t)\Big)$$

$$\leq 2t\Big([I(\mu_K) - 1/n] - [I(\mu_K) - \frac{1}{2n}] + 0(t)\Big).$$

Thus $I(\mu_t) < I(\mu_K)$ for t sufficiently small, contradicting the minimality of $I(\mu_K)$.

We prove item (2) by contradiction. Suppose $L_n \neq \emptyset$ for some n and take $z_0 \in L_n$; hence $p_{\mu_K}(z_0) > I(\mu_K) + 1/n$. By lsc of p_{μ_K}, there exists $r > 0$ with $p_{\mu_K}(z) > I(\mu_K) + 1/n$ on $\overline{B}(z_0, r)$. Also, since $z_0 \in$ suppμ_K, $m := \mu_K(\overline{B}(z_0, r)) > 0$. By item (1) and Proposition 2.12, $\mu_K(K_n) = 0$ for each n so that $p_{\mu_K} \geq I(\mu_K)$ μ_K-a.e. on K. Thus

$$I(\mu_K) = \int_K p_{\mu_K} d\mu_K = \int_{\overline{B}(z_0,r)} p_{\mu_K} d\mu_K + \int_{K \setminus \overline{B}(z_0,r)} p_{\mu_K} d\mu_K$$

$$\geq (I(\mu_K) + 1/n)m + I(\mu_K)(1-m) > I(\mu_K)$$

which is a contradiction.

□

There is an important result that will be useful in the weighted setting and which will generalize to the several complex variable setting. We will refer to it as a *global domination principle*; we will use it in the next section together with Frostman's theorem to relate V_K^* with p_{μ_K}.

Proposition 2.16. *Let $u \in L(\mathbb{C})$ and $v \in L^+(\mathbb{C})$ and suppose $u \leq v$ a.e.-$dd^c v$. Then $u \leq v$ on $\mathbb{C}$.*

Proof. We first give the proof in case u, v are continuous and afterwards give an independent proof of a case needed in the next section. Suppose the result is false; i.e., there exists $z_0 \in \mathbb{C}$ with $u(z_0) > v(z_0)$. Since $v \in L^+(\mathbb{C})$, by adding a constant to u, v we may assume $v(z) \geq \frac{1}{2}\log(1+|z|^2)$ in $\mathbb{C}$. By exercise 5 below, $\Delta\big[\frac{1}{2}\log(1+|z|^2)\big] > 0$ on $\mathbb{C}$. Fix $\delta,\ \epsilon > 0$ with $\delta < \epsilon/2$ in such a way that the set

$$S := \{z \in \mathbb{C} : u(z) + \frac{\delta}{2}\log(1+|z|^2) > (1+\epsilon)v(z)\}$$

contains z_0. In our setting, S is open; in the general case, by Corollary 1.5, S has positive Lebesgue measure. Moreover, since $\delta < \epsilon$ and $v \geq \frac{1}{2}\log(1+|z|^2)$, S is bounded. By Proposition 1.13, we conclude that

$$\int_S dd^c[u(z) + \frac{\delta}{2}\log(1+|z|^2)] \leq \int_S dd^c(1+\epsilon)v(z).$$

But $\int_S dd^c \frac{\delta}{2}\log(1+|z|^2) > 0$ since S has positive Lebesgue measure, so

$$(1+\epsilon)\int_S dd^c v > 0.$$

By hypothesis, for a.e.-$dd^c v$ points in $\mathrm{supp}(dd^c v) \cap S$ (which is not empty since $\int_S dd^c v > 0$), we have

$$(1+\epsilon)v(z) \leq u(z) + \frac{\delta}{2}\log(1+|z|^2) \leq v(z) + \frac{\delta}{2}\log(1+|z|^2),$$

i.e., $v(z) \leq \frac{1}{4}\log(1+|z|^2)$ since $\delta < \epsilon/2$. This contradicts the normalization $v \geq \frac{1}{2}\log(1+|z|^2)$. $\square$

We reformulate this in terms of logarithmic potentials and give a proof in this case. To make the connection clear, we mention without proof that if $v \in L^+(\mathbb{C})$ and $\mu = \frac{1}{2\pi}\Delta v$ has compact support, then $I(\mu) < \infty$. Moreover, if $u \in L(\mathbb{C})$ and $\nu = \frac{1}{2\pi}\Delta u$, then $\nu(\mathbb{C}) \leq \mu(\mathbb{C})$. These follow from the Jensen type formula: for $w \in SH(B(0,R'))$, if $R < R'$,

$$M_w(R) := \frac{1}{2\pi}\int_0^{2\pi} w(Re^{i\theta})d\theta = w(0) + \int_0^R \frac{n(t)}{t}dt \tag{2.13}$$

where $n(t) = \mu(B(0,t))$ and $\mu = \frac{1}{2\pi}\Delta w$.

Proposition 2.17. *Let μ, ν be finite, positive measures with compact support with $\nu(\mathbb{C}) \leq \mu(\mathbb{C})$ and $I(\mu) < \infty$. Suppose for some constant c we have*
$$p_\mu(z) \leq p_\nu(z) + c \ \mu - a.e.$$
Then $p_\mu(z) \leq p_\nu(z) + c$ for all $z \in \mathbb{C}$.

Proof. We consider the superharmonic function
$$u(z) := \min[p_\nu(z) + c, p_\mu(z)].$$
By the Riesz decomposition theorem, for any $r > 0$, on $D_r = \{z : |z| < r\}$ we have
$$u(z) = h_r(z) + \int_{D_r} \log \frac{1}{|z-t|} d\lambda(t), \ z \in D_r$$
where h_r is harmonic on D_r and λ is a positive Borel measure on $\mathbb{C}$ (given distributionally as $-\frac{1}{2\pi}\Delta u$). We claim that λ has compact support. To see this, note that since
$$p_\nu(z) = \nu(\mathbb{C}) \log \frac{1}{|z|} + 0(1/|z|), \ p_\mu(z) = \mu(\mathbb{C}) \log \frac{1}{|z|} + 0(1/|z|),$$
$\nu(\mathbb{C}) \leq \mu(\mathbb{C})$ implies $u(z) = p_\mu(z)$ for $|z|$ large, say $|z| > R$, and hence
$$p_\lambda(z) := \int_{\mathbb{C}} \log \frac{1}{|z-t|} d\lambda(t)$$
is harmonic for $|z| > R$. Thus $h_r|_{D_s} = h_s$ for $s < r$ and $h(z) := \lim_{r\to\infty} h_r(z)$ is well-defined and harmonic on $\mathbb{C}$. Hence λ has compact support and
$$u(z) = h(z) + p_\lambda(z), \ z \in \mathbb{C}.$$

Next, we claim that $\lambda(\mathbb{C}) = \mu(\mathbb{C})$. For $|z| > R$, since $u(z) = p_\mu(z)$, we have
$$\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty} [u(z) - p_\lambda(z)] = \lim_{|z|\to\infty} [p_\mu(z) - p_\lambda(z)]$$
$$= \lim_{|z|\to\infty} \int_{\mathbb{C}} \log \frac{1}{|z-t|} [d\mu(t) - d\lambda(t)].$$
If $\lambda(\mathbb{C}) < \mu(\mathbb{C})$, this limit equals $-\infty$; by the maximum principle, $h \equiv -\infty$, a contradiction. Similarly, if $\lambda(\mathbb{C}) > \mu(\mathbb{C})$, the above limit equals ∞ and applying the maximum principle to $-h$, we get a contradiction. We conclude that $\lambda(\mathbb{C}) = \mu(\mathbb{C})$ and $h \equiv 0$ so that
$$u(z) = p_\lambda(z) \text{ for } z \in \mathbb{C}.$$

Now note that $u = p_\lambda \leq p_\mu$ on $\mathbb{C}$ by definition of u. This gives

$$I(\lambda) = \int p_\lambda d\lambda \leq \int p_\mu d\lambda = \int p_\lambda d\mu$$
$$\leq \int p_\mu d\mu = I(\mu) < \infty$$

so that λ has finite energy and compact support. We now show that $I(\mu - \lambda) \leq 0$; appealing to Proposition 2.3, we have $I(\mu - \lambda) \geq 0$ and hence $I(\mu - \lambda) = 0$ from which we conclude, again by Proposition 2.3, that $\mu = \lambda$. Hence

$$p_\mu = p_\lambda = u \leq p_\nu + c, \ z \in \mathbb{C}.$$

The verification that $I(\mu-\lambda) \leq 0$ is a calculation which (finally!) uses the hypothesis that $p_\mu(z) \leq p_\nu(z) + c \ \mu -$ a.e.; i.e., $p_\mu(z) \leq p_\nu(z) + c$ on $\mathbb{C} \setminus E$ where $\mu(E) = 0$. Hence $u = p_\mu = p_\lambda$ on $\mathbb{C} \setminus E$. Then

$$I(\mu - \lambda) = \int [p_\mu - p_\lambda][d\mu - d\lambda] = \int_E [p_\mu - p_\lambda][d\mu - d\lambda]$$
$$= - \int_E [p_\mu - p_\lambda] d\lambda \leq 0$$

since $p_\mu \geq u = p_\lambda$ on $\mathbb{C}$ and $\mu(E) = 0$.

$\square$

Remark 2.18. Note in Proposition 2.16 some hypothesis on v stronger than $v \in L(\mathbb{C})$ is necessary, since, e.g., $u(z) = \log|z|$ and $v(z) = \log|z| + c$ satisfy the hypothesis but not the conclusion if $c < 0$. Also note that in Proposition 2.16 we do not assume $dd^c v$ (Δv) has compact support whereas in Proposition 2.17 we assume both μ and ν have compact support.

As an application of Frostman's theorem, we discuss a classical result of Brolin from complex dynamics (cf., Theorem 6.5.8 of [30]). The set-up begins with a polynomial $p(z)$ of degree $d > 1$ in $\mathbb{C}$. Writing $p^n = p \circ \cdots \circ p$ for the $n-$th iterate of p, the *Fatou set* or *attracting basin of* ∞ is the set

$$F := \{z \in \mathbb{C} : p^n(z) \to \infty \text{ as } n \to \infty\}$$

and the *Julia set* J is the boundary of F. Two standard examples are $p(z) = z^2$ (or $p(z) = z^d$ for any $d > 1$) in which case $F = \{z : |z| > 1\}$ and $J = \{z : |z| = 1\}$; and $p(z) = z^2 - 2$, in which case $F = \{z : z \notin [-2, 2]\}$ and $J = [-2, 2]$.

Note that in the case of $p(z) = z^d$ where $J = \{z : |z| = 1\}$ and $d\mu_J = \frac{1}{2\pi} d\theta$, we have $\text{supp}\mu_J = J$ and $I(\mu_J) = 0$. More generally, for a *monic* polynomial $p(z) = z^d + \cdots$, the Julia set J is nonpolar; $\text{supp}\mu_J = J$; and $I(\mu_J) = 0$. We refer the reader to [30], section 6.5 for verification of these facts. We show that we can recover μ_J via a pre-image process.

Theorem 2.19. **[Brolin]** *Fix $w \in J$ and define the sequence of discrete probability measures $\{\mu_n\}$ on J via*

$$\mu_n = \frac{1}{d^n} \sum_{p^n(z_j)=w} \delta_{z_j}.$$

Then $\mu_n \to \mu_J$ weak-.*

Proof. Note that $w \in J$ implies that $z_j \in J$ if $p^n(z_j) = w$ (exercise). Let

$$V_{\mu_n}(z) = \int_J \log|z - \zeta| d\mu_n(\zeta).$$

Writing $p^n(z) - w = \prod_{j=1}^{d^n} (z - z_j)$, we have

$$V_{\mu_n}(z) = \frac{1}{d^n} \sum_{j=1}^{d^n} \log|z - z_j| = \frac{1}{d^n} \log|p^n(z) - w|.$$

For $z \in J$, the points $\{p^n(z)\}$ and hence $\{p^n(z) - w\}$ remain bounded so we conclude that

$$\limsup_{n\to\infty} V_{\mu_n}(z) \leq 0 \text{ for } z \in J. \tag{2.14}$$

Now if $\{\mu_{n_j}\}$ is a subsequence of $\{\mu_n\}$, since $V_{\mu_J} = -p_{\mu_J} \geq I(\mu_J) = 0$ by Frostman's theorem, from Fatou's lemma and Fubini's theorem we have

$$\int_J [\limsup_{j\to\infty} V_{\mu_{n_j}}(z)] d\mu_J(z) \geq \limsup_{j\to\infty} \int_J V_{\mu_{n_j}}(z) d\mu_J(z)$$

$$= \limsup_{j\to\infty} \int_J V_{\mu_J}(z) d\mu_{n_j}(z) \geq 0.$$

From (2.14), we conclude that

$$\limsup_{j\to\infty} V_{\mu_{n_j}} = 0 \ \mu_J - a.e. \text{ on } J. \tag{2.15}$$

Recall that supp$\mu_J = J$. We use this fact to complete the proof by contradiction: suppose $\mu_n \not\to \mu_J$ weak-*. Then there exists a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$, a function $\phi \in C(J)$, and $\epsilon > 0$ with

$$|\int_J \phi d\mu_{n_j} - \int_J \phi d\mu_J| \geq \epsilon \tag{2.16}$$

for all j. Take a further subsequence, which we still denote by $\{\mu_{n_j}\}$, which converges weak-* to a measure $\mu \in \mathcal{M}(J)$. An argument similar to that used to prove (2.1) shows that

$$\limsup_{j\to\infty} V_{\mu_{n_j}}(z) \leq V_\mu(z) \text{ for } z \in \mathbb{C}.$$

Then (2.15) shows that $V_\mu(z) \geq 0$ μ_J-a.e on J. Since supp$\mu_J = J$ and V_μ is usc, we have $V_\mu(z) \geq 0$ on J. Thus

$$I(\mu) = \int_J [-V_\mu(z)]d\mu(z) \leq 0 = I(\mu_J).$$

By uniqueness of the energy minimizing measure, $\mu = \mu_J$. This contradicts (2.16). $\square$

We end this section with an L^2-version of the transfinite diameter. Let $\mathcal{P}_n$ denote the vector space of holomorphic polynomials of degree at most n. Given $n+1$ points $z_0, ..., z_n$, we make the following observations about $VDM(\lambda_0, ..., \lambda_n)$:

(1) for $j = 0, ..., n$, the map

$$z_j \to VDM(z_0, ..., z_n) = \prod_{j<k}(z_j - z_k) \in \mathcal{P}_n;$$

(2) for $j = 0, ..., n$, the polynomial

$$l_j(z) := \frac{\prod_{k\neq j}(z - z_k)}{\prod_{k\neq j}(z_j - z_k)} = \frac{VDM(z_0, ..., z_{j-1}, z, ..., z_n)}{VDM(z_0, ..., z_n)}$$

takes the value 1 at z_j at 0 at z_k if $k \neq j$.

From (2), for a compact set $K \subset \mathbb{C}$ and $n-$Fekete points $z_0, ..., z_n \in K$, it follows that

$$||l_j||_K = 1, \ j = 0, ..., n.$$

Let ν be a finite measure on K. We say that the pair (K, ν) satisfies a *Bernstein-Markov inequality* for holomorphic polynomials in $\mathbb{C}$ (or simply that ν is a Bernstein-Markov measure for K) if, given $\epsilon > 0$, there exists a constant $\tilde{M} = \tilde{M}(\epsilon)$ such that for all $n = 1, 2, ...$ and all $p_n \in \mathcal{P}_n$

$$||p_n||_K \leq \tilde{M}(1+\epsilon)^n ||p_n||_{L^2(\nu)}. \tag{2.17}$$

Equivalently, for all $p_n \in \mathcal{P}_n$,

$$||p_n||_K \leq M_n ||p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

Thus there is a strong comparability between L^2 and L^∞ norms. A simple example is the measure $d\nu = \frac{1}{2\pi} d\theta$ on $K = S^1 = \{z : |z| = 1\}$ (exercise 8). Note that $d\nu = d\mu_K$ in this case; indeed, for any non-polar compact set it turns out that (K, μ_K) satisfies a Bernstein-Markov inequality. We show that Bernstein-Markov measures exist on *any* compact set.

Proposition 2.20. *Let $K \subset \mathbb{C}$ be compact. There exists a finite measure μ which is a Bernstein-Markov measure on K.*

Proof. If K is a finite set, any measure putting positive mass at each point of K is a Bernstein-Markov measure on K. Suppose K contains infinitely many points. Let $z_0^{(n)}, ..., z_n^{(n)} \in K$ be a set of $n-$Fekete points for K at let $\mu_n := \frac{1}{n+1} \sum_{j=0}^n \delta_{z_j^{(n)}} \in \mathcal{M}(K)$. We show that

$$\mu := \frac{6}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \mu_n \in \mathcal{M}(K)$$

is a Bernstein-Markov measure on K. To this end, if $p_n \in \mathcal{P}_n$, we can write

$$p_n(z) = \sum_{j=0}^n p_n(z_j^{(n)}) l_j^{(n)}(z)$$

where

$$l_j^{(n)}(z) := \frac{VDM(z_0^{(n)}, ..., z_{j-1}^{(n)}, z, ..., z_n^{(n)})}{VDM(z_0^{(n)}, ..., z_n^{(n)})}.$$

From the observation above, $||l_j^{(n)}||_K = 1$ so that

$$||p_n||_K \leq \sum_{j=0}^n |p_n(z_j^{(n)})|.$$

On the other hand,

$$||p_n||_{L^2(\mu)} \geq ||p_n||_{L^1(\mu)} \geq 6\pi^2 \frac{1}{n^2} \int_K |p| d\mu_n$$
$$= 6\pi^2 \frac{1}{n^2} \frac{1}{n+1} \sum_{j=0}^{n} |p_n(z_j^{(n)})| \geq 6\pi^2 \frac{1}{n^2} \frac{1}{n+1} ||p_n||_K.$$

□

For now, we observe (see exercise 9 – use observation (1) above) that one can recover the transfinite diameter $\delta(K)$ in an L^2-fashion with such a measure.

Theorem 2.21. *Let K be compact and let (K, ν) satisfy a Bernstein-Markov inequality for holomorphic polynomials. Then*

$$\lim_{n\to\infty} Z_n^{1/n^2} = \delta(K)$$

where

$$Z_n = Z_n(K, \nu) := \tag{2.18}$$
$$\int_{K^{n+1}} |VDM(\lambda_0, ..., \lambda_n)|^2 d\nu(\lambda_0) \cdots d\nu(\lambda_n).$$

We will see the utility of this result, and generalizations of it, later on. The quantity Z_n is called the $n-$th *free energy* of (K, ν).

Exercises.

(1) Prove (2.1) using weak-* convergence of $\mu_n \times \mu_n$ to $\mu \times \mu$ and lowersemicontinuity of $z \to \log \frac{1}{|z-\zeta|}$. (*Hint: If you have trouble, see the start of the proof of Proposition 6.13 in section 6.*)
(2) Show that if $\{K_n\}$ are compact sets in $\mathbb{C}$ with $K_{n+1} \subset K_n$ for all n, then $\lim_{n\to\infty} I(\mu_{K_n}) = I(\mu_K)$ where $K = \cap_n K_n$. (*Hint: Use (2.1).*)
(3) Verify the claim in the proof of Theorem 2.19 that $w \in J$ implies that $z_j \in J$ if $p^n(z_j) = w$.
(4) Generally Fekete points of order n for a compact set K are not unique. In the case of the interval $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}$, they are unique. Find explicitly Fekete points $z_0 < z_1 < z_2 < z_3$ of order 3 for $[-1, 1]$.
(5) Compute $\Delta\left(\frac{1}{2}\log(1 + |z|^2)\right)$.
(6) Use (2.13) to show if $u \in L^+(\mathbb{C})$ and $\mu = \frac{1}{2\pi}\Delta u$, then $\mu(\mathbb{C}) = 1$.

(7) Verify that $\delta_{n+1}(K) \leq \delta_n(K)$ for $n = 1, 2, ...$ for any compact set $K \subset \mathbb{C}$. Conclude that the limit in (2.8) exists.

(8) Show that $d\nu = \frac{1}{2\pi} d\theta$ is a Bernstein-Markov measure on $K = S^1 = \{z : |z| = 1\}$.

(9) Recalling that the function $VDM(\lambda_0, ..., \lambda_n)$ is a holomorphic polynomial of degree at most n in each variable, prove Theorem 2.21. (*Hint: Apply the Bernstein-Markov property repeatedly; start with* $\lambda_0, ..., \lambda_n$ $n-$*Fekete points for* K.)

(10) *Extra Credit: Polar sets and energy.*
   (a) Find an example of a probability measure μ with compact support such that $I(\mu) < +\infty$ but μ puts no mass on polar sets.
   (b) Prove Proposition 2.16 (or Proposition 2.17) under the weaker hypothesis on μ that μ puts no mass on polar sets (instead of $I(\mu) < +\infty$).

3. Upper envelopes, extremal subharmonic functions and applications.

In the first section, we claimed that for any family $\{v_\alpha\}_{\alpha \in A} \subset SH(D)$ which is uniformly bounded above on any compact subset of D, the function

$$v(z) := \sup_\alpha v_\alpha(z)$$

is "nearly" shm in the sense that the usc regularization

$$v^*(z) := \limsup_{\zeta \to z} v(\zeta)$$

is shm in D. This fact is fairly straightforward (exercise 4). Here, v^* is the smallest usc majorant of v. Note the local uniform boundedness above is obviously needed: take, e.g., $D = \{z : |z| > 1\}$ and $v_n(z) = n \log |z|$. The following simple example shows that the set

$$\{z \in D : v(z) < v^*(z)\},$$

called a *negligible set*, need not be empty: let $D = B(0,1)$, let $\{v_\alpha\} = \{u_n\}$ where $u_n(z) = \frac{1}{n} \log |z|$; then, in $B(0,1)$, clearly $u(z) = \sup_n u_n(z) = 0$ for $0 < |z| < 1$ but $u(0) = -\infty$. Here, $u^*(z) \equiv 0$ and

$$\{z \in B(0,1) : u(z) < u^*(z)\} = \{0\}$$

which is admittedly "small". Indeed, we shall see that negligible sets are polar – and conversely.

As a preliminary step to the converse, we begin with a reason the notion of polarity is important: polar sets are removable sets for certain classes of functions. Recall the Riemann removable singularity theorem: if f is holomorphic in a punctured disk $B \setminus \{p\}$ and $|f|$ is bounded near p, then f can be defined at p to be holomorphic in B. In particular, the same result applies to harmonic functions, and even locally bounded above shm functions. More generally, the "size" of the removable set can be bigger but not too big: it can be a polar set.

Proposition 3.1. *Let u be shm on $D \setminus P$ where D is a bounded domain and P is a polar set. Suppose u is locally bounded above near P. Then u has a unique shm extension to D.*

Proof. We extend u to D by setting

$$u(z) := \limsup_{\zeta \to z,\ \zeta \in D \setminus P} u(\zeta). \tag{3.1}$$

Clearly this extension is usc in D. To see that u is shm in D, take any relatively compact subdomain D' in D and a harmonic function h on $\overline{D}'$ with $u \leq h$ on $\partial D'$. There exists v shm in D with $v = -\infty$ on P. For $\epsilon > 0$, $u - h + \epsilon v$ is shm on $D' \setminus P$ and equals $-\infty$ on $D' \cap P$; hence it is shm on D'. By the maximum principle,

$$u - h + \epsilon v \leq \sup_{\partial D'} \epsilon v \text{ on } D'.$$

Let $\epsilon \to 0$ to conclude $u \leq h$ on $D' \setminus P$. By the definition (3.1) of the extension of u to P and the continuity of h, it follows that $u \leq h$ on P as well.

Uniqueness follows from Corollary 1.5: two shm functions which agree a.e. are identical. □

Corollary 3.2. *Let h be harmonic on $D \setminus P$ where D is a bounded domain and P is a polar set. Suppose $|h|$ is locally bounded near P. Then h has a unique harmonic extension to D.*

Corollary 3.3. *If E is a bounded polar set, then there exists a negligible set N with $E \subset N$.*

Proof. Since E is polar, there exists u shm in a domain D containing E, $u \not\equiv -\infty$, with $E \subset \{z \in D : u(z) = -\infty\}$. On $D' \subset\subset D$ with $E \subset D'$, we can assume $u < 0$ (why?). Now take $\{v_\alpha\} = \{\alpha u\}$ for $0 < \alpha < 1$. Clearly for $z \in D'$ $v(z) := \sup_\alpha v_\alpha(z)$ satisfies

$$v(z) = 0 \text{ if } u(z) \neq -\infty \text{ and } v(z) = -\infty \text{ if } u(z) = -\infty.$$

By Proposition 3.1, $v^* \equiv 0$ and

$$E \subset \{z \in D' : u(z) = -\infty\} = \{z \in D' : v(z) < v^*(z)\}.$$

□

The converse is known as the *Brelot-Cartan Theorem.*

Theorem 3.4. *Let $\{v_\alpha\}_{\alpha \in A} \subset SH(D)$ be uniformly bounded above on compact subsets of D. Then $\{z \in D : v(z) < v^*(z)\}$ is polar.*

Proof. We give the proof in the case A is countable; the general case is left as exercise 3. Let $N := \{z \in D : v(z) < v^*(z)\}$. We can write N as a countable union of (Borel) sets of the form

$$E = E(a, z_0, r) := \{z \in B(z_0, r) : v(z) \leq a < v^*(z)\}$$

where $B := B(z_0, r)$ is a disk in D with a, r rational and $z_0 = x_0 + iy_0$ with x_0, y_0 rational. Since a countable union of Borel polar sets is polar, it suffices to show that such a set E is polar. Suppose not; then we find a compact, non-polar subset $K \subset E$ with equilibrium measure μ_K. Theorem 2.15 shows that $V_{\mu_K} > I(\mu_K)$ on the unbounded component of $\mathbf{C} \setminus K$; since $K \subset B$ we can take C sufficiently large so that the function

$$u(z) := C[V_{\mu_K}(z) - I(\mu_K)] + a = C[-p_{\mu_K}(z) - I(\mu_K)] + a$$

satisfies $\inf_{\partial B} u > \sup_{\partial B} v$ (recall the v_α are uniformly bounded above on compact subsets of D). Now u is harmonic outside K so for each v_α, $v_\alpha - u$ is subharmonic on $B \setminus K$. By construction, we have

$$\limsup_{z \to \zeta} [v_\alpha(z) - u(z)] \leq 0$$

for all $\zeta \in \partial(B \setminus K)$ – this is clear on ∂B and $v \leq a \leq u$ on K (Theorem 2.15 gives $V_{\mu_K}(z) \geq I(\mu_K)$ everywhere). Thus $v \leq u$ on $B \setminus K$. Again, since $v \leq a \leq u$ on K we have $v \leq u$ on all of B. Hence $v^* \leq u$ on B (why?) so that, in particular, $u > a$ on K, contradicting Theorem 2.15. □

Before we return to our general upper envelope constructions, we mention a beautiful and very general result of Choquet: if $\{v_\alpha\}_{\alpha \in A}$ is a family of real-valued functions defined on a separable metric space X which is uniformly bounded above on any compact subset of X, then one can extract a *countable* subfamily $\{u_n\} \subset \{v_\alpha\}$ with the property that

$$\big(\sup_\alpha v_\alpha\big)^* = \big(\sup_n u_n\big)^*.$$

To see this, set $v := \sup_\alpha v_\alpha$ and let $\{B_j\}$ be a countable basis of open sets for X. Then for each j we can take a sequence of points $\{x_{jk}\}_k \subset B_j$ with $\sup_{B_j} v = \sup_k v(x_{jk})$. Now for each pair j, k we can take a sequence of indices a_{jkl} with $v(x_{jk}) = \sup_l v_{a_{jkl}}(x_{jk})$. Now define $u := \sup_{j,k,l} v_{a_{jkl}}$; here we have a countable subfamily. Then

$$\sup_{B_j} u \geq \sup_k u(x_{jk}) \geq \sup_{k,l} v_{a_{jkl}}(x_{jk}) = \sup_k v(x_{jk}) = \sup_{B_j} v.$$

Thus $u^* \geq v^*$ and the conclusion follows.

We remark that if each v_α is continuous (or even lsc; i.e., $-v_\alpha$ is usc), then we can do better: we can find $\{u_n\}$ so that $\sup_\alpha v_\alpha = \sup_n u_n$ (exercise 2).

Now recall given a bounded domain D and $f \in C(\partial D)$ we formed the Perron envelope

$$U(0;f)(z) := \sup\{v(z) : v \in SH(D) : \limsup_{z\to\zeta} v(z) \leq f(\zeta)$$
$$\text{for all } \zeta \in \partial D\}.$$

Call

$$\mathcal{U} := \{v \in SH(D) : \limsup_{z\to\zeta} v(z) \leq f(\zeta) \text{ for all } \zeta \in \partial D\}.$$

Note that $U(0;f)$ is subharmonic in D for the following reason $U(0;f)^*$ is subharmonic in D (why?) and satisfies $\limsup_{z\to\zeta} U(0;f)^*(z) \leq f(\zeta)$ for all $\zeta \in \partial D$; hence $U(0;f)^* \in \mathcal{U}$ and so $U(0;f)^* \leq U(0;f)$. Thus equality holds.

Claim: *$U(0;f)$ is harmonic in D.*

To prove the claim, we show $U(0;f)$ is harmonic on any disk $B \subset D$. To this end, we first note that since any shm v is a decreasing limit of smooth shm functions, we can assume that each $v \in \mathcal{U}$ is continuous in D; and then by exercise 1, we can recover $U(0;f)$ as an upper envelope of a countable family of continuous functions $\{u_n\}$.

(1) By replacing $u_n \in \mathcal{U}$ by $v_n := \max[u_1, ..., u_n] \in \mathcal{U}$ we have $U(0;f)$ is an increasing sequence of continuous shm functions $\{v_n\}$.
(2) Replace each $v_n \in \mathcal{U}$ by its Poisson modification $\tilde{v}_n \in \mathcal{U}$ where $\tilde{v}_n = v_n$ outside B and $\tilde{v}_n = P_{v_n|_{\partial B}}$ on B. Then, on B, $U(0;f)$ is the monotone, increasing limit of *harmonic* functions.
(3) By Harnack's theorem (a monotone limit of harmonic functions in B either converges to a harmonic function or is identically $\pm\infty$), $U(0;f)$ is harmonic in B.

The key point here is (2): the family of shm functions $\mathcal{U}$ in the definition of $U(0;f)$ is closed under Poisson modification on disks in D. As another example of this type of argument, recall for $K \subset \mathbb{C}$ compact, we defined

$$V_K(z) = \sup\{u(z) : u \in L(\mathbb{C}),\ u \leq 0 \text{ on } K\}.$$

Let

$$\mathcal{U}_K := \{u \in L(\mathbb{C}),\ u \leq 0 \text{ on } K\}.$$

Then $\mathcal{U}_K$ is closed under Poisson modification on disks in $\mathbb{C} \setminus K$: for any $u \in \mathcal{U}_K$ and any disk $B \subset \mathbb{C} \setminus K$, the function $\tilde{u}$ defined as u in $\mathbb{C} \setminus B$

and as $P_{u|_{\partial B},B}$ in B belongs to $\mathcal{U}_K$. An appropriate modification of the above argument shows that, provided $\mathcal{U}_K$ is locally uniformly bounded above, we have that V_K is harmonic *outside of* K. In this case, since $V_K = 0$ on K (note $u \equiv 0 \in L(\mathbb{C})$) and $\{V_K < V_K^*\}$ is negligible and hence polar, $V_K^* = 0$ q.e. on K.

We can show that (2.11) holds in this setting; i.e., if $\mathcal{U}_K$ is locally uniformly bounded above then $\mu_K = \frac{1}{2\pi}\Delta V_K^*$; indeed, this will follow from the equality

$$V_K^* = -[p_{\mu_K} - I(\mu_K)] = V_{\mu_K} - I(\mu_K) \text{ on } \mathbb{C}. \tag{3.2}$$

As a preliminary to proving (3.2), we make some observations about V_K:

(1) If $K_1 \subset K_2$, then $V_{K_2} \le V_{K_1}$ (and $V_{K_2}^* \le V_{K_1}^*$);

(2) If $K = S^1 = \{z : |z| = 1\}$, then $V_K(z) = V_K^*(z) = \log^+|z|$.

To see this, let $u(z) := \log^+|z| \in L^+(\mathbb{C})$. This is harmonic outside K and vanishes on K; moreover, dd^cu is supported on K. For any $v \in L(\mathbb{C})$ with $v \le 0$ on K we have $v \le u$ on $K = \text{supp} dd^cu$ so by the global domination principle, Proposition 2.16, $v \le u$ in $\mathbb{C}$. More generally, the same argument shows:

(3) If $K = \overline{B(a,r)} = \{z : |z-a| \le r\}$ or $K = \partial B(a,r)$, then $V_K(z) = V_K^*(z) = \log^+\frac{|z-a|}{r}$.

Suppose we knew that

$$\mathcal{U}_K \text{ is locally uniformly bounded above } \iff K \text{ is not polar.} \tag{3.3}$$

From Theorem 2.15, for K non-polar $V_{\mu_K} \ge I(\mu_K)$ on $\mathbb{C}$ so that $V_{\mu_K} - I(\mu_K) \in L^+(\mathbb{C})$. We claim that $V_K^* \in L^+(\mathbb{C})$ as well. To see this, since $\mathcal{U}_K$ is locally uniformly bounded above, on a closed disk $\overline{B} = \overline{B(a,r)}$ we have $u \le M$ for all $u \in \mathcal{U}_K$; hence from (3)

$$V_K \le M + \log^+\frac{|z-a|}{r}$$

and the same inequality holds for V_K^*. Thus $V_K^* \in L(\mathbb{C})$. On the other hand, K is compact; hence $K \subset B(0,R)$ for R sufficiently large and hence by (1) and (3) above,

$$V_B(z) = \log^+\frac{|z|}{R} \le V_K(z) \le V_K^*(z).$$

Thus we have $V_K^* \in L^+(\mathbb{C})$.

Next, $V_{\mu_K} = I(\mu_K)$ q.e on K and, as observed earlier, $V_K^* = 0$ q.e. on K, so that

$$V_K^* = V_{\mu_K} - I(\mu_K) \text{ q.e. on } K.$$

Also, $\text{supp}(dd^c V_K^*) \subset K \cup P$ where P is polar; and $\text{supp}(dd^c(V_{\mu_K} - I(\mu_K))) \subset K$. By the domination principle Proposition 2.16,

$$V_K^* = V_{\mu_K} - I(\mu_K) \text{ on } \mathbb{C}. \tag{3.4}$$

Remark 3.5. Often the notation g_K is used for V_K^*, the *Green function* for K: it is characterized (uniquely) as the shm function in $\mathbb{C}$ which is in $L^+(\mathbb{C})$; harmonic in $\mathbb{C} \setminus K$; and equals 0 q.e. on K. We say K has a *classical Green function* if $g_K = 0$ on all of K.

We need to verify (3.3) – this is Proposition 3.7 – we first state and prove a very useful and general result, known as *Hartogs lemma*. We use a modification of the classical Dini lemma: let $\{f_n\}$ be a sequence of usc functions on a compact metric space X which decrease pointwise to a lsc function f. Then $f_n \to f$ uniformly on X. See exercise 10 for the modification utilized.

Lemma 3.6. *Let $\{u_j\}$ be a family of shm functions on a domain $D \subset \mathbb{C}$ which are locally uniformly bounded above in D. Suppose there exists $M < +\infty$ with*

$$\limsup_{j\to\infty} u_j(z) \le M \text{ for all } z \in D.$$

Given $\epsilon > 0$ and $K \subset D$ compact, there exists $j_0 = j_0(\epsilon, K)$ such that for $j \ge j_0$,

$$\sup_{z\in K} u_j(z) \le M + \epsilon.$$

Proof. Let $u(z) := \limsup_{j\to\infty} u_j(z)$ and $v_n(z) := \sup_{j\ge n} u_j(z)$. Then $v_n \downarrow u$. The functions v_n^* are shm and decrease pointwise to a shm function v on D. By the Brelot-Cartan theorem, $v_n = v_n^*$ q.e. and since a countable union of polar sets is polar (Corollary 2.14), $v = u$ q.e. Hence the shm functions v and u^* are equal q.e. and therefore a.e.; by Corollary 1.5 $v = u^*$ on D. Since $\{v_n^*\}$ form a decreasing sequence of shm functions with $v_n^* \le M$, by exercise 10, on any compact set $K \subset D$ the sequence $\{\max[v_n^*, M]\}$ converges uniformly to M so that $v \le M$ on K. Since $u_n \le v_n \le v_n^*$, the result follows. □

We saw that for the closed disk $B = \overline{B}(a, r) = \{z \in \mathbb{C} : |z - a| \le r\}$ we have $V_B(z) = V_B^*(z) = \max[\log|z - a|/r, 0]$. On the other hand, if

$K = \{p\}$ then for each n, $u_n(z) := \log|z-p| + n \leq V_K(z)$ showing that $V_K(z) = +\infty$ for $z \neq p$ and hence $V_K^* \equiv +\infty$. These examples illustrate the two cases of our next result.

Proposition 3.7. *Let $K \subset \mathbb{C}$ be compact. Either $V_K^* \equiv +\infty$, which occurs if K is polar, or else we have $V_K^* \in L^+(\mathbb{C})$.*

Proof. If V_K is locally bounded above, on a disk B, e.g., the unit disk, $V_K \leq M$; i.e., for all $u \in L(\mathbb{C})$ with $u \leq 0$ on K, we have $u - M \leq 0$ on B so that $u - M \leq V_B$ in $\mathbb{C}$ and hence $V_K \leq M + V_B$ in $\mathbb{C}$. Hence $V_K^* \in L(\mathbb{C})$. From the argument in the paragraph after (3.3) we know that, indeed, $V_K^* \in L^+(\mathbb{C})$.

If V_K is not locally bounded above, we claim that $P := \{z \in \mathbb{C} : V_K(z) < +\infty\}$ is polar. Since $V_K = 0$ on K, this shows, in particular, that K is polar. Thus assume V_K is not locally bounded above. Then there is a closed disk B and sequence $\{u_j\} \subset L(\mathbb{C})$ with $u_j \leq 0$ on K such that $M_j := \sup_B u_j \geq j$ for $j = 1, 2, ...$ It follows that

$$u_j(z) - M_j \leq V_B(z),\ z \in \mathbb{C},\ j = 1, 2, ...$$

We claim that from Hartogs lemma, there exists $z_0 \in \mathbb{C}$ with

$$\delta := \limsup_{j\to\infty} \exp\left(u_j(z_0) - M_j\right) > 0.$$

For if not, $\limsup_{j\to\infty} \exp\left(u_j(z) - M_j\right) \leq 0$ for all $z \in \mathbb{C}$. Hartogs lemma implies, e.g., that $\exp\left(u_j(z) - M_j\right) \leq 1/2$ for $z \in B$ and all j sufficiently large. But this contradicts the definition of $M_j := \sup_B u_j$.

Choose a subsequence $\{u_{j_k}\}$ so that

$$\delta = \lim_{k\to\infty} \exp\left(u_{j_k}(z_0) - M_{j_k}\right) \text{ and } M_{j_k} \geq 2^k$$

and define

$$w(z) := \sum_{k=1}^{\infty} 2^{-k}[u_{j_k}(z) - M_{j_k}]. \tag{3.5}$$

Check that $w(z_0) > -\infty$ (so $w \not\equiv -\infty$); w is shm in $\mathbb{C}$ (why?); and, indeed, $w \in L(\mathbb{C})$. We claim that $w = -\infty$ on P. For if $V_K(z) = M < +\infty$, we have $u_{j_k}(z) \leq M$ for all k and hence

$$\sum_k 2^{-k} u_{j_k}(z) < +\infty.$$

Thus

$$w(z) \le \sum_k 2^{-k} u_{j_k}(z) - \sum_k 1 = -\infty.$$

Hence P is polar. We leave as an exercise to show that $V_K^* \equiv +\infty$ in this case (see exercise 12).

Conversely, if K is polar, Proposition 2.9 shows that there exists $v = V_\mu \in L(\mathbb{C})$ with $v \not\equiv -\infty$ such that $K = \{z \in \mathbb{C} : v(z) = -\infty\}$. Then $\mathcal{U}_K$ contains the sequence of functions $v_n(z) := v(z) + n,\ n = 1, 2, ...$ and hence $\mathcal{U}_K$ is not locally uniformly bounded above.

□

Remark 3.8. An analysis of the proof yields the slightly more general result: *let $\mathcal{U}$ be a family of functions in $L(\mathbb{C})$ and let $u(z) := \sup\{v(z) : v \in \mathcal{U}\}$. If $P := \{z \in \mathbb{C} : u(z) < +\infty\}$ is not polar, then $\mathcal{U}$ is locally bounded above and $u^* \in L(\mathbb{C})$.*

From (3.4), we obtain the following.

Corollary 3.9. *For $K \subset \mathbb{C}$ non-polar,*

$$\lim_{|z| \to \infty} [V_K^*(z) - \log |z|] = I(\mu_K) = -\log \delta(K).$$

We mention that our definition of the extremal function V_K associated to a compact set $K \subset \mathbb{C}$ extends to arbitrary subsets of $\mathbb{C}$:

Definition 3.10. Let $E \subset \mathbb{C}$. We define

$$V_E(z) := \sup\{u(z) : u \in L(\mathbb{C}),\ u \le 0 \text{ on } E\}$$

and we call $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$ the *global extremal function of E*.

The proof of Proposition 3.7 yields that *if E is bounded*, either $V_E^* \equiv +\infty$ (which occurs precisely when E is polar) or $V_E^* \in L(\mathbb{C})$ (indeed, in this case, $V_E^* \in L^+(\mathbb{C})$ (why?)) and V_E^* is harmonic on $\mathbb{C} \setminus \overline{E}$ (why?).

Definition 3.11. Let $E \subset \mathbb{C}$. We say E is $L-$polar if there exists $u \in L(\mathbb{C})$, $u \not\equiv -\infty$, with

$$E \subset \{z \in \mathbb{C} : u(z) = -\infty\}.$$

An immediate corollary of the argument in Proposition 3.7 is that *a countable union of $L-$polar sets is $L-$polar*: indeed, if $E := \cup_j E_j$ is a countable union of $L-$polar sets, by replacing E_j with $E_1 \cup \cdots \cup E_j$ we can assume $E_1 \subset E_2 \subset ...$ Then take $u_j \in L(\mathbb{C})$, $u_j \not\equiv -\infty$, with

$u_j|_{E_j} = -\infty$. Let $M_j := \sup_{\overline{B(0,1)}} u_j$. By Hartogs lemma there exists $z_0 \in \mathbb{C}$ with

$$\limsup_{j\to\infty} e^{u_j(z_0)-M_j} = \delta > 0.$$

Taking a subsequence $\{u_{j_k}\} \subset \{u_j\}$ such that

$$\lim_{j\to\infty} e^{u_{j_k}(z_0)-M_{j_k}} = \delta > 0$$

and such that $M_{j_k} \geq 2^k$, one checks that

$$w(z) := \sum_k \frac{1}{2^k}[u_{j_k}(z) - M_{j_k}] \in L(\mathbb{C})$$

with $w|_E = -\infty$ and $w \not\equiv -\infty$.

This leads one to suspect that $L-$polar sets coincide with polar sets (clearly $L-$polar sets are polar) and this is the case. We simply remark that to see that a polar set E is $L-$polar, by taking $E_j := E \cap B(0,j)$ and using the fact that a countable union of $(L-)$polar sets is $(L-)$polar, it suffices to show each E_j is $L-$polar so from the beginning one can assume E is bounded. Moreover if E is also closed, i.e., E is compact, the result follows from Proposition 2.9.

Corollary 3.12. *If $E \subset \mathbb{C}$ is bounded and $F \subset \mathbb{C}$ is (L−)polar, then $V^*_{E\cup F} = V^*_E$.*

Proof. Clearly $E \subset E \cup F$ implies $V^*_{E\cup F} \leq V^*_E$. For the reverse inequality, take $v \in L(\mathbb{C})$ with $v = -\infty$ on F; since E is bounded, we may assume $v \leq 0$ on E. Then if $u \in L(\mathbb{C})$ with $u \leq 0$ on E,

$$(1-\epsilon)u + \epsilon v \in L(\mathbb{C}) \text{ and } (1-\epsilon)u + \epsilon v \leq 0 \text{ on } E \cup F.$$

Thus $(1-\epsilon)u + \epsilon v \leq V_{E\cup F} \leq V^*_{E\cup F}$ in $\mathbb{C}$. Letting $\epsilon \to 0$, we obtain $u \leq V^*_{E\cup F}$ on $\mathbb{C} \setminus \{v = -\infty\}$. In particular, $u \leq V^*_{E\cup F}$ a.e. in $\mathbb{C}$ and hence on all of $\mathbb{C}$.

□

Corollary 3.13. *If $\{E_j\}$ are increasing; i.e., $E_1 \subset E_2 \ldots$ and $E := \cup_j E_j$ is bounded, then*

$$\lim_{j\to\infty} V^*_{E_j} = V^*_E.$$

The proof is left as Exercise (13). See also Exercise (14) – and see [22] for more on these topics.

How "big" can polar sets be? We saw that polar sets must have Lebesgue measure zero, and indeed, a polar set must have zero Hausdorff dimension so it can't be too big. On the other hand, we saw that countable sets are polar; but there do exist uncountable polar sets. Examples can be constructed from certain generalized Cantor sets. We refer the reader to [30].

There is a notion of "thinness" of a set, which is very closely related to polarity. Recall from exercise 4 of section 1, if u is shm in D, then for each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) = u(z)$. Let $S \subset \mathbb{C}$ and $z \in \overline{S \setminus \{z\}}$. We say that S is *thin* at z if there exists u shm on a neighborhood of z_0 with

$$\limsup_{\zeta \to z,\ \zeta \in S \setminus \{z\}} u(\zeta) < u(z).$$

(For consistency, if $\zeta \notin \overline{S}$, we say that S is thin at ζ). It can be shown that *an F_σ polar set S is thin at each point, and, conversely, a set S which is thin at every point of itself must be polar.* We refer the reader to section 3.8 of [30] for details.

Exercises.

(1) Prove Corollary 3.2.

(2) Verify the modification of Choquet's result: let $\{v_\alpha\}_{\alpha \in A}$ be a family of real-valued continuous functions defined on a separable metric space X which is uniformly bounded above on any compact subset of X, then one can extract a countable subfamily $\{u_n\} \subset \{v_\alpha\}$ with the property that

$$\sup_\alpha v_\alpha = \sup_n u_n.$$

(3) Prove Theorem 3.4 in the case where A is uncountable.

(4) Let $\{v_\alpha\} \subset SH(D)$ be uniformly bounded above on any compact subset of D and define $v(z) := \sup_\alpha v_\alpha(z)$. Show that $v^*(z) := \limsup_{\zeta \to z} v(\zeta)$ is shm in D.

(5) Given a bounded domain D, and a point $z_0 \in D$, define

$$G(z; z_0) := \sup\{u(z) : u \in SH(D),\ u \le 0,$$
$$u(z) - \log|z - z_0| \text{ bounded as } z \to z_0\}$$

the *Green function* for D with pole at z_0. Show that $G(z; z_0)$ is harmonic in $D \setminus \{z_0\}$.

(6) Find a formula for $G(z;z_0)$ if $D = B(0,1)$ and $|z_0| < 1$. (*Hint: First do the case $z_0 = 0$ and for $z_0 \neq 0$ find a holomorphic self-map of $B(0,1)$ taking z_0 to 0*).

(7) Given a bounded domain D and a subset $E \subset D$, define

$$\omega(z,E,D) := \sup\{u(z) : u \in SH(D),\ u \leq 0,\ u|_E \leq -1\},$$

the *relative extremal function* for E relative to D. Show that if $\omega^*(z,E,D) \not\equiv 0$ then $\omega^*(z,E,D)$ is harmonic in $D \setminus \overline{E}$.

(8) Find a formula for $\omega(z,E,D)$ if $D = B(0,R)$ and $E = B(0,r)$, for $r < R$.

(9) Prove the *two-constants theorem*: for $E \subset D$, if u is shm in D satisfies $u \leq M$ in D and $u \leq m < M$ on E, then for $z \in D$,

$$u(z) \leq M(1 + \omega^*(z,E,D)) - m\omega^*(z,E,D).$$

(*Remark: If you apply this result to $u = \log|f|$ where f is holomorphic in D, $|f| \leq M'$ on D and $|f| \leq m'$ on E you get a generalization of the "three-circles" theorem from complex analysis.*)

(10) Let $\{f_n\}$ be a sequence of usc functions on a compact metric space X which decrease pointwise to a lsc function f. Let $\phi \geq f$ be continuous. Then $\max[f_n, \phi] \to \phi$ uniformly on X.

(11) Verify the "why?" in the proof of Proposition 3.7; i.e., prove the shm of w in equation 3.5.

(12) Let $E \subset \mathbb{C}$ with $V_E = +\infty$ q.e. Show that $V_E^* \equiv +\infty$.

(13) Prove Corollary 3.13.

(14) *Extra Credit:* Are Corollaries 3.12 and 3.13 valid if E is not bounded? Prove or find a counterexample.

4. Polynomial approximation and interpolation in $\mathbb{C}$.

There is a close relation between the smoothness of a function f and the speed at which f may be approximated by polynomials. To state results of this type we introduce, for any continuous complex-valued function f on any compact set K in the plane $\mathbb{C}$, the *approximation numbers*

$$d_n = d_n(f, K) \equiv \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\},$$

where recall $\mathcal{P}_n$ is the vector space of complex polynomials in z of degree at most n. The Weierstrass approximation theorem states that $\lim_{n\to\infty} d_n = 0$ for any continuous function f on $[-1, 1]$, and it is natural to ask for additional conditions on f which guarantee that d_n converges rapidly to zero. A beautiful result of this type is the classical theorem of Bernstein, which states that f extends to a holomorphic function on an open neighborhood of $[-1, 1]$ in $\mathbb{C}$ if and only if d_n satisfies an exponential decay estimate

$$d_n \leq C\rho^n \quad \text{for some constants } C > 0 \text{ and } \rho \in (0, 1).$$

In fact, a sharp version of the Bernstein theorem relates the constant ρ to the size of the open neighborhood of $[-1, 1]$ to which f can be extended. Walsh [33] later gave an important extension of the Bernstein theorem in which the interval $[-1, 1]$ is replaced by certain compact subsets of $\mathbb{C}$. The theorems of Bernstein and Walsh serve as a link between the classical ideas of approximation theory and some higher-dimensional problems concerning holomorphic functions of several complex variables.

An elementary approach to the theorems of Bernstein and Walsh is to regard them as statements about the error in truncating geometrically convergent series expansions. As the simplest example, consider first the closed unit disk $\overline{\Delta} = \{z : |z| \leq 1\}$ in $\mathbb{C}$, and suppose that f is holomorphic on a neighborhood of $\overline{\Delta}$. To be specific, we assume that f is holomorphic on the open disk $\{z : |z| < R\}$, where $R > 1$, and we ask to what extent the size of the radius R determines the rate of decay of the approximation numbers $d_n(f, \overline{\Delta})$. To study this, we recall that the Taylor expansion $\sum a_k z^k$ for f about the origin converges absolutely and uniformly on compact subsets of $\{z : |z| < R\}$ to f. Applying the Cauchy estimates to f on $\{z : |z| < r\}$, where $1 < r < R$, we obtain $|a_n| \leq M/r^n$ with $M = \sup\{|f(z)| : |z| \leq r\}$. Letting $p_n(z) = \sum_{k=0}^n a_k z^k$ be the n-th Taylor polynomial for f, it follows that $d_n(f,\overline{\Delta}) \le ||f-p_n||_{\overline{\Delta}} \le \dfrac{M}{r^n(r-1)}$. This implies that $\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/r$, and we may now let $r \uparrow R$ to conclude that

$$\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/R.$$

This proves the following equivalence in one direction.

Theorem 4.1. *Let f be continuous on $\overline{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$, and $R > 1$. Then*

$$\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/R \tag{4.1}$$

if and only if f is the restriction to $\overline{\Delta}$ of a function holomorphic in $\{z \in \mathbb{C} : |z| < R\}$.

Proof. We have already proved "if". To prove "only if" we will use the fact that any polynomial $p(z)$ satisfies the *Bernstein-Walsh inequality*

$$|p(z)| \le ||p||_{\overline{\Delta}}\, \rho^{\deg\ p}, \qquad |z| \le \rho; \tag{4.2}$$

this estimate follows from applying Lemma 4.2 below, with $g_{\overline{\Delta}}(z) \equiv \log|z|$, so for the moment we assume (4.2) and complete the proof of the theorem. Let f be a continuous function on $\overline{\Delta}$ such that (4.1) holds; we will show that if p_n is a polynomial of degree $\le n$ satisfying $d_n = ||f-p_n||_{\overline{\Delta}}$, then the series $p_0+\sum_1^\infty(p_n-p_{n-1})$ converges uniformly on compact subsets of $\{z : |z| < R\}$ to a holomorphic function F which agrees with f on $\overline{\Delta}$. To do this, we choose R' with $1 < R' < R$; by hypothesis the polynomials p_n satisfy

$$||f-p_n||_{\overline{\Delta}} \le \frac{M}{R'^n}, \qquad n = 0,1,2,..., \tag{4.3}$$

for some $M > 0$. We now let $1 < \rho < R'$, and apply (4.2) to the polynomial $p_n - p_{n-1}$ to obtain

$$\sup_{|z|\le\rho} |p_n(z)-p_{n-1}(z)| \le \rho^n||p_n-p_{n-1}||_{\overline{\Delta}} \le \rho^n(||p_n-f||_{\overline{\Delta}}+||f-p_{n-1}||_{\overline{\Delta}})$$

$$\le \rho^n\frac{M(1+R')}{R'^n}.$$

Since ρ and R' were arbitrary numbers satisfying $1 < \rho < R' < R$, we conclude that $p_0 + \sum_1^\infty(p_n - p_{n-1})$ is locally uniformly Cauchy on $\{z : |z| < R\}$, and hence converges locally uniformly on $\{z : |z| < R\}$

to a holomorphic function F; from (4.3) we see that $F \equiv f$ on $\overline{\Delta}$, so the theorem is proved. $\square$

For more general compact sets $K \subset \mathbb{C}$, we will see the importance of the function V_K from (2.10). We begin with a lemma.

Lemma 4.2. (Bernstein-Walsh property) *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. Suppose that $\mathbb{C} \setminus K$ has a classical Green function g_K; i.e., there is a continuous function $g_K : \mathbb{C} \to [0, +\infty)$ which is identically equal to zero on K, harmonic on $\mathbb{C} \setminus K$, and has a logarithmic singularity at infinity in the sense that $g_K(z) - \log |z|$ is harmonic at infinity. Then*

$$g_K(z) \equiv \max \left\{ 0, \sup_p \left\{ \frac{1}{\deg p} \log |p(z)| \right\} \right\}, \tag{4.4}$$

where the supremum is taken over all non-constant polynomials p such that $||p||_K \le 1$. In particular, $g_K = V_K$ and, if $R > 1$ and

$$D_R \equiv \{z : V_K(z) < \log R\}, \tag{4.5}$$

then

$$|p(z)| \le ||p||_K R^{\deg p}, \qquad z \in D_R. \tag{4.6}$$

The topological condition that $\mathbb{C} \setminus K$ is connected is equivalent to K being *polynomially convex*: this means that $K = \widehat{K}$ where

$$\widehat{K} \equiv \{z \in \mathbb{C} : |p(z)| \le ||p||_K, \ p \text{ polynomial}\}$$

is the *polynomial hull* of K (see the exercises). Note that using (2.12), i.e., the right-hand-side of (4.4), we have

$$V_K = V_{\widehat{K}}.$$

The extra condition that $\mathbb{C} \setminus K$ has a classical Green function g_K is referred to as *regularity* of K.

It is easy to prove a weak form of (4.4). In fact, if p is any nonconstant polynomial such that $||p||_K \le 1$, then the function $V \equiv \frac{1}{\deg p} \log |p| - g_K$ is subharmonic on $\mathbb{C} \setminus K$, bounded at ∞, and continuously assumes nonpositive values on ∂K. By the maximum principle we have $V \le 0$ on $\mathbb{C} \cup \{\infty\} - K$, which proves that $g_K(z)$ is greater than or equal to the right side of (4.4). To show that $g_K(z)$ is actually equal to the right side of (4.4), one can even construct a *sequence* of monic polynomials $\{p_n(z) = z^n + \cdots\}$ with $\deg p_n = n$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{|p_n(z)|}{||p_n||_K}\right) = g_K(z)$$

locally uniformly on $\mathbb{C} \cup \{\infty\} - \widehat{K}$ (cf., [33], section 4.4). Indeed, Proposition 2.6 gives an indication that a sequence of *Fekete polynomials* $p_n(z) = \prod_{j=1}^n (z - z_{nj})$ where $z_{n1}, ..., z_{nn}$ is a set of Fekete points of order $n-1$ for K will do since the corresponding sequence of Fekete measures $\{\mu_n\}$ converges weak-* to μ_K. We omit the verification, but see exercise 5 (in conjuction with Theorem 4.7). As a simple example, for the unit circle $K = \{z : |z| = 1\}$ (or its polynomial hull, the closed unit disk), $p_n(z) = z^n - 1,\ n = 2, 3, ...$ are Fekete polynomials with $||p_n||_K = 2$; and it is clear that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{|z^n - 1|}{2} \to \log|z|$$

locally uniformly in $\{z : |z| > 1\}$.

Note as a consequence, this proves the following.

Corollary 4.3. *Let K be a regular compact set in $\mathbb{C}$ and let*

$$\phi_n(z) := \sup\{|p(z)| : p \in \mathcal{P}_n,\ ||p||_K \leq 1\}.$$

Then

$$\frac{1}{n} \log \phi_n(z) \to V_K(z)$$

locally uniformly on $\mathbb{C}$.

Remark 4.4. For an arbitrary compact set $K \subset \mathbb{C}$, we have the pointwise convergence of $\frac{1}{n}\log \phi_n(z)$ to V_K on $\mathbb{C}$ since

$$\begin{aligned} V_K(z) &:= \sup\{u(z) : u \in L(\mathbb{C}),\ u \leq 0 \text{ on } K\} \\ &= \max\left\{0, \sup\left\{\frac{1}{\deg p} \log|p(z)| :\ ||p||_K\right\}\right\}. \end{aligned}$$

This fact is important to give connections with quantitative versions of polynomial approximation results, as we will see in Theorem 4.5. Moreover,

(1) the analogous result is valid in several complex variables, albeit with a more difficult (and less explicit) proof(s); and

(2) the restriction to *compact* sets in this equality is essential: since polynomials are continuous, clearly the function

$$\max\left\{0, \sup_p \left\{\frac{1}{\deg p}\log|p(z)|,\ ||p||_K\right\}\right\}$$

is the same for a set K and for its closure $\overline{K}$.

From Remark 4.4, a compact set is regular if and only if V_K is continuous. Indeed, this representation of V_K shows that it is *lowersemicontinuous* (being an upper envelope of *continuous* functions); hence, V_K is continuous if and only if $V_K = V_K^*$. Moreover, this latter condition is equivalent to $V_K^* = 0$ on K (why?). Any compact set K can be approximated from the outside by regular compacta; i.e., one can find $\{K_j\}$ regular with $K_{j+1} \subset K_j$ and $\cap_j K_j = K$. We can take, e.g., $K_j = \{z \in \mathbb{C} : \mathrm{dist}(z, K) \leq 1/j\}$. The fact that each K_j is regular can be seen by recalling from section 3 that for a closed unit disk $B = \overline{B}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$ we have $V_B(z) = V_B^*(z) = \max[\log|z - a|/r, 0]$. Now each $z_0 \in K_j$ belongs to a closed ball $\tilde{B} := \overline{B}(a, 1/j) \subset K_j$ and since

$$V_{K_j}(z) \leq V_{\tilde{B}}(z) = V_{\tilde{B}}^*(z) = \log^+ j|z - a|,$$

we have $V_{K_j}^*(z_0) = 0$. Hence $V_{K_j}^* \equiv 0$ on K_j.

We remark that for a general compact set $K \subset \mathbb{C}$, if one minimizes the supremum norm on K of monic polynomials of degree n; i.e., one takes

$$\tau_n(K) := \inf\{||p_n||_K : p_n(z) = z^n + \cdots\},$$

then

$$\lim_{n\to\infty} \tau_n(K)^{1/n} = \inf_{n\geq 1} \tau_n(K)^{1/n} = \delta(K). \tag{4.7}$$

Thus the *Chebyshev constant* $\lim_{n\to\infty} \tau_n(K)^{1/n}$ of K coincides with the transfinite diameter. A monic polynomial t_n with $||t_n||_K = \tau_n(K)$ is called a *Chebyshev polynomial* for K; such a polynomial exists (and is unique if K has at least n points). Note the fact that

$$\lim_{n\to\infty} \tau_n(K)^{1/n} = \inf_{n\geq 1} \tau_n(K)^{1/n} =: \tau(K)$$

(and, in particular, that the limit exists) follows since

$$||t_{n+m}||_K \leq ||t_n \cdot t_m||_K \leq ||t_n||_K \cdot ||t_m||_K,$$

showing that the sequence $\{\log \tau_n(K)\}$ is subadditive.

For the proof of the equality $\tau(K) = \delta(K)$ let

$$V_n = V_n(K) := \max_{z_0,\ldots,z_n \in K} |VDM(z_0, ..., z_n)|.$$

We first show

$$\tau_n(K) \leq V_n/V_{n-1} \leq (n+1)\tau_n(K). \tag{4.8}$$

Taking a Fekete polynomial $p_n(z) = \prod_{j=1}^{n}(z - z_{nj})$, by definition we have $||t_n||_K \leq ||p_n||_K$; but then for *any* $z \in K$, the $(n+1)-$tuple $z, z_{n1}, ..., z_{nn}$ is a candidate for a set of Fekete points of order n for K. Thus

$$|p_n(z)| \cdot \delta_{n-1}(K)^{\binom{n}{2}} = \prod_{j=1}^{n} |z - z_{nj}| \prod_{j<k} |z_{nj} - z_{nk}| \leq \delta_n(K)^{\binom{n+1}{2}}$$

and since $\delta_n(K) \leq \delta_{n-1}(K)$ (exercise 7 in section 2), we have

$$||t_n||_K \leq ||p_n||_K \leq \frac{\delta_n(K)^{\binom{n+1}{2}}}{\delta_{n-1}(K)^{\binom{n}{2}}} = V_n/V_{n-1} \leq \frac{\delta_n(K)^{\binom{n+1}{2}}}{\delta_n(K)^{\binom{n}{2}}} = \delta_n(K)^n$$

giving the left-hand inequality in (4.8) as well as

$$\limsup_{n\to\infty} \tau_n(K)^{1/n} \leq \delta(K).$$

Note we have also proved that $||p_n||_K^{1/n} \leq \delta_n(K)$ for the Fekete polynomials p_n.

To verify the right-hand inequality in (4.8), let $z_0, ..., z_n$ be Fekete points of order n for K; then

$$VDM(z_0, ..., z_n) = \det[z_i^j]_{i,j=0,1,\ldots,n} = \prod_{j<k}(z_j - z_k)$$

$$= \det \begin{bmatrix} 1 & z_0 & \ldots & z_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \ldots & z_n^n \end{bmatrix} = \det \begin{bmatrix} 1 & z_0 & \ldots & t_n(z_0) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \ldots & t_n(z_n) \end{bmatrix}$$

using elementary column operations and the fact that t_n is monic. Expanding the determinant, we get the estimate

$$V_n \leq (n+1)||t_n||_K V_{n-1},$$

as desired.

Now we multiply the inequalities in (4.8) for $n = 1, 2, ..., m$ and observe the telescoping:

$$V_1/V_0 \cdots V_m/V_{m-1} = V_m/V_0 = V_m$$

(define V_0 to be 1). Thus

$$(\tau_1 \cdots \tau_m)^{1/\binom{m+1}{2}} \le V_m^{1/\binom{m+1}{2}} \le [(m+1)!]^{1/\binom{m+1}{2}} (\tau_1 \cdots \tau_m)^{1/\binom{m+1}{2}}.$$

It suffices now to show that

$$(\tau_1 \cdots \tau_m)^{1/\binom{m+1}{2}} \to \tau.$$

But since $\lim_{m\to\infty} \tau_m^{1/m} = \tau$, the sequence

$$\tau_1, \tau_2^{1/2}, \tau_2^{1/2}, \tau_3^{1/3}, \tau_3^{1/3}, \tau_3^{1/3}, ...$$

in which $\tau_m^{1/m}$ is repeated m times, also converges to τ and the corresponding sequence

$$\log \tau_1, \frac{1}{2} \log \tau_2, \frac{1}{2} \log \tau_2, \frac{1}{3} \log \tau_3, \frac{1}{3} \log \tau_3, \frac{1}{3} \log \tau_3, ...$$

converges to $\log \tau$. The arithmetic mean of the first $\binom{m+1}{2}$ of these logarithmic terms coincides with $\log(\tau_1 \cdots \tau_m)^{1/\binom{m+1}{2}}$.

We return to the generalization of Theorem 4.1.

Theorem 4.5. (Walsh) *Let K be a compact subset of the plane such that $\mathbb{C}\backslash K$ is connected and has a Green's function g_K. Let $R > 1$, and define D_R by (4.5). Let f be continuous on K. Then*

$$\limsup_{n\to\infty} d_n(f, K)^{1/n} \le 1/R$$

if and only if f is the restriction to K of a function holomorphic in D_R.

To prove "only if" in this theorem we repeat the proof after the statement of Theorem 4.1, using the Bernstein-Walsh inequality (4.6). The proof of the "if" direction we are about to outline is one of the simplest to give, yet the most difficult to generalize; it uses polynomial interpolation to construct good approximators. The key ingredient we need is the Hermite remainder formula for interpolation of a holomorphic function of one variable. Let $z_1, ...z_n$ be n distinct points in the plane and let f be a function which is defined at these points. In Proposition 2.20 we utilized the polynomials $l_j(z) = \prod_{k\neq j}(z - z_k)/\prod_{k\neq j}(z_j - z_k),\ j = 1, ..., n$. These polynomials of degree $n-1$ satisfy $l_j(z_k) = \delta_{j,k}$ and are called the *fundamental Lagrange interpolating polynomials*, or FLIP's, associated to $z_1, ..., z_n$. We recall from Chapter 2 that we can also write

$$l_j(z) = \frac{VDM(z_1, ..., z_{j-1}, z, z_{j+1}, ..., z_n)}{VDM(z_1, ..., z_n)} \text{ (why?)}$$

and this form of a FLIP will generalize to $\mathbb{C}^N$, $N > 1$. Then the polynomial $p(z) = \sum_{j=1}^n f(z_j)l_j(z)$ is the unique polynomial of degree $n-1$ satisfying $p(z_j) = f(z_j)$, $j = 1, ..., n$; we call it the *Lagrange interpolating polynomial,* or LIP, associated to $f, z_1, ..., z_n$. Suppose now that Γ is a rectifiable Jordan curve such that the points $z_1, ..., z_n$ are inside Γ, and f is holomorphic inside and on Γ. We can estimate the error in our approximation of f by p at points inside Γ using the following formula.

Lemma 4.6. (Hermite Remainder Formula) *For any z inside Γ,*

$$f(z) - p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{(t-z)}\,dt,$$

where $\omega(z) = \prod_{k=1}^n (z - z_k)$.

Proof. The function

$$\widetilde{p}(z) \equiv \frac{1}{2\pi i}\int_\Gamma \Big[\frac{\omega(t)-\omega(z)}{t-z}\Big]\frac{f(t)}{\omega(t)}dt$$

is clearly a polynomial of degree $\leq n-1$. Using the Cauchy integral formula for f, we see that

$$f(z) - \widetilde{p}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{(t-z)}dt \tag{4.9}$$

for z inside Γ. In particular, for each k we have $f(z_k) - \widetilde{p}(z_k) = 0$, and hence $\widetilde{p} = p$. Now the lemma follows from (4.9). □

The proof of the "if" direction in Theorem 4.5 can now be completed using Lagrange interpolating polynomials for f at Fekete points of K and the Hermite remainder formula (exercise 4). We next give a fundamental result of Walsh. Let $\{z_{nj}\}$, $j = 0, ..., n$; $n = 1, 2, ...$ be an array of points. For each f defined in a neighborhood of this array, we can form the sequence of LIP's $\{p_n\}$ associated to f. We write $p_n = L_n f$ to denote the degree and the dependence on f; i.e., $L_n f$ is the LIP of degree n associated to $f, z_{n0}, ..., z_{nn}$, and we write l_{nj}, $j = 0, ..., n$ for the FLIP's associated to $\{z_{nj}\}$, $j = 0, ..., n$. Let $\omega_n(z) := \prod_{j=0}^n (z - z_{nj})$.

Theorem 4.7. *Let $K \subset \mathbb{C}$ be compact and regular with $\mathbb{C} \setminus K$ connected. Let $\{z_{nj}\}$ be an array of points in K. Then for any f which is holomorphic in a neighborhood of K, we have $L_n f \rightrightarrows f$ on K if and only if*

$$\lim_{n\to\infty} |\omega_n(z)|^{\frac{1}{n+1}} = \delta(K) \cdot e^{V_K(z)} \tag{4.10}$$

uniformly on compact subsets of $\mathbb{C} \setminus K$.

Condition (4.10) is equivalent to

$$\lim_{n\to\infty} ||\omega_n||_K^{1/n+1} = \delta(K)$$

(exercise 5). We will call the array $\{z_{nj}\}$ "good" – meaning good for polynomial interpolation of holomorphic functions – if condition (4.10) holds. To construct arrays satisfying (4.10), define

$$\Lambda_n \equiv \sup_{z\in K} \sum_{j=0}^{n} |l_{nj}(z)|$$

the n-th *Lebesgue constant* for the array. This is the norm of the linear operator

$$\mathcal{L}_n : C(K) \to \mathcal{P}_n \subset C(K)$$

defined by $\mathcal{L}_n(f) := L_n f$ where we equip $C(K)$ with the supremum norm (exercise). We observe that, from Theorem 4.5, if the array satisfies

$$\lim_{n\to\infty} \Lambda_n^{1/n} = 1, \tag{4.11}$$

then (4.10) holds. To see this, we take f holomorphic on a neighborhood of K, and we show that $L_n f \rightrightarrows f$ on K. To this end, we note that f is holomorphic in D_R for some $R > 1$ so by Theorem 4.5 we can find a sequence of polynomials $\{p_n\}$ with $\deg p_n \leq n$ and $||f - p_n||_K = 0(1/R^n)$. Since $L_n p_n = p_n$ (why?), we have

$$||f - L_n f||_K \leq ||f - p_n||_K + ||p_n - L_n f||_K$$
$$= ||f - p_n||_K + ||L_n(p_n - f)||_K \leq (1 + \Lambda_n)||f - p_n||_K$$

and the result follows.

Next, the condition (4.11) implies that the array is *asymptotically Fekete* in the sense that

$$\lim_{n\to\infty} |VDM(z_{n0}, ..., z_{nn})]^{1/\binom{n+1}{2}} := \delta(K). \tag{4.12}$$

(cf., [12]). Moreover, on pp. 462-463 in [12], it was observed that for an array $\{z_{nj}\} \subset K$ with

$$|VDM(z_{n0},...,z_{nn})| = c_n V_n(K)$$

where

$$0 < c_n < 1, \ \limsup_{n\to\infty} c_n^{1/n} < 1, \ \text{and} \ \lim_{n\to\infty} c_n^{1/l_n} = 1$$

(e.g., $c_n = v^n$ for $0 < v < 1$), property (4.12) holds but (4.11) does not. More precisely, we have the following.

Proposition 4.8. *Let $\{z_{nj}\}_{j=0,...,n;\ n=1,2,...} \subset K$ be an array of points. Suppose that*

$$\lim_{n\to\infty} \Big(\frac{V_n(K)}{|VDM(z_{n0},...,z_{nn})|}\Big)^{1/n} = 1.$$

Then (4.11) holds.

Proof. The result follows trivially from the observation that if

$$\frac{V_n(K)}{|VDM(z_{n0},...,z_{nn})|} \le a(n),$$

then $\Lambda_n \le (n+1)\cdot a(n)$. This observation is a consequence of the fact that each FLIP can be written as

$$l_{nj}(z) \equiv \frac{VDM(z_{n0},...,z,...,z_{nn})}{VDM(z_{n0},...,z_{nn})}$$

so that

$$|l_{nj}(z)| \le a(n)\frac{|VDM(z_{n0},...,z,...,z_{nn})|}{V_n(K)}.$$

Since $|VDM(z_{n0},...,z,...,z_{nn})| \le V_n(K)$ for each $z \in K$, we have $||l_{nj}||_K \le a(n)$. □

Indeed, both the conditions (4.11) and (4.12) imply that the sequence of discrete measures

$$\mu_n := \frac{1}{n+1}\sum_{j=0}^{n} \delta_{z_{nj}}$$

converge weak-* to μ_K.

Proposition 4.9. *Let $K \subset \mathbb{C}$ be compact with $\delta(K) > 0$. For any array $\{z_{nj}\} \subset K$ satisfying (4.12), $\mu_n \to \mu_K$ weak-*.*

We will prove a more general version of this result in section 6 (Proposition 6.13). To summarize, we have the following (see [12] for more details).

Proposition 4.10. *Let $K \subset \mathbb{C}$ be compact, regular, and polynomially convex. Consider the following four properties which an array $\{z_{nj}\}_{j=0,...,n;\ n=1,2,...} \subset K$ may or may not possess:*

1. $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$*;*
2. $\lim_{n\to\infty} |VDM(z_{n0},...,z_{nn})|^{\frac{1}{\binom{n+1}{2}}} = \delta(K)$*;*
3. $\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} \delta_{z_{nj}} = \mu_K$ *weak-*;*
4. $L_n f \rightrightarrows f$ *on K for each f holomorphic on a neighborhood of K.*

Then (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) *and there are counterexamples to each of the reverse implications.*

We end this section with a construction, due to Edrei and Leja, of a *sequence* of points $\{z_j\}$ in a compact set K with the property that the array $\{z_{nj}\} = \{z_j\}$ satisfies (4.12) and hence, if K is regular with $\mathbb{C} \setminus K$ connected, (4.10) holds. Let z_0 be any point in K, and, having chosen $z_1, ..., z_{n-1} \in K$, we choose $z_n \in K$ such that

$$\max_{z\in K} \prod_{j=0}^{n-1} |z - z_j| = \prod_{j=0}^{n-1} |z_n - z_j|. \tag{4.13}$$

The proof that (4.12) holds is outlined in exercise 8.

Exercises.

1. Prove that for $K \subset \mathbb{C}$ compact, $\widehat{K} = K$ if and only if $\mathbb{C} \setminus K$ is connected.
2. For a compact set $K \subset \mathbb{C}$:
   (a) Determine $\widehat{K}$ if $K = \{z : |z| = 1\}$.
   (b) Determine $\widehat{K}$ if $K = \{z : a \le |z| \le b\}$ where $0 < a < b$.
   (c) Show that if $K = \widehat{K}$ then $\mathbb{C} \setminus K$ is connected.
   (d) Note that if $\mathbb{C} \setminus K$ is connected, then Runge's theorem states that any f analytic on a neighborhood of K can be uniformly approximated on K by polynomials. (Theorem 4.5 is a quantitative version of this). Use this to prove the converse to (c): if $\mathbb{C} \setminus K$ is connected, then $K = \widehat{K}$. (*Hint*: If $z_0 \in \mathbb{C}\setminus K$, then $K \cup \{z_0\}$ also has connected complement.

Take a sequence $z_n \to z_0$ and consider $f_n(z) = \frac{1}{z-z_n}$ which is holomorphic on a neighborhood of $K \cup \{z_0\}$. Now use Runge to find a polynomial p with $|p(z_0)| > \max_{\zeta \in K} |p(\zeta)|$).

(3) Suppose that $\mathbb{C} \setminus K$ is connected and has a Green function, and assume that $(\mathbb{C} \cup \{\infty\}) \setminus K$ is simply connected. Prove that for $z \notin K$, $g_K(z) = \log |\phi(z)|$ where ϕ is a conformal map of $(\mathbb{C} \cup \{\infty\}) \setminus K$ onto $\{z : |z| > 1\}$ with $\phi(\infty) = \infty$. Use this result to find $g_{[-1,1]}$.

(4) Use the Hermite remainder formula to prove the "if" direction of Theorem 4.5.

(5) Prove that the condition

$$\lim_{n\to\infty} |\omega_n(z)|^{\frac{1}{n+1}} = \delta(K) \cdot e^{V_K(z)}$$

uniformly on compact subsets of $\mathbb{C} \setminus K$ is equivalent to

$$\lim_{n\to\infty} ||\omega_n||_K^{\frac{1}{n+1}} = \delta(K).$$

(6) Use the Hermite remainder formula to prove the following: given *any* array $\{z_{nj}\}$ in the closed unit disk $\overline{D} = \{z : |z| \leq 1\}$, if f is analytic in $D_R = \{z : |z| < R\}$ where $R > 3$, then $\{L_n f\}$ converge uniformly to f on $\overline{D}$.

(7) Use the previous exercise to prove the following: given *any* bounded array $\{z_{nj}\}$ in $\mathbb{C}$, if f is an entire function, then the sequence of LIP's $\{L_n f\}$ converges uniformly on compact subsets of $\mathbb{C}$ to f.

(8) Verify that a Leja sequence for K defined in (4.13) satisfies (4.12) using the following outline:
   (a) Show for any monic polynomial $p_n(z) = z^n + \cdots$, $||p_n||_K \geq \delta(K)^n$ (you may assume (4.7)).
   (b) Verify that, for the Leja sequence $\{z_j\}_{j=0,1,...}$,

$$V_{n+1}(K) \geq |VDM(z_0, ..., z_n)| \geq ||\omega_n||_K \cdot ||\omega_{n-1}||_K \cdots ||\omega_0||_K$$

   where $\omega_j(z) = \prod_{i=0}^{j}(z - z_i)$.
   (c) Combine parts (a) and (b).

(9) *EXTRA extra credit*: Prove that if $K \subset \mathbb{C}$ is not polar, then there exists a regular compact subset $K' \subset K$. This is a deep theorem of Ancona [1].

5. Random polynomials in $\mathbb{C}$.

Consider *random polynomials* $p_n(z) = \sum_{j=0}^n a_j z^j$ where the coefficients $a_0, ..., a_n$ are i.i.d. complex Gaussian random variables with $\mathbf{E}(a_j) = \mathbf{E}(a_j a_k) = 0$ and $\mathbf{E}(a_j \overline{a}_k) = \delta_{jk}$; i.e., each a_j has a distribution

$$\phi(t)dm(t) = \frac{1}{\pi} e^{-|t|^2} dm(t)$$

where dm denotes Lebesgue measure on $\mathbb{C}$. Thus we get a probability measure $Prob_n$ on $\mathcal{P}_n$, the polynomials of degree at most n, identified with $\mathbb{C}^{n+1}$, where, for $G \subset \mathbb{C}^{n+1}$,

$$\begin{aligned} Prob_n(G) &= \int_G \phi(a_0) \cdots \phi(a_n) dm(a_0) \cdots dm(a_n) \\ &= \frac{1}{\pi^{n+1}} \int_G e^{-\sum_{j=0}^n |a_j|^2} dm(a_0) \cdots dm(a_n). \end{aligned}$$

Note this measure $Prob_n$ is invariant under unitary maps of $\mathbb{C}^{n+1}$.

We form the product probability space of sequences of polynomials:

$$\mathcal{P} := \otimes_{n=1}^\infty (\mathcal{P}_n, Prob_n) = \otimes_{n=1}^\infty (\mathbb{C}^{n+1}, Prob_n).$$

Write $p_n(z) = a_n \sum_{j=1}^n (z - \zeta_j)$ and call $\tilde{Z}_{p_n} := \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j}$ the *normalized zero measure* of p_n. Note $\tilde{Z}_{p_n} = \Delta \frac{1}{n} \log |p_n|$ where $\Delta \log |z| = \delta_0$ (warning: in this section, we ignore the 2π). What can we say about asymptotics of

- $\{\frac{1}{n} \log |p_n|\}$ for random sequences $\{p_n\} \in \mathcal{P}$?
- $\mathbf{E}(\tilde{Z}_{p_n})$?

Here, $\mathbf{E}(\tilde{Z}_{p_n})$ is a measure defined, for $\psi \in C_c(\mathbb{C})$, as

$$\big(\mathbf{E}(\tilde{Z}_{p_n}), \psi\big)_{\mathbb{C}} := \int_{\mathbb{C}^{n+1}} (\tilde{Z}_{p_n}, \psi)_{\mathbb{C}} \; dProb_n(a^{(n)})$$

where $a^{(n)} = (a_0, ..., a_n)$ and $(\tilde{Z}_{p_n}, \psi)_{\mathbb{C}} = \frac{1}{n} \sum_{j=1}^n \psi(\zeta_j)$.

Note that $\{z^j\}_{j=0,\ldots,n} := \{b_j^{(n)}(z)\}_{j=0,\ldots,n}$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\mu_{S^1})$ where $\mu_{S^1} = \frac{1}{2\pi} d\theta$ on $S^1 = \{z : |z| = 1\}$.

Proposition 5.1. $\lim_{n\to\infty} \mathbf{E}(\tilde{Z}_{p_n}) = \mu_{S^1}$.

Proof. We begin by observing that

$$S_n(z,w) := \sum_{j=0}^n b_j^{(n)}(z) \overline{b_j^{(n)}(w)}$$

is the reproducing kernel for point evaluation at z on $\mathcal{P}_n$: if $p_n(z) = \sum_{j=0}^n a_j b_j^{(n)}(z) = \sum_{j=0}^n a_j z^j$, then

$$\int_{S^1} p_n(w) S_n(z,w) d\mu_{S^1}(w) = p_n(z).$$

On the diagonal $w = z$, we have $S_n(e^{i\theta}, e^{i\theta}) = n+1$ and

$$S_n(z,z) = \sum_{j=0}^n |z|^{2j} = \frac{1-|z|^{2n+2}}{1-|z|^2} \text{ Thus:}$$

$$\frac{1}{2n} \log S_n(z,z) = \frac{1}{2n} \log \frac{1-|z|^{2n+2}}{1-|z|^2} \to \log^+ |z| = \max[0, \log |z|]$$

locally uniformly on $\mathbb{C}$. Note that $\Delta \log^+ |z| = \mu_{S^1}$; thus

$$\Delta\big(\frac{1}{2n} \log S_n(z,z)\big) \to \mu_{S^1}.$$

Write $|p_n(z)| = |\sum_{j=0}^n a_j b_j^{(n)}(z)| =: | < a^{(n)}, b^{(n)}(z) >_{\mathbb{C}^{n+1}} |$

$$= S_n(z,z)^{1/2} | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} |$$

where

$$u^{(n)}(z) := \frac{b^{(n)}(z)}{||b^{(n)}(z)||} = \frac{b^{(n)}(z)}{S_n(z,z)^{1/2}}.$$

Then for $\psi \in C_c(\mathbb{C})$ (recall $\tilde{Z}_{p_n} = \Delta \frac{1}{n} \log |p_n|$)

$$\big(\mathbf{E}(\tilde{Z}_{p_n}), \psi\big)_{\mathbb{C}} = \int_{\mathbb{C}^{n+1}} \big(\Delta \frac{1}{n} \log |p_n(z)|, \psi(z)\big)_{\mathbb{C}} \, dProb_n(a^{(n)})$$

$$= \int_{\mathbb{C}^{n+1}} \big(\Delta \frac{1}{2n} \log |S_n(z,z)|, \psi(z)\big)_{\mathbb{C}} dProb_n(a^{(n)})$$

$$+ \int_{\mathbb{C}^{n+1}} \big(\Delta \frac{1}{n} \log | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} |, \psi(z)\big)_{\mathbb{C}} \, dProb_n(a^{(n)}).$$

The first term goes to $\int_{S^1} \psi d\mu_{S^1}$ as $n \to \infty$ and the second term can be rewritten:

$$\int_{\mathbb{C}^{n+1}} \big(\frac{1}{n} \log | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} |, \Delta\psi(z)\big)_{\mathbb{C}} \, dProb_n(a^{(n)})$$

$$= \int_{\mathbb{C}} \Delta\psi(z) \big[\frac{1}{n} \int_{\mathbb{C}^{n+1}} \log | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} | \, dProb_n(a^{(n)})\big] dm(z)$$

(Fubini). By unitary invariance of $dProb_n(a^{(n)})$,

$$I_n(u^{(n)}(z)) := \int_{\mathbb{C}^{n+1}} \log | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} | \, dProb_n(a^{(n)})$$

$$= \int_G \frac{1}{\pi^{n+1}} \log | < a^{(n)}, u^{(n)}(z) >_{\mathbb{C}^{n+1}} | e^{-\sum_{j=0}^n |a_j|^2} dm(a_0) \cdots dm(a_n)$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} \log |a_0| e^{-|a_0|^2} dm(a_0)$$

(here we map the unit vector $u^{(n)}(z) \to (1, 0, ..., 0) \in \mathbb{C}^{n+1}$) is a constant for unit vectors $u^{(n)}(z)$, independent of n (and z). Thus the second term is $0(1/n)$ and

$$\lim_{n\to\infty} \mathbf{E}(\tilde{Z}_{p_n}) = \mu_{S^1}.$$

□

Note if we simply change $\{z^j\}_{j=0,...,n}$ to $\{(z-a)^j\}_{j=0,...,n}$, i.e., we write our random polynomials as $p_n(z) = \sum_{j=0}^n a_j (z-a)^j$, the same analysis gives $\frac{1}{2n} \log S_n(z, z) \to \log^+ |z - a|$ so

$$\lim_{n\to\infty} \mathbf{E}(\tilde{Z}_{p_n}) = \frac{1}{2\pi} d\theta \text{ on } S^1 - a$$

where $S^1 - a = \{z : |z - a| = 1\}$.

For $K \subset \mathbb{C}$ compact, recall we have the extremal function

$$V_K(z) := \max[0, \sup\{\frac{1}{deg(p)} \log |p(z)| : p \in \cup_n \mathcal{P}_n, \ ||p||_K \leq 1\}].$$

Note $V_{S^1-a}(z) = \log^+ |z - a|$. If V_K is continuous, as is the case with $K = S^1 - a$, defining

$$\phi_n(z) := \sup\{|p(z)| : p \in \mathcal{P}_n, \ ||p||_K \leq 1\},$$

we recall from Corollary 4.3 that

$$\frac{1}{n} \log \phi_n(z) \to V_K(z) \text{ locally uniformly on } \mathbb{C}. \tag{5.1}$$

Let $\mu_K := \Delta V_K$, the equilibrium measure for K. Note $\mu_{S^1-a} = \frac{1}{2\pi} d\theta$ on $S^1 - a$. We can recover V_K and μ_K via L^2-methods. If τ is a measure on K such that

$$||p||_K \leq M_n ||p||_{L^2(\tau)} \tag{5.2}$$

for all polynomials $p \in \mathcal{P}_n$, where $n = 1, 2, ...$ and

$$\limsup_{n\to\infty} M_n^{1/n} = \limsup_{n\to\infty} \big([\max_{z\in K} S_n(z,z)]^{1/2}\big)^{1/n} = 1,$$

we recall that τ is called a Bernstein-Markov (BM) measure for K. For such τ, we show that

$$\frac{1}{n+1} \leq \frac{S_n(z,z)}{\phi_n(z)^2} \leq M_n^2(n+1) \tag{5.3}$$

It follows from (5.1) and (5.3) that if V_K is continuous (BM) is *equivalent* to

$$\lim_{n\to\infty} \frac{1}{2n} \log S_n(z,z) = V_K(z) \text{ locally uniformly on } \mathbb{C}.$$

To prove (5.3), the left-hand inequality follows from the reproducing property of $S_n(z,w)$ (and is valid for any τ whose support is not a finite set). Let p be a polynomial of degree at most n with $||p||_K \leq 1$. Then

$$|p(z)| = |\int_K p(w) S_n(z,w) d\tau(w)| \leq \int_K |S_n(z,w)| d\tau(w)$$

$$\leq \int_K S_n(z,z)^{1/2} S_n(w,w)^{1/2} d\tau(w) = S_n(z,z)^{1/2} \int_K S_n(w,w)^{1/2} d\tau(w)$$

$$\leq S_n(z,z)^{1/2} ||1||_{L^2(\tau)} \cdot ||S_n(w,w)||_{L^2(\tau)} \leq S_n(z,z)^{1/2} \cdot (n+1)^{1/2}.$$

Since $\phi_n(z) = \sup\{|p(z)| : p \in \mathcal{P}_n, \ ||p||_K \leq 1\}$, taking the supremum over all such p gives the left-hand inequality. The right-hand inequality uses (BM) applied to an orthonormal basis $\{b_j^{(n)}\}_{j=0,\ldots,n}$ in $L^2(\tau)$ for $\mathcal{P}_n$. We have $||b_j^{(n)}||_K \leq M_n$ so that $|b_j^{(n)}(z)|/M_n \leq \phi_n(z)$ and

$$S_n(z,z) = \sum_{j=0}^{n} |b_j^{(n)}(z)|^2 \leq (n+1) \cdot M_n^2 \cdot [\phi_n(z)]^2.$$

Now the *exact* same proof of Proposition 5.1 shows the following.

Proposition 5.2. *Let $K \subset \mathbb{C}$ be compact with V_K continuous and let τ be a (BM) measure on K. Consider random polynomials $p_n(z) = \sum_{j=0}^n a_j b_j^{(n)}(z)$ where $\{b_j^{(n)}\}_{j=0,\ldots,n}$ form an orthonormal basis in $L^2(\tau)$ for $\mathcal{P}_n$ and where the coefficients $a_0, ..., a_n$ are i.i.d. complex Gaussian random variables with $\mathbf{E}(a_j) = \mathbf{E}(a_j a_k) = 0$ and $\mathbf{E}(a_j \bar{a}_k) = \delta_{jk}$. With $\mathcal{P} := \otimes_{n=1}^\infty (\mathcal{P}_n, Prob_n)$ as before,*

$$\lim_{n\to\infty} \mathbf{E}(\tilde{Z}_{p_n}) = \mu_K.$$

Remark 5.3. This is a *universality* result; regardless of the choice of (BM) measure τ (and hence orthonormal basis $\{b_j^{(n)}\}_{j=0,...,n}$), the limiting expectation is always μ_K. One can severely weaken the hypothesis that the coefficients $a_0, ..., a_n$ are i.i.d. complex Gaussian; the same is true for Theorem 5.4 below.

We can also prove an almost surely convergence result for the subharmonic functions $\{\frac{1}{n}\log|p_n|\}$.

Theorem 5.4. *For $K \subset \mathbb{C}$ compact with V_K continuous and τ a (BM) measure on K, consider random polynomials $p_n(z) = \sum_{j=0}^n a_j b_j^{(n)}(z)$ where $\{b_j^{(n)}\}_{j=0,...,n}$ form an orthonormal basis in $L^2(\tau)$ for $\mathcal{P}_n$ and where the coefficients $a_0, ..., a_n$ are i.i.d. complex Gaussians. Then almost surely in $\mathcal{P}$ we have*

$$\big(\limsup_{n\to\infty} \frac{1}{n}\log|p_n(z)|\big)^* = V_K(z)$$

pointwise for all $z \in \mathbb{C}$ and $\frac{1}{n}\log|p_n| \to V_K$ in $L^1_{loc}(\mathbb{C})$. Hence almost surely

$$\Delta(\frac{1}{n}\log|p_n|) \to \mu_K.$$

We remind the reader that Bernstein-Markov measures exist in abundance; e.g., if K is regular then μ_K is a Bernstein-Markov measure. Even discrete measures can be Bernstein-Markov measures; indeed, in the proof of Proposition 2.20, we used Fekete points to prove that for $K \subset \mathbb{C}$ an arbitrary compact set there exists a measure $\nu \in \mathcal{M}(K)$ such that (K, ν) satisfies a Bernstein-Markov property.

We can easily generalize Theorem 5.4 to the case where each a_j has distribution $\phi(t)dm(t)$ where ϕ has the properties that for some $T > 0$, we have

$$|\phi(z)| \leq T \text{ for all } z \in \mathbb{C}; \tag{5.4}$$

$$|\int_{|z|\geq R} \phi(z)dm_2(z)| \leq T/R^2 \text{ for all } R \text{ sufficiently large.} \tag{5.5}$$

We will require the Borel-Cantelli lemma.

Lemma 5.5. *Let $\{E_n\} \subset \mathcal{F}$ be a sequence of events on some probability space $(\Omega, \mathcal{F}, \Pr)$. If the sum of the probabilities of the E_n is finite, i.e.,*

$$\sum_{n=1}^{\infty} \Pr(E_n) < \infty,$$

then the probability that infinitely many of them occur is 0*:*

$$\Pr\left(\limsup_{n\to\infty} E_n\right) = \Pr\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = 0.$$

We proceed with the proof of Theorem 5.4.

Proof. Let $\{w^{(n)} := (w_{n0}, ..., w_{nn})\}_{n=1,2,...}$ be a sequence of non-zero vectors $w^{(n)} \in \mathbb{C}^{n+1}$. Identify $\mathcal{P} = \otimes_{n=1}^{\infty}\mathcal{P}_n \simeq \otimes_{n=1}^{\infty}\mathbb{C}^{n+1}$. Let

$$\mathcal{A} := \{\{a^{(n)} := (a_{n0}, ..., a_{nn})\}_{n=1,2,...} \in \mathcal{P} : \frac{|< a^{(n)}, w^{(n)} >|}{||w^{(n)}||} \geq 1/n^2$$
$$\text{for } n \text{ sufficiently large}\}$$

and

$$\mathcal{A}' := \{\{a^{(n)} := (a_{n1}, ..., a_{nn})\}_{n=1,2,...} \in \mathcal{P} : \frac{|< a^{(n)}, w^{(n)} >|}{||w^{(n)}||} \leq n^2$$
$$\text{for } n \text{ sufficiently large}\}$$

(think: $w^{(n)} = b^{(n)}(z)$ so $\frac{w^{(n)}}{||w^{(n)}||} = u^{(n)}(z) = \frac{b^{(n)}(z)}{||b^{(n)}(z)||}$). Then $\mathcal{A}, \mathcal{A}'$ are of probability one in $\mathcal{P}$ from Borel-Cantelli and properties of the complex Gaussian.

To see this, by rescaling we may assume $||w^{(n)}|| = 1$. We first work with $\mathcal{A}$ and consider

$$Prob_n\{a^{(n)} \in \mathbb{C}^{n+1} : |< a^{(n)}, w^{(n)} >| \leq 1/n^2\}$$

$$= \int_{|<a^{(n)},w^{(n)}>|\leq 1/n^2} \phi(a_{n0})\cdots\phi(a_{nn})dm_2(a_{n0})\cdots dm_2(a_{nn}). \tag{5.6}$$

We may assume $|w_{n0}| \geq 1/\sqrt{n}$ and we make the complex-linear change of coordinates on $\mathbb{C}^{n+1}$ given by:

$$\alpha_0 := a_{n0}w_{n0} + \cdots + a_{nn}w_{nn},\ \alpha_1 = a_{n1}, \cdots, \alpha_n = a_{nn}.$$

Then (5.6) becomes

$$\int_{\mathbb{C}^n}\int_{|\alpha_0|\leq 1/n^2} \frac{1}{|w_{n0}|^2}\phi(\frac{\alpha_0 - \alpha_1 w_{n1} - \cdots - \alpha_n w_{nn}}{w_{n0}})\phi(\alpha_1)\cdots\phi(\alpha_n)$$
$$dm_2(\alpha_0)\cdots dm_2(\alpha_n).$$

Using (5.4) this is bounded above by

$$n|\int_{|\alpha_0|\leq 1/n^2} Tdm_2(\alpha_1)| \leq \pi T/n^3.$$

The result follows from Lemma 5.5. Note that the set $\mathcal{A}$ depends on $\{w^{(n)}\}$ but for each $\{w^{(n)}\}$ the corresponding set is of probability one in $\mathcal{P}$.

For $\mathcal{A}'$ (which contains the *same* set of probability one for each $\{w^{(n)}\}$), and with $||w^{(n)}|| = 1$,

$$| < a^{(n)}, w^{(n)} > | \leq ||a^{(n)}|| \cdot ||w^{(n)}|| = ||a^{(n)}||.$$

We have

$$Prob_n\{a^{(n)} \in \mathbb{C}^{n+1} : ||a^{(n)}|| \geq n^2\} = Prob_n\{a^{(n)} \in \mathbb{C}^n : \sum_{j=0}^{n} |a_{nj}|^2 \geq n^4\}$$

$$\leq Prob_n\{a^{(n)} \in \mathbb{C}^{n+1} : |a_{nj}| \geq n^{3/2} \text{ for some } j = 0, ..., n\}$$

$$= nProb_n\{a^{(n)} \in \mathbb{C}^{n+1} : |a_{n1}| \geq n^{3/2}\} \leq n\frac{T}{n^3} = \frac{T}{n^2}$$

by (5.5). The result again follows from Lemma 5.5.

The conclusion of these estimates is that for ϕ satisfying (5.4) and (5.5), with probability one in $\mathcal{P}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log | < a^{(n)}, w^{(n)} > | \leq \limsup_{n\to\infty} \frac{1}{n} \log ||w^{(n)}|| \tag{5.7}$$

for all $\{w^{(n)}\}$. For each $\{w^{(n)}\}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log | < a^{(n)}, w^{(n)} > | \geq \liminf_{n\to\infty} \frac{1}{n} \log ||w^{(n)}|| \tag{5.8}$$

with probability one in $\mathcal{P}$; i.e., for each $\{w^{(n)}\}$, the set

$$\{\{a^{(n)} := (a_{n1}, ..., a_{nn})\}_{n=1,2,...} \in \mathcal{P} : (5.8) \text{ holds}\}$$

depends on $\{w^{(n)}\}$ but is always of probability one.

Thus, given $z \in \mathbb{C}$, take

$$w^{(n)} := b^{(n)}(z) = (b_0^{(n)}(z), ..., b_n^{(n)}(z)) \in \mathbb{C}^{n+1};$$

almost surely in $\mathcal{P}$

$$\limsup_{n\to\infty} \frac{1}{n} \log |p_n(z)| \leq V_K(z) \tag{5.9}$$

for all $z \in \mathbb{C}$.

Fix a countable dense subset $\{z_t\}_{t\in S}$ of $\mathbb{C}$. For each z_t, almost surely in $\mathcal{P}$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log |p_n(z_t)| \geq V_K(z_t). \tag{5.10}$$

A countable intersection of sets of probability one is a set of probability one; thus (5.10) holds almost surely in $\mathcal{P}$ for each $z_t,\ t \in S$. Define

$$H(z) := \bigl(\limsup_{n\to\infty} \frac{1}{n} \log |p_n(z)|\bigr)^*.$$

From (5.9), $H(z) \leq V_K^*(z)$ for all $z \in \mathbb{C}$ and H is subharmonic. By (5.10), $H(z_t) \geq V_K(z_t)$ for all $t \in S$. Now given $z \in \mathbb{C}$ at which V_K is continuous, let $S' \subset S$ with $\{z_t\}_{t\in S'}$ converging to z. Then,

$$V_K(z) = \lim_{t\in S',\ z_t\to z} V_K(z_t) \leq \limsup_{t\in S',\ z_t\to z} H(z_t) \leq H(z).$$

Thus $H(z) = V_K(z)$ for all $z \in \mathbb{C}$ at which V_K is continuous. But V_K is continuous a.e. in $\mathbb{C}$; since $H(z) = V_K(z)$ a.e. they are equal everywhere.

The $L^1_{loc}(\mathbb{C})$ result follows from

- a similar argument to show almost surely in $\mathcal{P}$, for *any* subsequence J of positive integers,

  $$\bigl(\limsup_{n\in J} \frac{1}{n} \log |p_n(z)|\bigr)^* = V_K(z)$$

  for all $z \in \mathbb{C}$; and
- Hartogs' lemma (Lemma 3.6): the sequence $\{\frac{1}{n} \log |p_n|\}$ is locally bounded above (since V_K is);

and a proof by contradiction. We refer the reader to [18] for details. $\square$

6. Weighted potential theory in $\mathbb{C}$.

Let $K \subset \mathbb{C}$ be closed and let w be an *admissible* weight function on K: w is a nonnegative, uppersemicontinuous function with $\{z \in K : w(z) > 0\}$ nonpolar − hence K is not polar. If K is unbounded, we require that w satisfies the growth property

$$|z|w(z) \to 0 \text{ as } |z| \to \infty, \ z \in K. \tag{6.1}$$

We write $Q := -\log w$ and denote the collection of lowersemicontinuous Q of this form as $\mathcal{A}(K)$. Then (6.1) can be written as

$$Q(z) - \log|z| \to +\infty \text{ as } |z| \to \infty, \ z \in K. \tag{6.2}$$

Associated to K, Q is a *weighted energy minimization problem*: for a probability measure τ on K, consider the weighted energy

$$I^w(\tau) := \int_K \int_K \log \frac{1}{|z-t|w(z)w(t)} d\tau(t)d\tau(z) = I(\tau) + 2\int_K Q d\tau$$

and find $\inf_\tau I^w(\tau)$ where the infimum is taken over all probability measures τ with compact support in K. This is often referred to as *a logarithmic energy minimization in the presence of an external field Q*. The case $w \equiv 1$ on K; i.e., $Q \equiv 0$, is the "unweighted" case (here we need K to be compact). The existence and uniqueness of a weighted energy minimizing measure $\mu_{K,Q}$, i.e., $\mu_{K,Q} \in \mathcal{M}(K)$ and

$$I^w(\mu_{K,Q}) = \inf_{\tau \in \mathcal{M}(K)} I^w(\tau) =: V_w,$$

follows as in the unweighted case. Moreover, condition (6.1) implies that we need only consider measures with support in a fixed compact set.

Proposition 6.1. *For $\epsilon > 0$ sufficiently small,*

$$\inf_{\tau \in \mathcal{M}(K)} I^w(\tau) = \inf_{\tau \in \mathcal{M}(K^\epsilon)} I^w(\tau)$$

where $K^\epsilon := \{z \in K : w(z) \geq \epsilon\}$.

Proof. We first show for $\epsilon > 0$ sufficiently small

$$\log \frac{1}{|z-t|w(z)w(t)} > V_w + 1 \text{ if } (z,t) \notin K^\epsilon \times K^\epsilon. \tag{6.3}$$

Indeed, it suffices to show that for any sequence $\{(z_n, t_n)\} \subset K \times K$ with

$$\lim_{n\to\infty} \min[w(z_n), w(t_n)] = 0,$$

$$\lim_{n\to\infty} \log \frac{1}{|z_n - t_n| w(z_n) w(t_n)} = +\infty.$$

Without loss of generality we may assume $z_n \to z,\ t_n \to t$ where either or both of z, t may be the point at infinity in the extended complex plane. If both $z, t \in \mathbb{C}$, the result is clear. If, e.g., $z = \infty$ and $t \in \mathbb{C}$, since $|z_n - t_n| w(z_n) \to 0$ from (6.1), the result follows in this case. If $z = t = \infty$, we use the estimate $|z_n - t_n| \leq 2\max(|z_n|, |t_n|)$ and (6.1) to conclude.

Using this ϵ, we show that for any $\mu \in \mathcal{M}(K)$ with $\mathrm{supp}(\mu) \cap (\mathbb{C} \setminus K^\epsilon) \neq \emptyset$ and $I^w(\mu) < V_w + 1$ there exists $\tilde{\mu} \in \mathcal{M}(K^\epsilon)$ such that $I^w(\tilde{\mu}) < I^w(\mu)$. Indeed, take $\tilde{\mu} := \frac{\mu|_{K^\epsilon}}{\mu(K^\epsilon)}$. Note that $\mu(K^\epsilon) > 0$ from $I^w(\mu) < V_w + 1$ and (6.3). Then

$$I^w(\mu) = \Big(\int_{K^\epsilon}\int_{K^\epsilon} + \int\int_{\mathbb{C}^2 \setminus K^\epsilon \times K^\epsilon}\Big) \log \frac{1}{|z-t| w(z) w(t)} d\mu(t) d\mu(z)$$

$$> [\mu(K^\epsilon)]^2 I^w(\tilde{\mu}) + \big(1 - [\mu(K^\epsilon)]^2\big)(V_w + 1).$$

Using $I^w(\mu) < V_w + 1$ gives the result.

$\square$

We next prove a weighted version of Theorem 2.15. Given a closed set $K \subset \mathbb{C}$ and $Q \in \mathcal{A}(K)$, let

$$F := F(K, Q) := I^w(\mu_{K,Q}) - \int_K Q d\mu_{K,Q} = V_w - \int_K Q d\mu_{K,Q}.$$

Note if $Q \equiv 0$, then $F = I(\mu_K)$ for K compact (and non-polar). If $Q \not\equiv 0$,

$$F = I(\mu_{K,Q}) + 2\int_K Q d\mu_{K,Q} - \int_K Q d\mu_{K,Q}$$

$$= I(\mu_{K,Q}) + \int_K Q d\mu_{K,Q} = \int_K [p_{\mu_{K,Q}} + Q] d\mu_{K,Q}.$$

Theorem 6.2. *Given a closed set $K \subset \mathbb{C}$ and $Q \in \mathcal{A}(K)$,*

$$p_{\mu_{K,Q}} + Q \geq F \text{ q.e. on } K; \tag{6.4}$$

$$p_{\mu_{K,Q}} + Q \leq F \text{ on } S_w := supp(\mu_{K,Q}). \tag{6.5}$$

In particular, $p_{\mu_{K,Q}} + Q = F$ q.e. on S_w.

Note that Q is only defined on K while $F - p_{\mu_{K,Q}} = F + V_{\mu_{K,Q}}$ is defined in all of $\mathbb{C}$.

Proof. We prove (6.4) and leave (6.5) for the reader. Let $\mu := \mu_{K,Q}$ and $U := p_{\mu_{K,Q}} + Q = p_\mu + Q$. From Proposition 6.1 we may assume K is compact (note to prove (6.4) q.e. on K it suffices to prove it q.e. on K^ϵ for each ϵ). Since p_μ, Q are lsc, so is U and hence

$$\{z \in K : U(z) \leq a\}$$

is closed for all $a \in \mathbb{R}$.

For the sake of obtaining a contradiction, we assume

$$A := \{z \in K : U(z) < F\}$$

is not polar. Thus we can find n_0 sufficiently large so that the compact set

$$E_1 := \{z \in K : U(z) \leq F - 1/n_0,\ |z| \leq n_0\}$$

is not polar. Next, since

$$\int_K U d\mu = I(\mu) + \int_K Q d\mu = F,$$

there exists a compact subset $E_2 \subset S_w$ with $m := \mu(E_2) > 0$ and such that $U(z) > F - \frac{1}{2n_0}$ for $z \in E_2$. Note that $E_1 \cap E_2 = \emptyset$.

Since E_1 is not polar, there exists a positive measure σ on E_1 with $\sigma(E_1) = m$ such that $I(\sigma) < \infty$. Indeed, $I^w(\sigma) < \infty$ since $Q = -\log w \leq -\log \epsilon$ on K^ϵ (and we assume $K = K^\epsilon$ for some ϵ). Then the signed measure σ_1 on K defined as

$$\sigma \text{ on } E_1;\ -\mu|_{E_2} \text{ on } E_2;\ 0 \text{ otherwise}$$

has total mass 0. It is easy to see that for any $0 \leq t < 1$, the measure $\mu + t\sigma_1 \in \mathcal{M}(K)$ and has compact support. We show that for $t > 0$ sufficiently small,

$$I^w(\mu + t\sigma_1) < I^w(\mu),$$

a contradiction. Indeed,

$$I^w(\mu + t\sigma_1) = I(\mu + t\sigma_1) + 2\int_K Q d(\mu + t\sigma_1)$$

$$= I(\mu) + 2 < \mu, t\sigma_1 > +2\int_K Q d(\mu + t\sigma_1) + 0(t^2)$$

$$= I^w(\mu) + 2t\int_K U d\sigma_1 + 0(t^2)$$

$$\leq I^w(\mu) + 2t[m(F - 1/n_0) - m(F - \frac{1}{2n_0})] + 0(t^2)$$

$$= I^w(\mu) - \frac{2tm}{2tn_0} + 0(t^2) < I^w(\mu)$$

for $t > 0$ sufficiently small.

□

We note, for future use, that $p_{\mu_{K,Q}} \in L^\infty_{loc}(\mathbb{C})$. Indeed, this function is lsc and hence locally bounded below; from the theorem, $p_{\mu_{K,Q}} \leq F - Q \leq M$ (constant) on S_w and hence by Proposition 1.10, $p_{\mu_{K,Q}} \leq M$ on $\mathbb{C}$.

As an application of Theorem 6.2 we prove the existence of *balayage measures*. Given a bounded domain $G \subset \mathbb{C}$ and a finite Borel measure ν with compact support in G, the *balayage problem* is to find a finite measure $\widehat{\nu}$ on ∂G with the same total mass as ν such that

$$p_{\widehat{\nu}} = p_\nu \text{ q.e. on } \partial G.$$

In the case of an unbounded domain $G \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we weaken this condition to

$$p_{\widehat{\nu}} = p_\nu + c \text{ q.e. on } \partial G$$

for some constant c.

To give some examples, consider first $G = B(0,1)$ and $\nu = \delta_0$, a point mass at the origin. Since $-p_\nu(z) = \log|z| = 0$ on $S^1 = \partial B(0,1)$, clearly we can take $\widehat{\nu} = \frac{1}{2\pi}d\theta$ since $-p_{\widehat{\nu}}(z) = \log^+|z| = 0$ on S^1. Note that for any $h : \overline{B(0,1)} \to \mathbb{R}$ which is continuous on $\overline{B(0,1)}$ and harmonic on $B(0,1)$ we have trivially that

$$\int_G h d\nu = h(0) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})d\theta = \int_{\partial G} h d\widehat{\nu}.$$

More generally, for any point z_0 in the open unit disk $B(0,1)$, if $\nu = \delta_{z_0}$ we can take $d\widehat{\nu} = \frac{1}{2\pi}\frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2}d\theta$ and the same conclusions hold.

For an unbounded example, take $G = \{z : |z| > 1\}$ and for $R > 1$ take $d\nu = \frac{1}{2\pi}d\theta$ on $|z| = R$. then

$$-p_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|z - Re^{i\theta}|d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|z/R - e^{i\theta}|d\theta + \log R = \log^+|z|/R + \log R$$

which equals $\log R$ on $|z| = 1$. One checks easily that $\widehat{\nu} = \frac{1}{2\pi}d\theta$ on $|z| = 1$ satisfies

$$p_{\widehat{\nu}} = p_\nu + \log R \text{ on } \partial G.$$

Proposition 6.3. *Let $G \subset \mathbb{C}$ be a bounded domain and let ν be a finite Borel measure with compact support in G. Then there exists a unique measure $\widehat{\nu}$ with support in ∂G satisfying*

(1) $\widehat{\nu}(\partial G) = \nu(G)$;
(2) $p_{\widehat{\nu}}$ *is bounded on* ∂G; *and*
(3) $p_{\widehat{\nu}} = p_\nu$ *q.e. on* ∂G.

Moreover

(1) $p_{\widehat{\nu}} \leq p_\nu$ *on* $\mathbb{C}$;
(2) $p_{\widehat{\nu}} = p_\nu$ *on* $\mathbb{C} \setminus \overline{G}$; *and*
(3) *for any* $h : \overline{G} \to \mathbb{R}$ *which is continuous on* $\overline{G}$ *and harmonic on* G *we have*
$$\int_G h d\nu = \int_{\partial G} h d\widehat{\nu}.$$

The uniqueness is due to

(1) $\widehat{\nu}(\partial G) = \nu(G)$;
(2) $p_{\widehat{\nu}}$ is bounded on ∂G.

Indeed, take $G = \{z : 0 < |z| < 1\}$, the punctured disk, and $\nu = \frac{1}{2\pi} d\theta$ on $|z| = 1/2$. It is easy to see that, setting $\mu := \frac{1}{2\pi} d\theta$ on $|z| = 1$, for any $0 \leq a \leq 1$, the convex combination $\mu_a := a\delta_0 + (1-a)\mu$ works as $\widehat{\nu}$ in the sense that $p_{\widehat{\nu}} = p_\nu$ q.e. on ∂G; but only for $a = 0$ do we have $p_{\widehat{\nu}}$ is bounded on ∂G. Similarly, for any $a > 0$, $\mu_a := a\mu$ works as $\widehat{\nu}$ in the sense that $p_{\widehat{\nu}} = p_\nu$ q.e. on ∂G; but only for $a = 1$ do we have $\widehat{\nu}(\partial G) = \nu(G)$.

Before proving the proposition, we recall the global domination principle (GDP) in the form of Proposition 2.17: for μ, ν finite measures of the same total mass with compact support and $I(\mu) < \infty$, if for some constant c we have
$$p_\mu(z) \leq p_\nu(z) + c \ \mu - \text{a.e.}$$
then $p_\mu(z) \leq p_\nu(z) + c$ for all $z \in \mathbb{C}$.

Proof. We may assume $\nu(G) = 1$ and we seek $\widehat{\nu} \in \partial G$ satisfying $p_{\widehat{\nu}}$ bounded on ∂G and $p_{\widehat{\nu}} = p_\nu$ q.e. on ∂G. Let $K := \partial G$ and note that K is not polar since G is bounded. Define $Q(z) := -p_\nu(z)$. Observe this is continuous on K (and hence is admissible).

We claim that $\mu_{K,Q} = \widehat{\nu}$. To see this, we have $\widehat{\nu} \in \mathcal{M}(K)$ and from our earlier observation, $p_{\mu_{K,Q}} \in L^\infty_{loc}(\mathbb{C})$ so that $p_{\mu_{K,Q}}$ is bounded on K.

By Theorem 6.2,

$$p_{\mu_{K,Q}} \geq p_\nu + F \text{ q.e. on } K \text{ and}$$
$$p_{\mu_{K,Q}} \leq p_\nu + F \text{ on } S_w.$$

Since $I(\mu_{K,Q}) < \infty$, by Proposition 2.17,

$$p_{\mu_{K,Q}} \leq p_\nu + F \text{ on } \mathbb{C}.$$

But then we have

$$p_{\mu_{K,Q}} = p_\nu + F \text{ q.e. on } K$$

and we are left to show that $F = 0$. Integrating this equality with respect to the *unweighted* equilibrium measure μ_K for K and recalling that $p_{\mu_K} = I(\mu_K)$ q.e. on K, we get

$$\int_K p_{\mu_{K,Q}} d\mu_K = \int_K p_{\mu_K} d\mu_{K,Q} = I(\mu_K)$$

while

$$\int_K (p_\nu + F) d\mu_K = \int_K p_{\mu_K} d\nu + F = I(\mu_K) + F,$$

giving the result. We leave the rest to the reader. □

Remark 6.4. Recall for $K = S^1$ (or $K = \overline{B(0,1)}$) the Chebyshev polynomials are the monomials $t_n(z) = z^n$ while the polynomials $F_n(z) = z^n - 1$ are Fekete polynomials. Comparing the normalized zero measures μ_n of these two sequences, for the Fekete polynomials we get

$$\mu_n \to \frac{1}{2\pi} d\theta = d\mu_K \text{ weak-*}$$

while for the Chebyshev polynomials we get

$$\mu_n = \delta_0 \text{ for all } n.$$

Here, if $G = \{z : |z| > 1\}$, μ_K *is the balayage measure of* δ_0. More generally, we state the following, without proof.

Proposition 6.5. *Let $K \subset \mathbb{C}$ be compact and non-polar and let G be the unbounded component of $\mathbb{C} \setminus K$. Let $\{p_n(z) = z^n + \cdots\}$ be a sequence of monic polynomials with $\lim_{n\to\infty} ||p_n||_K^{1/n} = \delta(K)$. Let $\{\mu_n\}$ be the sequence of normalized zero measures of $\{p_n\}$. Then any weak-* subsequential limit σ of $\{\mu_n\}$ satisfies $\widehat{\sigma} = \mu_K$.*

Returning to the potential $p_{\mu_{K,Q}}$ of the weighted equilibrium measure, we note that

(1) $F - p_{\mu_{K,Q}} = F + V_{\mu_{K,Q}} \in L(\mathbb{C})$;
(2) $F + V_{\mu_{K,Q}} = Q$ q.e. on K;
(3) $S_w = \mathrm{supp}(\mu_{K,Q})$ is a compact subset of K.

In the unweighted case, if $K \subset \mathbb{C}$ is compact and non-polar, $\mu_K \in \mathcal{M}(K)$ with

$$V_{\mu_K} - I(\mu_K) \in L(C) \text{ and } V_{\mu_K} - I(\mu_K) = 0 \text{ q.e. on } K.$$

Then we saw that $V_{\mu_K} - I(\mu_K) = V_K^*$ where

$$V_K(z) := \sup\{u(z) : u \in L(\mathbb{C}),\ u \leq 0 \text{ on } K\}.$$

We have a weighted version of this. We define

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}),\ u \leq Q \text{ on } K\}.$$

For $K \subset \mathbb{C}$ compact, we say K is locally regular if for each $z \in K$ the unweighted Green function for the sets $K \cap \overline{B(z,r)},\ r > 0$ are continuous at z. Here $B(z,r)$ denotes the Euclidean disk with center z and radius r. In this one-variable setting, local regularity of K is equivalent to (global) regularity; i.e., $V_K = V_K^*$ is continuous. If K is regular and Q is continuous, then $V_{K,Q}$ is continuous. We have the elementary fact that for such K and Q,

$$V_{K,Q}(z) = V_{K,Q}^*(z) \leq Q(z) \text{ on } K. \tag{6.6}$$

We relate $V_{K,Q}^*$ and $F + V_{\mu_{K,Q}}$.

Proposition 6.6. *$V_{K,Q}^* = F + V_{\mu_{K,Q}}$ on $\mathbb{C}$.*

This will follow from the GDP in the form of Proposition 2.17 applied to $V_{K,Q}^*, F + V_{\mu_{K,Q}}$ after we verify that $V_{K,Q}^*$ satisfies properties analogous to (1)-(3) of $F + V_{\mu_{K,Q}}$. Note we know that $I(\mu_{K,Q}) < \infty$ since $I^w(\mu_{K,Q}) < \infty$.

To this end, recall Remark 3.8: if $\mathcal{U}$ is a family of functions in $L(\mathbb{C})$ and $u(z) := \sup\{v(z) : v \in \mathcal{U}\}$, then either $P := \{z \in \mathbb{C} : u(z) < +\infty\}$ is polar or $\mathcal{U}$ is locally bounded above and $u^* \in L(\mathbb{C})$. Since in our weighted setting, K is not polar so that K^ϵ is not polar for ϵ sufficiently small, we use this to show

$$\mathcal{U} := \{u(z) : u \in L(\mathbb{C}),\ u \leq Q \text{ on } K\}$$

is locally uniformly bounded above. Indeed, if $v \in \mathcal{U}$ then $v \leq -\log \epsilon$ on K^ϵ which shows that

$$v \leq -\log \epsilon + V_{K^\epsilon} \text{ on } \mathbb{C},$$

yielding the result. Thus $V_{K,Q}^* \in L(\mathbb{C})$.

Next, note that $\mathcal{U}$ is closed under Poisson modification on disks disjoint from K. Thus $V_{K,Q}^*$ is *harmonic* outside K; i.e., $\mathrm{supp}(\Delta V_{K,Q}^*) \subset K$. Indeed, we show more:

Proposition 6.7. *With K, Q as above*

$$supp(\Delta V_{K,Q}^*) \subset S_w^* := \{z \in K : V_{K,Q}^*(z) \geq Q(z)\}.$$

Proof. Let $z_0 \in K \setminus S_w^*$. Then $V_{K,Q}^*(z_0) < Q(z_0)$. By usc of $V_{K,Q}^*$ and lsc of Q, there exists $r > 0$ with

$$\sup_{z \in B(z_0,r)} V_{K,Q}^*(z) < \inf_{z \in B(z_0,r)} Q(z).$$

Let

$$u(z) := V_{K,Q}^*(z) \text{ if } z \notin B(z_0,r);\ u(z) := P_{V_{K,Q}^*|_{\partial B(z_0,r)}}(z),\ z \in B(z_0,r).$$

Then $u \in L(\mathbb{C})$ with $u \leq Q$ on K so that

$$u \leq V_{K,Q} \leq V_{K,Q}^* \text{ in } \mathbb{C};$$

but by construction, $u \geq V_{K,Q}^*$ in $\mathbb{C}$ and equality holds. Thus $\Delta V_{K,Q}^* = 0$ on $B(z_0, r)$.

□

A useful observation is that if $K_1 \subset K_2$ are closed subsets of $\mathbb{C}$ with $Q \in \mathcal{A}(K_1) \cap \mathcal{A}(K_2)$, then $V_{K_1,Q} \geq V_{K_2,Q}$. We use this, together with (6.2),

$$Q(z) - \log|z| \to +\infty \text{ as } |z| \to \infty,\ z \in K,$$

to prove:

Proposition 6.8. *With K, Q as above, for R sufficiently large,*

$$V_{K,Q} = V_{K \cap \overline{B(0,R)},Q}.$$

Proof. Since K is not polar, $K \cap \overline{B(0,R)}$ is not polar for R large; and, similarly, $\{z \in K \cap \overline{B(0,R)} : w(z) > 0\}$ is not polar for R large. In particular, $V_{K \cap \overline{B(0,R)},Q}^* \in L(\mathbb{C})$ for R large so that there exists c with

$$V_{K \cap \overline{B(0,R)},Q}^* \leq \log|z| + c,\ z \in \mathbb{C}.$$

Take $R' > R$ large so that

$$Q(z) > \log|z| + c + 1 \text{ for } z \in K \setminus B(0,R')$$

and then we have

$$V^*_{K\cap\overline{B(0,R')},Q} \le V^*_{K\cap\overline{B(0,R)},Q} \le \log|z| + c,\ z\in\mathbb{C}.$$

Then for any $u \in L(\mathbb{C})$ with $u \le Q$ on $K \cap \overline{B(0,R')}$, we have $u \le V^*_{K\cap\overline{B(0,R')},Q}$ so from the above inequalities $u \le Q$ on K. Hence

$$V^*_{K\cap\overline{B(0,R')},Q} \le V^*_{K,Q}.$$

The reverse inequality is clear.

□

We conclude that our function $V^*_{K,Q}$ satisfies:

(1) $V^*_{K,Q} \in L(\mathbb{C})$;
(2) $V^*_{K,Q} \le Q$ q.e. on K and $V^*_{K,Q} = Q$ q.e. on $S^*_w \subset K$;
(3) $\mathrm{supp}(\mu_{K,Q})$ is a compact subset of K.

Hence from the GDP we have $V^*_{K,Q} = F + V_{\mu_{K,Q}}$ on $\mathbb{C}$. Moreover,

$$S_w = \mathrm{supp}(\mu_{K,Q}) \subset S^*_w = \{z \in K : V^*_{K,Q}(z) \ge Q(z)\}; \tag{6.7}$$

indeed, $V^*_{K,Q} = Q$ on $\mathrm{supp}(\mu_{K,Q})$ except perhaps for a polar set.

We next give a type of converse to the weighted Frostman Theorem 6.2.

Proposition 6.9. *Let $\mu \in \mathcal{M}(K)$ have compact support. If there exists a constant c such that*

$$p_\mu(z) + Q(z) \ge c \text{ q.e. on } K \text{ and}$$

$$p_\mu(z) + Q(z) \le c \text{ on } supp(\mu),$$

then $\mu = \mu_{K,Q}$.

Note that the hypothesis implies that

$$\int_K (p_\mu + Q)d\mu = I(\mu) + \int_K Qd\mu \le c$$

so that $I(\mu), \int_K Qd\mu, I^w(\mu) < \infty$.

Proof. We write

$$\mu_{K,Q} = \mu + (\mu_{K,Q} - \mu).$$

Then

$$I^w(\mu) \ge I^w(\mu_{K,Q}) = I^w(\mu) + I(\mu_{K,Q} - \mu) + 2R$$

with

$$R := \int_K \big[\int_K \log \frac{1}{|x-y|} d\mu(y) + Q(x)\big] d(\mu_{K,Q} - \mu)(x)$$
$$= \int_K (p_\mu(x) + Q(x)) d(\mu_{K,Q} - \mu)(x).$$

Making use of the inequalities in the hypotheses, we conclude that

$$R \geq C \int_K d\mu_{K,Q} - C \int_K d\mu = 0.$$

Recall that $I(\mu_{K,Q} - \mu) \geq 0$ with equality if and only if $\mu_{K,Q} = \mu$. Thus

$$I^w(\mu) \geq I^w(\mu_{K,Q}) \geq I^w(\mu)$$

so that equality holds throughout, and $I^w(\mu) = I^w(\mu_{K,Q})$, from which follows $\mu = \mu_{K,Q}$.

$\square$

Let's look at some concrete examples of weighted equilibrium measures $\mu_{K,Q}$ and associated weighted extremal functions $V^*_{K,Q}$. We will see immediately that the presence of the weight Q (the "external field") affects the support S_w of $\mu_{K,Q}$.

Example 6.10. Let $K = \overline{B(0,1)}$. In this case, we know the unweighted equilibrium measure is $\mu_K = \frac{1}{2\pi} d\theta$ on $S^1 = \partial B(0,1)$. Now take $Q(z) = |z|^2$; i.e., $w(z) = e^{-|z|^2}$. We claim that $V^*_{K,Q} = V_{K,Q}$ (this follows since Q is continuous and K is (locally) regular) and

$$V_{K,Q}(z) = |z|^2 \text{ if } |z| \leq 1/\sqrt{2} \text{ while}$$
$$V_{K,Q}(z) = \log|z| + 1/2 - \log 1/\sqrt{2} \text{ if } |z| \geq 1/\sqrt{2}.$$

In particular, $S_w = \overline{B(0, 1/\sqrt{2})}$.

To see this, call $V(z)$ the function defined by the right-hand sides. Then clearly $V \leq V_{K,Q}$ since $V \in L(\mathbb{C})$ and $V \leq Q$ on K. Now ΔV is supported on $\overline{B(0,1/\sqrt{2})}$ and $I(\Delta V) < \infty$ (note ΔV is simply Lebesgue measure). Since $V = Q$ on this set, $V_{K,Q} \leq V$ on the support of ΔV. By the GDP (Proposition 2.17), $V_{K,Q} \leq V$ on $\mathbb{C}$.

The motivation for the definition of V comes from two things: first, in this setting, the weighted extremal function $V_{K,Q}$ should be radial; i.e., it should be a function of $r = |z|$. Next, we try to construct a function which equals Q until the full "mass" of ΔQ is used up: here, the total mass of $\Delta|z|^2$ on $\overline{B(0,1/\sqrt{2})}$ is one. Then we know $V_{K,Q}$ grows like $\log|z|$; so we "match up" the univariate functions r^2 and $\ln r + c$ so that no extra mass occurs on the matching circle: note the derivatives $2r$ and $1/r$ agree if $r = 1/\sqrt{2}$.

Indeed, taking $K = \mathbb{C}$ – or any closed set containing $\overline{B(0,1/\sqrt{2})}$ – and the same weight function $Q(z) = |z|^2$, one obtains the same weighted extremal function $V_{K,Q}$. This last result is a special case of the following: let $Q(z) = Q(|z|) = Q(r)$ be a radially symmetric weight function on $\mathbb{C}$ which is convex on $r > 0$. Let r_0 be the smallest number for which $Q'(r) > 0$ for all $r > r_0$ and let R_0 be the smallest solution of $R_0 Q'(R_0) = 1$. Then $S_w = \{z : r_0 \leq |z| \leq R_0\}$ and $d\mu_{K,Q}(r) = \frac{1}{2\pi}\big(rQ'(r)\big)' dr d\theta$. This is part of Theorem IV.6.1 of [32].

Example 6.11. A "real" version of the last example is to take $K = \mathbb{R}$ and $Q(x) = x^2$ (so $w(x) = e^{-x^2}$). In this case, it can be shown, using Proposition 6.9, that

$$d\mu_{K,Q}(x) = c\sqrt{4 - x^2}dx$$

(where c is chosen to make this a probability measure). See [32] for details. This is the well-known *semicircle law* associated with random matrix theory, specifically, the Gaussian unitary ensemble (GUE). Here, $S_w = [-2, 2]$. We will discuss this later.

Example 6.12. Let's return to the case of $K = \overline{B(0,1)}$ but now take $Q(z) = -|z|^2$; i.e., $w(z) = e^{|z|^2}$. In this case, if $u \in L(\mathbb{C})$ with $u \leq 0$ on $\overline{B(0,1)}$, then, in particular, $u \leq -1$ on $S^1 = \partial\overline{B(0,1)}$. But then by the maximum principle, $u \leq -1$ on all of $\overline{B(0,1)}$. It follows readily that $V_{K,Q}(z) = V_K(z) - 1 = \log^+|z| - 1$ and $\mu_{K,Q} = \frac{1}{2\pi}d\theta$ on S^1.

The associated discrete problem leads to the *weighted transfinite diameter of K with respect to w*:

$$\delta^w(K) := \lim_{n\to\infty}\Big[\max_{\lambda_i\in K}|VDM(\lambda_0,...,\lambda_n)|w(\lambda_0)^n\cdots w(\lambda_n)^n\Big]^{1/\binom{n+1}{2}}. \tag{6.8}$$

Here $VDM(\zeta_1,...,\zeta_n) = \det[\zeta_i^{j-1}]_{i,j=1,...,n} = \prod_{j<k}(\zeta_j - \zeta_k)$ is the classical Vandermonde determinant. The proof that the limit exists is similar to the unweighted case and is left as exercise 2. Points $\lambda_0,...,\lambda_n \in K$ for which

$$|VDM(\lambda_0,...,\lambda_n)|w(\lambda_0)^n\cdots w(\lambda_n)^n$$

$$= |\det \begin{bmatrix} 1 & \lambda_0 & \dots & \lambda_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^n \end{bmatrix} | \cdot w(\lambda_0)^n \cdots w(\lambda_n)^n$$

is maximal are called *weighted Fekete points of order* n. For future use, we write

$$\delta_n^w(K) := \bigl[\max_{\lambda_i \in K} |VDM(\lambda_0, ..., \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n\bigr]^{1/\binom{n+1}{2}}. \tag{6.9}$$

We have

$$\inf_\tau I^w(\tau) = -\log \delta^w(K). \tag{6.10}$$

We will prove the following fact, which says that for any doubly indexed array of points $\{z_k^{(n_j)}\}_{k=1,...,n_j;\ j=1,2,...}$ in E which is *asymptotically Fekete with respect to the weight* w, the limiting measures

$$d\mu_{n_j} := \frac{1}{n_j} \sum_{k=1}^{n_j} \delta_{z_k^{(n_j)}} \tag{6.11}$$

have the same weak-* limit, the weighted equilibrium measure $d\mu_{K,Q}$. To avoid some technical points, we give the proof for K compact.

Proposition 6.13. *Let $K \subset \mathbb{C}$ be compact and let w be an admissible weight on K. If, for a subsequence of positive integers $\{n_j\}$ with $n_j \uparrow \infty$, the points $z_1^{(n_j)}, ..., z_{n_j}^{(n_j)} \in K$ are chosen so that*

$$\lim_{j\to\infty} \bigl[|VDM(z_1^{(n_j)}, ..., z_{n_j}^{(n_j)})|^2 w(z_1^{(n_j)})^{2n_j} \cdots w(z_{n_j}^{(n_j)})^{2n_j}\bigr]^{1/n_j^2} = \delta^w(K),$$

then $d\mu_{n_j} \to d\mu_{K,Q}$ weak- where $d\mu_{n_j}$ is defined in (6.11).*

Proof. Take a subsequence of the measures $\{\mu_{n_j}\}$ which converges weak-* to a probability measure σ on K. We use the same notation for the subsequence and the original sequence. We show that $I^w(\sigma) = -\log \delta^w(K)$; by uniqueness of the weighted energy minimizing measure (6.10) we will then have $\sigma = \mu_{K,Q}$. First of all, choose continuous admissible weight functions $\{w_m\}$ with $w_m \downarrow w$ (recall w is usc!) and $w_m \geq \alpha_m > 0$ on K and for a real number M let

$$h_{M,m}(z,t) := \min[M, \log \frac{1}{|z-t| w_m(z) w_m(t)}] \leq \log \frac{1}{|z-t| w_m(z) w_m(t)}$$

and

$$h_M(z,t) := \min[M, \log \frac{1}{|z-t|w(z)w(t)}] \leq \log \frac{1}{|z-t|w(z)w(t)}.$$

Then $h_{M,m} \leq h_M$. By the Stone-Weierstrass theorem, every continuous function on $K \times K$ can be uniformly approximated by finite sums of the form $\sum_j f_j(z)g_j(t)$ where f_j, g_j are continuous on K; hence $\mu_{n_j} \times \mu_{n_j} \to \sigma \times \sigma$ and we have

$$I^w(\sigma) = \lim_{M\to\infty} \lim_{m\to\infty} \int_K \int_K h_{M,m}(z,t) d\sigma(z) d\sigma(t)$$

$$= \lim_{M\to\infty} \lim_{m\to\infty} \lim_{j\to\infty} \int_K \int_K h_{M,m}(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t)$$

$$\leq \lim_{M\to\infty} \limsup_{j\to\infty} \int_K \int_K h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t)$$

since $h_{M,m} \leq h_M$. Now

$$h_M(z_k^{(n_j)}, z_l^{(n_j)}) \leq \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}$$

if $k \neq l$ and hence

$$\int_K \int_K h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t) \leq$$

$$\frac{1}{n_j} M + (\frac{1}{n_j^2 - n_j}) [\sum_{k \neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}].$$

By assumption, given $\epsilon > 0$,

$$(\frac{1}{n_j^2 - n_j}) [\sum_{k \neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}] \leq -\log[\delta^w(K) - \epsilon]$$

for $j \geq j(\epsilon)$; in particular, $w(z_k^{(n_j)}) > 0$ for such j and hence

$$I^w(\sigma) \leq \lim_{M\to\infty} \limsup_{j\to\infty} \frac{1}{n_j} M - \log[\delta^w(K) - \epsilon] = -\log[\delta^w(K) - \epsilon]$$

for all $\epsilon > 0$; i.e., $I^w(\sigma) = -\log \delta^w(K)$. □

A weighted polynomial on K is a function of the form $w(z)^n p_n(z)$ where p_n is a holomorphic polynomial of degree at most n. As in the unweighted case, the weighted extremal function $V_{K,Q}$ can be obtained by using only polynomials; i.e.,

$$V_{K,Q}(z) = \sup\{\frac{1}{deg(p)} \log|p(z)| : p \text{ polynomial}, \ ||w^{deg(p)}p||_K \leq 1\}.$$

For $K \subset \mathbb{C}$ closed and w admissible on K, it follows that

$$||w^{deg(p)}p||_K < \infty, \ \forall \ p \in \cup_n \mathcal{P}_n$$

($\lim_{|z|\to\infty, \ z\in K} |z|w(z) = 0$ implies $\lim_{|z|\to\infty, \ z\in K} |z|^n(w(z))^n = 0$). It it easily verified that $p_n \to ||w^n p_n||_K$ is a norm on $\mathcal{P}_n$ for each n. Going back to Example 6.10 where $K = \mathbb{C}$ and $Q(z) = |z|^2$, consider $p_n(z) = z^n$ and the corresponding weighted polynomial $z^n e^{-n|z|^2}$. It is easily checked that the supremum of $|z|^n e^{-n|z|^2}$ on $\mathbb{C}$ is attained on $|z| = 1/\sqrt{2}$. This is an example of a weighted phenomenon: the "sup norm" of a weighted polynomial on K is the same as its "sup norm" on S_w, the support of the weighted equilibrium measure.

Proposition 6.14. *Let $K \subset \mathbb{C}$ be closed and w admissible on K. If $p_n \in \mathcal{P}_n$ and $w(z)^n|p_n(z)| \leq M$ q.e. on S_w, then $w(z)^n|p_n(z)| \leq M$ q.e. on K.*

Proof. The hypothesis $w(z)^n|p_n(z)| \leq M$ q.e. on S_w can be rewritten as

$$\frac{1}{n} \log \frac{|p_n(z)|}{M} \leq Q(z) \text{ q.e. on } S_w.$$

But recall that $V_{K,Q}^* = Q$ q.e. on S_w, so

$$\frac{1}{n} \log \frac{|p_n(z)|}{M} \leq V_{K,Q}^*(z) \text{ q.e. on } S_w.$$

By the global domination principle, this inequality holds pointwise on all of $\mathbb{C}$. This gives a *weighted Bernstein-Walsh inequality*:

$$|p_n(z)| \leq M e^{nV_{K,Q}^*(z)}, \ z \in \mathbb{C}$$

and, rewriting, for $z \in K$,

$$w(z)^n|p_n(z)| \leq M e^{n[V_{K,Q}^*(z) - Q(z)]}.$$

Since $V_{K,Q}^* \leq Q$ q.e. on K, the result follows. □

Let μ be a measure with support in K such that

$$\text{supp}(\mu) \cap \{z \in K : w(z) > 0\}$$

contains infinitely many points. We do not assume that $\mathrm{supp}(\mu)$ is compact nor do we assume $\mu(K) < \infty$. Then we can define, for $n = 1, 2, ...$ weighted $L^2(\mu)$ norms on $\mathcal{P}_n$: for $p_n \in \mathcal{P}_n$,

$$(6.12) \qquad ||p_n||_{L^2(w^{2n}d\mu)} = ||w^n p_n||_{L^2(\mu)} = [\int_K |p_n(z)|^2 w(z)^{2n} d\mu(z)]^{1/2}$$

provided this is finite. If K is not compact, we need to assume strong enough decay of $w(z)$ as $|z| \to \infty$ to insure that $||w^n p_n||_{L^2(\mu)} < \infty$ for $p_n \in \mathcal{P}_n$. Then (6.12) defines a norm on $\mathcal{P}_n$ and hence there exists M_n such that

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\mu)}, \ \forall \ p_n \in \mathcal{P}_n.$$

In Example 6.10, where $K = \mathbb{C}$ and $Q(z) = |z|^2$, we can take $d\mu = dm$, Lebesgue measure on $\mathbb{C}$. In this case, note that although $\mu(\mathbb{C}) = \infty$, for each n, the measures $d\mu_n = w(z)^{2n} dm(z)$ have finite total mass and $||w^n p_n||_{L^2(\mu)} < \infty$ for $p_n \in \mathcal{P}_n$. Similarly, in Example 6.11, where $K = \mathbb{R}$ and $Q(x) = x^2$, we can take $d\mu = dx$, Lebesgue measure on $\mathbb{R}$. Again, $\mu(\mathbb{R}) = \infty$ but for each n, the measures $d\mu_n = w(x)^{2n} dm(x)$ have finite total mass and $||w^n p_n||_{L^2(\mu)} < \infty$ for $p_n \in \mathcal{P}_n$.

We can construct an orthonormal basis $\{b_0^{(n)}, ..., b_n^{(n)}\}$ for $\mathcal{P}_n$ in the weighted L^2-space $L^2(w^{2n} d\mu)$ – note that because of the varying power of the weight there is in general no relation between $\{b_0^{(m)}, ..., b_n^{(m)}\}$ and $\{b_0^{(n)}, ..., b_n^{(n)}\}$ for $m \neq n$. Then

$$S_n^{\mu,w}(z,\zeta) := \sum_{j=0}^n b_j^{(n)}(z) \overline{b_j^{(n)}(\zeta)}$$

is the reproducing kernel for point evaluation at z: for $p_n \in \mathcal{P}_n$,

$$p_n(z) = \int_K p_n(\zeta) S_n^{\mu,w}(z,\zeta) w(\zeta)^{2n} d\mu(\zeta).$$

Let

$$\phi_{K,Q,n}(z) := \sup\{|p(z)| : p \in \mathcal{P}_n, \ ||w^n p_n||_K \leq 1\}.$$

As in the unweighted case,

$$\lim_{n\to\infty} \frac{1}{n} \log \phi_{K,Q,n}(z) = V_{K,Q}(z)$$

pointwise in $\mathbb{C}$ and if $V_{K,Q}$ is continuous then the convergence is locally uniform in $\mathbb{C}$. Again, as in the unweighted case, we have the double inequality

$$\frac{1}{(n+1)M_n^2} \leq \frac{[\phi_{K,Q,n}(z)]^2}{S_n^{\mu,w}(z,z)} \leq n+1.$$

Definition 6.15. We say μ is a *weighted Bernstein-Markov measure for* K,Q if for all for all $p_n \in \mathcal{P}_n$,

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\mu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1. \tag{6.13}$$

Example 6.16. For $K = \mathbb{C}$ and $Q(z) = |z|^2$, $d\mu = dm$, Lebesgue measure on $\mathbb{C}$ is a weighted Bernstein-Markov measure for K,Q. Indeed, from Proposition 6.10, it suffices to verify that $d\mu_{K,Q} = dm$ on $S_w = \overline{B(0,1/\sqrt{2})}$ is a weighted Bernstein-Markov measure for K,Q. Similarly, for $K = \mathbb{R}$ and $Q(x) = x^2$, $d\mu = dx$, Lebesgue measure on $\mathbb{R}$ is a weighted Bernstein-Markov measure for K,Q. Again, from Proposition 6.10, it suffices to verify that dx on $S_w = [-2,2]$ is a weighted Bernstein-Markov measure for K,Q.

Suppose $K \subset \mathbb{C}$ is closed; Q is an admissible weight on K; and μ is a weighted Bernstein-Markov measure for K,Q.

(1) If $V_{K,Q}$ is continuous, then

$$\lim_{n\to\infty} \frac{1}{2n} \log S_n^{\mu,w}(z,z) = V_{K,Q}(z)$$

locally uniformly in $\mathbb{C}$ and hence

$$\Delta \frac{1}{2n} \log S_n^{\mu,w} \to \mu_{K,Q}$$

weak-*. This is all that is needed to get weighted versions of Proposition 5.2 and Theorem 5.4.

(2) Define, analogous to (2.18),

$$Z_n = Z_n(K,w,\mu) :=$$

$$\int_{K^{n+1}} |VDM(\lambda_0,...,\lambda_n)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} d\mu(\lambda_0) \cdots d\mu(\lambda_n).$$

We have a weighted version of Theorem 2.21.

Theorem 6.17. *If μ is a weighted Bernstein-Markov measure for K,Q then*

$$\lim_{n\to\infty} Z_n^{1/n^2} = \delta^w(K).$$

Let's return to Example 6.11, $K = \mathbb{R}$ and $Q(x) = x^2$ (so $w(x) = e^{-x^2}$). In this case,

$$Z_n = \int_{\mathbb{R}^{n+1}} \prod_{j<k} (x_j - x_k)^2 e^{-2n \sum_{j=0}^n x_j^2} dx_0 \cdots dx_n.$$

We can define a probability measure P_n on $\mathbb{R}^{n+1}$ via, for $A \subset \mathbb{R}^{n+1}$,

$$P_n(A) := \frac{1}{Z_n} \int_A \prod_{j<k} (x_j - x_k)^2 e^{-2n \sum_{j=0}^n x_j^2} dx_0 \cdots dx_n.$$

In the GUE setting, this is the probability that a random $(n+1) \times (n+1)$ Hermitian matrix has (real) eigenvalues $a_0, a_1, ..., a_n \in \mathbb{R}$ with $(a_0, ..., a_n) \in A$.

Example 6.18. As an application of weighted potential theory, we consider the theory of incomplete polynomials. For simplicity, we work on the real interval $K = [0, 1]$. Given $0 < \theta < 1$, a $\theta-$incomplete polynomial is a polynomial of the form

$$p_N(x) = \sum_{k=s_N}^{N} a_k x^k$$

where $s_N/N \to \theta$ as $N \to \infty$. Thus such a polynomial is "missing" a fraction θ of its lowest degree terms. Taking $N = \frac{n}{1-\theta}$, we see that these incomplete polynomials are closely related to weighted polynomials $w(x)^n p_n(x)$ where $w(x) = x^{\frac{\theta}{1-\theta}}$. One can prove that $S_w = [\theta^2, 1]$. It turns out that *a continuous function f on $[0, 1]$ is the uniform limit of incomplete polynomials if and only if f vanishes on $[0, \theta^2]$ if and only if f is the uniform limit of weighted polynomials $w(x)^n p_n(x)$.* This is a special case of the general *weighted approximation problem*: given $K \subset \mathbb{C}$ closed and an admissible weight w on K, which $f \in C(K)$ can be uniformly approximated on K by a sequence of weighted polynomials $\{w^n p_n\}$? For details, see Chapter VI, section 1 of [32].

Note that the proofs of many of the results in the weighted situation are similar to their analogues in the unweighted case. We will see that in the case of $\mathbb{C}^N$, $N > 1$, the weighted theory is essential to prove results even in the unweighted case.

Exercises.

(1) Prove (6.5) of Theorem 6.2.

(2) Following the "unweighted" proof, verify that the limit
$$\lim_{n\to\infty} \delta_n^w(K) = \delta^w(K)$$
exists for a nonpolar set K and an admissible weight function w on K. Here $\delta_n^w(K)$ is defined in (6.9).

(3) Using the previous exercise, and observing that the function $VDM(\lambda_0, ..., \lambda_n)w(\lambda_0)^n \cdots w(\lambda_n)^n$ is a weighted polynomial of degree at most n in each variable, prove Theorem 6.17.

(4) Verify the formula for the remainder R in the proof of Proposition 6.9.

(5) Prove the weighted version of Corollary 4.3: let K be a regular compact set, let $w = e^{-Q}$ be continuous, and for $n = 1, 2, ...,$ define
$$\Phi_{K,Q,n}(z) := \sup\{|p(z)| : ||w^{\deg p} p||_K \leq 1,\ p \in \mathcal{P}_n\}.$$
Then
$$\frac{1}{n} \log \Phi_{K,Q,n} \to V_{K,Q}$$
locally uniformly on $\mathbb{C}$.

7. Plurisubharmonic functions in $\mathbb{C}^N$, $N>1$ and the complex Monge-Ampère operator.

Let D be a domain in $\mathbb{C}^N$. A complex-valued function $f : D \to \mathbb{C}$ is called *holomorphic* and we write $f \in \mathcal{O}(D)$ if f is holomorphic in each variable $z_1, ..., z_N$ separately. Apriori, if one assumes that $f \in C^1(D)$, holomorphicity is equivalent to f satisfying the system of partial differential equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \ j = 1, ..., N \tag{7.1}$$

where, for $z_j = x_j + iy_j$,

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}).$$

It turns out that the hypothesis that $f \in C^1(D)$ is superfluous. A *holomorphic mapping* $F : D' \to D$ where D' is a domain in $\mathbb{C}^m$ is a mapping $F = (f_1, ..., f_N)$ where each $f_i : D' \to \mathbb{C}$ is holomorphic. Our main interest is in the class of *plurisubharmonic* (psh) functions: a real-valued function $u : D \to [-\infty, +\infty)$ defined on a domain $D \subset \mathbb{C}^N$ is plurisubharmonic in D and we write $u \in PSH(D)$ if the following two conditions are satisfied:

(1) u is uppersemicontinuous on D and
(2) $u|_{D\cap l}$ is subharmonic (shm) on components of $D \cap l$ for each complex line (one-dimensional (complex) affine space) l.

Remark 7.1. It is unknown if (2) implies (1); i.e., it is unknown whether condition (1) is superfluous.

Given $z \in \mathbb{C}^N$ and $b \in \mathbb{C}^N \setminus \{0\}$, the complex line $l = l_{z,b}$ through z in the direction b is the set

$$l = \{z + tb : t \in \mathbb{C}\} = \{(z_1 + tb_1, ..., z_N + tb_N) : t \in \mathbb{C}\}.$$

Thus:

(1) $f \in \mathcal{O}(D)$ implies $f|_{D\cap l} \in \mathcal{O}(D \cap l)$ since $t \to f(z + tb)$ is holomorphic (where defined) by the chain rule:

$$\frac{\partial}{\partial \bar{t}} f(z + tb) = \sum_{j=1}^{N} [\frac{\partial f}{\partial z_j}\frac{\partial z_j}{\partial \bar{t}} + \frac{\partial f}{\partial \bar{z}_j}\frac{\partial \bar{z}_j}{\partial \bar{t}}] = 0$$

since $\frac{\partial z_j}{\partial \bar{t}} = \frac{\partial f}{\partial \bar{z}_j} = 0$.

(2) This shows that $f \in \mathcal{O}(D)$ implies $u := \log|f| \in PSH(D)$ since $t \to u(z+tb)$ is subharmonic where defined.

(3) If $u \in C^2(D)$, then $u \in PSH(D)$ if and only if for each $z \in D$ and $b \in \mathbb{C}^N$, the Laplacian of $t \mapsto u(z+tb)$ is nonnegative at $t=0$; i.e., *the complex Hessian*

$$H(u)(z) := [\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)]$$

of u is positive semidefinite on D:

$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) b_j \bar{b}_k \geq 0$$

Exercise 1, using chain rule calculations as in (1), will verify this.

The $\mathbb{R}^m$ analogue of (3) is that for $f : D \subset \mathbb{R}^m \to \mathbb{R}$ with $f \in C^2(D)$, f is convex in D, i.e., $f_{D \cap l}$ is convex on $D \cap l$ for each real line l, if and only if the real Hessian

$$[\frac{\partial^2 f}{\partial x_j \partial x_k}(x)]$$

of f is positive semidefinite on D. For example, $f(x) = x_1^2 + \cdots + x_m^2$ is convex in $\mathbb{R}^m$. The complex analogue of this function in $\mathbb{C}^N$ is

$$u(z) := |z|^2 := |z_1|^2 + \cdots + |z_N|^2.$$

Here $H(u)(z) = I_N$, the $N \times N$ identity matrix. Indeed, this function is *strictly plurisubharmonic* on $\mathbb{C}^N$.

Definition 7.2. Let $u \in PSH(D)$.

(1) If $u \in C^2(D)$ and the complex Hessian $H(u)$ is positive *definite* on D, we say that u is *strictly* psh in D.

(2) More generally, we say u is *strongly* psh in D if for all $D' \Subset D$ there exists $c = c(D) > 0$ such that

$$u(z) - c|z|^2 \in PSH(D).$$

Note that if $u \in PSH(D) \cap C^2(D)$, then the trace of $H(u)$

$$trH(u)(z) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j}(z) = 4\Delta u(z)$$

is nonnegative so that u is subharmonic in the $\mathbb{R}^{2N}-$sense in D. The converse is false, even if u is $\mathbb{R}^{2N}-$harmonic: consider, e.g., $u(z_1, z_2) =$

$|z_1|^2 - |z_2|^2$. Of course, a psh function $u(z_1, ..., z_N)$ is, in particular, shm in each complex variable z_j when all of the others are fixed; Exercise 10 asks about the converse (hint: the converse is false).

From the definition of psh, and the properties of shm functions on domains in $\mathbb{C}$, many analogous properties follow readily for psh functions. Let $D \subset \mathbb{C}^N$ be a domain and consider the following properties of $PSH(D)$:

(1) $PSH(D)$ forms a convex cone; i.e., if $u, v \in PSH(D)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in PSH(D)$.

(2) $u \in PSH(D)$ implies, since u is $\mathbb{R}^{2N}-$subharmonic, that $u \in L^1_{loc}(D)$. Indeed, it turns out that $u \in L^p_{loc}(D)$ for all $1 \leq p < \infty$ (see [24]).

(3) Analogous to the univariate case, smoothing a psh function u by convolving with a *radial* regularizing kernel $\chi(z_1, ..., z_N) = \chi(|z_1|, ..., |z_N|)$ gives a plurisubharmonic function (on a smaller domain), so that given u psh in a domain D, we can find a decreasing sequence $\{u_j\}$ of smooth psh functions, $u_j = u * \chi_{1/j}$ defined on $\{z \in D : \text{dist}(z, \partial D) > 1/j\}$ with $\lim_j u_j = u$ in D. This allows us, as in the subharmonic case, to verify properties for smooth psh functions and then pass to the limit.

(4) Recall that for $u \in SH(\mathbb{C})$,

$$M_u(r) := \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta$$

is a convex, increasing function of r; in the multivariate case, if $u \in PSH(\mathbb{C}^N)$, then

$$M_u(r_1, ..., r_N)) := (\frac{1}{2\pi})^N \int_0^{2\pi} \cdots \int_0^{2\pi} u(r_1 e^{i\theta_1}, ..., r_N e^{i\theta_N}) d\theta_1 \cdots d\theta_N$$

is a convex, increasing function of $(r_1, ..., r_N)$. In particular, if $u \in PSH(\mathbb{C}^N)$ and u is bounded above, then u is constant; and the *Lelong class*

$$L(\mathbb{C}^N) := \{u \in PSH(\mathbb{C}^N) : u(z) - \log |z| = 0(1), \ |z| \to \infty\}$$

are the psh functions in $\mathbb{C}^N$ of minimal growth. In particular, if $p(z) := p(z_1, ..., z_N)$ is a holomorphic polynomial of degree $d \geq 1$, then

$$u(z) := \frac{1}{d} \log |p(z)| \in L(\mathbb{C}^N).$$

(5) If $F : D \subset \mathbb{C}^N \to D' \subset \mathbb{C}^M$ is a holomorphic mapping with $F(D) = D'$ and $u \in PSH(D')$, then $u \circ F \in PSH(D)$. It suffices to verify this for $M = N = 1$ and for $u \in C^2(D')$ (why?) and this follows from the calculation
$$\frac{\partial^2}{\partial z \partial \overline{z}}(u \circ F)(z) = \frac{\partial^2}{\partial w \partial \overline{w}} u(w) \cdot |F'(z)|^2$$
where $w = F(z)$.

Regarding this last property, we have the following characterization of plurisubharmonicity.

Proposition 7.3. *A function $u : D \to [-\infty, +\infty)$ is psh if and only if for all holomorphic mappings $F : D' \to D$ where $D' \subset \mathbb{C}^m$ either $u \circ F$ is shm in D' (in the $\mathbb{R}^{2m}$ sense) or $u \circ F \equiv -\infty$.*

Proof. If $u \in PSH(D) \cap C^2(D)$, the holomorphicity of $F = (f_1, ..., f_N)$ and the chain rule (use (7.1) for each f_j) show that the complex Hessian of $u \circ F$ is positive semidefinite in D'; i.e., $u \circ F \in PSH(D')$ (and hence shm in D' in the $\mathbb{R}^{2m}$ sense). For arbitrary $u \in PSH(D)$, take a decreasing sequence $\{u_j\}$ of smooth psh functions, $u_j = u * \chi_{1/j}$ defined on $\{z \in D : \mathrm{dist}(z, \partial D) > 1/j\}$ with $\lim_j u_j = u$ in D and apply the previous result to $\{u_j\}$; then, since a decreasing sequence of psh functions is psh or identically minus infinity, the result follows.

The converse is trivial since one can take the holomorphic maps $t \to a + tb$ for $a \in D,\ b \in \mathbb{C}^n \setminus \{0\}$, and $t \in \mathbb{C}$ with $|t|$ sufficiently small. □

Indeed, it turns out that $u : D \to [-\infty, +\infty)$ *is psh if and only if* $u \circ A$ *is* $\mathbb{R}^{2N}$*−subharmonic in* $A^{-1}(D)$ *for every complex linear isomorphism* $A : \mathbb{C}^N \to \mathbb{C}^N$.

The limit function $u(z) := \lim_{n\to\infty} u_n(z)$ of a decreasing sequence $\{u_n\} \subset PSH(D)$ is psh in D (we may have $u \equiv -\infty$); while for any family $\{v_\alpha\} \subset PSH(D)$ (resp., sequence $\{v_n\} \subset PSH(D)$) which is uniformly bounded above on any compact subset of D, the functions
$$v(z) := \sup_\alpha v_\alpha(z) \text{ and } w(z) := \limsup_{n\to\infty} v_n(z)$$
are "nearly" psh: the usc regularizations
$$v^*(z) := \limsup_{\zeta\to z} v(\zeta) \text{ and } w^*(z) := \limsup_{\zeta\to z} w(\zeta)$$
are psh in D. Analogous to the univariate case, a set of the form

$$N := \{z \in D : v(z) := \sup_\alpha v_\alpha(z) < v^*(z)\} \tag{7.2}$$

where $\{v_\alpha\} \subset PSH(D)$ is called a *plurinegligible* set; and $E \subset \mathbb{C}^N$ is *pluripolar* if there exists u psh, $u \not\equiv -\infty$ with $E \subset \{u(z) = -\infty\}$. The proof that any polar set is negligible in Corollary 3.3 carries over to show any pluripolar set is plurinegligible; the converse is true but is a very deep result of Bedford and Taylor [4].

The precise definition of pluripolar is a local one: E is *pluripolar* if for each $z \in E$ there exists a neighborhood U of z and a psh function u in U with $E \cap U \subset \{z \in U : u(z) = -\infty\}$. For example, any analytic subvariety V of $\mathbb{C}^N$ is pluripolar as locally $V = \{f_1 = \cdots = f_m = 0\}$ for holomorphic f_j; whence $u = \log[|f_1|^2 + \cdots + |f_m|^2]$ works. The *first problem of Lelong* was to determine whether (locally) pluripolar sets, as defined above, were *globally* pluripolar; i.e., if E is pluripolar, can one find u psh on a neighborhood of E with $E \subset \{u = -\infty\}$? Indeed, one can; u can be taken to be psh on all of $\mathbb{C}^N$; and we can even find such a $u \in L(\mathbb{C}^N)$. We remark that:

(1) Nonpluripolar sets can be small: Take a nonpolar Cantor set $E \subset \mathbb{R} \subset \mathbb{C}$ of Hausdorff dimension 0 (see [30] for a construction). Then $E \times \cdots \times E$ is nonpluripolar in $\mathbb{C}^N$ (in general, $E_1 \times \cdots \times E_j \subset \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_j}$ is nonpluripolar in $\mathbb{C}^{m_1+\cdots+m_j}$ if and only if $E_k \subset \mathbb{C}^{m_k}$ is nonpluripolar for $k = 1, ..., j$; cf., exercise 7) and has Hausdorff dimension 0.

(2) *Pluripolar sets can be big*: A complex hypersurface $S = \{z : f(z) = 0\}$ for a holomorphic function f is a pluripolar set (take $u = \log|f|$) which has Hausdorff dimension $2N - 2$. Recall that a psh function is, in particular, subharmonic (in the $\mathbb{R}^{2N}$ sense); hence a pluripolar set is Newtonian polar. For such sets is known that the Hausdorff dimension cannot exceed $2N - 2$.

(3) *Size doesn't matter*: In $\mathbb{C}^2$, the totally real plane $\mathbb{R}^2 = \{(z_1, z_2) : \mathrm{Im} z_1 = \mathrm{Im} z_2 = 0\}$ is nonpluripolar (why?) but the *complex* plane $\mathbb{C} = \{(z_1, 0) : z_1 \in \mathbb{C}\}$ is pluripolar (take $u = \log|z_1|$). Also, there exist C^∞ arcs in $\mathbb{C}^N$ which are not pluripolar; while such a real-analytic arc must be pluripolar (why?).

One can easily construct examples of nonpluripolar sets $E \subset \mathbb{C}^N$ which intersect every affine complex line in finitely many points (hence these intersections are polar in these lines). Indeed, take

$$E := \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1 + z_2^2) = \mathrm{Re}(z_1 + z_2 + z_2^2) = 0\}.$$

Then for any complex line $L := \{(z_1, z_2) : a_1z_1 + a_2z_2 = b\},\ a_1, a_2, b \in \mathbb{C}$, $E \cap L$ is the intersection of two real quadrics and hence consists of at most four points. However, E is a totally real, two-(real)-dimensional submanifold of $\mathbb{C}^2$ and hence – as is the case with $\mathbb{R}^2 = \mathbb{R}^2 + i0 \subset \mathbb{C}^2$ in 3. – is not pluripolar. Thus pluripolarity cannot be detected by "slicing" with complex lines. In this example, E intersects the one-(complex)-dimensional analytic variety $A := \{(z_1, z_2) : z_1 + z_2^2 = 0\}$ in a nonpolar set. Nevertheless, one can construct a nonpluripolar set E in $\mathbb{C}^N,\ N > 1$, which intersects every one-dimensional complex analytic subvariety in a polar set [23].

The *second problem of Lelong* was to decide whether plurinegligible sets (recall (7.2)) were pluripolar: The positive solution of both of these problems comes fairly quickly utilizing results of Bedford and Taylor on the *relative capacity* $C(E, D)$ of a subset E of a bounded domain D in $\mathbb{C}^N$. We will define this notion in the next chapter.

We list a few more useful properties of $PSH(D)$:

(1) If ϕ is a real-valued, convex increasing function of a real variable, and u is psh in D, then so is $\phi \circ u$. Thus, e.g., if $u \in PSH(D)$ then $e^u \in PSH(D)$.
(2) $u, v \in PSH(D)$ implies $\max(u, v) \in PSH(D)$.
(3) Since psh functions are $\mathbb{R}^{2N}-$shm, they satisfy a maximum principle: if $D \Subset \mathbb{C}^N$ and $u \in PSH(D)$ with $\limsup_{z\to\zeta} u(z) \leq M$ for all $\zeta \in \partial D$, then $u \leq M$ in D.
(4) A version of the gluing lemma holds: if $D' \subset D$, $u \in PSH(D')$ and $v \in PSH(D)$ with $\limsup_{z\to\zeta} u(z) \leq v(\zeta)$ for $\zeta \in \partial D' \cap D$, then

$$\tilde{u}(z) := \max[u(z), v(z)],\ z \in D';\ \tilde{u}(z) := v(z),\ z \in D \setminus D'$$

is psh in D.

However, unlike logarithmic potential theory in the plane, in which case subharmonic functions are those locally integrable functions u with Laplacian $\Delta u \geq 0$ in the sense of distributions, the differential operator of paramount importance in $\mathbb{C}^N$ if $N > 1$ is a non-linear operator, the so-called *complex Monge-Ampère operator*. We proceed with an introduction to this topic.

If $u \in C^1(D)$ is real or complex valued, we write the $1-$form

$$du = \sum_{j=1}^{N} \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^{N} \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j =: \partial u + \overline{\partial} u$$

as the sum of a form ∂u of *bidegree* $(1,0)$ and a form $\overline{\partial} u$ of bidegree $(0,1)$ where, recall,

$$\frac{\partial u}{\partial z_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}); \ \frac{\partial u}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j});$$

and we have

$$dz_j = dx_j + idy_j; \ d\overline{z}_j = dx_j - idy_j.$$

For a complex-valued $f \in C^1(D)$, one easily checks that f is holomorphic in D if and only if $\overline{\partial} f = 0$ in D (see also exercise 15 at the end of this section). We also define

$$d^c u := i(\overline{\partial} u - \partial u).$$

Note that if $u \in C^2(D)$,

$$dd^c u = 2i\partial\overline{\partial} u = 2i \sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \overline{\partial} z_k} dz_j \wedge d\overline{z}_k$$

so that the coefficients of the $2-$form $dd^c u$ form the $N \times N$ complex Hessian matrix

$$H(u) := [\frac{\partial^2 u}{\partial z_j \overline{\partial} z_k}]_{j,k=1}^{N},$$

of u. We saw that if $u \in C^2(D)$, then $u \in PSH(D)$ if and only if $H(u)$ is positive semi-definite at each point of D; i.e., $dd^c u$ is a *positive form of bidegree* $(1,1)$; more generally it turns out that if u is only usc and locally integrable on D, then $u \in PSH(D)$ if and only if $dd^c u$ is a *positive current*. For a brief overview of differential forms in $\mathbb{C}^N$ and currents – differential forms with distribution coefficients – see Appendix A.

We remark that if $u \in C^2(D) \cap PSH(D)$, then the determinant of $H(u)$ is a nonnegative function on u. Elementary linear algebra shows that

$$(dd^c u)^N := dd^c u \wedge \cdots \wedge dd^c u = c_N \det H(u) dV$$

where $dV = (\frac{1}{2i})^N dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_N \wedge d\overline{z}_N$ is the volume form on $\mathbb{C}^N$ and c_N is a dimensional constant. For $u \in C^2(D)$, we thus obtain an absolutely continuous measure, $(dd^c u)^N$, the *complex Monge-Ampère* *measure* associated to u. To elaborate in $\mathbb{C}^2$ with variables (z,w), for a C^1 function u,

$$\partial u := \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial w}dw, \quad \overline{\partial} u := \frac{\partial u}{\partial \overline{z}}d\overline{z} + \frac{\partial u}{\partial \overline{w}}d\overline{w}.$$

For a C^2 function u,

$$dd^c u = 2i\big[\frac{\partial^2 u}{\partial z \partial \overline{z}}dz\wedge d\overline{z} + \frac{\partial^2 u}{\partial w \partial \overline{w}}dw\wedge d\overline{w} + \frac{\partial^2 u}{\partial z \partial \overline{w}}dz\wedge d\overline{w} + \frac{\partial^2 u}{\partial \overline{z} \partial w}d\overline{z}\wedge dw\big]$$

and

$$(dd^c u)^2 = 16\big[\frac{\partial^2 u}{\partial z \partial \overline{z}}\frac{\partial^2 u}{\partial w \partial \overline{w}} - \frac{\partial^2 u}{\partial z \partial \overline{w}}\frac{\partial^2 u}{\partial w \partial \overline{z}}\big]\frac{i}{2}dz \wedge d\overline{z} \wedge \frac{i}{2}dw \wedge d\overline{w}$$

is indeed a positive constant times the determinant of the complex Hessian of u times the volume form on $\mathbb{C}^2$. Thus if u is also psh, $(dd^c u)^2$ is a positive measure which is absolutely continuous with respect to Lebesgue measure. Note that for real-valued u,

$$\frac{\partial^2 u}{\partial z \partial \overline{w}} = \overline{\frac{\partial^2 u}{\partial w \partial \overline{z}}}.$$

As an elementary example, take $u(z,w) = |z|^2 + |w|^2 = z\overline{z} + w\overline{w}$. Then

$$dd^c u = 2idz \wedge d\overline{z} + 2idw \wedge d\overline{w}$$

and

$$(dd^c u)^2 = 16 \cdot \frac{i}{2}dz \wedge d\overline{z} \wedge \frac{i}{2}dw \wedge d\overline{w}.$$

Bedford and Taylor, and, independently, Sadullaev, have shown how to associate a positive measure (not necessarily absolutely continuous) to any locally bounded plurisubharmonic function u in such a way that, among other things, this *Monge-Ampère measure* associated to u, denoted $(dd^c u)^N$, is

(1) continuous under decreasing limits – if $\{u_j\}$ form a decreasing sequence of locally bounded psh functions with $u_j \downarrow u$ and u is psh and locally bounded, then
$$(dd^c u_j)^N \to (dd^c u)^N$$
weakly as measures; and it is

(2) continuous under a.e. increasing limits – if $\{u_j\}$ form a sequence of locally bounded psh functions with $u_j \uparrow u$ a.e., and u is psh and locally bounded, then
$$(dd^c u_j)^N \to (dd^c u)^N$$

weakly as measures.

In particular, since, as with subharmonic functions, given a general psh function u on a domain D, the standard smoothings $u_j := u * \chi_{1/j}$ decrease to u, this gives us a way of (in principle) computing $(dd^c u)^N$. Indeed, the "correct" domain of definition of the complex Monge-Ampère operator on a domain D in $\mathbb{C}^N$ should be this: $u \in PSH(D)$ is in the domain of definition of the complex Monge-Ampère operator on D if there exists a locally finite measure μ on D such that for any relatively compact subdomain D' in D and any sequence $\{u_j\} \in PSH(D') \cap C^2(D')$ with $u_j \downarrow u$ on D, we have $(dd^c u_j)^N \to \mu|_{D'}$ weak-*. Then we define $(dd^c u)^N$ to be the measure μ. The problem is to give a more concrete description of this domain of definition. This was achieved through work of Cegrell and his school culminating in the definitive answer due to Blocki [8].

For a general psh function, $dd^c u$ is a $(1,1)-$current; i.e., a $(1,1)-$form with distribution coefficients. Hence the wedge product $dd^c u \wedge dd^c u$ does not, apriori, make sense as we would be multiplying distributions or measures. Bedford and Taylor [3] gave an inductive way to define $(dd^c u)^k,\ k = 1, ..., N$, for $u \in L^\infty_{loc}(D) \cap PSH(D)$. We give their definition of $(dd^c u)^2$ in $\mathbb{C}^2$ for u psh and locally bounded in D.

We first recall that a psh function u in D is an usc function u in D which is subharmonic on components of $D \cap l$ for complex affine lines l. In particular, u is a locally integrable function in D such that $dd^c u$ is a $(1,1)$ current. The derivatives are to be interpreted in the distribution sense and are actually measures; i.e., they act on compactly supported continuous functions. For a $(1,1)$ current $T = i \sum_{j,k=1}^N T_{jk} dz_j \wedge d\overline{z}_k$ with coefficients T_{jk} that are continuous functions on D, T is *positive* if the matrix $[T_{jk}(z)]$ is positive semidefinite at all $z \in D$. Thus $dd^c u$ for $u \in C^2(D) \cap PSH(D)$ is a positive $(1,1)$ current. More generally, a $(1,1)$ current T on a domain D in $\mathbb{C}^2$ is positive if T applied to $i\beta \wedge \overline{\beta}$ is a positive distribution for all $(1,0)$ forms $\beta = adz + bdw$ with $a, b \in C_0^\infty(D)$ (smooth functions having compact support in D). Writing the action of a current T on a form ψ as $< T, \psi >$, this means that

$$< T, \phi(i\beta \wedge \overline{\beta}) > \geq 0 \quad \text{for all } \phi \in C_0^\infty(D) \text{ with } \phi \geq 0.$$

For a general psh function, $dd^c u$ is a positive $(1,1)-$current. As an example, take $u(z,w) = \log|z|$ in $\mathbb{C}^2$. Then the $(1,1)$ current

$$T = dd^c u = i\pi\delta_0(z)dz \wedge d\bar{z}$$

is a current of integration on the complex line $E = \{(z,w) : z = 0\}$. Here we have written $dd^c u$ as a $(1,1)$ form where the coefficient $\delta_0(z)$ is a distribution, the point mass at $z = 0$ in the complex $z-$plane. More generally, if f is holomorphic and $u = \log|f|$, then, locally, $dd^c u$ is the current of integration on the complex hypersurface $\{f = 0\}$. For a discussion of currents and the general definition of positivity, we refer the reader to [25], section 3.3 or [7].

Following [3], we now define $(dd^c u)^2$ for a psh u in D if $u \in L^\infty_{loc}(D)$ using the fact that $dd^c u$ is a positive $(1,1)$ current with measure coefficients. First note that if u were of class C^2, given $\phi \in C_0^\infty(D)$, we have

$$(7.3) \qquad \int_D \phi(dd^c u)^2 = -\int_D d\phi \wedge d^c u \wedge dd^c u \text{ (exercise 14)}$$

$$= \int_D du \wedge d^c\phi \wedge dd^c u = \int_D u dd^c\phi \wedge dd^c u$$

since all boundary integrals vanish. The applications of Stokes' theorem are justified if u is smooth; for arbitrary $u \in PSH(D) \cap L^\infty_{loc}(D)$, these formal calculations serve as motivation to *define* $(dd^c u)^2$ as a positive measure (precisely, a positive current of bidegree $(2,2)$ and hence a positive measure) via

$$< (dd^c u)^2, \phi >:= \int_D u dd^c\phi \wedge dd^c u.$$

This defines $(dd^c u)^2$ as a $(2,2)$ current (acting on $(0,0)$ forms; i.e., test functions) since $udd^c u$ has measure coefficients. We refer the reader to [3] or [25] (p. 113) for the verification of the positivity of $(dd^c u)^2$.

In some sense, the complex Monge-Ampère measure associated to a locally bounded psh function is a "minimal" Laplacian. Bellman's principle states that if B is a positive semidefinite Hermitian $N \times N$ matrix, then

$$(\det B)^{1/N} = \frac{1}{N} \inf_A \text{trace}(AB)$$

where the infimum is taken over all positive definite Hermitian $N \times N$ matrices A with $\det A = 1$. Hence, given such a matrix $A = [a_{jk}]$, let

$$\Delta_A := \frac{1}{N}\sum_{j,k=1}^{N} a_{jk}\frac{\partial^2}{\partial z_j \partial \overline{z}_k}.$$

Then $(dd^c u)^N = \inf_A [\Delta_A u]^N$ if $u \in C^2(D)$.

Exercises.

(1) Verify that for $u \in C^2(D)$, $z \in D$, and $a \in \mathbb{C}^N$ the Laplacian of $t \mapsto u(z + ta)$ (for $t \in \mathbb{C}$ with $z + ta \in D$) is equal to a positive multiple of

$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) a_j \overline{a}_k.$$

(2) Prove that if u is psh in a domain $D \subset \mathbb{C}^N$, then u is shm as a function on a domain in $\mathbb{R}^{2N}$; i.e., u is usc in D and $\Delta u \geq 0$ in the sense of distributions.

(3) If $N > 1$, find a function u which is shm in $\mathbb{C}^N = \mathbb{R}^{2N}$ but which is not psh in $\mathbb{C}^N$. Can you find such a u which is *harmonic* in $\mathbb{C}^N = \mathbb{R}^{2N}$?

(4) Find a harmonic function h in $\mathbb{R}^2$ and a real linear isomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h \circ T$ is not subharmonic in $\mathbb{R}^2$.

(5) Verify that if ϕ is a real-valued, convex increasing function of a real variable, and $u \in C^2(D)$ is psh in D, then $\phi \circ u$ is psh in D. (Note, in particular, that e^u is psh in D).

(6) *Gluing psh functions*. Let u, v be psh in open sets U, V where $U \subset V$ and assume that $\limsup_{\zeta \to z} u(\zeta) \leq v(z)$ for $z \in V \cap \partial U$. Show that the function w defined to be $w = \max(u, v)$ in U and $w = v$ in $V \setminus U$ is psh in V.

(7) Let $E = E_1 \times E_2 \subset \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$. Show that E is pluripolar in $\mathbb{C}^2$ if and only if at least one of E_1, E_2 is polar in $\mathbb{C}$.

(8) Is $\{(z_1, z_2) \in \mathbb{C}^2 : \text{Im} z_1 = \text{Im} z_2 = 0\}$ pluripolar? Why or why not?

(9) Is $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$ pluripolar? Why or why not?

(10) *Extra Credit*. A psh function $u(z_1, ..., z_n)$ is, in particular, shm in each complex variable z_j when all of the others are fixed. Is the converse true? See [25] for help.

(11) Let $D \subset \mathbb{C}^N = \mathbb{R}^{2N}$ be a bounded, smoothly bounded domain and let ρ be a smooth *defining function* for D: ρ is defined and smooth on a neighborhood of $\overline{D}$; $D = \{z : \rho(z) < 0\}$; and $\nabla\rho \neq 0$ on ∂D.
   (a) Show that $\nabla\rho \neq 0$ on ∂D is equivalent to $d\rho \neq 0$ on ∂D and the tangent space $T_p(\partial D)$ at any point $p \in \partial D$ is given by $\{v \in \mathbb{C}^N : d\rho(v) = 0\}$.
   (b) Show that the coefficient functions of $d^c\rho$ at $p \in \partial D$ define a tangent vector to ∂D at p.
   (c) As an example, take $\rho(z) = |z_1|^2 + \cdots + |z_N|^2 - 1$. Then D is the unit ball. Compute $T_p(\partial D)$ for $p = (1, 0, ..., 0)$ and the coefficient functions of $d^c\rho$ at this point.

(12) *An illustrative example.* In $\mathbb{C}^2$, let $u(z, w) = \frac{1}{2}\log(|z|^2 + |w|^2)$. This psh function is smooth away from $(0, 0)$. Prove that

$$(dd^c u)^2 = 0 \text{ on } \mathbb{C}^2 \setminus \{0\}.$$

(Note u is not locally bounded near $(0, 0)$ but it turns out that one can define $(dd^c v)^2$ for psh v with compact singularities and here $(dd^c u)^2 = (2\pi)^2\delta_{(0,0)}$.)

(13) In $\mathbb{C}^2$, let $v(z, w) = \frac{1}{2}\log(|z|^2 + |w|^4)$. This psh function is smooth away from $(0, 0)$. Prove that $(dd^c v)^2 = 0$ on $\mathbb{C}^2 \setminus \{0\}$. (Here, it turns out that $(dd^c v)^2 = 2(2\pi)^2\delta_{(0,0)}$.)

(14) In (7.3), verify the equality

$$-\int_D d\phi \wedge d^c u \wedge dd^c u = \int_D du \wedge d^c\phi \wedge dd^c u.$$

(15) For a complex-valued $f \in C^1(D)$, write $f = u + iv$ where u, v are real-valued. Show that f is holomorphic in D if and only if $d^c u = dv$ in D.

8. Perron-Bremmermann envelopes, extremal plurisubharmonic functions and applications.

Let D be a bounded domain in $\mathbb{C}^N$ and let $f \in C(\partial D)$ be real-valued and continuous. We form the *Perron-Bremmermann envelope*

$$u(z) = u_{f,D}(z) := \tag{8.1}$$

$$\sup\{v(z) : v \in PSH(D),\ \limsup_{z\to\zeta} v(z) \leq f(\zeta) \text{ for all } \zeta \in \partial D\}.$$

Then $u^*(z) = \limsup_{\zeta\to z} u(z) \in PSH(D)$ since the family

$$\mathcal{U} := \{v \in PSH(D) : \limsup_{z\to\zeta} v(z) \leq f(\zeta) \text{ for all } \zeta \in \partial D\}$$

is uniformly bounded above in D (by $||f||_{\partial D}$). If $N = 1$, $u(z) = u^*(z)$ is harmonic in D; and if $N > 1$, by analogous reasoning, if we replace "$PSH(D)$" by "$\mathbb{R}^{2N}-$subharmonic in D" then again, $u(z) = u^*(z)$ is $\mathbb{R}^{2N}-$harmonic in D. In general, what type of function is u? One very special possibility is the following.

Definition 8.1. A function $u : D \to \mathbb{R}$ is *pluriharmonic* in D, and we write $u \in PH(D)$, if $u \in C^2(D)$ and $dd^c u = 0$ in D; i.e., $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \equiv 0$ in D for all $j, k = 1, ..., N$. Equivalently, $u \in C^2(D)$ and $u|_{D\cap l}$ is harmonic on (components of) $D \cap l$ for all complex lines l.

Pluriharmonic functions – which, a posteriori are C^∞ and even real-analytic – are very special; locally, such a function is the real part of a holomorphic function. The converse statement, that the real and imaginary parts of a holomorphic function are pluriharmonic, follows from exercise 15 of section 7. Unfortunately, $u = u_{f,D}$ need not be pluriharmonic:

Example 8.2. Let $B = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$ and take any subharmonic function $s = s(z)$ which is continuous on the closed unit disk. Setting $f(z, w) := s(z)$ we claim that $u(z, w) = s(z)$. Thus, for any s which is not harmonic, u is not pluriharmonic in B.

Clearly $s(z) \leq u(z, w)$; for the reverse inequality, take $v \in \mathcal{U}$ and fix $(z^0, w^0) \in B$. The intersection of B with the complex line $l = \{z = z^0\}$ is the disk

$$\{(z^0, w) : |w|^2 < 1 - |z^0|^2\}.$$

The function $w \to v(z^0, w)$ is subharmonic in the disk $D = \{w : |w|^2 < 1 - |z^0|^2\}$ and for $\zeta \in \partial D$,

$$\limsup_{w \to \zeta} v(z^0, w) \leq v^*(z^0, \zeta) \leq s(z^0)$$

(note $s(z^0)$ is a constant). By the (univariate) maximum principle, $v(z^0, w) \leq s(z^0)$ for all $w \in D$; in particular, $v(z^0, w^0) \leq s(z^0)$.

The function $u(z, w) = s(z)$ in this example is a *maximal* psh function in B.

Definition 8.3. We call $u \in PSH(D)$ *maximal* if, for any relative compact subdomain D' and any $v \in PSH(D')$ which is usc on $\overline{D}'$, if $u \geq v$ on $\partial D'$, then $u \geq v$ in D'.

If $N = 1$, $u \in SH(D)$ is maximal if and only if u is harmonic in D. If $N > 1$ and if u is harmonic (in the $\mathbb{R}^{2N}$ sense; i.e., $\Delta u \geq 0$), and $u \in PSH(D)$, then u is clearly maximal. In this case, (exercise 2) u is pluriharmonic in D. The argument in the example shows that if $s(z)$ is shm in $\Omega \subset \mathbb{C}$, then $u(z, w) := s(z)$ is a maximal psh function in $\Omega \times \mathbb{C}$. Indeed, more generally, if $u(z_1, ..., z_N) \in PSH(D)$ is independent of one or more of the variables $z_1, ..., z_N$, then u is maximal in D (exercise 3). In particular, this shows that maximal psh functions need not even be continuous!

If $u \in PH(D)$, then $dd^c u \equiv 0$ in D; u is maximal in D; and, trivially, $(dd^c u)^n \equiv 0$ in D. The function $u(z) = \log |z|$ is not maximal in $\mathbb{C}^N$ (compare with $\log^+ |z|$ on the unit ball) but it is maximal on $\mathbb{C}^N \setminus \{0\}$ (why?). Note that $(dd^c u)^n \equiv 0$ on $\mathbb{C}^N \setminus \{0\}$. We show the following:

Proposition 8.4. *Let $u \in C^2(D)$ be psh. If u is maximal in D then $\det H(u) \equiv 0$ in D; i.e., $(dd^c u)^N = 0$ in D.*

Proof. Suppose u is maximal in D but $\det H(u) \not\equiv 0$ in D. We can find a point $z_0 \in D$ such that for each $a \in \mathbb{C}^N \setminus \{0\}$

$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z_0) a_j \overline{a}_k > 0.$$

This strict inequality persists for all $z \in \overline{B}(z_0, r)$ for small $r > 0$ (why?). By compactness of $\overline{B}(z_0, r)$ we can find $c > 0$ with

$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) a_j \overline{a}_k \geq c \sum_{j=1}^{N} |a_j|^2 \tag{8.2}$$

for $z \in \overline{B}(z_0, r)$ and for each $a \in \mathbb{C}^N \setminus \{0\}$. From (8.2), the function $v(z)$ defined to be $v(z) = u(z)$ on $D \setminus \overline{B}(z_0, r)$ and $v(z) = u(z) + c(r^2 - |z - z_0|^2)$ on $\overline{B}(z_0, r)$ is psh in D. Moreover, v agrees with u on $\partial\overline{B}(z_0, r)$; and we have $v(z_0) > u(z_0)$, contradicting maximality of u. $\square$

The converse is also true (see Corollary 8.8). Note this generalizes the univariate situation where $\det H(u) \equiv 0$ simply says that $\Delta u = 0$. For a function $u \in PSH(D) \cap C^2(D)$, it is easy to see why $\det H(u) = 0$ implies that u is maximal: at each point $z_0 \in D$, $H(u)$ has a zero eigenvalue; assuming, as we do for simplicity, that $(dd^c u)^{N-1} \neq 0$, by a complex form of the Frobenius theorem we can find an analytic disk through z_0 on which u is harmonic. That is, there exists a holomorphic mapping f from the unit disk in $\mathbb{C}$ into D with $u(0) = z_0$ such that $u \circ f$ is harmonic on D. Any psh function v is subharmonic on this disk; if u dominates v on the boundary of the disk, then u dominates v in the disk.

More generally: *In the case where $u \in L^\infty_{loc}(D) \cap PSH(D)$, u is maximal in D if and only if $(dd^c u)^N = 0$ in D.* This will be a consequence of a pluripotential-theoretic version of a comparison principle. It will follow that *solutions of a Dirichlet problem for the complex Monge-Ampère operator are maximal.*

We now outline the procedure of solving the *Dirichlet problem for the complex Monge-Ampère operator in the unit ball B in $\mathbb{C}^N$.* Let f be a continuous, real-valued function on ∂B. We seek a psh function u in B, $u \in C(\overline{B})$, with $u = f$ on ∂B and $(dd^c u)^N = 0$ in B; i.e., u is maximal in B. Bedford and Taylor proved existence and uniqueness of the solution u (in the slightly more general setting where B is a so-called *strictly pseudoconvex* domain). We caution the reader that no matter how smooth f is, the solution u is generally not in $C^2(B)$ (although $u \in C^{1,1}(B)$ if $f \in C^2(\partial B)$; see (4) below). To construct u, one forms the Perron-Bremmermann envelope

$$u(z) = u_{f,B}(z)$$

$$:= \sup\{v(z) : v \in PSH(B) : \limsup_{z \to \zeta} v(z) \leq f(\zeta) \text{ for all } \zeta \in \partial B\}.$$

The proof that u works consists of the following steps:

(1) *$u \in PSH(B)$ and $u = f$ on ∂B*:
Proof of (1): We first show $u = u^*$ in B. Take h harmonic (in the $\mathbb{R}^{2N}-$sense) in B with $h = f$ on ∂B; clearly $u \leq h$

in B since each competitor v is shm and satisfies $v \leq h$. It is classical that h is continuous on $\overline{B}$ hence $u^* \leq h$ in $\overline{B}$ so that, in particular, since u^* is psh and satisfies $\limsup_{z\to\zeta} u^*(z) \leq f(\zeta)$ for all $\zeta \in \partial B$, $u^* \leq u$ in B and equality holds. In particular, $u = u^* \in PSH(B)$.

Now we show $u^* = f$ on ∂B. Fix $z_0 \in \partial B$ and $\epsilon > 0$ and define, where $< z, z' > := \sum_{j=1^N} z_j \overline{z}'_j$,

$$v(z) := c\big[\mathrm{Re} < z, z_0 > -1\big] + f(z_0) - \epsilon \in C(\overline{B})$$

where $c > 0$ is chosen to insure $v \leq f$ on ∂B. Note that v is a competitor for u and, by construction, $v(z_0) = f(z_0) - \epsilon$; thus

$$\liminf_{z\to z_0} u(z) \geq \liminf_{z\to z_0} v(z) \geq f(z_0),$$

yielding the result. Here, the function $b(z) := \mathrm{Re} < z, z_0 > -1$ is a psh *barrier* for ∂B at z_0: $b \in PSH(B) \cap C(\overline{B})$ with $b(z_0) = 0 > b(z)$ for $z \in B$.

(2) *u is maximal in B;*

Proof of (2): If $G \subset\subset B$, v is usc on $\overline{G}$, psh on G and $v \leq u$ on ∂G, then by the gluing lemma for psh functions, the function V defined as $V = \max(u, v)$ in G and $V = u$ in $B \setminus G$ is psh and is a competitor for u; thus, in particular, $v \leq u$ in G.

(3) $u \in C(\overline{B})$; i.e., if $u_* = u^* = f \in C(\partial B)$, then $u \in C(B)$.

This is a theorem of J. B. Walsh (cf., Theorem 3.1.4 [25]); it uses the notion of psh barriers. Here we use the notation $u_*(\zeta) := \liminf_{z\to\zeta} u(z)$.

Proof of (3): Since we know $u = u^*$ in B so that u is usc in B, it suffices to show $u = u_*$ in B so that u is lsc in B. To this end, fix $z_0 \in B$ and $\epsilon > 0$. We show: *there exists $\delta > 0$ such that $u(\widehat{z}) > u(z_0) - \epsilon$ for $\widehat{z} \in B$ with $|\widehat{z} - z_0| < \delta$.*

We begin by observing that since ∂B is compact and $u_* = u^* = f$ on ∂B, there exists $\delta > 0$ such that

$$z \in B,\ w \in \partial B \text{ with } |z - w| \leq \delta \text{ implies } |u(z) - f(w)| \leq \epsilon. \tag{8.3}$$

Take $\widehat{z} \in B$ with $|\widehat{z} - z_0| < \delta/2$ and let $\widehat{B} := B - (z_0 - \widehat{z})$ (translate of B). Then

$$v(z) := \max[u(z), u(z + (z_0 - \widehat{z})) - 2\epsilon],\ z \in B \cap \widehat{B};\ u(z),\ z \in B \setminus \widehat{B}$$

is psh in B by the Gluing lemma. Here we use (8.3) which implies $v = u$ on a neighborhood of $B \cap \widehat{B}$.

We show $v \leq u$ in B; then, at $\widehat{z}$, we have

$$u(\widehat{z}) \geq v(\widehat{z}) \geq u(z_0) - 2\epsilon$$

which verifies the italicised statement. The inequality $v \leq u$ is clear in $B \setminus \widehat{B}$. Suppose $z \in B \cap \widehat{B}$ and $dist(z, \partial B) \leq \delta/2$. Take $w \in \partial B$ with $|z - w| \leq \delta/2$. Then

$$|z + (z_0 - \widehat{z}) - w| \leq |z - w| + |z_0 - \widehat{z}| \leq \delta/2 + \delta/2 = \delta$$

and again by (8.3), $u(z + (z_0 - \widehat{z})) - f(w)| \leq \epsilon$; i.e.,

$$u(z + z_0 - \widehat{z}) - 2\epsilon \leq f(w) - \epsilon \leq u(z).$$

Thus $v(z) \leq u(z)$ in a neighborhood of ∂B. By maximality of u, $v \leq u$ in B.

(4) *If $f \in C^2(\partial B)$, then $u \in C^{1,1}(B)$*:
This is very clever; it uses *automorphisms* of B to show, e.g., that given $\epsilon > 0$, there exists $C > 0$ such u satisfies an estimate of the form

$$u(z + h) - 2u(z) + u(z - h) \leq C|h|^2$$

for $|z| \leq 1 - \epsilon$ and $|h| \leq \epsilon/2$.

(5) $(dd^c u)^N = 0$ on B:
This is first proved under the assumption that $u \in C^{1,1}(B)$ which follows if $f \in C^{1,1}(\partial B)$. The general case follows by approximating $f \in C(\partial B)$ by a decreasing sequence $f_j \in C^2(\partial B)$, giving rise to a corresponding sequence $\{u_j\}$ which decrease and converge uniformly to u; since $(dd^c u_j)^N = 0$ and the complex Monge-Ampère operator is continuous under decreasing limits, we have $(dd^c u)^N = 0$.

A nice exposition of the details of steps (3)-(5) can be found in chapter 4 of [25]; for a more streamlined argument of the entire proof see [7]. Here is an interesting example, due to Gamelin, of $f \in C^\infty(\partial B)$ – indeed, here we will have $f \in C^\omega(\partial B)$! – with $u \notin C^2(B)$. Take, for $N = 2$,

$$f(z, w) = (|z|^2 - 1/2)^2 = (|w|^2 - 1/2)^2.$$

Then

$$u(z, w) = \big(\max[0, |z|^2 - 1/2, |w|^2 - 1/2]\big)^2$$

satisfies $(dd^c u)^2 = 0$ in B and $u = f$ on ∂B, but $u \notin C^2(B)$.

We remark that it is already easy to see from (1)-(3) that a general maximal psh function is locally a decreasing limit of continuous maximal functions:

Proposition 8.5. *Let u be psh and maximal in a domain $D \subset \mathbb{C}^N$. For any ball B with $\overline{B} \subset D$, there exist $\{u_j\}$ continuous in $\overline{B}$ and psh and maximal in B with $u_j \downarrow u$ in B.*

Proof. By smoothing, we can find $\{v_j\}$ psh and smooth in a neighborhood G of $\overline{B}$ with $G \subset D$ and $v_j \downarrow u$ in G. Now define u_j on $\overline{B}$ by $u_j = u_{u_j|_{\partial B}, B}$ in $\overline{B}$ (recall the notation in (8.1)). $\square$

We now turn to versions of the comparison and domination principles in our pluripotential settings. We first state the comparison principle (compare with Proposition 1.13).

Proposition 8.6. *Let u, v be psh and bounded in a bounded, open set $D \subset \mathbb{C}^N$. Suppose* $\liminf_{z \to \zeta}[u(z) - v(z)] \geq 0$ *for all $\zeta \in \partial D$. Then*

$$\int_{\{u<v\}} (dd^c v)^N \leq \int_{\{u<v\}} (dd^c u)^N. \tag{8.4}$$

Proof. We give the proof for u, v continuous on $\overline{D}$. Here we may assume $u = v$ on ∂D and $D = \{u < v\}$. Given $\epsilon > 0$, let $v_\epsilon := \max[v - \epsilon, u]$ and note that $v_\epsilon = u$ near ∂D. We use this to show

$$\int_D (dd^c v_\epsilon)^N = \int_D (dd^c u)^N. \tag{8.5}$$

To this end, let $\phi \in C_0^\infty(D)$ with $\phi \equiv 1$ on a neighborhood of the closure of $\{z \in D : v_\epsilon(z) > u(z)\}$. Then

$$\int_D \phi (dd^c v_\epsilon)^N = \int_D v_\epsilon dd^c \phi \wedge (dd^c v_\epsilon)^{N-1} \text{ (why?)}$$

$$= \int_D u dd^c \phi \wedge (dd^c u)^{N-1} = \int_D \phi (dd^c u)^N$$

(note $u = v_\epsilon$ on the support of $dd^c \phi$). This proves (8.5).

Now we use the fact that v_ϵ increase to v so that $(dd^c v_\epsilon)^N \to (dd^c v)^N$ as positive measures in D. Take $\psi \in C_0(D)$ with $0 \leq \psi \leq 1$ and observe that

$$\int_D \psi (dd^c v)^N = \lim_{\epsilon \to 0} \int_D \psi (dd^c v_\epsilon)^N \leq \lim_{\epsilon \to 0} \int_D (dd^c v_\epsilon)^N = \int_D (dd^c u)^N,$$

the last equality by (8.5). This holds for any such ψ; hence

$$\int_D (dd^c v)^N \leq \int_D (dd^c u)^N.$$

□

Using Proposition 8.6 we can prove a domination principle for psh functions. As a preliminary remark, we mention that if $a, b \in PSH(D) \cap L^\infty_{loc}(D)$, then $a + b \in PSH(D) \cap L^\infty_{loc}(D)$ and

$$(dd^c(a+b))^N \geq (dd^c a)^N + (dd^c b)^N$$

as positive measures. Indeed, if $N = 2$, the Bedford-Taylor theory shows how to define, e.g., $dd^c a \wedge T$ as a positive measure (positive $(2,2)-$current) for any positive $(1,1)-$current T: for $\phi \in C_0^\infty(D)$,

$$(dd^c a \wedge T)(\phi) := (aT)(dd^c \phi)$$

which we write as

$$\int_D \phi dd^c a \wedge T := \int_D a dd^c \phi \wedge T.$$

In particular, $dd^c a \wedge dd^c b$ is a positive measure and

$$(dd^c(a+b))^2 = (dd^c a)^2 + (dd^c b)^2 + 2dd^c a \wedge dd^c b \geq (dd^c a)^2 + (dd^c b)^2.$$

Proposition 8.7. *Let u, v be psh and bounded in a bounded domain $D \subset \mathbb{C}^N$. Suppose $\liminf_{z \to \zeta}[v(z) - u(z)] \geq 0$ for all $\zeta \in \partial D$ and assume that*

$$(dd^c u)^N \geq (dd^c v)^N \text{ in } D.$$

Then $v \geq u$ in D.

Proof. Assume not, i.e., suppose $\{z \in D : u(z) > v(z)\} \neq \emptyset$. We can choose $\epsilon, \delta > 0$ small so that we have

$$u(z) + \epsilon|z|^2 - \delta < u(z) \text{ in } D,$$

hence

$$\liminf_{z \to \zeta}[v(z) - (u(z) + \epsilon|z|^2 - \delta)] \geq 0 \text{ for all } \zeta \in \partial D,$$

and so that

$$S := \{z \in D : u(z) + \epsilon|z|^2 - \delta > v(z)\} \neq \emptyset.$$

If u were continuous, S is open; in the general case, S still has positive Lebesgue measure (why?). By Proposition 8.6

$$\int_S (dd^c(u+\epsilon|z|^2-\delta))^N \leq \int_S (dd^c v)^N.$$

By hypothesis, $\int_S (dd^c v)^N \leq \int_S (dd^c u)^N$. On the other hand, since S has positive Lebesgue measure, $\int_S (dd^c|z|^2)^N > 0$ and

$$\int_S (dd^c(u+\epsilon|z|^2-\delta))^N \geq \int_S (dd^c u)^N + \epsilon^N \int_S (dd^c|z|^2)^N > \int_S (dd^c u)^N,$$

a contradiction. □

Corollary 8.8. *If $v \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ and $(dd^c v)^N = 0$ in Ω, then v is maximal in Ω.*

Proof. Take $D \Subset \Omega$ and $u \in PSH(\overline{D})$ with $u \leq v$ on ∂D. Since v is bounded in D, to show $u \leq v$ in D we may restrict ourselves to $u \in PSH(\overline{D})$ which are bounded: clearly we need only consider u that are bounded from below on D; and $\overline{D}$ is compact so u is bounded above (since it is usc) on $\overline{D}$. Thus $(dd^c u)^N$ is well-defined as a positive measure and $(dd^c u)^N \geq (dd^c v)^N = 0$ in D. By Proposition 8.6 we conclude that $u \leq v$ in D.

□

Given these results on the Dirichlet problem and maximal psh functions, many important notions and results in pluripotential theory can be proved in ways analogous to those in classical logarithmic potential theory. We now describe some *extremal* psh functions modeled on their one-variable counterparts.

Recall the class of plurisubharmonic functions u in $\mathbb{C}^N$ of *logarithmic growth*, i.e., such that $u(z) \leq \log|z| + C$, $|z| \to \infty$ where $C = C(u)$, is called the class $L = L(\mathbb{C}^N)$. The functions $\frac{1}{\deg p}\log|p(z)|$ for a polynomial p clearly belong to L. For any Borel set E, set

$$V_E(z) := \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } E\} \tag{8.6}$$

and we call $V_E^*(z)$ the *L-extremal function of E*. We generally restrict our attention to compact sets $K \subset \mathbb{C}^N$. The function V_K is lower semicontinuous, but it need not be upper semicontinuous. The proof of Proposition 3.7 carries over to show that the upper semicontinuous regularization

$$V_K^*(z) = \limsup_{\zeta \to z} V_K(\zeta)$$

of V_K is either identically $+\infty$ or else V_K^* is plurisubharmonic once we show that for a closed Euclidean ball $K = \{z \in \mathbb{C}^N : |z-a| \leq R\}$ we have $V_K(z) = V_K^*(z) = \max[0, \log|z-a|/R]$. The case $V_K^* \equiv +\infty$ occurs precisely when K is pluripolar. In the other case, as in the univariate situation (Proposition 3.7), we have $V_K^* \in L^+(\mathbb{C}^N)$ where

$$L^+(\mathbb{C}^N) := \{u \in L(\mathbb{C}^N) : u(z) \geq \log^+|z| + C\}$$

where $C = C(u)$. Note that $L^+(\mathbb{C}^N) \subset L^\infty_{loc}(\mathbb{C}^N)$ so that $(dd^cV_K^*)^N$ is a positive measure if K is not pluripolar.

We have the same version of the global domination principle, Proposition 2.16:

Proposition 8.9. *Let $u \in L(\mathbb{C}^N)$ and $v \in L^+(\mathbb{C}^N)$ and suppose $u \leq v$ a.e.-$(dd^cv)^N$. Then $u \leq v$ on $\mathbb{C}^N$.*

Proof. We give the proof in case u, v are continuous. Suppose the result is false; i.e., there exists $z_0 \in \mathbb{C}^N$ with $u(z_0) > v(z_0)$. Since $v \in L^+(\mathbb{C}^N)$, by adding a constant to u, v we may assume $v(z) \geq \frac{1}{2}\log(1+|z|^2)$ in $\mathbb{C}^N$. Note that $(dd^c\big[\frac{1}{2}\log(1+|z|^2)\big])^N > 0$ on $\mathbb{C}^N$ (exercise 8). Fix $\delta,\ \epsilon > 0$ with $\delta < \epsilon/2$ in such a way that the set

$$S := \{z \in \mathbb{C} : u(z) + \frac{\delta}{2}\log(1+|z|^2) > (1+\epsilon)v(z)\}$$

contains z_0. In our setting, S is open; in the general case, S has positive Lebesgue measure. Moreover, since $\delta < \epsilon$ and $v \geq \frac{1}{2}\log(1+|z|^2)$, S is bounded. By Proposition 8.6, we conclude that

$$\int_S (dd^c[u(z) + \frac{\delta}{2}\log(1+|z|^2)])^N \leq \int_S (dd^c(1+\epsilon)v(z))^N.$$

But $\int_S (dd^c\frac{\delta}{2}\log(1+|z|^2))^N > 0$ since S has positive Lebesgue measure, so

$$(1+\epsilon)\int_S (dd^cv)^N > 0.$$

By hypothesis, for a.e.-$(dd^cv)^N$ points in $\mathrm{supp}(dd^cv)^N \cap S$ (which is not empty since $\int_S (dd^cv)^N > 0$), we have

$$(1+\epsilon)v(z) \leq u(z) + \frac{\delta}{2}\log(1+|z|^2) \leq v(z) + \frac{\delta}{2}\log(1+|z|^2),$$

i.e., $v(z) \leq \frac{1}{4}\log(1+|z|^2)$ since $\delta < \epsilon/2$. This contradicts the normalization $v \geq \frac{1}{2}\log(1+|z|^2)$. □

We return to the example of a closed Euclidean ball $K = \{z \in \mathbb{C}^N : |z-a| \leq R\}$ and we show $V_K(z) = V_K^*(z) = \max[0, \log|z-a|/R]$. Let's verify this for $a = 0$ and $R = 1$; i.e., for the closed unit ball $K = \{z \in \mathbb{C}^N : |z| \leq 1\}$, we show $V_K(z) = V_K^*(z) = \log^+|z|$. Clearly $V_K(z) \geq \log^+|z|$ since $\log^+|z| \in L$ and is 0 on K. Indeed, observe that $\log^+|z| \in L^+(\mathbb{C}^N)$ and clearly the support of $(dd^c \log^+|z|)^N$ is in $\{z : |z| = 1\}$ (indeed, by symmetry $(dd^c \log^+|z|)^N$ is a multiple of normalized surface area measure on the unit sphere). For the reverse inequality, take any $u \in L$ with $u \leq 0$ on K. Then $u(z) \leq \log^+|z|$ on the support of $(dd^c \log^+|z|)^N$ and hence by Proposition 8.9, $u(z) \leq \log^+|z|$ on all of $\mathbb{C}^N$.

If K is not pluripolar, then $(dd^c V_K^*)^N$ is the $\mathbb{C}^N$ analogue of the univariate equilibrium measure. We show that, indeed, for any non-pluripolar compact set K, the total mass of $(dd^c V_K^*)^N$ is the same.

Proposition 8.10. *If $u \in L(\mathbb{C}^N) \cap L^\infty_{loc}(\mathbb{C}^N)$, then*

(1) $\int_{\mathbb{C}^N} (dd^c u)^N \leq \int_{\mathbb{C}^N} (dd^c \log^+|z|)^N =: c_N$.
(2) *If $v \in L^+(\mathbb{C}^N)$ then $\int_{\mathbb{C}^N} (dd^c v)^N = c_N$.*

Proof. It suffices to show that for all $u \in L(\mathbb{C}^N) \cap L^\infty_{loc}(\mathbb{C}^N)$ and all $v \in L^+(\mathbb{C}^N)$ we have

$$\int_{\mathbb{C}^N} (dd^c u)^N \leq \int_{\mathbb{C}^N} (dd^c v)^N.$$

To verify this inequality, given such u, v it suffices to show for all $K \Subset \mathbb{C}^N$ and all $\alpha > 1$ sufficiently close to 1 we have

$$\int_K (dd^c u)^N \leq \alpha^N \int_{\mathbb{C}^N} (dd^c v)^N.$$

Thus we fix K and α with $1 < \alpha < 2$. Since adding a constant to u, v does not affect their Monge-Ampère measures, and since K is compact and u, v are bounded on K, we may assume

$$0 \leq 2v \leq u \text{ on } K.$$

Since $u \in L(\mathbb{C}^N)$ and $v \in L^+(\mathbb{C}^N)$, for R sufficiently large we have $u \leq \alpha v$ if $|z| > R$. This shows that $\{z \in \mathbb{C}^N : \alpha v(z) < u(z)\}$ is bounded. Moreover, since $0 \leq 2v \leq u$ on K and $\alpha < 2$ we have

$$K \subset \{z \in \mathbb{C}^N : \alpha v(z) < u(z)\}.$$

We apply Proposition 8.6 to αu and v to obtain

$$\int_{\{\alpha v<u\}}(dd^c u)^N \leq \int_{\{\alpha v<u\}}(dd^c\alpha v)^N = \alpha^N\int_{\{\alpha v<u\}}(dd^c v)^N.$$

Thus

$$\int_K (dd^c u)^N \leq \int_{\{\alpha v<u\}}(dd^c u)^N \leq \alpha^N\int_{\{\alpha v<u\}}(dd^c v)^N \leq \alpha^N\int_{\mathbb{C}^N}(dd^c v)^N.$$

□

We say that K is *L-regular* if V_K is continuous. We will soon see that

(1) $V_K(z) = \max\left\{0, \sup_p\left\{\frac{1}{\deg p}\log|p(z)| : \ p \text{ poly.}, \ ||p||_K \leq 1\right\}\right\}$ which shows that V_K is always lowersemicontinuous.
(2) We say that K is *L-regular* if V_K is continuous; thus from (1) K is $L-$regular precisely when $V_K = V_K^*$. As in the univariate case, this holds if and only if $V_K^* \equiv 0$ on K.
(3) Any compact set K can be approximated from above by a decreasing sequence of L-regular sets; e.g.,

$$K_{1/n} := \{z : \operatorname{dist}(z, K) \leq 1/n\}.$$

The fact that each $K_{1/n}$ is $L-$regular can be seen as in section 4 by utilizing the fact observed above that a closed Euclidean ball has this property. This approximation often allows us to reduce proofs to the case when K is $L-$regular.
(4) For a product set $K = K_1 \times \cdots \times K_N$ of planar compact sets $K_j \subset \mathbb{C}$,

$$V_K(z_1, \ldots, z_N) = \max_{j=1,\ldots,N} V_{K_j}(z_j). \tag{8.7}$$

In particular, for a closed polydisk

$$P := \{(z_1, \ldots, z_N) : |z_j - a_j| \leq r_j, \ j = 1, \ldots, N\},$$

$$V_P(z_1, \ldots, z_N) = \max_{j=1,\ldots,N}[0, \log|z_j - a_j|/r_j].$$

Regarding (3), note that $V_{K_{1/n}} = V_{K_{1/n}}^*$ and we show that $V_{K_{1/n}} \uparrow V_K^*$ a.e. (indeed, $V_{K_{1/n}} \uparrow V_K^*$ q.e. where this now means everywhere except a pluripolar set). Hence, if K is nonpluripolar, $(dd^c V_{K_{1/n}})^N \to (dd^c V_K^*)^N$ weak-*. We begin with a general fact about $\mathbb{R}^m$ negligible sets.

Proposition 8.11. *$\mathbb{R}^m$ negligible sets have $\mathbb{R}^m$ Lebesgue measure zero; i.e., if $\{u_\alpha\}$ is a locally uniformly bounded above family of shm functions on $D \subset \mathbb{R}^m$, then $\{z \in D : u(x) := \sup_\alpha u_\alpha(x) < u^*(x)\}$ has $\mathbb{R}^m$ Lebesgue measure zero,*

Proof. Note that u satisfies the subaveraging property and is locally integrable in D but may fail to be usc in D. Thus the smoothings $U_\delta := u * \chi_\delta$ are shm in D_δ and it is straightforward to see that $U_\delta \downarrow u^*$ everywhere in D while $U_\delta \downarrow u$ a.e. in D. □

In particular, this shows that plurinegligible sets in $\mathbb{C}^N$ have $\mathbb{R}^{2N}$ Lebesgue measure zero. Returning to the situation in (3), it is easy to see that $V_{K_{1/n}} \uparrow V_K$ on all of $\mathbb{C}^N$. For $V_{K_{1/n}} \leq V_K$ implies $\lim_{n\to\infty} V_{K_{1/n}} \leq V_K$ in $\mathbb{C}^N$; for the reverse inequality, if $u \in L(\mathbb{C}^N)$ with $u \leq 0$ on K then given $\epsilon > 0$ we have $K_{1/n} \subset \{z \in \mathbb{C}^N : u(z) < \epsilon\}$ for $n > n(\epsilon)$. Hence

$$u - \epsilon \leq V_{K_{1/n}} \leq \lim_{n\to\infty} V_{K_{1/n}}.$$

From Proposition 8.11, $V_K = V_K^*$ a.e. and the result follows.

We verify item (4); i.e., (8.7) in the case $N = 2$ using the following result due to Cegrell (cf., [7]).

Proposition 8.12. *Let $D \subset \mathbb{C}^2$ and let $u, v \in PSH(D) \cap L^\infty_{loc}(D)$. Then*

$$(dd^c \max(u,v))^2 = dd^c \max(u,v) \wedge dd^c(u+v) - dd^c u \wedge dd^c v. \tag{8.8}$$

To verify that $V_K(z_1,z_2) = \max[V_{K_1}(z_1), V_{K_2}(z_2)]$, call $v(z_1,z_2) := \max[V_{K_1}(z_1), V_{K_2}(z_2)]$. By approximation, we can assume K_1, K_2 are regular and K is $L-$regular (which would follow from the formula). Clearly $v \leq V_K$. To show the reverse inequality, we proceed in steps:

(1) $V_K \leq v$ on $(\mathbb{C} \times K_2) \cup (K_1 \times \mathbb{C})$.
Proof of (1): Fixing $z_2^0 \in K_2$, the function $z_1 \to V_K(z_1, z_2^0)$ belongs to $L(\mathbb{C})$ and is nonpositive on K_1. Hence $V_K(z_1, z_2^0) \leq V_{K_1}(z_1) = v(z_1, z_2^0)$ for $z_1 \in \mathbb{C}$. Fixing $z_1^0 \in K_1$ gives the inequality on $K_1 \times \mathbb{C}$.

(2) For all $a > 1$ we have $V_K \leq av$ on $\mathbb{C}^2$.
Proof of (2): From (1), we have $V_K \leq av$ on $(\mathbb{C}\times K_2)\cup(K_1\times\mathbb{C})$. Moreover, since $v \in L^+(\mathbb{C}^2)$, for $|(z_1,z_2)| \geq R$ for $R = R(a)$ sufficiently large we have $V_K \leq av$. Let $D := \{z : |z| < R\} \setminus [(\mathbb{C}\times K_2)\cup(K_1\times\mathbb{C})]$. We use (8.8) to show v (and hence av) is

maximal in D; since $V_K \leq av$ on ∂D we conclude that $V_K \leq av$ in D, finishing the proof. Now

$$(dd^c v)^2 = dd^c v \wedge [dd^c V_{K_1}(z_1) + dd^c V_{K_2}(z_2)] - dd^c V_{K_1}(z_1) \wedge dd^c V_{K_2}(z_2).$$

But V_{K_1} is harmonic in $\mathbb{C} \setminus K_1$ so that $dd^c V_{K_1}(z_1) \equiv 0$ on D; similarly, V_{K_2} is harmonic in $\mathbb{C} \setminus K_2$ so that $dd^c V_{K_2}(z_2) \equiv 0$ on D. Hence $(dd^c v)^2 = 0$ in D and v is maximal in D.

We give the proof of (8.8):

Proof. From properties of the complex Monge-Ampère operator and approximation, we may assume u, v are smooth. Let $w := \max(u, v)$. Thus we want to show

$$(dd^c w)^2 = dd^c w \wedge dd^c(u + v) - dd^c u \wedge dd^c v.$$

This can be rewritten in a more symmetric fashion as

$$dd^c(w - u) \wedge (dd^c(w - v) = 0.$$

Given $\epsilon > 0$, let $w_\epsilon := \max[u + \epsilon, v]$. We show

$$dd^c(w_\epsilon - u) \wedge (dd^c(w - v) = 0.$$

Since $w_\epsilon \downarrow w$ as $\epsilon \downarrow 0$, $dd^c(w_\epsilon) \to dd^c w$ and

$$dd^c(w_\epsilon - u) \wedge (dd^c(w - v) \to dd^c(w - u) \wedge (dd^c(w - v),$$

giving the result. Now $D = \{u < v\} \cup \{u + \epsilon > v\}$ and each of these sets is open. On $\{u < v\}$, $w = v$ so that $dd^c(w - v) \equiv 0$. On $\{u + \epsilon > v\}$, $w_\epsilon = u + \epsilon$ so that $w_\epsilon - u = \epsilon$ and hence $dd^c(w_\epsilon - u) \equiv 0$.

□

Remark 8.13. In this product case where K_1, K_2 are regular and K is $L-$regular, from the formula $V_K(z_1, z_2) = \max[V_{K_1}(z_1), V_{K_2}(z_2)]$, it can be shown that

$$(dd^c V_K)^2 = dd^c V_{K_1} \wedge dd^c V_{K_2} = \mu_{K_1} \times \mu_{K_2},$$

the product of the univariate equilibrium measures of K_1 and K_2. For example, if $K := \{(z_1, z_2) : |z_j - a_j| \leq r_j,\ j = 1, 2\}$, $(dd^c V_K)^2$ is (a multiple of) Haar measure on the torus $\{(z_1, z_2) : |z_j - a_j| = r_j,\ j = 1, 2\}$.

As discussed in item (1) after Proposition 8.10, a generalization of the one-variable Green function g_K for $K \subset \mathbb{C}^N$ compact is

$$\tilde{V}_K(z) := \max\left\{0, \sup_p \left\{\frac{1}{\deg p} \log |p(z)|\right\}\right\} \tag{8.9}$$

where the supremum is taken over all non-constant polynomials p with $||p||_K \leq 1$. We define the *polynomial hull* of K as

$$\widehat{K} \equiv \{z \in \mathbb{C}^N : |p(z)| \leq ||p||_K, \ p \text{ polynomial}\}.$$

Clearly $\tilde{V}_K = \tilde{V}_{\widehat{K}}$ and if $K = \widehat{K}$ we say K is polynomially convex. It turns out $\widehat{K}$ can just as well be constructed as a "hull" with respect to continuous psh functions; i.e., for D a neighborhood of $\widehat{K}$ (e.g., a sufficiently large ball or all of $\mathbb{C}^N$,

$$\widehat{K} = \widehat{K}_{PSH(D)} := \{z : u(z) \leq \sup_{\zeta \in K} u(\zeta) \text{ for all } u \in PSH(D) \cap C(D)\}.$$

For compact sets K, the upper envelope

$$V_K(z) := \sup\{u(z) : u \in L, \ u \leq 0 \text{ on } K\}$$

as defined in (8.6) coincides with that in (8.9). We sketch a proof of this. An important feature of the proof is the correspondence between psh functions in $L(\mathbb{C}^N)$ and "homogeneous" psh functions in $\mathbb{C}^{N+1}$. We remind the reader of the standard correspondence between polynomials p_d of degree d in N variables and homogeneous polynomials H_d of degree d in $N+1$ variables via

$$p_d(z_1, \ldots, z_N) \mapsto H_d(w_0, \ldots, w_N) := w_0^d p_d(w_1/w_0, \ldots, w_N/w_0).$$

Clearly $\tilde{V}_K(z) \leq V_K(z)$ and to prove the reverse inequality, by approximating K from above we may assume K is L-regular. We consider $h(z,w)$ defined for $(z,w) \in \mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$ as follows:

$$h(z,w) := |w| \exp V_K(z/w) \text{ if } w \neq 0;$$

$$h(z,w) := \limsup_{(z',w') \to (z,0)} h(z',w') \text{ if } w = 0.$$

This is a nonnegative *homogeneous* psh function in $\mathbb{C}^{N+1}$; i.e., we have $h(tz,tw) = |t| h(z,w)$ for $t \in \mathbb{C}$. We say that the function $\log h$ is *logarithmically homogeneous*: $\log h(tz,tw) = \log |t| + \log h(z,w)$. Fix a point $(z_0, w_0) \neq (0,0)$ with $z_0/w_0 \notin K$ and fix $0 < \epsilon < 1$. Using the fact that the polynomial hull coincides with the hull with respect to continuous psh functions, it follows that the compact set

$$E := \{(z,w) \in \mathbb{C}^{N+1} : h(z,w) \leq (1-\epsilon) h(z_0,w_0)\}$$

is polynomially convex. Moreover, E is *circled*: $(z,w) \in E$ implies $(e^{it}z, e^{it}w) \in E$ for all real t.

Claim. *Given a compact, circled set $E \subset \mathbb{C}^N$ and a polynomial $p_d = h_d + h_{d-1} + \cdots + h_0$ of degree d written as a sum of homogeneous polynomials, we have $||h_j||_E \leq ||p_d||_E,\ j = 0, \ldots, d$.*

From the Claim, whose proof is left as an exercise, the polynomial hull of our circled set E is the same as the hull obtained using only homogeneous polynomials. Since $E = \widehat{E}$ and $(z_0, w_0) \notin E$, we can find a homogeneous polynomial h_s of degree s with $|h_s(z_0, w_0)| > ||h_s||_E$. Define

$$p_s(z, w) := \frac{h_s(z, w)}{||h_s||_E} \cdot [(1 - \epsilon)h(z_0, w_0)]^s.$$

Then $|p_s(z, w)|^{1/s} \leq |h(z, w)|$ for $(z, w) \in \partial E$ and by homogeneity of $|p_s|^{1/s}$ and h we have $|p_s|^{1/s} \leq h$ in all of $\mathbb{C}^{N+1}$. At (z_0, w_0), we have

$$|p_s(z_0, w_0)|^{1/s} > (1 - \epsilon)h(z_0, w_0);$$

since $\epsilon > 0$ was arbitrary, as was the point (z_0, w_0) (provided $z_0/w_0 \notin K$), we get that

$$h(z, w) = \sup_s\{|p_s(z, w)|^{1/s} : p_s \text{ homogeneous of degree } s,\ |p_s|^{1/s} \leq |h|\}.$$

At $w = 1$, we obtain

$$\exp V_K(z) = h(z, 1)$$
$$= \sup_s\{|Q_s(z)|^{1/s} : Q_s \text{ of degree } s,\ |Q_s|^{1/s} \leq \exp V_K\}$$

which proves the result (note $V_K = 0$ on K).

From now on, we write V_K for the (unregularized) $L-$extremal function of a compact set K and we verify that:

Claim: If K is a nonpluripolar compact set, then V_K^* is maximal in $\mathbb{C}^N \setminus K$; i.e., $(dd^c V_K^*)^N = 0$ in $\mathbb{C}^N \setminus K$. Hence

$$\mu_K := \frac{1}{(2\pi)^N}(dd^c V_K^*)^N \tag{8.10}$$

is a positive measure on K (indeed, $\mu_K \in \mathcal{M}(K)$) and is called the *extremal measure* for K.

To prove the Claim, we begin with

$$V_K(z) = \sup\{u(z) : u \in L : u \leq 0 \text{ on } K\}.$$

From the existence on a ball B of a psh function $u \in C(\overline{B})$ with $u = f$ on ∂B for $f \in C(\partial B)$ and $(dd^c u)^N = 0$ in B, for f only usc on ∂B we can approximate f from above by $f_j \in C(\partial B)$ and construct u_j

satisfying $u_j \in C(\overline{B})$ with $u_j = f_j$ on ∂B and $(dd^c u_j)^N = 0$ in B; hence $u = \lim_{j\to\infty} u_j$ is maximal in B and $\limsup_{z\to\zeta} u(z) \leq f$ on ∂B. Using exercise 6 in section 7 (the gluing lemma for psh functions), we see that the class $\mathcal{U}$ of $v \in L$ with $v \leq 0$ on K is a Perron-Bremermann family; i.e., see step (2) below. Thus:

(1) From Choquet's lemma, we can recover V_K as an upper envelope of a countable family $\{u_n\}$; by replacing u_n by $v_n := \max[u_1, ..., u_n]$ we have V_K as an increasing sequence of psh functions $\{v_n\}$.
(2) Fix a ball $B \subset \mathbb{C}^N \setminus K$ and replace each v_n by its Perron-Bremermann modification $u_{v_n|_{\partial B}, B}$ on B. The function $\tilde{v}_n$ defined to be $u_{v_n|_{\partial B}, B}$ on B and v_n on $\mathbb{C}^N \setminus B$ is again in $\mathcal{U}$. Hence, on B, V_K is the monotone, increasing limit of maximal psh functions; i.e., we have $(dd^c \tilde{v}_n)^N = 0$ on B.
(3) By continuity of the complex Monge-Ampère operator under *increasing* limits for locally bounded psh functions (cf., [4]), $(dd^c V_K^*)^N = 0$ in B. This holds for each $B \subset \mathbb{C}^N \setminus K$.

Clearly for K compact, $V_K = V_{\widehat{K}}$ and $K \subset \widehat{K}$. The polynomial hull $\widehat{K}$ is the maximal ideal space of the uniform algebra $P(K)$ consisting of all complex-valued continuous functions f on K such that f is the uniform limit of a sequence of holomorphic polynomials on K. The *Shilov boundary* S_K of $P(K)$ is the smallest closed subset of K such that $||f||_{S_K} = ||f||_K$ for all $f \in P(K)$. Thus $V_{S_K} = V_K$; $S_K \subset K$; $\widehat{S_K} = \widehat{K}$; and for K nonpluripolar, the support of $(dd^c V_K^*)^N$ is contained in S_K. As an example, the extremal functions for a closed polydisk

$$K_1 := \{(z_1, ..., z_N) : |z_j - a_j| \leq r_j, j = 1, ..., N\}$$

and the torus

$$K_2 := \{(z_1, ..., z_N) : |z_j - a_j| = r_j, j = 1, ..., N\}$$

are the same. Here, $\widehat{K}_2 = K_1$ and $S_{K_1} = K_2$.

We next turn to a definition of a *relative extremal function* of a subset E of a bounded domain D in $\mathbb{C}^N$. For E a subset of D, define

$$\omega(z, E, D) := \sup\{u(z) : u \text{ psh in } D,\ u \leq 0 \text{ in } D,\ u|_E \leq -1\}.$$

The usc regularization $\omega^*(z, E, D)$ is called the *relative extremal function* of E relative to D (recall exercise 7 of section 3 for the univariate version of this).

As an example, take $K = \{z \in \mathbb{C}^N : |z| \leq r\}$ and $D = \{z \in \mathbb{C}^N : |z| < R\}$ with $R > r$. One can check that

$$\omega(z,K,D) = \frac{\log^+ \frac{|z|}{r} - \log \frac{R}{r}}{\log \frac{R}{r}} = \frac{1}{\log(R/r)}[\log^+ \frac{|z|}{r} - \log \frac{R}{r}].$$

Indeed, call $v(z) := \frac{\log^+ \frac{|z|}{r} - \log \frac{R}{r}}{\log \frac{R}{r}}$. Clearly $\omega(z,K,D) \geq v(z)$ since $v \in PSH(D)$ with $v = -1$ on K and $v \leq 0$ on D. For the reverse inequality, if $u \in PSH(D)$ with $u \leq 0$ in D and $u \leq -1$ on K, we claim that $u \leq v$ on D. For note that v is maximal in $D \setminus K$ since $(dd^c v)^N = 0$ there. Since $v \geq u$ on $\partial(D \setminus K)$ we have $v \geq u$ on $D \setminus K$. On K, $u \leq -1 = v$ and the claim follows. Hence $\omega(z,K,D) \leq v(z)$. Thus

$$(dd^c \omega(z,K,D))^N = \frac{1}{(\log(R/r))^N} \cdot (dd^c \log^+ \frac{|z|}{r})^N.$$

The function $\log^+ \frac{|z|}{r}$ we recognize as the $L-$extremal function V_K of K.

We show that pluripolarity of $E \subset D$ is characterized by triviality of $\omega^*(z,E,D)$.

Proposition 8.14. *Either $\omega^*(z) = \omega^*(z,E,D) \equiv 0$ in D or else ω^* is a nonconstant psh function in D satisfying $(dd^c\omega^*)^N = 0$ in $D \setminus \overline{E}$. We have $\omega^* \equiv 0$ if and only if E is pluripolar.*

Proof. If $\omega^*(z^0) = 0$ at some point $z^0 \in D$, then $\omega^* \equiv 0$ in D by the maximum principle for shm functions on domains in $\mathbb{R}^{2N}$. By Proposition 8.11, $\omega(z,E,D) = 0$ a.e. in D. Fix a point z' with $\omega(z',E,D) = 0$ and take a sequence of psh functions u_j in D with $u_j \leq 0$ in D, $u_j|_E \leq -1$, and $u_j(z') \geq -1/2^j$. Then $u(z) := \sum u_j(z)$ is psh in D (the partial sums form a decreasing sequence of psh functions) with $u(z') \geq -1$ (so $u \not\equiv -\infty$) and $u|_E = -\infty$; thus E is pluripolar.

Conversely, if E is pluripolar, there exists u psh in D with $u|_E = -\infty$; since D is bounded we may assume $u \leq 0$ in D. Then $\epsilon u \leq \omega(z,E,D)$ in D for all $\epsilon > 0$ which implies that $\omega(z,E,D) = 0$ for $z \in D$ where $u(z) \neq -\infty$. Since pluripolar sets have measure zero (why?), $\omega(z,E,D) = 0$ a.e. in D and hence $\omega^*(z,E,D) \equiv 0$ in D.

The proof that $(dd^c\omega^*)^N = 0$ in $D \setminus \overline{E}$ in case E is nonpluripolar goes along the same lines as the proof for V_K in the Claim. $\square$

The relative extremal function and the notion of *relative capacity* were key tools in the proof of Bedford and Taylor of Josefson's result that locally pluripolar sets are globally pluripolar. Here, for E a Borel subset of D,

$$C(E,D) := \sup\{\int_E (dd^c u)^N : u \text{ psh in } D,\ 0 \le u \le 1 \text{ in } D\}$$

is the *relative capacity* of E relative to D. See [25] or [7] for more details.

Exercises.

(1) Let $u \in C^2(D)$ where D is a domain in $\mathbb{C}^N$. Prove that u is pluriharmonic in D if and only if $dd^c u = 0$ in D.

(2) Let $u \in C^\infty(D)$ where D is a domain in $\mathbb{C}^N$. Prove that if u is psh in D and harmonic considered as a function in $D \subset \mathbb{R}^{2N}$, then u is pluriharmonic in D.

(3) If $u(z_1, ..., z_N) \in PSH(D)$ is independent of one or more of the variables $z_1, ..., z_N$, show that u is maximal in D.

(4) Let $D \subset \mathbb{R}^N$ be a domain. Show that $u : D \to \mathbb{R}$ is convex if and only if $U(z_1, ..., z_N) := u(\mathrm{Re} z_1, ..., \mathrm{Re} z_N) : D + i\mathbb{R}^N \subset \mathbb{C}^N \to \mathbb{R}$ is psh.

(5) Let $L : (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \to \mathbb{R}^2$ be defined as

$$L(z_1, z_2) = (\log|z_1|, \log|z_2|).$$

Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 on D and let $u := f \circ L$.

(a) Show that u is psh (where defined) if f is convex.

(b) Find a formula for $(dd^c u)^2$ in terms of the *real* Hessian of f.

(6) Verify that for the set

$$E := \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1 + z_2^2) = \mathrm{Re}(z_1 + z_2 + z_2^2) = 0\}$$

any complex line $L := \{(z_1, z_2) : a_1 z_1 + a_2 z_2 = b\},\ a_1, a_2, b \in \mathbb{C}$ intersects E in at most four points.

(7) Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be holomorphic. Show that

$$G(f) := \{(z, f(z)) : z \in D\}$$

is pluripolar. (A deep result of Shcherbina states that for *continuous* f on D, f is holomorphic if and only if $G(f)$ is pluripolar).

(8) Show that $(dd^c\big[\frac{1}{2}\log(1 + |z|^2)\big])^N > 0$ on $\mathbb{C}^N$; i.e., $\frac{1}{2}\log(1 + |z|^2)$ is stricly psh in $\mathbb{C}^N$.

(9) Show that $u(z,w) = \big(\max[0, |z|^2 - 1/2, |w|^2 - 1/2]\big)^2$ satisfies $(dd^c u)^2 = 0$ in B; $u \in C^{1,1}(B)$; but $u \notin C^2(B)$.

(10) Prove the claim that for a compact, circled set $E \subset \mathbb{C}^N$ and a polynomial $p_d = h_d + h_{d-1} + \cdots + h_0$ of degree d written as a sum of homogeneous polynomials, $||h_j||_E \leq ||p_d||_E$, $j = 0, \ldots, d$. (*Hint: Fix a point* $b \in E$ *at which* $|h_j(b)| = ||h_j||_E$ *and use Cauchy's estimates on* $\lambda \mapsto p_d(\lambda b) = \sum_{j=0}^{d} \lambda^j h_j(b)$).

9. Transfinite diameter and polynomial interpolation in $\mathbb{C}^N$.

We have seen that, as in $\mathbb{C}$, for a compact set $K \subset \mathbb{C}^N$, either $V_K^* \equiv +\infty$, in which case K is pluripolar, or else $V_K^* \in L^+(\mathbb{C}^N)$. In the latter case, the measure $\mu_K = \frac{1}{(2\pi)^N}(dd^c V_K^*)^N$ plays the role of the equilibrium measure. However, since the complex Monge-Ampère operator is nonlinear, there is no natural notion of energy of measures which μ_K minimizes. Nevertheless, there is an analogue of the notion of transfinite diameter, and this turns out to be a nonnegative set function on compact sets which is zero precisely on the pluripolar sets. We highlight the main points of the fundamental work of Zaharjuta [34]. We begin by considering a function Y from the set of multiindices $\alpha \in \mathbf{N}^N$ to the nonnegative real numbers satisfying:

$$Y(\alpha+\beta) \leq Y(\alpha)\cdot Y(\beta) \text{ for all } \alpha,\ \beta \in \mathbf{N}^N. \tag{9.1}$$

We call a function Y satisfying (9.1) *submultiplicative*; we have two main examples below. Let $e_1(z), ..., e_j(z), ...$ be a listing of the monomials $\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1}\cdots z_N^{\alpha_N}\}$ in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha = \alpha(i) = (\alpha_1, ..., \alpha_N) \in \mathbf{N}^N$, but with $\deg e_i = |\alpha(i)|$ nondecreasing. We write $|\alpha| := \sum_{j=1}^N \alpha_j$.

We define the following integers:

(1) $m_d^{(N)} = m_d :=$ the number of monomials $e_i(z)$ of degree at most d in N variables;
(2) $h_d^{(N)} = h_d :=$ the number of monomials $e_i(z)$ of degree exactly d in N variables;
(3) $l_d^{(N)} = l_d :=$ the sum of the degrees of the m_d monomials $e_i(z)$ of degree at most d in N variables.

We have the following relations:

$$m_d^{(N)} = \binom{N+d}{d};\ h_d^{(N)} = m_d^{(N)} - m_{d-1}^{(N)} = \binom{N-1+d}{d} \tag{9.2}$$

and

(9.3)
$$h_d^{(N+1)} = \binom{N+d}{d} = m_d^{(N)};\ l_d^{(N)} = N\binom{N+d}{N+1} = (\frac{N}{N+1})\cdot d m_d^{(N)}.$$

The elementary fact that the dimension of the space of homogeneous polynomials of degree d in $N+1$ variables equals the dimension of the space of polynomials of degree at most d in N variables will be useful. Finally, we let

$$r_d^{(N)} = r_d := dh_d^{(N)} = d(m_d^{(N)} - m_{d-1}^{(N)})$$

which is the sum of the degrees of the h_d monomials $e_i(z)$ of degree exactly d in N variables. We observe that

$$l_d^{(N)} = \sum_{k=1}^{d} r_k^{(N)} = \sum_{k=1}^{N} kh_k^{(N)}. \tag{9.4}$$

Let $K \subset \mathbb{C}^N$ be compact. Here are two natural constructions of families of Chebyshev-type constants associated to K:

(1) *Chebyshev constants*: Define the class of polynomials

$$P_i = P(\alpha(i)) := \{e_i(z) + \sum_{j<i} c_j e_j(z)\};$$

and the Chebyshev constants

$$Y_1(\alpha) := \inf\{||p||_K : p \in P_i\}.$$

We write $t_{\alpha,K} := t_{\alpha(i),K}$ for a Chebyshev polynomial; i.e., $t_{\alpha,K} \in P(\alpha(i))$ and $||t_{\alpha,K}||_K = Y_1(\alpha)$.

(2) *Homogeneous Chebyshev constants*: Define the class of homogeneous polynomials

$$P_i^{(H)} = P^{(H)}(\alpha(i)) := \{e_i(z) + \sum_{j<i,\ \deg(e_j)=\deg(e_i)} c_j e_j(z)\};$$

and the homogeneous Chebyshev constants

$$Y_2(\alpha) := \inf\{||p||_K : p \in P_i^{(H)}\}.$$

We write $t_{\alpha,K}^{(H)} := t_{\alpha(i),K}^{(H)}$ for a homogeneous Chebyshev polynomial; i.e., $t_{\alpha,K}^{(H)} \in P^{(H)}(\alpha(i))$ and $||t_{\alpha,K}^{(H)}||_K = Y_2(\alpha)$.

Let Σ denote the standard $(N-1)-$simplex in $\mathbb{R}^N$; i.e.,

$$\Sigma = \{\theta = (\theta_1, ..., \theta_N) \in \mathbb{R}^N : \sum_{j=1}^{N} \theta_j = 1,\ \theta_j \geq 0,\ j = 1, ..., N\},$$

and let

$$\Sigma^0 := \{\theta \in \Sigma :\ \theta_j > 0,\ j = 1, ..., N\}.$$

Given a submultiplicative function $Y(\alpha)$, define, as with the above examples, a new function

$$\tau(\alpha) := Y(\alpha)^{1/|\alpha|}. \tag{9.5}$$

An examination of lemmas 1, 2, 3, 5, and 6 in the fundamental paper by Zaharjuta [34] shows that (9.1) is the only property of the numbers $Y(\alpha)$ needed to establish those lemmas. To summarize, we have the following results for $Y : \mathbf{N}^N \to \mathbb{R}^+$ satisfying (9.1) and the associated function $\tau(\alpha)$ in (9.5).

Lemma 9.1. *For all $\theta \in \Sigma^0$, the limit*

$$T(Y,\theta) := \lim_{\alpha/|\alpha| \to \theta} Y(\alpha)^{1/|\alpha|} = \lim_{\alpha/|\alpha| \to \theta} \tau(\alpha)$$

exists.

We call $T(Y,\theta)$ a *directional Chebyshev constant* in the direction θ.

Lemma 9.2. *The function $\theta \to T(Y,\theta)$ is log-convex on Σ^0 (and hence continuous).*

Lemma 9.3. *Given $b \in \partial\Sigma$,*

$$\liminf_{\theta \to b,\ \theta \in \Sigma^0} T(Y,\theta) = \liminf_{i \to \infty,\ \alpha(i)/|\alpha(i)| \to b} \tau(\alpha(i)).$$

Lemma 9.4. *Let $\theta(k) := \alpha(k)/|\alpha(k)|$ for $k = 1, 2, ...$ and let Q be a compact subset of Σ^0. Then*

$$\limsup_{|\alpha| \to \infty} \{\log \tau(\alpha(k)) - \log T(Y(\theta(k))) : |\alpha(k)| = \alpha,\ \theta(k) \in Q\} = 0.$$

Lemma 9.5. *Define*

$$\tau(Y) := \exp\Big[\frac{1}{meas(\Sigma)} \int_{\Sigma} \log T(Y,\theta) d\theta\Big]$$

Then

$$\lim_{d \to \infty} \frac{1}{h_d} \sum_{|\alpha| = d} \log \tau(\alpha) = \log \tau(Y);$$

i.e., using (9.5),

$$\lim_{d \to \infty} \Big[\prod_{|\alpha| = d} Y(\alpha)\Big]^{1/dh_d} = \tau(Y).$$

One can incorporate all of the $Y(\alpha)'$s for $|\alpha| \leq d$; this is the content of the next result.

Theorem 9.6. *We have*

$$\lim_{d\to\infty}\Big[\prod_{|\alpha|\le d} Y(\alpha)\Big]^{1/l_d} \text{ exists and equals } \tau(Y).$$

Proof. Define the geometric means

$$\tau_d^0 := \Big(\prod_{|\alpha|=d} \tau(\alpha)\Big)^{1/h_d},\ d = 1, 2, ...$$

The sequence

$$\log\tau_1^0, \log\tau_1^0, ...(r_1 \text{ times}), ..., \log\tau_d^0, \log\tau_d^0, ...(r_d \text{ times}), ...$$

converges to $\log\tau(Y)$ by the previous lemma; hence the arithmetic mean of the first $l_d = \sum_{k=1}^d r_k$ terms (see (9.4)) converges to $\log\tau(Y)$ as well. Exponentiating this arithmetic mean gives

$$\Big(\prod_{k=1}^d (\tau_k^0)^{r_k}\Big)^{1/l_d} = \Big(\prod_{k=1}^d \prod_{|\alpha|=k} \tau(\alpha)^k\Big)^{1/l_d} = \Big(\prod_{|\alpha|\le d} Y(\alpha)\Big)^{1/l_d} \tag{9.6}$$

and the result follows. □

Returning to our examples (1) and (2), example (1) was the original setting of Zaharjuta [34] which he utilized to prove the existence of the limit in the definition of the *transfinite diameter* of a compact set $K\subset\mathbb{C}^N$. For $\zeta_1, ..., \zeta_n \in \mathbb{C}^N$, let

$$\begin{aligned} VDM(\zeta_1, ..., \zeta_n) &= \det[e_i(\zeta_j)]_{i,j=1,...,n} \\ &= \det\begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \dots & e_n(\zeta_n)\end{bmatrix}\end{aligned} \tag{9.7}$$

be a generalized Vandermonde determinant, in analogy with the univariate case, and for a compact subset $K\subset\mathbb{C}^N$ let

$$V_n = V_n(K) := \max_{\zeta_1,...,\zeta_n\in K} |VDM(\zeta_1, ..., \zeta_n)|.$$

Then

$$\delta(K) = \lim_{d\to\infty} V_{m_d}^{1/l_d} \tag{9.8}$$

is the *transfinite diameter* of K; Zaharjuta [34] showed that the limit exists by showing that one has

$$\delta(K) = \exp\Big[\frac{1}{\text{meas}(\Sigma)}\int_{\Sigma^0} \log\tau(K,\theta)d\theta\Big] \tag{9.9}$$

where $\tau(K,\theta) = T(Y_1,\theta)$ from (1); i.e., the right-hand-side of (9.9) is $\tau(Y_1)$. This follows from Theorem 9.6 for $Y = Y_1$ and the estimate

$$\Big(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\Big)^{1/l_d} \leq V_{m_d}^{1/l_d} \leq (m_d!)^{1/l_d}\Big(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\Big)^{1/l_d}$$

in [34] (compare the estimate (9.6)). We make two comments:

(1) Clearly if a compact set K is contained in an algebraic subvariety of $\mathbb{C}^N$ then $\delta(K) = 0$ (why?). It turns out that *for* $K \subset \mathbb{C}^N$ *compact,* $\delta(K) = 0$ *if and only if* K *is pluripolar* [28].

(2) A set of points $z_1^{(d)}, ..., z_{m_d}^{(d)} \in K$ with

$$V_{m_d} = V_{m_d}(K) = |VDM(z_1^{(d)}, ..., z_{m_d}^{(d)})|$$

is called a set of *Fekete points of order* d *for* K. An interesting question is whether, for a nonpluripolar compact set K, we have $\mu_d := \sum_{j=1}^{m_d} \delta_{z_j^{(d)}} \to \mu_K$ weak-*. We discuss this in the final section.

For a compact *circled* set $K \subset \mathbb{C}^N$; i.e., $z \in K$ if and only if $e^{i\phi}z \in K$, $\phi \in [0, 2\pi]$, one need only consider homogeneous polynomials in the definition of the directional Chebyshev constants $\tau(K,\theta)$. In other words, in the notation of (1) and (2), $Y_1(\alpha) = Y_2(\alpha)$ for all α so that

$$T(Y_1,\theta) = T(Y_2,\theta) \text{ for circled sets } K.$$

This is because for such a set, if we write a polynomial p of degree d as $p = \sum_{j=0}^{d} H_j$ where H_j is a homogeneous polynomial of degree j, then, from the Cauchy integral formula, $||H_j||_K \leq ||p||_K,\ j = 0, ..., d$ (see the Claim and exercise 10 in the previous section). Moreover, a slight modification of Zaharjuta's arguments proves the existence of the limit of appropriate roots of maximal *homogeneous* Vandermonde determinants; i.e., the homogeneous transfinite diameter $d^{(H)}(K)$ of a compact set. From the above remarks, it follows that

$$\text{for circled sets } K,\ \delta(K) = d^{(H)}(K). \tag{9.10}$$

We will use this in the next section. Since we will be using the homogeneous transfinite diameter, we amplify the discussion. We relabel the standard basis monomials $\{e_i^{(H,d)}(z) = z^{\alpha(i)} = z_1^{\alpha_1} \cdots z_N^{\alpha_N}\}$ where $|\alpha(i)| = d,\ i = 1, ..., h_d$, we define the $d-$homogeneous Vandermonde determinant

$$VDMH_d((\zeta_1,...,\zeta_{h_d}) := \det\left[e_i^{(H,d)}(\zeta_j)\right]_{i,j=1,...,h_d}. \tag{9.11}$$

Then

$$d^{(H)}(K) = \lim_{d\to\infty}\left[\max_{\zeta_1,...,\zeta_{h_d}\in K}|VDMH_d(\zeta_1,...,\zeta_{h_d})|\right]^{1/dh_d} \tag{9.12}$$

is the homogeneous transfinite diameter of K; the limit exists and equals

$$\exp\left[\frac{1}{\text{meas}(\Sigma)}\int_{\Sigma^0}\log T(Y_2,\theta)d\theta\right]$$

where $T(Y_2,\theta)$ comes from the homogeneous Chebyshev constants of item (2).

A useful fact is that

$$\delta(K) = \delta(\widehat{K}) \text{ and } d^{(H)}(K) = d^{(H)}(\widehat{K}) \tag{9.13}$$

for K compact where recall

$$\widehat{K} := \{z \in \mathbb{C}^N : |p(z)| \leq ||p||_K, \text{ all polynomials } p\}$$

is the polynomial hull of K.

We turn to a discussion of polynomial approximation and interpolation in several complex variables. If the compact set $K \subset \mathbb{C}^N$ is L-regular, then for each $R > 1$ we define the open sets

$$D_R \equiv \{z : V_K(z) < \log R\}; \tag{9.14}$$

then we clearly have, from (8.9), the Bernstein-Walsh inequality

$$|p(z)| \leq ||p||_K R^{\deg p}, \qquad z \in D_R \tag{9.15}$$

for every polynomial p in $\mathbb{C}^N$.

Recall that a compact set $K \subset \mathbb{C}^N$ is called polynomially convex if K coincides with $\widehat{K}$. With these definitions, Theorem 4.5 goes over *exactly* to several complex variables:

Theorem 9.7. *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$. Let $R > 1$, and let D_R be defined by (9.14). Let f be continuous on K. Then*

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq 1/R$$

if and only if f is the restriction to K of a function holomorphic in D_R.

Here, recall that for $f \in C(K)$,

$$d_n = d_n(f, K) \equiv \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}.$$

For the rest of this section, we use n instead of d to index the degree of polynomials to avoid notational issues with the distance "d_n". To prove "only if" we may repeat the proof after the statement of Theorem 4.1, since K satisfies the Bernstein-Walsh inequality (9.15). The "if" proof, although not hard, requires some deeper knowledge of several complex variables.

We can utilize Lagrange interpolation in this higher-dimensional setting. Choose m_n points $A_n = \{a_{n1}, ..., a_{nm_n}\} \subset K$ and form the Vandermonde determinant

$$V_n(A_n) \equiv \det[e_i(a_{nj})]_{i,j=1,...,m_n}.$$

If $V_n(A_n) \neq 0$, we can form the FLIP's

$$l_{nj}(x) \equiv \frac{V_n(a_{n1}, ..., x, ..., a_{nm_n})}{V_n(A_n)}, \qquad j = 1, ..., m_n. \tag{9.16}$$

In the one (complex) variable case, we get *cancellation* in this ratio so that the formulas for the FLIP's simplify. In general, we still have $l_{nj}(a_{ni}) = \delta_{ji}$ and $l_{nj} \in \mathcal{P}_n$ since l_{nj} is a linear combination of $e_1, .., e_{m_n}$. Note that for a set of Fekete points of order n, we have $||l_{nj}||_K = 1$ for $j = 1, ..., m_n$ (why?). For f defined on K,

$$(L_n f)(x) \equiv \sum_{j=1}^{m_n} f(a_{nj}) l_{nj}(x)$$

is the Lagrange interpolating polynomial (LIP) for f at the points A_n. We call

$$\Lambda_n \equiv \sup_{x \in K} \sum_{j=1}^{m_n} |l_{nj}(x)|$$

the n-th *Lebesgue constant* for K, A_n. As in section 4, this is the norm of the linear operator

$$\mathcal{L}_n : C(K) \to \mathcal{P}_n \subset C(K)$$

defined by $\mathcal{L}_n(f) := L_n f$ where we equip $C(K)$ with the supremum norm. For a set of Fekete points of order n, we have $\Lambda_n \leq m_n$. We say that K is *determining* for $\bigcup \mathcal{P}_n$ if whenever $h \in \bigcup \mathcal{P}_n$ satisfies $h = 0$ on K, it follows that $h \equiv 0$. For these sets we can find points A_n for each n with $V_n(A_n) \neq 0$. We have the following elementary result, similar to the proof in one variable that arrays satisfying (4.11) yield good polynomial approximants to holomorphic functions.

Theorem 9.8. *Let K be determining for $\bigcup \mathcal{P}_n$ and let $A_n \subset K$ satisfy $V_n(A_n) \neq 0$ for each n. Given f bounded on K, if $\limsup \Lambda_n^{1/n} = 1$, then $\limsup ||f - L_n f||_K^{1/n} = \limsup d_n^{1/n}$.*

Proof. Fix $\epsilon > 0$ and choose, for each n, a polynomial $p_n \in \mathcal{P}_n$ with $||f - p_n||_K^{1/n} \leq d_n^{1/n} + \epsilon$. Since $p_n \in \mathcal{P}_n$, we have $L_n p_n = p_n$ and

$$||f - L_n f||_K = ||f - p_n + L_n p_n - L_n f||_K$$

$$\leq ||f - p_n||_K + \Lambda_n ||f - p_n||_K = (1 + \Lambda_n)||f - p_n||_K.$$

Using the hypothesis $\limsup \Lambda_n^{1/n} = 1$, we obtain the conclusion. □

Immediately from Theorems 9.7 and 9.8 we have

Corollary 9.9. *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$ and let $\{A_n\} \subset K$ satisfy $\limsup \Lambda_n^{1/n} = 1$. Then for any f holomorphic on a neighborhood of K, $L_n f \to f$ uniformly on K.*

As in the univariate case, for $K \subset \mathbb{C}^N$ compact, $L-$regular and polynomially convex, we can consider the following four properties which an array $\{a_{nj}\}_{j=1,...,m_n;\ n=1,2,...} \subset K$ may or may not possess:

(1) $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$;
(2) $\lim_{n\to\infty} |VDM(a_{n1}, ..., a_{nm_n})|^{\frac{1}{l_n}} = \delta(K)$;
(3) $\lim_{n\to\infty} \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{a_{nj}} = \mu_K$ weak-*;
(4) $L_n f \rightrightarrows f$ on K for each f holomorphic on a neighborhood of K.

Corollary 9.9 shows that (1) $\implies$ (4); the univariate proof that (1) $\implies$ (2) generalizes to the multivariate setting; and a recent deep result in [6], which we give as Corollary 11.13 in section 10, shows that (2) $\implies$ (3). The reference [12] includes counterexamples to most other implications. A major problem with Lagrange interpolation of holomorphic functions in $\mathbb{C}^N,\ N > 1$, is the lack of a Hermite remainder formula. Together with the fact that one needs to insure, for each n, that the points $a_{n1}, ..., a_{nm_n}$ one chooses satisfy $VDM(a_{n1}, ..., a_{nm_n}) \neq 0$ (unisolvence), one might seek other polynomial interpolation procedures.

A more promising type of interpolation procedure has been successfully applied to many approximation problems by Tom Bloom and his collaborators. A natural extension of Lagrange interpolation to $\mathbb{R}^N$, $N > 1$ was discovered by P. Kergin (a student of Bloom) in his thesis. Indeed, Kergin interpolation acting on ridge functions (a univariate function composed with a linear form) is Lagrange interpolation. The Kergin interpolation polynomials generalize to the case of C^m functions in $\mathbb{R}^N$ both the Lagrange interpolation polynomials and those of Hermite.

As brief motivation, given $f \in C^m([0,1])$, say, and given $m+1$ points $t_0 < \cdots < t_m \in [0,1]$, if one constructs the Lagrange interpolating polynomial $L_m f$ for f at these points, then there exist (at least) $m-1$ points between pairs of successive t_j at which f' and $(L_m f)'$ agree; then there exist (at least) $m-2$ points between triples of successive t_j at which f'' and $(L_m f)''$ agree, etc. Given a set $\mathbf{A} = [A_0, A_1, \ldots, A_m] \subset \mathbb{R}^N$ of $m+1$ points and f a function of class C^m on a neighborhood of the convex hull of these points, there exists a unique polynomial $\mathcal{K}_A(f) = \mathcal{K}_A(f)(x_1, \ldots, x_N)$ of total degree m such that $\mathcal{K}_A(f)(A_j) = f(A_j)$, $j = 0, 1, \ldots, m$, and such that for every integer r, $0 \leq r \leq m-1$, every subset J of $\{0, 1, \ldots, m\}$ with cardinality equal to $r+1$, and every homogeneous differential operator Q of order r with constant coefficients, there exists ξ belonging to the convex hull of the (A_j), $j \in J$, such that $Qf(\xi) = Q\mathcal{K}_A(f)(\xi)$. In [9], Bloom gives a proof of this result by using a formula due to Micchelli and Milman [29] which gives an explicit expression for $\mathcal{K}_A(f)$. If $f = u + iv$ is holomorphic in a convex region D in $\mathbb{C}^N$, and if $\mathbf{A} = [A_0, A_1, \ldots, A_m] \subset D \subset \mathbb{C}^N = \mathbb{R}^{2N}$, then we can construct $\mathcal{K}_A(u)$ and $\mathcal{K}_A(v)$. It turns out (cf., [20]) that $\mathcal{K}_A(u) + i\mathcal{K}_A(v)$ is a holomorphic polynomial.

An alternate description, which we give in the holomorphic setting, is as follows (cf., [14]). Let D be a $\mathbb{C}$-convex domain in $\mathbb{C}^N$, i.e., the intersection of D with any complex line is connected and simply connected. Note that in $\mathbb{R}^N$ this is the same condition as convexity if we replace "complex line" by "real line." For any set $\mathbf{A} = [A_0, \ldots, A_d]$ of (not necessarily distinct) $d+1$ points in D there exists a unique linear projector $\mathcal{K}_{\mathbf{A}} : \mathcal{O}(D) \to \mathcal{P}_d$ (recall that $\mathcal{O}(D)$ is the space of holomorphic functions on D and $\mathcal{P}_d$ is the space of polynomials of N complex variables of degree less than or equal to d) such that

(1) $\mathcal{K}_{\mathbf{A}}(f)(A_j) = f(A_j)$ for $j = 0, \cdots, d$,
(2) $\mathcal{K}_{\mathbf{A}}(g \circ \lambda) = \mathcal{K}_{\lambda(\mathbf{A})}(g) \circ \lambda$ for every affine map $\lambda : \mathbb{C}^N \to \mathbb{C}$ and $g \in \mathcal{O}(\lambda(D))$, where $\lambda(\mathbf{A}) = (\lambda(A_0), \ldots, \lambda(A_d))$,

(3) $\mathcal{K}_{\mathbf{A}}$ is independent of the ordering of the points in $\mathbf{A}$, and
(4) $\mathcal{K}_{\mathbf{B}} \circ \mathcal{K}_{\mathbf{A}} = \mathcal{K}_{\mathbf{B}}$ for every subsequence $\mathbf{B}$ of $\mathbf{A}$.

The operator $\mathcal{K}_{\mathbf{A}}$ is called the Kergin interpolating operator with respect to $\mathbf{A}$.

Set $\mathcal{K}_d := \mathcal{K}_{\mathbf{A}_d}$ with $\mathbf{A}_d = [A_{d0}, \ldots, A_{dd}]$ and A_{dj} in a compact subset K of $D \subset \mathbb{C}^N$ for every $j = 0, \ldots, d$ and $d = 1, 2, 3, \ldots$. *Under what conditions on the array $\{\mathbf{A}_d\}_{d=1,2,\ldots}$ is it true that $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\overline{D}$?* Bloom and Calvi [14] attacked this problem with the aid of an integral representation formula for the remainder $f - \mathcal{K}_d(f)$ proved by M. Andersson and M. Passare [2]. Their solution reads as follows. Assume that the measures $\mu_d = (d+1)^{-1} \sum_{j=0}^{d} \delta_{A_{dj}}$ converge weak-* as $d \to \infty$ to a measure μ. In one variable, the answer comes from potential theory: one considers the logarithmic potential

$$V_\mu(z) := \int_K \log|z - \zeta| d\mu(\zeta)$$

and the required condition is that

$$\{z \in \mathbb{C} : V_\mu(z) \leq \sup_K V_\mu\} \subset D.$$

For $N > 1$, given a linear form $p : \mathbb{C}^N \to \mathbb{C}$, define $\mu^p = p_* \mu$ as the push-forward of μ to $\mathbb{C}$ via p, i.e., for $f \in C_0(\mathbb{C})$,

$$\mu^p(f) := \int_{\mathbb{C}} f d\mu^p = \mu(f \circ p) := \int_{\mathbb{C}^N} (f \circ p) d\mu.$$

Set

$$\Psi_\mu(p, z) := \mu^p(\log|z - \cdot|) = \int_{\mathbb{C}} \log|z - \zeta| d\mu^p(\zeta),$$

and let $M_\mu(p)$ be the maximum of $z \mapsto \Psi_\mu(p, z)$ on $p(K)$. *If D has C^2 boundary and $\{z \in \mathbb{C} : \Psi_\mu(p, z) \leq M_\mu(p)\} \subset p(D)$ for every linear form p on $\mathbb{C}^N$, then $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\overline{D}$.*

We call an array $\{\mathbf{A}_d\}_{d=1,2,\ldots}$ *extremal* for K if $\mathcal{K}_d(f)$ converges to f uniformly on K for each f holomorphic in a neighborhood of K. Of course, $\mathcal{K}_d(f)$ should make sense; i.e., f should be defined, e.g., in the convex (or more generally, the $\mathbb{C}-$convex) hull of K. In the setting of compact, convex subsets K of $\mathbb{R}^N$, Bloom and Calvi proved the following striking result.

Theorem 9.10. [15] *Let $K \subset \mathbb{R}^N$, $N \geq 2$, be a compact, convex set with nonempty interior. Then K admits extremal arrays if and only if $N = 2$ and K is the region bounded by an ellipse.*

For the Andersson-Passare remainder formula one needs an integral formula with a *holomorphic* kernel; moreover, one with a kernel that is the composition of a univariate function with an affine function. Together with property (2) of the Kergin interpolating operator, this allows a reduction of the multivariate problem to a univariate setting. For an outline of these items, see [26].

Exercises.

(1) Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1,\ z_2 = 0\}$. What is $\delta(K)$? Give a proof of your answer.

(2) Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1,\ |z_2| \leq 1\}$. What is $\delta(K)$? Give a proof of your answer.

(3) *Extra Credit.* Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : (\mathrm{Re} z_1)^2 + (\mathrm{Re} z_2)^2 \leq 1,\ \mathrm{Im} z_1 = \mathrm{Im} z_2 = 0\}$. What is $\delta(K)$? Give a proof of your answer.

(4) Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : 0 \leq |z_1| \leq |z_2| \leq 1\}$. Find $\widehat{K}$.

(5) Let $\{A_n\}$ be a Fekete array for K; i.e., for each $n = 1, 2, ...$, the points $A_n = \{a_{n1}, ..., a_{nm_n}\} \subset K$ form a set of Fekete points of order n for K. Prove that $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$.

(6) Verify (9.13) that for K compact, $\delta(K) = \delta(\widehat{K})$ and $d^{(H)}(K) = d^{(H)}(\widehat{K})$. (*Hint: Compare the supremum norms of the Chebyshev polynomials $t_{\alpha,K}$, $t_{\alpha,\widehat{K}}$ and those of the homogeneous Chebyshev polynomials $t^{(H)}_{\alpha,K}$, $t^{(H)}_{\alpha,\widehat{K}}$*).

10. Weighted pluripotential theory in $\mathbb{C}^N$, $N > 1$, Bergman functions and L^2-theory.

As in the univariate case, in weighted pluripotential theory in $\mathbb{C}^N$ for $N > 1$ one restricts to closed but possibly unbounded sets. Again for $K \subset \mathbb{C}^N$ closed we let $\mathcal{A}(K)$ denote the collection of lowersemicontinuous $Q := -\log w$ where w is a nonnegative, usc function on K with $\{z \in K : w(z) > 0\}$ nonpluripolar; if K is unbounded, we require

$$|z|w(z) \to 0 \text{ as } |z| \to \infty, \ z \in K. \tag{10.1}$$

We define the weighted extremal function or weighted pluricomplex Green function $V_{K,Q}^*(z) := \limsup_{\zeta \to z} V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^N), \ u \leq Q \text{ on } K\}.$$

We have $V_{K,Q}^* \in L(\mathbb{C}^N)$. In the unbounded case, we again remind the reader that property (10.1) is equivalent to

$$Q(z) - \log|z| \to \ +\infty \text{ as } |z| \to \infty \text{ through points in } K;$$

hence $V_{K,Q}$ is well-defined and equals $V_{K \cap \mathcal{B}_R, Q}$ for $R > 0$ sufficiently large where $\mathcal{B}_R = \{z : |z| \leq R\}$ (Definition 2.1 and Lemma 2.2 of Appendix B in [32]). Thus $V_{K,Q}^* \in L_{loc}^\infty(\mathbb{C}^N)$ and $(dd^c V_{K,Q}^*)^N$ is a well-defined positive measure and the support

$$S_w := \text{supp}(\mu_{K,Q})$$

of the *weighted extremal measure*

$$\mu_{K,Q} := \frac{1}{(2\pi)^N}(dd^c V_{K,Q}^*)^N$$

is compact (recall the definition of μ_K in (8.10)). The proof of (6.7), adjusted using the solution of the Dirichlet problem for the complex Monge-Ampère equation on a ball, shows that

$$S_w \subset S_w^* := \{z \in K : V_{K,Q}^*(z) \geq Q(z)\}. \tag{10.2}$$

Moreover,

$$V_{K,Q}^* = Q \text{ q.e. on } S_w$$

(i.e., $V_{K,Q}^* = Q$ on $S_w \setminus F$ where F is pluripolar); and if $u \in L(\mathbb{C}^N)$ satisfies $u \leq Q$ q.e. on S_w then $u \leq V_{K,Q}^*$ on $\mathbb{C}^N$. Indeed,
(10.3)

$$V_{K,Q}(z) = \sup\{\frac{1}{deg(p)} \log|p(z)| : ||w^{deg(p)}p||_{S_w} \leq 1, \ p \text{ polynomial}\}$$

and

$$||w^{deg(p)}p||_{S_w} = ||w^{deg(p)}p||_K.$$

Theorem 2.8 of Appendix B in [32] includes the slightly stronger statement that

$$V_{K,Q}^*(z) = \big[\sup\{\frac{1}{deg(p)}\log|p(z)| : ||w^{deg(p)}p||_K^* \leq 1,\ p \text{ polynomial}\}\big]^*$$

where

$$||w^{deg(p)}p||_K^* := \inf\{||w^{deg(p)}p||_{K\setminus F} : F \subset K \text{ pluripolar}\}.$$

The unweighted case is when K is compact and $w \equiv 1$ ($Q \equiv 0$); we then write $V_K := V_{K,0}$ to be consistent with the previous notation.

We say K is locally regular if for each $z \in K$ we have $V_{K\cap\overline{B(z,r)}}^*(z) = 0$ for $r > 0$ where $B(z,r) = \{w : |w-z| < r\}$. Local regularity implies $L-$regularity but the converse is not necessarily true in the multivariate setting. If K is locally regular and Q is continuous on K then $V_{K,Q}$ is continuous on $\mathbb{C}^N$.

A natural definition of a weighted transfinite diameter uses weighted Vandermonde determinants. Let $K \subset \mathbb{C}^N$ be compact and let w be an admissible weight function on K. Given $\zeta_1, ..., \zeta_{m_d} \in K$, let

$$W(\zeta_1, ..., \zeta_{m_d}) := VDM(\zeta_1, ..., \zeta_{m_d})w(\zeta_1)^d \cdots w(\zeta_{m_d})^d$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{m_d}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d}(\zeta_1) & e_{m_d}(\zeta_2) & \dots & e_{m_d}(\zeta_{m_d}) \end{bmatrix} \cdot w(\zeta_1)^d \cdots w(\zeta_{m_d})^d$$

be a *weighted Vandermonde determinant.* Define a *$d-th$ order weighted Fekete set for K and w* to be a set of m_d points $\zeta_1, ..., \zeta_{m_d} \in K$ with the property that

$$W_{m_d} = W_{m_d}(K) := |W(\zeta_1, ..., \zeta_{m_d})| = \sup_{\xi_1,...,\xi_{m_d}\in K} |W(\xi_1, ..., \xi_{m_d})|.$$

In analogy with the univariate notation, we also set

$$\delta_d^w(K) := W_{m_d}^{1/l_d}.$$

Define

$$\delta^w(K) := \limsup_{d\to\infty} W_{m_d}^{1/l_d} = \limsup_{d\to\infty} \delta_d^w(K). \tag{10.4}$$

We will show in Proposition 10.1 that $\lim_{d\to\infty} W_{m_d}^{1/l_d}$ (the weighted analogue of (9.8)) exists.

Proposition 10.1. *Let $K \subset \mathbb{C}^N$ be a compact set with an admissible weight function w. The limit*

$$\lim_{d\to\infty}\Big[\max_{\lambda^{(i)}\in K}|VDM(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})|\cdot w(\lambda^{(1)})^d\cdots w(\lambda^{(m_d^{(N)})})^d\Big]^{1/l_d^{(N)}}$$

exists (*and equals* $\delta^w(K)$).

Proof. Following [11], we define the circled set

$$F = F(K,w) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in K,\ |t| = w(\lambda)\}.$$

We first relate weighted Vandermonde determinants for K with homogeneous Vandermonde determinants for the compact set

$$F(D) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in K,\ |t| \le w(\lambda)\}. \tag{10.5}$$

Note that $F \subset \overline{F} \subset F(D) \subset \widehat{\overline{F}}$ (cf., [11], (2.4)) where $\widehat{\overline{F}}$ is the polynomial hull of $\overline{F}$ (recall (9.13)); thus

$$d^{(H)}(\overline{F}) = d^{(H)}(F(D)). \tag{10.6}$$

To this end, for each positive integer d, choose

$$m_d^{(N)} = \binom{N+d}{d}$$

(recall (9.2)) points $\{(t_i,z^{(i)})\}_{i=1,...,m_d^{(N)}} = \{(t_i,t_i\lambda^{(i)})\}_{i=1,...,m_d^{(N)}}$ in $F(D)$ and form the $d-$homogeneous Vandermonde determinant

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}},z^{(m_d^{(N)})})).$$

We extend the lexicographical order of the monomials in $\mathbb{C}^N$ to $\mathbb{C}^{N+1}$ by letting t precede any of $z_1,...,z_N$. Writing the standard basis monomials of degree d in $\mathbb{C}^{N+1}$ as

$$\{t^{d-j}e_k^{(H,d)}(z) : j = 0,...,d;\ k = 1,...,h_j\};$$

i.e., for each power $d-j$ of t, we multiply by the standard basis monomials of degree j in $\mathbb{C}^N$, and dropping the superscript (N) in $m_d^{(N)}$, we have the $d-$homogeneous Vandermonde matrix

$$\begin{bmatrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}e_2(z^{(1)}) & t_2^{d-1}e_2(z^{(2)}) & \dots & t_{m_d}^{d-1}e_2(z^{(m_d)}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d}(z^{(1)}) & e_{m_d}(z^{(2)}) & \dots & e_{m_d}(z^{(m_d)}) \end{bmatrix}$$

$$= \begin{bmatrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \dots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \dots & (z_N^{(m_d)})^d \end{bmatrix}.$$

Factoring t_i^d out of the $i-$th column, we obtain

$$VDMH_d((t_1, z^{(1)}), ..., (t_{m_d}, z^{(m_d)})) = t_1^d \cdots t_{m_d}^d \cdot VDM(\lambda^{(1)}, ..., \lambda^{(m_d)});$$

thus, writing $|A| := |\det A|$ for a square matrix A,

$$\left| \begin{matrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \dots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \dots & (z_N^{(m_d)})^d \end{matrix} \right| \tag{10.7}$$

$$= |t_1|^d \cdots |t_{m_d}|^d \left| \begin{matrix} 1 & 1 & \dots & 1 \\ \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N^{(1)})^d & (\lambda_N^{(2)})^d & \dots & (\lambda_N^{(m_d)})^d \end{matrix} \right|,$$

where $\lambda_k^{(j)} = z_k^{(j)}/t_j$ provided $t_j \neq 0$. By definition of $F(D)$, since $(t_i, z^{(i)}) = (t_i, t_i\lambda^{(i)}) \in F(D)$, we have $|t_i| \leq w(\lambda^{(i)})$. Clearly the maximum of

$$|VDMH_d((t_1, z^{(1)}), ..., (t_{m_d}, z^{(m_d)}))|$$

over points in $F(D)$ will occur when all $|t_j| = w(\lambda^{(j)}) > 0$ (recall w is an admissible weight) so that from (10.7)

$$\max_{(t_i, z^{(i)}) \in F(D)} |VDMH_d((t_1, z^{(1)}), ..., (t_{m_d}, z^{(m_d)}))| =$$

$$\max_{\lambda^{(i)} \in K} |VDM(\lambda^{(1)}, ..., \lambda^{(m_d)})| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d)})^d.$$

As mentioned in the discussion of (9.12) the limit

$$\lim_{d\to\infty} \big[\max_{(t_i, z^{(i)}) \in F(D)} |VDMH_d((t_1, z^{(1)}), ..., (t_{m_d}, z^{(m_d)}))| \big]^{1/dh_d^{(N+1)}}$$

$$=: d^{(H)}(F(D))$$

exists; thus the limit

$$\lim_{d\to\infty} \big[\max_{\lambda^{(i)} \in K} |VDM(\lambda^{(1)}, ..., \lambda^{(m_d)})| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d)})^d \big]^{1/l_d^{(N)}} := \delta^w(K)$$

exists. $\square$

Corollary 10.2. *For $K \subset \mathbb{C}^N$ a nonpluripolar compact set with an admissible weight function w and*

$$F = F(K,w) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in K,\ |t| = w(\lambda)\},$$

$$\delta^w(K) = d^{(H)}(\overline{F})^{\frac{N+1}{N}} = \delta(\overline{F})^{\frac{N+1}{N}}. \tag{10.8}$$

Proof. The first equality follows from the proof of Proposition 10.1 using (10.6) and the relation

$$l_d^{(N)} = (\frac{N}{N+1}) \cdot dh_d^{(N+1)}$$

(see (9.3)). The second equality is (9.10). $\square$

Remark 10.3. One can define another notion of a weighted transfinite diameter by defining *weighted directional Chebyshev constants* $\tau^w(K,\theta)$ and setting

$$d^w(K) := \exp\big[\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau^w(K,\theta) d\theta\big].$$

There is a relationship between $\delta^w(K)$ and $d^w(K)$:

$$\delta^w(K) = \big(\exp[-\int_K Q(dd^c V^*_{K,Q})^N]\big)^{1/N} d^w(K)$$

(cf., [17]).

Given a compact set $K \subset \mathbb{C}^N$ and a measure ν on K, we say that (K,ν) satisfies the Bernstein-Markov inequality if, as in the univariate case, there is a strong comparability between L^2 and L^∞ norms of holomorphic polynomials on K. Precisely, for all $p_d \in \mathcal{P}_d$,

$$||p_d||_K \leq M_d ||p_d||_{L^2(\nu)} \text{ with } \limsup_{d\to\infty} M_d^{1/d} = 1;$$

equivalently, given $\epsilon > 0$, there exists a constant $\tilde{M} = \tilde{M}(\epsilon)$ such that

$$||p_d||_K \leq \tilde{M}(1+\epsilon)^d ||p_d||_{L^2(\nu)}.$$

If K is $L-$regular, (K,μ_K) satisfies the Bernstein-Markov inequality where μ_K is the extremal measure $\frac{1}{(2\pi)^N}(dd^c V_K)^N$ from (8.10). One can even find a Bernstein-Markov measure ν which is rather "sparse" in the sense that there exists a countable subset $K' \subset K$ with $\nu(K') = \nu(K)$. The next result is the multivariate generalization of Proposition 2.20:

any compact set admits a Bernstein-Markov measure. As in the proof of Proposition 2.20, the construction below provides a "sparse" example.

Proposition 10.4. *Let $K \subset \mathbb{C}^N$ be an arbitrary compact set. Then there exists a measure $\nu \in \mathcal{M}(K)$ such that (K,ν) satisfies a Bernstein-Markov property.*

Proof. To construct ν, we first observe that if K is a finite set, any measure ν which puts positive mass at each point of K will work. If K has infinitely many points, for each $k = 1, 2, \ldots$ let $m_k = \dim \mathcal{P}_k(K)$, the holomorphic polynomials on $\mathbb{C}^N$ restricted to K. Then $\lim_{k\to\infty} m_k = \infty$ and $m_k \leq \binom{N+k}{k} = 0(N^k)$. For each k, let

$$\mu_k := \frac{1}{m_k}\sum_{j=1}^{m_k} \delta(z_j^{(k)})$$

where $\{z_j^{(k)}\}_{j=1,\ldots,m_k}$ is a set of Fekete points of order k for K relative to the vector space $\mathcal{P}_k(K)$; i.e., if $\{e_1, \ldots, e_{m_k}\}$ is any basis for $\mathcal{P}_k(K)$,

$$\left|\det[e_i(z_j^{(k)})]_{i,j=1,\ldots,m_k}\right| = \max_{q_1,\ldots,q_{m_k}\in K}\left|\det[e_i(q_j)]_{i,j=1,\ldots,m_k}\right|. \tag{10.9}$$

Define

$$\nu := c\sum_{k=3}^{\infty}\frac{1}{k(\log k)^2}\mu_k$$

where $c > 0$ is chosen so that $\nu \in \mathcal{M}(K)$. If $p \in \mathcal{P}_k(K)$, we have

$$p(z) = \sum_{j=1}^{m_k} p(z_j^{(k)}) l_j^{(k)}(z)$$

where $l_j^{(k)} \in \mathcal{P}_k(K)$ with $l_j^{(k)}(z_k^{(k)}) = \delta_{jk}$. We have $||l_j^{(k)}||_K = 1$ from (10.9) and hence

$$||p||_K \leq \sum_{j=1}^{m_k} |p(z_j^{(k)})|.$$

On the other hand,

$$||p||_{L^2(d\nu)} \geq ||p||_{L^1(d\nu)} \geq \frac{c}{k(\log k)^2}\int_K |p| d\mu_k$$

$$= \frac{c}{km_k(\log k)^2}\sum_{j=1}^{m_k} |p(z_j^{(k)})|.$$

Thus we have

$$||p||_K \leq \frac{km_k(\log k)^2}{c}||p||_{L^2(d\nu)}.$$

□

We return to the setting of Theorem 9.7, i.e., K is a polynomially convex L-regular compact set in $\mathbb{C}^N$. Given a measure ν such that (K,ν) satisfies a Bernstein-Markov property, we show that best $L^2(\nu)$-approximating polynomials to certain functions $f \in C(K)$ – which are in principle easy to calculate – have asymptotic behavior similar to best supremum norm polynomial approximants. It will be convenient to let n denote the degree of a polynomial $p_n \in \mathcal{P}_n$ since we recall the notation

$$d_n = d_n(f,K) = \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}.$$

Proposition 10.5. *Let K be a polynomially convex L-regular compact set in $\mathbb{C}^N$ and let ν be a measure supported on K such that (K,ν) satisfies the Bernstein-Markov property. If $f \in C(K)$ satisfies*

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} = \rho < 1,$$

and if $\{p_n\}$ is a sequence of best $L^2(\nu)$-approximants to f, then

$$\limsup_{n\to\infty} ||f - p_n||_K^{1/n} = \rho.$$

Proof. Note the hypothesis implies that f extends to be holomorphic on a neighborhood of K by Theorem 9.7. For simplicity we take $\nu(K) = 1$. The proof follows trivially from the fact that if $\rho < r < 1$ and $\{q_n\}$ are best sup-norm approximating polynomials, so that $||f - q_n||_K \leq Mr^n$ for some M (independent of n), then

$$||f - p_n||_{L^2(\nu)} \leq ||q_n - f||_{L^2(\nu)} \leq ||q_n - f||_K \leq Mr^n.$$

Thus we have $||p_n - p_{n-1}||_{L^2(\nu)} \leq Mr^n(1 + 1/r)$ which shows that $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges to f in $L^2(\nu)$ and pointwise ν-a.e. to f on K. By the Bernstein-Markov property, for each $\epsilon < 1/r - 1$ there exists $\tilde{M} > 0$ with

$$||p_n - p_{n-1}||_K \leq \tilde{M}(1+\epsilon)^n||p_n - p_{n-1}||_{L^2(\nu)} \leq \tilde{M}[(1+\epsilon)r]^n M(1 + 1/r)$$

showing that $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges uniformly to a continuous function g on K (holomorphic on the interior of K). Since f and g are continuous and $g = f$ ν-a.e. on K, $g = f$ on K. Then

$$||f - p_n||_K = ||\sum_{k=n+1}^{\infty}(p_k - p_{k-1})||_K \leq \tilde{M}[(1+\epsilon)r]^{n+1}M\frac{(1+1/r)}{[1-(1+\epsilon)r]}$$

showing that $\limsup_{n\to\infty}||p_n - f||_K^{1/n} \leq (1+\epsilon)r$. □

We recall briefly the basic theory of reproducing kernels on a Hilbert space in the context of the Hilbert space H_n consisting of elements in $\mathcal{P}_n$ equipped with the L^2-norm associated to a (probability) measure ν with compact support K. This was used in Chapter 5 in the univariate situation. We presume that the measure is "thick" enough so that $||p||^2_{L^2(\nu)} := \int_K |p|^2 d\nu = 0$ for $p \in \mathcal{P}_n$ implies $p \equiv 0$. Then for each $z \in K$, the linear functional of point evaluation $z \to p(z)$ is continuous as a map from H_n to $\mathbb{C}$ (why?). Thus, by the Riesz representation theorem, this functional is given by taking an inner product (in the norm of H_n) with a fixed element $Q_z \in \mathcal{P}_n$; i.e.,

$$p(z) = \int_K p\overline{Q}_z d\nu \text{ for } p \in \mathcal{P}_n.$$

Define $K_n^{\nu}(z,w) := \overline{Q}_z(w)$. One can check that if $\{q_j^{(n)}\}_{j=1,\ldots,m_n}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to $L^2(\nu)$, then

$$K_n^{\nu}(z,w) = \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)$$

(note here that $m_n = \binom{N+n}{n}$)). Indeed, observing that for any $p \in \mathcal{P}_n$ we have

$$p(z) = \sum_{j=1}^{m_n}\big(\int_K p(w)\overline{q}_j^{(n)}(w)d\nu(w)\big)q_j^{(n)}(z),$$

we see that

$$\int_K p(w)\big(\sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)\big)d\nu(w) =$$

$$\sum_{j=1}^{m_n}\big(q_j^{(n)}(z)\int_K p(w)\overline{q}_j^{(n)}(w)d\nu(w)\big) = p(z),$$

verifying that $\overline{Q}_z(w) = \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)$. Restricting this reproducing kernel to the diagonal $\{z = w\}$, we call

$$B_n^\nu(z) := K_n^\nu(z,z) = \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2$$

the $n-th$ *Bergman function* of K, ν.

It is known if (K, ν) satisfies the Bernstein-Markov inequality that

$$\lim_{n\to\infty} \frac{1}{2n} \log B_n^\nu(z) = V_K(z) \tag{10.10}$$

locally uniformly on $\mathbb{C}^N$ (cf., [19]). As an example, recall in Chapter 5, we considered $K = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disk in $\mathbb{C}$, and $\nu = \frac{1}{2\pi}d\theta = \mu_K$. Then the monomials $1, z, ..., z^n$ give an orthonormal basis for $\mathcal{P}_n$ in $L^2(\nu)$, and thus

$$B_n^\nu(z) = \sum_{j=0}^{n} |z|^{2j} = \frac{|z|^{2n+2} - 1}{|z|^2 - 1}.$$

Clearly, then, $\lim_{n\to\infty} \frac{1}{2n} \log B_n^\nu(z) = \log^+ |z|$ locally uniformly.

What happens in the weighted situation? For $K \subset \mathbb{C}^N$ compact, $w = e^{-Q}$ an admissible weight function on K, and ν a measure on K, we say that the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property if there is a strong comparability between L^2 and L^∞ norms of *weighted* polynomials on K; precisely, for all $p_n \in \mathcal{P}_n$, writing $||w^n p_n||_K := \sup_{z\in K} |w(z)^n p_n(z)|$ and $||w^n p_n||^2_{L^2(\nu)} := \int_K |p_n(z)|^2 |w(z)|^{2n} d\nu(z)$,

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

If K is locally regular and w is continuous, taking $\nu = (dd^c V_{K,Q})^N$ we have (K, ν, Q) satisfies a weighted Bernstein-Markov property (cf., [11]). Now if (K, ν, Q) satisfies a weighted Bernstein-Markov property we have that

$$\lim_{n\to\infty} \frac{1}{2n} \log K_n^{\nu,w}(z,z) = V_{K,Q}(z) \tag{10.11}$$

locally uniformly on $\mathbb{C}^N$ where

$$K_n^{\nu,w}(z,\zeta) := \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q_j^{(n)}(\zeta)}.$$

and

$$B_n^{\nu,w}(z) := K_n^{\nu,w}(z,z)w(z)^{2n} := \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 w(z)^{2n} \tag{10.12}$$

is the $n-th$ *Bergman function* of K, w, ν (cf., [10]). Here, $\{q_j^{(n)}\}_{j=1,\ldots,m_n}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to the weighted L^2-norm $p \to ||w^n p_n||_{L^2(\nu)}$. A sketch of the proof of (10.11) and/or (10.10) runs as for (5.3) in Chapter 5: first, one shows that if

$$\Phi_{K,Q,n}(z) := \sup\{|p(z)| : ||w^{\deg p} p||_K \leq 1,\ p \in \mathcal{P}_n\},$$

then

$$\frac{1}{n} \log \Phi_{K,Q,n} \to V_{K,Q}$$

locally uniformly on $\mathbb{C}^N$ (see Corollary 4.3 and exercise 5 of section 6 for univariate versions). Next, one verifies the inequality

$$\frac{[\Phi_{K,Q,n}(z)]^2}{m_n} \leq K_n^{\nu,w}(z,z) \leq m_n \cdot M_n^2 [\Phi_{K,Q,n}(z)]^2.$$

The left-hand inequality follows simply from the reproducing property of the kernel function $K_n^{\nu,w}(z,\zeta)$; i.e., for any $p \in \mathcal{P}_n$,

$$p(z) = \int_K K_n^{\nu,w}(z,\zeta) p(\zeta) w(\zeta)^{2n} d\nu(\zeta),$$

and the Cauchy-Schwartz inequality; it is the right-side inequality which utilizes the weighted Bernstein-Markov property. Indeed, for an element $q_j^{(n)} \in \mathcal{P}_n$ in the orthonormal basis,

$$||w^n q_j^{(n)}||_K \leq M_n \text{ and } \frac{|q_j^{(n)}(z)|}{||w^n q_j^{(n)}||_K} \leq \Phi_{K,Q,n}(z)$$

imply

$$|q_j^{(n)}(z)| \leq M_n \Phi_{K,Q,n}(z)$$

so that

$$K_n^{\nu,w}(z,z) = \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 \leq m_n \cdot M_n^2 [\Phi_{K,Q,n}(z)]^2.$$

These results were proved in the unweighted case, i.e., (10.10), by Bloom and Shiffman [19] and in the general (weighted) case, i.e., (10.11), by Bloom [10].

From the local uniform convergence in (10.11) follows the weak-* convergence of the Monge-Ampère measures

$$[dd^c \frac{1}{2n} \log K_n^{\nu,w}(z,z)]^N \to (dd^c V_{K,Q}^*)^N \text{ weak-}*\,.$$

One of the main results in the next section is a much stronger version of "Bergman asymptotics": if (K,ν,w) satisfies a weighted Bernstein-Markov inequality, then

$$\frac{1}{m_n} B_n^{\nu,w} d\nu \to \mu_{K,Q} := \frac{1}{(2\pi)^N} (dd^c V_{K,Q}^*)^N \text{ weak-}*\,.$$

This was proved in the one variable case ($N=1$) in [16]. Returning to the (unweighted) example $K = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\nu = \frac{1}{2\pi} d\theta = \mu_K$, here $B_n^\nu(z) = \sum_{j=0}^n |z|^{2j} = n+1$ for $|z|=1$; i.e., for $z \in \text{supp}(\nu)$. Thus $\frac{1}{m_n} B_n^\nu d\nu = \frac{1}{n+1}(n+1) d\nu = d\nu$.

We continue with a multivariate version of Theorem 6.17, the relation of the weighted Bernstein-Markov property and weighted transfinite diameter. Here, we use the notation

$$G_n^{\nu,w} := \left[\int_K \overline{e_i(z)} e_j(z) w(z)^{2n} d\nu \right] \in \mathbb{C}^{m_n \times m_n} \tag{10.13}$$

for the Gram matrix of the standard basis monomials $e_i \in \mathcal{P}_n$ with respect to the measure ν and weight w (see Exercise 6 for the relationship between our generalized Vandermonde matrices and a particular Gram matrix). Recall that

$$l_n = \sum_{j=1}^{m_n} deg(e_j) = \frac{Nnm_n}{N+1}.$$

Thus, in the formulas below, $\frac{N+1}{2Nnm_n}$ is simply $\frac{1}{2l_n}$.

Proposition 10.6. *Let $K \subset \mathbb{C}^N$ be a compact set and let w be an admissible weight function on K. If ν is a measure on K with (K,ν,Q) satisfying a weighted Bernstein-Markov property, then*

$$\lim_{n\to\infty} \frac{N+1}{2Nnm_n} \cdot \log \det G_n^{\nu,w} = \log \delta^w(K).$$

Proof. Note first that $\det G_n^{\nu,w} = \prod_{j=1}^{m_n} ||r_j||^2_{L^2(w^{2n}\nu)}$ where $\{r_1,...,r_{m_n}\}$ are an orthogonal basis of $\mathcal{P}_n$ obtained by applying Gram-Schmidt to the standard basis monomials of $\mathcal{P}_n$. Note the r_j are not necessarily orthonormal: here, $r_j = e_j + \sum_{k<j} c_k^{(j)} e_k$ for some $c_k^{(j)} \in \mathbb{C}$. Defining, analogous to (2.18),

$$Z_n := Z_n(K, w, \nu)$$

$$:= \int_K \cdots \int_K |VDM(z_1, ..., z_{m_n})|^2 w(z_1)^{2n} \cdots w(z_{m_n})^{2n} d\nu(z_1) \cdots d\nu(z_{m_n})$$

we show that

$$\lim_{n\to\infty} Z_n^{\frac{N+1}{2Nnm_n}} = \delta^w(K).$$

To see this, clearly

$$Z_n \leq \delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \nu(K)^{m_n}. \tag{10.14}$$

On the other hand, taking points $x_1, ..., x_{m_n}$ achieving the maximum in $\delta_n^w(K)$, we have, upon applying the weighted Bernstein-Markov property to the weighted polynomial

$$z_1 \to VDM(z_1, x_2 ..., x_{m_n}) w(z_1)^n \cdots w(x_{m_n})^n,$$

$$\delta_n^w(K)^{\frac{2Nnm_n}{N+1}} = |VDM(x_1, ..., x_{m_n})|^2 w(x_1)^{2n} \cdots w(x_{m_n})^{2n}$$

$$\leq M_n^2 \int_K \cdots \int_K |VDM(z_1, x_2 ..., x_{m_n})|^2 w(z_1)^{2n} \cdots w(x_{m_n})^{2n} d\nu(z_1).$$

Repeating this argument in each variable we obtain

$$\delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \leq M_n^{2m_n} Z_n. \tag{10.15}$$

Note that (10.14) and (10.15) give

$$Z_n \leq \delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \nu(K)^N \leq \nu(K)^N M_n^{2m_n} Z_n.$$

Since $[\nu(K)^N M_n^{2m_n}]^{\frac{N+1}{2Nnm_n}} \to 1$, using (10.4)

$$\lim_{n\to\infty} Z_n^{\frac{N+1}{2Nnm_n}}$$

exists and equals

$$\lim_{n\to\infty} \delta_n^w(K)^{\frac{N+1}{Nnm_n}}.$$

Using elementary row operations in $|VDM(z_1, ..., z_{m_n})|^2$ in the integrand of Z_n, we can replace the monomials $\{e_j\}$ by the orthogonal basis $\{r_1, ..., r_{m_n}\}$ and obtain

$$Z_n = m_n! \prod_{j=1}^{m_n} ||r_j||^2_{L^2(w^{2n}\nu)}.$$

Putting everything together gives the result. Note that

$$Z_n = m_n! \cdot \det(G_n^{\nu,w})$$

(see (10.17) below). $\square$

Definition 10.7. If a probability measure μ has the property that

$$\det(G_n^{\mu',w}) \leq \det(G_n^{\mu,w}) \tag{10.16}$$

for all other probability measures μ' on K then μ is said to be an *optimal measure* of degree n for K and w.

Note we have fixed the usual monomial basis to compute our Gram matrices but it is an easy exercise to show that the notion of optimal measure is independent of the basis we choose. We continue with some algebraic preliminaries relating Gram determinants, Bergman functions, and generalized Vandermonde determinants, whose proofs we leave as exercises.

Lemma 10.8. *Suppose that $\mu \in \mathcal{M}(K)$ and that w is an admissible weight. Then*

$$\det(G_n^{\mu,w}) = \frac{1}{m_n!}\int_{K^{m_n}} |VDM(z_1,\cdots,z_{m_n})|^2 \tag{10.17}$$

$$\cdot w(z_1)^{2n}\cdots w(z_{m_n})^{2n} d\mu(z_1)\cdots d\mu(z_{m_n}) = \frac{Z_n}{m_n!}$$

and

$$B_n^{\mu,w}(z) = \frac{m_n}{Z_n}\int_{K^{m_n-1}} |VDM(z,z_2,\cdots,z_{m_n})|^2 \tag{10.18}$$

$$\cdot w(z)^{2n} w(z_2)^{2n}\cdots w(z_{m_n})^{2n} d\mu(z_2)\cdots d\mu(z_{m_n}).$$

A similar argument to the proof of Proposition 10.6 shows that the Gram determinants associated to a sequence of weighted optimal measures also converges to $\delta^w(K)$ (exercise 8). In this proposition, we again compute the Gram determinant with respect to the standard basis monomials.

Proposition 10.9. *Let K be compact and w an admissible weight function. For $n = 1, 2, ...$, let μ_n be an optimal measure of order n for K and w. Then*

$$\lim_{n\to\infty} \det(G_n^{\mu_n,w})^{\frac{N+1}{2Nnm_n}} = \delta^w(K).$$

The connection between (weighted) optimal measures and (weighted) Bergman functions is the following.

Proposition 10.10. *Let w be an admissible weight on K. A probability measure μ is an optimal measure of degree n for K and w if and only if*

$$\max_{z\in K} B_n^{\mu,w}(z) = m_n. \tag{10.19}$$

For the proof of Proposition 10.10, cf., [21]. As a corollary, we obtain the following key property of optimal measures.

Lemma 10.11. *Suppose that μ is an optimal measure of degree n for K and w Then*

$$B_n^{\mu,w}(z) = m_n, \quad a.e.\ \mu.$$

Proof. On the one hand, by Proposition 10.10

$$\max_{z\in K} B_n^{\mu,w}(z) = m_n,$$

while on the other hand, by orthonormality of the $q_j^{(n)}$ in (10.12) (with $\nu = \mu$)

$$\int_K B_n^{\mu,w}\, d\mu = \int_K \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 w(z)^{2n}\, d\mu(z) = m_n,$$

and the result follows. □

Exercises.

(1) Give a proof of (10.2) analogous to the univariate proof of (6.7) using the solution to the Dirichlet problem for the complex Monge-Ampère operator in a ball.

(2) Suppose K is the closed unit ball and Q is continuous on K and *plurisuperharmonic* on the interior of K (i.e., $-Q$ is psh). What can you say about S_w?

(3) Suppose K is the closed unit ball and Q is continuous on K and is a maximal psh function on the interior of K. What can you say about S_w?

(4) Find $V_{K,Q}$ for K the closed unit ball and $Q(z) = -|z|^2$.

(5) Verify that the dimension of the space of homogeneous polynomials of degree d in $\mathbb{C}^{N+1}$ equals the dimension of the space of polynomials of degree at most d in $\mathbb{C}^N$.

(6) For m_n distinct points $a_1, ..., a_{m_n}$, let $V = V(a_1, ..., a_{m_n}) := [e_i(a_j)]_{i,j=1,...,m_n}$ and let V^* denote the conjugate transpose of V. Show that
$$\frac{1}{m_n} VV^* = G_n^{\nu_n}$$
where $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n} \delta_{a_j}$.

(7) Verify equations (10.17) and (10.18) of Lemma 10.8.

(8) Prove Proposition 10.9. (*Hint: Use the fact that* $\det(G_n^{\nu_n,w}) \leq \det(G_n^{\mu_n,w})$ *where* $\nu_n = \frac{1}{m_n}\sum_{k=1}^{m_n} \delta_{x_k}$ *and* $x_1, ..., x_{m_n}$ *are points in* K *achieving the maximum in* $\delta_n^w(K)$.)

11. Recent results in pluripotential theory.

In this final section, we outline proofs of the strong Bergman asymptotic result mentioned in the previous section as well as the analogue of Proposition 6.13 for asymptotic weighted Fekete arrays in $\mathbb{C}^N$. These results are based on work of R. Berman and S. Boucksom. As the reader will see, the weighted theory is essential even if one only wants these results in the unweighted case.

Given a compact set $K \subset \mathbb{C}$, a discretization of the logarithmic energy minimization problem $\inf_{\mu\in\mathcal{M}(K)} I(\mu)$ led to the notion of transfinite diameter $\delta(K)$. In the nonpolar case, the energy-minimizing measure is given by $\mu_K = \frac{1}{2\pi}\Delta V_K^*$. Thus, in a sense, the notion of logarithmic energy relates $\delta(K)$ with V_K^*. How can we relate these two notions in $\mathbb{C}^N$, $N > 1$ without a notion of energy of a measure?

Corollary 11.10 below provides part of the answer; but Theorem 11.7 is the key. We begin by defining a special functional on the class $L^+(\mathbb{C}^N)$. The strictly psh function $u_0(z) := \frac{1}{2}\log(1+|z|^2)$ belongs to this class. For $u \in L^+(\mathbb{C}^N)$ we define

$$E(u) := \frac{1}{N+1}\int_{\mathbb{C}^N}\sum_{j=0}^{N}(u-u_0)(dd^cu)^j\wedge(dd^cu_0)^{N-j}. \tag{11.1}$$

Note that $u - u_0$ is (globally) bounded; the "mixed Monge-Ampère measures" $(dd^cu)^j \wedge (dd^cu_0)^{N-j}$ are all positive measures with the same total mass $c_N = (2\pi)^N = \int_{\mathbb{C}^N}(dd^cu_0)^N$ so that $|E(u)| < \infty$. Indeed, for any $u, v \in L^+(\mathbb{C}^N)$, $(dd^cu)^j \wedge (dd^cv)^{N-j}$ is a positive measures with total mass c_N. The functional E is a primitive for the complex Monge-Ampère operator in a sense that will be made precise in Proposition 11.2. In the univariate case; i.e., $N = 1$,

$$E(u) = \frac{1}{2}\int_{\mathbb{C}}(u-u_0)dd^c(u+u_0). \tag{11.2}$$

The motivation for definition (11.1) comes from complex geometry. Let X be an $N-$dimensional, compact, Kähler manifold with Kähler form ω normalized so that $\int_X \omega^N = 1$. We use ω^N as a volume form on X. Define the class of $\omega-$*PSH functions*

$$PSH(X,\omega) := \{\phi \in L^1(X) : \phi \text{ usc},\ dd^c\phi + \omega \geq 0\}.$$

Here, we write $dd^c\phi + \omega \geq 0$ to mean that this is a positive $(1,1)$ current on X. Note these functions are automatically bounded above by uppersemicontinuity; they are not necessarily bounded from below. In the case where $X = \mathbb{P}^N$ with the usual Kähler form ω we can identify the $\omega-$PSH functions with the Lelong class $L(\mathbb{C}^N)$, i.e.,

$$PSH(X,\omega) \approx L(\mathbb{C}^N),$$

and the bounded $\omega-$PSH functions with the subclass $L^+(\mathbb{C}^N)$:

$$PSH(X,\omega) \cap L^\infty(X) \approx L^+(\mathbb{C}^N).$$

Indeed, if $\phi \in PSH(X,\omega)$, then

$$u(z) = u(z_1,...,z_N) := \phi([1 : z_1 : \cdots : z_N]) + \frac{1}{2}\log(1+|z|^2) \in L(\mathbb{C}^N);$$

conversely, if $u \in L(\mathbb{C}^N)$ then

$$\phi([1 : z_1 : \cdots : z_N]) := u(z_1,...,z_N) - \frac{1}{2}\log(1+|z|^2)$$

is a well-defined function on $\mathbb{P}^N \setminus H_\infty$ (where $H_\infty = \mathbb{P}^N \setminus \mathbb{C}^N$ is the hyperplane at infinity); this function is bounded above near H_∞ and extends (by taking lim sup) to H_∞ to give an element of $PSH(X,\omega)$; i.e.,

$$\phi([0 : z_1 : \cdots : z_N]) = \limsup_{t\to\infty,\ t\in\mathbb{C}} [u(tz) - \frac{1}{2}\log(1+|tz|^2)].$$

Clearly if $u \in L^+(\mathbb{C}^N)$ then ϕ is bounded. Note that $u_0(z) = \frac{1}{2}\log(1+|z|^2)$ is a local potential for ω; i.e., on $\mathbb{C}^N \subset \mathbb{P}^N$, $\omega = dd^c u_0$. Thus the identification between $u \in L(\mathbb{C}^N)$ and $\phi \in PSH(X,\omega)$ can simply, but imprecisely be remembered as "$u = \phi + u_0$."

For $\phi \in PSH(X,\omega) \cap L^\infty(X)$, define

$$E(\phi) := \frac{1}{N+1}\sum_{j=0}^{N}\int_X \phi(\omega + dd^c\phi)^j \wedge \omega^{N-j}.$$

With the correspondence $u = \phi + u_0$ between $u \in L^+(\mathbb{C}^N)$ and $\phi \in PSH(X,\omega) \cap L^\infty(X)$ and the relation $\omega = dd^c u_0$ on $\mathbb{C}^N$ this is equivalent to (11.1).

Next, for $Q \in \mathcal{A}(K)$, define

$$P(Q) = P_K(Q) := V^*_{K,Q}.$$

We record some straightforward properties of this operator P.

Proposition 11.1. *The operator $P : \mathcal{A}(K) \to L^+(\mathbb{C}^N)$ is increasing and concave: for $0 \le t \le 1$ and $Q_1, Q_2 \in \mathcal{A}(K)$,*

$$P(Q_1) \le P(Q_2) \text{ if } Q_1 \le Q_2 \text{ and}$$

$$P(tQ_1 + (1-t)Q_2) \ge tP(Q_1) + (1-t)P(Q_2).$$

The first statement is simply that $V^*_{K,Q_1} \le V^*_{K,Q_2}$ on $\mathbb{C}^N$ if $Q_1 \le Q_2$ on K while the second follows if one can verify that

$$u(z) := tV^*_{K,Q_1}(z) + (1-t)V^*_{K,Q_2}(z) \le V^*_{K,tQ_1+(1-t)Q_2}(z)$$

for $z \in \mathbb{C}^N$ (see exercise 1).

The composition of the E and P operators is Gateaux differentiable; this non-obvious result (Theorem 11.7) was proved by Berman and Boucksom in [5] and is the key to many recent results in (weighted) pluripotential theory. We begin with the statement that E is a primitive for the complex Monge-Ampère operator.

Proposition 11.2. *The functional $E : L^+(\mathbb{C}^N) \to \mathbb{R}$ is (Gateaux) differentiable; i.e., for $u, v \in L^+(\mathbb{C}^N)$, the function*

$$f(t) := E((1-t)u + tv)$$

is differentiable for $0 \le t \le 1$ with

$$f'(t) = \int_{\mathbb{C}^N} (v-u)\big(dd^c(1-t)u + tv\big)^N. \tag{11.3}$$

In particular,

$$f'(0) := \lim_{t \downarrow 0^+} \frac{f(t) - f(0)}{t} = \int_{\mathbb{C}^N} (v-u)(dd^c u)^N. \tag{11.4}$$

We first give the proof in the case $N = 1$ where (recall (11.2))

$$E(u) = \frac{1}{2} \int_{\mathbb{C}} (u - u_0) dd^c(u + u_0).$$

We need a lemma whose proof we omit.

Lemma 11.3. *Let $a, b, c, d \in L^+(\mathbb{C})$. Then*

$$\int_{\mathbb{C}} (a-b) dd^c(c-d) = -\int_{\mathbb{C}} d(a-b) \wedge d^c(c-d) = \int_{\mathbb{C}} (c-d) dd^c(a-b).$$

Remark 11.4. We must consider *differences* of $L^+(\mathbb{C})$ functions in Lemma 11.3; e.g.,

$$\int_{\mathbb{C}} (a-b)dd^c \neq \int_{\mathbb{C}} cdd^c(a-b)$$

as the example of $b = c = \log^+|z|,\ a = \log^+|z| + 1$ shows.

Proposition 11.5. *The functional $E : L^+(\mathbb{C}) \to \mathbb{R}$ is (Gateaux) differentiable; i.e., for $u, v \in L^+(\mathbb{C})$, the function $f(t) := E((1-t)u + tv)$ is differentiable for $0 \leq t \leq 1$ with*

$$f'(t) = \int_{\mathbb{C}} (v-u)\big(dd^c(1-t)u + tv\big). \tag{11.5}$$

In particular,

$$f'(0) := \lim_{t \downarrow 0^+} \frac{f(t) - f(0)}{t} = \int_{\mathbb{C}} (v-u)(dd^c u). \tag{11.6}$$

Proof. We show (11.6). Letting $w := v - u$,

$$f(t) - f(0) = E(u + tw) - E(u)$$

$$= \frac{1}{2} \int_{\mathbb{C}} \big[(u + tw - u_0)(dd^c(u+tw) + dd^c u_0) - (u - u_0)(dd^c u + dd^c u_0)\big]$$

$$= t \cdot \frac{1}{2} \int_{\mathbb{C}} \big[w dd^c(u + u_0) + (u - u_0) dd^c w\big] + 0(t^2)$$

$$= t \cdot \int_{\mathbb{C}} w(dd^c u) + 0(t^2).$$

Here we have used Lemma 11.3.

Formula (11.5) follows by a standard argument: apply (11.6) to $g(s) := f(t+s)$ at $s = 0$. Since

$$g(s) = f(t+s) = E(u + t(v-u) + s(v-u)),$$

$$f'(t) = g'(0) = \int_{\mathbb{C}} (v-u)\big(dd^c(1-t)u + tv\big).$$

□

We sketch the proof of Proposition 11.2, the multivariate case. We begin with the observation that if again we set $w := v - u$, then

$$\sum_{j=0}^{N} [dd^c(u+tw)]^j \wedge (dd^c u_0)^{N-j} - \sum_{j=0}^{N} (dd^c u)^j \wedge (dd^c u_0)^{N-j}$$

$$= t\sum_{j=0}^{N} j[dd^c w \wedge (dd^c u)^{j-1} \wedge (dd^c u_0)^{N-j} + 0(t^2).$$

Then (all integrals are over $\mathbb{C}^N$)

$$(N+1)\big(E(u+t(v-u)-E(u)\big) = (N+1)\big(E(u+tw)-E(u)\big)$$

$$= \int [u+tw-u_0]\sum_{j=0}^{N}[dd^c(u+tw)]^j \wedge (dd^c u_0)^{N-j}$$

$$- \int (u-u_0)\sum_{j=0}^{N}(dd^c u)^j \wedge (dd^c u_0)^{N-j}$$

$$= t\int (u-u_0)\sum_{j=0}^{N} j[dd^c w \wedge (dd^c u)^{j-1} \wedge (dd^c u_0)^{N-j} + 0(t^2)$$

$$+ \int tw\sum_{j=0}^{N}[dd^c(u+tw)]^j \wedge (dd^c u_0)^{N-j}$$

$$= t\big[\int (u-u_0)\sum_{j=0}^{N} j[dd^c w \wedge (dd^c u)^{j-1} \wedge (dd^c u_0)^{N-j}]$$

$$+ \int w\sum_{j=0}^{N}(dd^c u)^j \wedge (dd^c u_0)^{N-j}\big] + 0(t^2)$$

$$= t\big[\int w\sum_{j=0}^{N} j[dd^c(u-u_0) \wedge (dd^c u)^{j-1} \wedge (dd^c u_0)^{N-j}]$$

$$+ \int w\sum_{j=0}^{N}(dd^c u)^j \wedge (dd^c u_0)^{N-j}\big] + 0(t^2).$$

In the last step we have used an "integration by parts" formula generalizing Lemma 11.3 involving *differences* of functions in $L^+(\mathbb{C}^N)$; to wit: for $A,B,C,D \in L^+(\mathbb{C}^N)$ and $u_1,...,u_{N-1} \in L^+(\mathbb{C}^N)$ (so that $T := dd^c u_1 \wedge \cdots \wedge dd^c u_{N-1}$ is a positive closed $(N-1,N-1)$ current), we have

$$\int_{\mathbb{C}^N}(A-B)(dd^c C - dd^c D) \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{N-1}$$

$$= \int_{\mathbb{C}^N}(C-D)(dd^c A - dd^c B) \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{N-1}.$$

Now check that

$$\sum_{j=0}^{N} j\,dd^c(u-u_0)\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j}+\sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}$$
$$=(N+1)(dd^cu)^N$$

(try the case $N=2$!) and the result follows.

Note that the formula for f' – and its proof – does not involve u_0. Thus given any $\tilde{u}\in L^+(\mathbb{C}^N)$ we may define $\mathcal{E}:L^+(\mathbb{C}^N)\times L^+(\mathbb{C}^N)\to\mathbb{R}$ via

$$\mathcal{E}(u,\tilde{u})=\frac{1}{N+1}\int_{\mathbb{C}^N}\sum_{j=0}^{N}(u-\tilde{u})(dd^cu)^j\wedge(dd^c\tilde{u})^{N-j}.$$

Thus $E(u)=\mathcal{E}(u,u_0)$. Fixing $\tilde{u}$, defining, for $u,v\in L^+(\mathbb{C}^N)$

$$\tilde{f}(t):=\mathcal{E}(u+t(v-u),\tilde{u}),$$

we have

$$\tilde{f}'(t)=\int_{\mathbb{C}^N}(v-u)\big(dd^c(1-t)u+tv\big)^N$$

and

$$\tilde{f}'(0)=\int_{\mathbb{C}^N}(v-u)(dd^cu)^N.$$

We use this to show the following properties of our functional E.

Proposition 11.6. *The functional $E:L^+(\mathbb{C}^N)\to\mathbb{R}$ is increasing and concave; i.e., if $u,v\in L^+(\mathbb{C}^N)$ then*

1. *if $u\geq v$ we have $E(u)\geq E(v)$; and*
2. *For $0\leq t\leq 1$,*

$$E(tu+(1-t)v)\geq tE(u)+(1-t)E(v).$$

Proof. We observe that the functional $\mathcal{E}$ satisfies

1. $\mathcal{E}(a,b)=-\mathcal{E}(b,a)$ (antisymmetry) and
2. $\mathcal{E}(a,b)+\mathcal{E}(b,c)+\mathcal{E}(c,a)=0$ (cocycle property).

Indeed, the antisymmetry is obvious from the definition. To verify the cocycle property, consider

$$g(t):=\mathcal{E}(a+t(c-a),b)+\mathcal{E}(b,a)\text{ and}$$
$$h(t):=\mathcal{E}(a+t(c-a),c)+\mathcal{E}(c,a).$$

The antisymmetry gives $g(0) = h(o) = 0$. From Proposition 11.2,

$$g'(t) = \int_{\mathbb{C}^N} (c-a)(dd^c(a+t(c-a)))^N = h'(t)$$

for all t; hence $g(1) = h(1)$ – this is the cocycle property.

We apply the cocycle property to prove E is increasing. Let $u, v \in L^+(\mathbb{C}^N)$ with $u \geq v$. Then

$$E(u) - E(v) = \mathcal{E}(u, u_0) - \mathcal{E}(v, u_0) = \mathcal{E}(u, u_0) + \mathcal{E}(u_0, v) = -\mathcal{E}(v, u)$$

$$= \mathcal{E}(u, v) = \frac{1}{N+1} \int_{\mathbb{C}^N} \sum_{j=0}^{N} (u-v)(dd^c u)^j \wedge (dd^c v)^{N-j} \geq 0.$$

To verify the concavity of E, consider the case $N = 1$ for simplicity. Let

$$G(t) := E(tu + (1-t)v) = E(v + t(u-v)).$$

By Proposition 11.2,

$$G'(t) = \int_{\mathbb{C}} (u-v) dd^c (v + t(u-v)).$$

Then

$$G''(t) = \int_{\mathbb{C}} (u-v) dd^c (u-v) = -\int_{\mathbb{C}} d(u-v) \wedge d^c(u-v) \leq 0$$

where we have used Lemma 11.3.

□

The statement of the main differentiabilty result is rather surprising.

Theorem 11.7. **[Berman-Boucksom]** *The functional defined for a nonpluripolar compact set $K \subset \mathbb{C}^N$ as the composition $E \circ P$ is Gateaux differentiable; i.e., for $Q \in \mathcal{A}(K)$, $F(t) := (E \circ P)(Q + tv)$ is differentiable for all $v \in C(K)$ and $t \in \mathbb{R}$. Furthermore,*

$$F'(0) = \int_K v(dd^c P(Q))^N. \tag{11.7}$$

The proof of Theorem 11.7 utilizes a global version of the comparison principle from section 7: for $u, v \in L^+(\mathbb{C}^N)$,

$$\int_{\{u<v\}} (dd^c v)^N \leq \int_{\{u<v\}} (dd^c u)^N, \tag{11.8}$$

as well as the properties of the E and P operators in Proposition 11.1. The proof of (11.8) is outlined in exercise 2. We give the proof of the theorem in the case $N = 1$.

Proposition 11.8. *Let $N = 1$. The functional $E \circ P$ is Gateaux differentiable; i.e., for $K \subset \mathbb{C}$ nonpolar and $Q \in \mathcal{A}(K)$,*

$$F(t) := (E \circ P)(Q + tv)$$

is differentiable for all $v \in C(K)$ and $t \in \mathbb{R}$. Furthermore,

$$F'(t) = \int_K v dd^c P(Q + tv).$$

In particular,

$$F'(0) = \int_K v dd^c P(Q). \tag{11.9}$$

Proof. We will verify (11.9); the formula for $F'(t)$ follows. It suffices to show

$$F(t) - F(0) = \int_K [P(Q + tv) - P(Q)] dd^c P(Q) + o(t) \tag{11.10}$$

and

$$\lim_{t \to 0} \int_{D(0) \setminus D(t)} dd^c P(Q) = 0 \tag{11.11}$$

where $D(t) := \{P(Q + tv) = Q + tv\} \subset K$. For given (11.10) and (11.11), we have

$$F(t) - F(0) = \int_{D(0) \setminus D(t)} [P(Q + tv) - P(Q)] dd^c P(Q)$$

$$+ \int_{D(0) \cap D(t)} [P(Q + tv) - P(Q)] dd^c P(Q) + o(t)$$

$$= \int_{D(0) \setminus D(t)} [P(Q + tv) - P(Q)] dd^c P(Q) + t \int_{D(0) \cap D(t)} v dd^c P(Q) + o(t)$$

$$= \int_{D(0) \setminus D(t)} [P(Q + tv) - P(Q) - tv] dd^c P(Q) + t \int_{D(0)} v dd^c P(Q) + o(t).$$

Since $P(Q + tv) - P(Q) - tv = 0(t)$ on $D(0) \setminus D(t)$, (11.9) follows from (11.11).

To prove (11.11) we use the comparison principle (11.8) in the univariate case. Extend v to a continuous function on $\mathbb{C}$ with compact support (which we still call v). Then $dd^cv = \mu^+ - \mu^-$ where μ^+, μ^- are compactly supported Borel measures. Choose $M > 0$ sufficiently large so that $v + M\psi$ is strictly shm; i.e., $dd^c(v + M\psi) > 0$, where

$$\psi := \frac{1}{2}[u_0 - p_{\mu^-}]$$

and we assume, for simplicity of notation, that μ^- is a probability measure (recall $-p_{\mu^-}(z) := \int_{\mathbb{C}} \log|z - \zeta| d\mu^-(\zeta)$). Then

$$D(0) \setminus D(t) \subset S := \{P(Q + tv) < P(Q) + tv\}$$

and

$$S = \{P(Q + tv) + tM\psi < P(Q) + t(v + M\psi)\} \text{ with } D(t) \cap S = \emptyset.$$

Thus

$$\int_{D(0)\setminus D(t)} dd^cP(Q) \leq \int_S dd^cP(Q) \leq \int_S dd^c[P(Q) + t(v + M\psi)]$$

$$\leq \int_S dd^c[P(Q + tv) + tM\psi] = \int_S dd^cP(Q + tv) + 0(t) = 0(t).$$

Here the inequality

$$\int_S dd^cP(Q) \leq \int_S dd^c[P(Q) + t(v + M\psi)] \leq \int_S dd^c[P(Q + tv) + tM\psi]$$

uses the comparison, principle, (11.8), since each of $(\frac{1}{1+tM})[P(Q)+t(v+M\psi)]$ and $(\frac{1}{1+tM})[P(Q + tv) + tM\psi]$ belong to $L^+(\mathbb{C})$.

To prove (11.10), let $G(t) := \int_K [P(Q + tv) - P(Q)]dd^cP(Q)$. Since $G(0) = 0$, we want to show that

$$\lim_{t\to 0} \frac{F(t) - F(0)}{t} = \lim_{t\to 0} \frac{G(t) - G(0)}{t} = A$$

where $A = \int_K v(dd^cP(Q))$. We verify this for $t > 0$ with $t \downarrow 0^+$; the proof for $t < 0$ is similar. We observe that by concavity of P, G is concave and $\lim_{t\downarrow 0^+} \frac{G(t)-G(0)}{t}$ exists; moreover, the argument showing how (11.9) follows from (11.10) and (11.11) proves that this limit is indeed A. Thus we must show $\lim_{t\downarrow 0^+} \frac{F(t)-F(0)}{t}$ exists and equals A. We get

$$\limsup_{t\downarrow 0^+} \frac{F(t) - F(0)}{t} \leq A$$

from concavity and differentiability of E. Precisely, this yields

$$E(P(Q+tv)) \leq E(P(Q)) + \int_{\mathbb{C}} [P(Q+tv) - P(Q)] dd^c P(Q)$$

in other words, $F(t) - F(0) \leq G(t) - G(0)$.

To show

$$\liminf_{t \downarrow 0^+} \frac{F(t) - F(0)}{t} \geq A,$$

we begin by appealing to the differentiability of G at $t = 0$. Given $\epsilon > 0$, we can take $\delta > 0$ so that $\frac{G(\delta)-G(0)}{\delta} \geq A - \epsilon$; i.e.,

$$\int_{\mathbb{C}} [P(Q+\delta v) - P(Q)] dd^c P(Q) \geq (A - \epsilon)\delta.$$

Differentiability of E yields, for $t > 0$ sufficiently small,

$$E\big(P(Q)) + t[P(Q+\delta v) - P(Q)]\big) - E(P(Q))$$

$$\geq t\Big[\int_{\mathbb{C}} [P(Q+\delta v) - P(Q)] dd^c P(Q) - \delta\epsilon\Big].$$

Combining these last two inequalities, we have

$$E\big((1-t)P(Q) + tP(Q+\delta v)\big) \geq E(P(Q)) = t\delta A - 2t\delta\epsilon.$$

Finally, the concavity of P and monotonicity of E yield the result. □

Theorem 11.7 is one ingredient used to relate the weighted transfinite diameter $\delta^w(K)$ with $E(V^*_{K,Q})$, the Aubin-Mabuchi energy of the weighted extremal function. Indeed, the key result of [5] can be stated as follows (we refer to [5] or [27] for the proof).

Theorem 11.9. *Given a weighted Bernstein-Markov measure $\nu \in \mathcal{M}(K)$ for K, Q,*

$$\lim_{n\to\infty} -\frac{N+1}{2Nnm_n} \cdot \log\det G_n^{\nu,w} = \frac{N+1}{c_N N}\big(E(V^*_{K,Q}) - E(V_T)\big)$$

where $T = T^N = \{z = (z_1, ..., z_N) : |z_1| = \cdots = |z_N| = 1\}$.

Here (and for the rest of this section) the Gram matrix is taken with respect to the standard basis monomials $\{e_j\}_{j=1,\ldots,m_n}$ which give an orthonormal basis for $\mathcal{P}_n$ in $L^2(\frac{1}{c_N}(dd^c V_T)^N)$. Also, recall $\frac{N+1}{2Nnm_n} = \frac{1}{2l_n}$ and $c_N = (2\pi)^N$.

Recalling Proposition 10.6, if ν is a measure as in Theorem 11.9, then

$$\lim_{n\to\infty} \frac{N+1}{2Nnm_n} \cdot \log\det G_n^{\nu,w} = \log \delta^w(K).$$

Thus, as a corollary we get the following formula originally proved (in a slightly different formulation) by R. Rumely [31].

Corollary 11.10. *For $K \subset \mathbb{C}^N$ compact and nonpluripolar, and $Q \in \mathcal{A}(K)$,*

$$-\log \delta^w(K) = \frac{N+1}{c_N N}\big(E(V_{K,Q}^*) - E(V_T)\big).$$

This gives a surprising relationship between the weighted transfinite diameter $\delta^w(K)$ and $E(V_{K,Q}^*)$. Theorem 11.7 and Corollary 11.10 are two ingredients used to obtain the following general result.

Proposition 11.11. *Let $K \subset \mathbb{C}^N$ be compact with admissible weight w. Let $\{\mu_n\}$ be a sequence of probability measures on K with the property that*

$$\lim_{n\to\infty} \frac{N+1}{2Nnm_n} \cdot \log\det G_n^{\mu_n,w} = \log \delta^w(K). \tag{11.12}$$

Then

$$\frac{1}{m_n} B_n^{\mu_n,w} d\mu_n \to \mu_{K,Q} = \frac{1}{(2\pi)^N}(dd^c V_{K,Q}^*)^N \ \textit{weak-}* . \tag{11.13}$$

In particular, from Proposition 11.11 and Proposition 10.6 we have a general strong Bergman asymptotic result.

Corollary 11.12. **[Strong Bergman Asymptotics]** *If (K,μ,w) satisfies a weighted Bernstein-Markov inequality, then*

$$\frac{1}{m_n} B_n^{\mu,w} d\mu \to \mu_{K,Q} \ \textit{weak-}* .$$

Another consequence of Proposition 11.11 is the analogue of Proposition 6.13 on asymptotic weighted Fekete arrays in $\mathbb{C}^N$.

Corollary 11.13. **[Asymptotic Weighted Fekete Points]** *Let $K \subset \mathbb{C}^N$ be compact with admissible weight w. For each n, take points $x_1^{(n)}, x_2^{(n)}, \cdots, x_{m_n}^{(n)} \in K$ for which*

$$\lim_{n\to\infty} \big[|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n\big]^{\frac{(N+1)}{Nnm_n}}$$

$$= \delta^w(K) \tag{11.14}$$

(*asymptotically weighted Fekete points*) *and let* $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$. *Then* $\mu_n \to \mu_{K,Q}$ *weak* $-*$.

Proof. Note that the hypothesis (11.14) is equivalent to (11.12) by observing (10.17) with $\mu = \mu_n$. By direct calculation, we have $B_n^{\mu_n,w}(x_j^{(n)}) = m_n$ for $j = 1, ..., m_n$ and hence a.e. μ_n on K. Indeed, this property holds for *any* discrete, equally weighted measure $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$ with

$$|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n \neq 0$$

(exercise 3). The result follows immediately from Proposition 11.11, specifically, equation (11.13). $\square$

Finally, using Lemma 10.11 and Proposition 10.9 in conjuction with Proposition 11.11, we conclude that a sequence of weighted optimal measures converges to $\mu_{K,Q}$.

Corollary 11.14. [**Weighted Optimal Measures**] *Let* $K \subset \mathbb{C}^N$ *be compact with admissible weight* w. *For each* n, *let* μ_n *be an optimal measure of degree* n *for* K *and* w. *Then* $\mu_n \to \mu_{K,Q}$ *weak* $-*$.

We proceed with an outline of the steps utilized to prove Proposition 11.11. Let w be an admissible weight function on K and fix $u \in C(K)$. Following the ideas in [1], [2], [3], [4], [5] we consider the perturbed weight $w_t(z) := w(z)\exp(-tu(z))$, $t \in \mathbb{R}$. For the moment, we let $\{\mu_n\}$ be any sequence of measures in $\mathcal{M}(K)$. We set

$$f_n(t) := -\frac{1}{2l_n}\log\det(G_n^{\mu_n,w_t}). \tag{11.15}$$

We have the following (see Lemma 6.4 in [5]).

Lemma 11.15. *We have*

$$f_n'(t) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w_t}(z)d\mu_n.$$

In particular,

$$f_n'(0) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w}(z)d\mu_n$$

and if $B_n^{\mu_n,w} = m_n$ *a.e.* μ_n,

$$f_n'(0) = \frac{N+1}{N}\int_K u(z)d\mu_n. \tag{11.16}$$

Before we give the proof, an illustrative example can be given if $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j}$. Then $B_n^{\mu_n,w}(x_j) = m_n$ for $j = 1,...,m_n$ (recall exercise 3) so

$$\log\det(G_n^{\mu_n,w_t})$$
$$= \log\bigl(|W(x_1,...,x_{m_n})|^2 e^{-2ntu(x_1)}\cdots e^{-2ntu(x_{m_n})}\bigr)$$

implies

$$\frac{d}{dt}\log\det(G_n^{\mu_n,w_t}) = \frac{d}{dt}\bigl(-2tn\sum_{j=1}^{N}u(x_j)\bigr)$$
$$= -2n\sum_{j=1}^{m_n}u(x_j) = -2nm_n\int_K u(z)\frac{1}{m_n}B_n^{\mu_n,w}(z)d\mu_n.$$

Note that in this case, $\frac{d}{dt}\log\det(G_n^{\mu_n,w_t})$ is a constant, independent of t; hence $\frac{d^2}{dt^2}\log\det(G_n^{\mu_n,w_t}) \equiv 0$ – see Lemma 11.16.

Proof. The proof we offer here is based on the integral formulas of Lemma 10.8.

By (10.17) we may write

$$f_n(t) = -\frac{1}{2l_n}\log(F_n) + \frac{1}{2l_n}\log(m_n!)$$

where $l_n = (\frac{N}{N+1})nm_n$ and

$$F_n(t) := \int_{K^{m_n}} V\exp(-tU)d\mu$$

and

$$V := V(z_1, z_2, \cdots, z_{m_n}) = |VDM(z_1,\cdots,z_{m_n})|^2 w(z_1)^{2n}\cdots w(z_{m_n})^{2n},$$
$$U := U(z_1, z_2, \cdots, z_{m_n}) = 2n(u(z_1) + \cdots + u(z_{m_n})),$$
$$d\mu := d\mu_n(z_1)d\mu_n(z_2)\cdots d\mu_n(z_{m_n}).$$

Further, by (10.18) for $w = w_t$ and $\mu = \mu_n$, we have

$$B_n^{\mu_n,w_t}(z)$$
$$= \frac{m_n}{Z_n}\int_{K^{m_n-1}} V(z, z_2, z_3, \cdots, z_{m_n})\exp(-tU)d\mu_n(z_2)\cdots d\mu_n(z_{m_n})$$

where

$$Z_n = Z_n(t) := m_n!\det(G_n^{\mu_n,w_t}) = \int_{K^{m_n}} V\ \exp(-tU)d\mu.$$

Note that $Z_n(t) = F_n(t)$. Now

$$f_n'(t) = -\frac{1}{2l_n}\frac{F_n'(t)}{F_n(t)}$$

and we may compute

$$F_n'(t) = \int_{K^{m_n}} V(-U)\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n})$$

$$= -2n\int_{K^{m_n}}(u(z_1)+\cdots+u(z_{m_n}))V\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n}).$$

Notice that the integrand is symmetric in the variables and hence we may "de-symmetrize" to obtain

$$F_n'(t)$$

$$= -2nm_n\int_{K^{m_n}} u(z_1)V(z_1,\cdots,z_{m_n})\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n})$$

so that, integrating in all but the z_1 variable, we obtain

$$F_n'(t) = -2nm_n\int_K u(z)B_n^{\mu_n,w_t}(z)\frac{Z_n}{n}d\mu_n(z).$$

Thus, using the fact that $Z_n(t) = F_n(t)$, we obtain

$$f_n'(t) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w_t}(z)d\mu_n(z)$$

as claimed. In particular,

$$f_n'(0) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w}(z)d\mu_n$$

and if $B_n^{\mu_n,w} = m_n$ a.e. μ_n, we recover (11.16):

$$f_n'(0) = \frac{N+1}{N}\int_K u(z)d\mu_n.$$

□

The next result was proved in a different way in [6], Lemma 2.2, and also in [13], Lemma 3.6.

Lemma 11.16. *The functions $f_n(t)$ are concave.*

Proof. We show that $f_n''(t) \le 0$. With the notation used in the proof of Lemma 11.15,

$$f_n''(t) = \frac{1}{2l_n} \frac{(F_n'(t))^2 - F_n''(t)}{F_n^2(t)}$$

and

$$\begin{aligned} F_n'(t) &= -\frac{1}{m_n!} \int_{K^{m_n}} UV \exp(-tU) d\mu, \\ F_n''(t) &= \frac{1}{m_n!} \int_{K^{m_n}} U^2 V \exp(-tU) d\mu. \end{aligned}$$

We must show that $(F_n'(t))^2 - F_n''(t) \ge 0$. Now, for a fixed t, we may mulitply V by a constant so that

$$\int_{K^{m_n}} V \exp(-tU) d\mu = 1.$$

Let $d\gamma := V \exp(-tU) d\mu$. Then by the above formulas for F_n' and F_n'', we must show that

$$\int_{K^{m_n}} U^2 d\gamma \ge \left(\int_{K^{m_n}} U \, d\gamma \right)^2,$$

but this is a simple consequence of the Cauchy-Schwarz inequality. □

The following "calculus lemma" is essential for the proof of Proposition 11.11.

Lemma 11.17. (Berman and Boucksom [5]**)** *Let $f_n(t)$ be a sequence of concave functions on $\mathbb{R}$ and $g(t)$ a function on $\mathbb{R}$. Suppose that*

$$\liminf_{n\to\infty} f_n(t) \ge g(t), \quad \forall t \in \mathbb{R}$$

and that

$$\lim_{n\to\infty} f_n(0) = g(0).$$

Suppose further that the f_n and g are differentiable at $t = 0$. Then

$$\lim_{n\to\infty} f_n'(0) = g'(0).$$

Here we really need differentiability at $t = 0$; one-sided differentiability is not sufficient.

With these preliminaries, we now prove Proposition 11.11.

Proof. Recall we are assuming the measures $\{\mu_n\}$ satisfy (11.12):

$$\lim_{n\to\infty} \frac{N+1}{2Nnm_n} \cdot \log\det G_n^{\mu_n,w} = \log \delta^w(K)$$

and we want to show (11.13):

$$\frac{1}{m_n} B_n^{\mu_n,w} d\mu_n \to \mu_{K,Q} = \frac{1}{(2\pi)^N}(dd^c V_{K,Q}^*)^N \text{ weak-}*\,.$$

For $u \in C(K)$ we again set $w_t(z) := w(z)\exp(-tu(z))$ which corresponds to $Q_t := Q + tu$ and $f_n(t)$ as in (11.15). From (11.12), for $t = 0$, $w_0 = w$ we have

$$\lim_{n\to\infty} f_n(0) = -\log(\delta^w(K)).$$

From Corollary 11.10 and Theorem 11.7, setting $g(t) = -\log(\delta^{w_t}(K))$,

$$g'(0) = \frac{N+1}{N(2\pi)^N}\int_K u(z)(dd^c V_{K,Q}^*)^N. \tag{11.17}$$

Now note that for each fixed t, the measure μ_n is a candidate for the optimal measure for K and w_t. If follows from Definition 10.7 that

$$\det(G_n^{\mu_n,w_t}) \le \det(G_n^{\mu_n^t,w_t})$$

where we denote an optimal measure for K and w_t by μ_n^t. Hence (see (11.15))

$$f_n(t) \ge -\frac{1}{2m_n}\log(\det(G_n^{\mu_n^t,w_t}))$$

and consequently from Proposition 10.9 we have

$$\liminf_{n\to\infty} f_n(t) \ge -\log(\delta^{w_t}(K)) = g(t).$$

It now follows from Lemma 11.17 that

$$\lim_{n\to\infty} f_n'(0) = g'(0).$$

In other words, by Lemma 11.15,

$$\begin{aligned}\lim_{n\to\infty}\frac{N+1}{N}\int_K u(z)d\mu_n &= \frac{N+1}{N(2\pi)^N}\int_K u(z)(dd^c V_{K,Q}^*)^N \\ &= \frac{N+1}{N}\int_K u(z)d\mu_{K,Q},\end{aligned}$$

and hence $\mu_n \to \mu_{K,Q}$ weak$-*$. $\square$

The reader can consult [27] for a self-contained discussion of the results in this section.

Exercises.

(1) Prove that the operator $P : \mathcal{A}(K) \to L^+(\mathbb{C}^N)$ is increasing and concave: for $0 \leq t \leq 1$ and $Q_1, Q_2 \in \mathcal{A}(K)$,

$$P(Q_1) \leq P(Q_2) \text{ if } Q_1 \leq Q_2 \text{ and}$$

$$P(tQ_1 + (1-t)Q_2) \geq tP(Q_1) + (1-t)P(Q_2).$$

(2) Prove (11.8) using the following outline:
   (a) We can assume $u \geq 0$ (why?). For $\epsilon > 0$, apply (8.4) to $(1+\epsilon)u$ and v on the bounded set $\{(1+\epsilon)u < v\}$.
   (b) Show that $\bigcup_{j=1}^{\infty}\{(1+1/j)u < v\} = \{u < v\}$.
   (c) Apply (a) with $\epsilon = 1/j$ and conclude using (b) and monotone convergence.

(3) Verify that $B_n^{\mu_n,w}(x_j^{(n)}) = m_n$ for $j = 1, ..., m_n$ for any discrete, equally weighted measure $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$ with

$$|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n \neq 0.$$

(*Hint: Show that the orthonormal polynomials are given by* $q_j^{(n)}(z) = \frac{\sqrt{m_n}\, l_{nj}(z)}{w(x_j^{(n)})^{2n}}$ *where* l_{nj} *is the FLIP associated to* $x_j^{(n)}$ (*recall* (*9.16*)).

(4) Prove Lemma 11.17.

12. Appendix A: Differential forms and currents in $\mathbb{C}^N$.

We introduce some standard material on differential forms and currents. We may identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ via the mapping

$$(z_1, \ldots, z_N) = (x_1 + iy_1, \ldots, x_N + iy_N) \mapsto (x_1, y_1, \ldots, x_N, y_N).$$

We have for $k = 1, \ldots, N$, the complex differentials

$$dz_k = dx_k + idy_k, \quad d\overline{z}_k = dx_k - idy_k.$$

We also recall the following notation—for a multi-index $I = (i_1, \ldots, i_p)$ we write

$$|I| = p \text{ (the multi-index length)},$$

$$dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}\,, \quad d\overline{z}^I = d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p}.$$

The standard volume form in $\mathbb{C}^N \sim \mathbb{R}^{2N}$ is defined by

$$dV_{2N} := \left(\frac{i}{2}\right)^N dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_N \wedge d\overline{z}_N = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_N \wedge dy_N.$$

Let D be a domain in $\mathbb{C}^N$ and k a nonnegative integer, $k \leq 2N$. A complex differential k-form on D can be written as

$$\omega = \Sigma'_{|I|+|J|=k} \omega_{IJ}\, dz^I \wedge d\overline{z}^J$$

for some coefficient functions $\omega_{IJ} \in C^\infty(D, \mathbb{C}) := C^\infty(D)$. Here the 'prime' ($'$) indicates that we sum over increasing multi-indices only: if $I = (i_1, \ldots, i_p)$, then $1 \leq i_1 < i_2 < \cdots < i_p \leq N$. The *norm of* ω is given pointwise by

$$|\omega| = \left(\Sigma'_{|I|+|J|=k} |\omega_{IJ}|^2\right)^{\frac{1}{2}}.$$

It measures at each point of D the Euclidean norm of the k-form with respect to the orthonormal basis $\{dz^I \wedge d\overline{z}^J\}_{|I|+|J|=k}$.

We write $\bigwedge^k(D, \mathbb{C})$ to denote the complex vector space of (smooth) k-forms on D. The 0-forms, by convention, are the functions in $C^\infty(D, \mathbb{C})$. The space $\bigwedge^k(D, \mathbb{C})$ has some important subspaces. Given nonnegative integers p, q with $p + q = k$, we define $\bigwedge^{p,q}(D, \mathbb{C})$, the *forms of bidegree (p,q)*, as the set of all k-forms ω that can be written as

$$\omega = \Sigma'_{|I|=p, |J|=q} \omega_{IJ}\, dz^I \wedge d\overline{z}^J.$$

In pluripotential theory we often consider only the spaces $\bigwedge^{p,p}(D, \mathbb{C})$, where $0 \leq p \leq N$ is a nonnegative integer. Note that $\bigwedge^{2N}(D, \mathbb{C}) =$

$\bigwedge^{N,N}(D,\mathbb{C})$. For such differential forms, we will define the notion of *positivity.*

Definition 12.1. An (N,N)-form ω on D is called *positive* if $\omega = \tau dV_{2N}$ for some function $\tau : D \to [0,\infty)$.

A (p,p)-form α is called *elementary strongly positive* if there are linearly independent complex linear mappings $\eta_j : \mathbb{C}^N \to \mathbb{C},\ j = 1,\dots,p$ such that

$$\alpha = \frac{i}{2}\, d\eta_1 \wedge d\overline{\eta}_1 \wedge \cdots \wedge \frac{i}{2}\, d\eta_p \wedge d\overline{\eta}_p.$$

A form ω is called *strongly positive* if $\omega = \sum \lambda_j \omega_j$ for m non-negative numbers $\lambda_1,\dots,\lambda_m$ and elementary strongly positive forms $\omega_1,\dots,\omega_m$, where m is a positive integer.

A (p,p)-form ω is called *positive* if for any strongly positive $(N-p, N-p)$-form η, the (N,N)-form $\omega \wedge \eta$ is positive.

As an example, the standard Kähler form in $\mathbb{C}^N$ is defined by $\beta := \frac{i}{2}\sum_{j=1}^N dz_j \wedge d\overline{z}_j$. For a positive integer $p \leq N$, $\beta^p = \beta \wedge \cdots \wedge \beta$ (p times) is a positive (p,p)-form. In particular, $\beta^N = N! dV_{2N}$.

We denote by $\mathcal{D}^k(D,\mathbb{C})$ the subspace of $\bigwedge^k(D,\mathbb{C})$ made up of those forms whose coefficients are in $C_0^\infty(D,\mathbb{C}) := C_0^\infty(D)$. They are called the *test forms of degree* k. Note that $\mathcal{D}^0(D,\mathbb{C}) = C_0^\infty(D,\mathbb{C})$, the usual *test functions* of distribution theory. The *test forms of bidegree* (p,q), $\mathcal{D}^{p,q}(D,\mathbb{C})$, are defined similarly.

We equip $\mathcal{D}^0(D,\mathbb{C})$ with the topology characterized by the following convergence property: given test functions $\{\phi_j\}_{j=1}^\infty, \phi$ then $\phi_j \to \phi$ if there exists a compact set $K \subset D$ such that

(1) $supp(\phi_j), supp(\phi) \subset K$
(2) The functions ϕ_j converge uniformly to ϕ on K, and the derivatives (of all orders) of ϕ_j converge uniformly to the corresponding derivatives of ϕ.

The topology on $\mathcal{D}^k(D,\mathbb{C})$ is characterized by the property that given forms $\{\omega_j\}, \omega$ in $\mathcal{D}^k(D,\mathbb{C})$, then $\omega_j \to \omega$ if and only if each coefficient of ω_j converges to the corresponding coefficient of ω in the above sense.

Definition 12.2. A *current T of degree k* is a linear functional on $\mathcal{D}^{2N-k}(D,\mathbb{C})$, i.e., an element of the dual space $\big(\mathcal{D}^{2N-k}(D,\mathbb{C})\big)'$. We will use the *dual pairing* notation $<T,\phi>$ to indicate the action of a current T on a test form ϕ.

We furnish the space of currents $\big(\mathcal{D}^{2N-k}(D,\mathbb{C})\big)'$ with the weak* topology, which is characterized by the property that given currents $\{T_j\}, T$ then $T_j \to T$ if and only if $< T_j, \phi > \to < T, \phi >$ for all $\phi \in \mathcal{D}^{2N-k}(D,\mathbb{C})$.

If T is a k-current and ψ is a smooth m-form with $k+m \leq 2N$, then we define the $(k+m)$-current $T \wedge \psi$ by the formula

$$<T \wedge \psi \,,\, \phi>:=<T \,,\, \psi \wedge \phi>, \quad \phi \in \mathcal{D}^{N-k-m}(D,\mathbb{C}). \tag{12.1}$$

Remark 12.3. One can extend the definition of differential k-forms to a larger class by allowing the forms to have *distribution coefficients*. Denoting the set of such forms by $\mathcal{D}'^k(D,\mathbb{C})$, it turns out that $\mathcal{D}'^k(D,\mathbb{C}) = \big(\mathcal{D}^{2N-k}(D,\mathbb{C})\big)'$. Similarly, we can also define $\mathcal{D}'^{p,q}(D,\mathbb{C})$ to be the (p,q)-forms with distribution coefficients. Then we also have $\mathcal{D}'^{p,q}(D,\mathbb{C}) = \big(\mathcal{D}^{N-p,N-q}(D,\mathbb{C})\big)'$, the currents of bidegree (p,q).

A distribution T, considered as a 0-current, acts on a test $2N$-form $\phi = \phi_{2N} dV_{2N}$ by the formula

$$<T,\phi>:= (T,\phi_{2N}), \tag{12.2}$$

where the pairing $(\cdot,\cdot)$ on the right-hand side of 12.2 is the usual pairing of a distribution with a test function. If T is a k-current in $\mathbb{C}^N$ that can be written as $T = T_0\omega$ where T_0 is a distribution and ω is a k-form, then by (12.1) and (12.2), T acts on a test form ϕ of degree $2N-k$ as follows:

$$<T,\phi> = <T_0 \,,\, \omega \wedge \phi> = (T_0 \,,\, [\omega \wedge \phi]_{2N}).$$

In the above equation we use the subscript $2N$ to denote the coefficient of dV_{2N} in a $2N$-form on $\mathbb{C}^N$.

We will generalize the notion of positivity in Definition 12.1 to currents; first, we recall the notion of a positive distribution.

Definition 12.4. A *positive distribution* is a distribution S such that for any test function ϕ with range in $[0,\infty)$, we have $(S,\phi) \in [0,\infty)$.

Definition 12.5. For $k \leq N$, a (k,k)-current T is called *positive* if for every strongly positive $(N-k,N-k)$-form ω, $T \wedge \omega = \tau dV_{2N}$ for some positive distribution τ.

Remark 12.6. A positive distribution can be extended to a linear functional on $C_0(D,\mathbb{C})$. The Riesz representation theorem says that for any continuous linear functional A on $C_0(D)$, there exists a unique measure μ such that $(A, \phi) = \int_D \phi \, d\mu$ for any $\phi \in C_0(D)$. The measure μ thus obtained is called a *Radon measure*. We may therefore identify positive distributions with Radon measures. If T is a current which can be written in the form $T = \mu\omega$, where ω is a k-form and μ is a Radon measure, then the action of T on a test $(2N-k)$-form ϕ is given by

$$<T\,,\,\phi> = <\mu\,,\,\omega \wedge \phi> = \int [\omega \wedge \phi]_{2N} \, d\mu.$$

13. Appendix B: Exercises on distributions.

(1) If $g \in L^1_{loc}(\mathbb{R})$, we define the distribution $\mathcal{L}_g$ via

$$\mathcal{L}_g(\phi) = \int_{\mathbb{R}} \phi(x)g(x)dx$$

for $\phi \in C_0^\infty(\mathbb{R})$.

   (a) Show that if $\{g_n\} \subset L^1_{loc}(\mathbb{R})$ and $g_n \to g$ in $L^1_{loc}(\mathbb{R})$, then $\mathcal{L}_{g_n} \to \mathcal{L}_g$ as distributions.

   (b) Verify that if $g \in C^1(\mathbb{R})$ then $\mathcal{L}_g' = \mathcal{L}_{g'}$.

(2) If $g_1, g_2 \in L^1_{loc}(\mathbb{R})$ and $g_1 = g_2$ a.e., then clearly $\mathcal{L}_{g_1} = \mathcal{L}_{g_2}$ as distributions. Prove the converse: let $g_1, g_2 \in L^1_{loc}(\mathbb{R})$; suppose that

$$\mathcal{L}_{g_1}(\phi) = \mathcal{L}_{g_2}(\phi) \text{ for all } \phi \in C_0^\infty(\mathbb{R});$$

and show that $g_1 = g_2$ a.e. (*Hint: Clearly* $g_1 * \chi_{1/j} = g_2 * \chi_{1/j}$ *for all* $j = 1, 2, ...$ *where* $\chi(x) = \chi(|x|) \geq 0$ *with* $\chi \in C_0^\infty(\mathbb{R})$ *and* $\int_{\mathbb{R}} \chi(x)dx = 1$. *Thus it suffices to show that* $g_i * \chi_{1/j} \to g_i$, $i = 1, 2$ *in* $L^1_{loc}(\mathbb{R})$ *as* $j \to \infty$).

(3) Let $f(x) = |x|$.

   (a) Show that if ϕ is a C^1-function (ϕ is differentiable and ϕ' is continuous) which is identically zero outside of an interval; e.g., $\phi(x) = 0$ if $|x| > M$ for some M, then

$$\int_{\mathbb{R}} \phi'(x)f(x)dx = -\int_{\mathbb{R}} \phi(x)f'(x)dx.$$

   (b) Show that if ϕ is a C^2-function (ϕ'' is continuous) which is identically zero outside of an interval; e.g., $\phi(x) = 0$ if $|x| > M$ for some M, then

$$\int_{\mathbb{R}} \phi''(x)f(x)dx = 2\phi(0).$$

   This shows in particular that as distributions,

$$\mathcal{L}''_{|x|} = 2\delta_0(x)$$

   where $\delta_0(x)$ is the delta function at 0; i.e., the distribution whose action on a test function $\phi(x)$ gives $\phi(0)$.

(4) We defined the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ by $\mathcal{L}'(f) := -\mathcal{L}(f')$ and the product of a distribution $\mathcal{L}$ and a smooth function g by

$$(g \cdot \mathcal{L})(f) := \mathcal{L}(gf).$$

(a) Using this definition, find the distribution $x \cdot \delta_0(x)$; i.e., describe its action on a test function $f(x)$.
(b) Using this definition, and the definition of distributional deriviative, find the distribution $x \cdot \delta_0'(x)$; i.e., describe its action on a test function $f(x)$.
(c) Using this definition, and the definition of distributional deriviative, find the distribution $x^2 \cdot \delta_0''(x)$; i.e., describe its action on a test function $f(x)$.

(5) We recall again the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ in one variable is defined by $\mathcal{L}'(f) := -\mathcal{L}(f')$.
(a) Suppose g is piecewise smooth on $\mathbb{R}$, differentiable on $\mathbb{R} \setminus \{0\}$, and has a (possible) jump discontinuity at 0; i.e., $g(0+) := \lim_{x\to 0^+} g(x)$ and $g(0-) := \lim_{x\to 0^-} g(x)$ exist but (perhaps) are different. Find the distribution $\mathcal{L}_g'$; i.e., describe its action on a test function $f(x)$.
(b) Let $g(x)$ be the Heaviside function $H(x)$; i.e., $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. What does your answer to (a) give for the action of $\mathcal{L}_H'$ on a test function $f(x)$?
(c) Compare the distributions $\mathcal{L}_{g_1}'$ and $\mathcal{L}_{g_2}'$ where

$g_1(x) = 0$ for $x \leq 0$ and $g_1(x) = x^2$ for $x \geq 0$ and

$g_2(x) = -1$ for $x < 0$ and $g_2(x) = x^2$ for $x \geq 0$;

i.e., describe each one's action on a test function $f(x)$.

(6) Suppose $\mathcal{L}$ is a distribution with $\mathcal{L}' = 0$, i.e., $\mathcal{L}'(f) = 0$ for all $f \in C_0^\infty(\mathbb{R})$. What can you conclude about $\mathcal{L}$?

(7) Let $g(x, y) = H(x)H(y) : \mathbb{R}^2 \to \mathbb{R}$ where H is the (univariate) Heaviside function; i.e., $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. Then g determines a distribution $\mathcal{L}_g$ on $C_0^\infty(\mathbb{R}^2)$ by

$$\mathcal{L}_g(f) := \int\int f(x, y)g(x, y)dA(x, y).$$

Determine the distribution $\Delta\mathcal{L}_g$; i.e., describe its action on a test function $f(x, y)$.

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