Separately analytic functions

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Preface

Chapter 1

Introduction

Let us begin with the following (elementary) problem.

(S-$\mathcal{C}$) We are given two domains $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ and a function

$$f : D \times G \longrightarrow \mathbb{R}$$

that is *separately continuous* on $D \times G$, i.e.:

- $f(a, \cdot)$ is continuous on G for arbitrary $a \in D$,
- $f(\cdot, b)$ is continuous on D for arbitrary $b \in G$.

We ask whether the above conditions imply that f is continuous on $D \times G$.

It is well known that the answer is negative. However, recall that the answer was not known for instance to A. Cauchy (1), who in 1821 in his *Cours d'Analyse* claimed that f must be continuous (cf. [Pio 1985-86], [Pio 1996], [Pio 2000]). According to C.J. Thomae (cf. [Tho 1870], p. 13, [Tho 1873], p. 15) (2), the error had been first discovered by E. Heine (3). As an counterexample may serve the function

$$p = q = 1,\ D = G = \mathbb{R}, \quad f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}, \tag{*}$$

which was already known to G. Peano in 1884 (4) (cf. [Gen 1884], p. 173).

Since the answer is in general negative, one can ask how big is the set $\mathcal{S}_{\mathcal{C}}(f)$ of discontinuity points $(a, b) \in D \times G$ of a separately continuous function f. A partial answer was first given in 1899 by R. Baire (5) ([Bai 1899], see also [Rud 1981]), who proved that every separately continuous function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is of the *first Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ such that $f_k \longrightarrow f$ pointwise on $\mathbb{R}^2$. Consequently, if $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is separately continuous, then f is Borel measurable and $\mathcal{S}_{\mathcal{C}}(f)$ must be of the *first Baire category*, i.e. $\mathcal{S}_{\mathcal{C}}(f) \subset \bigcup_{k=1}^{\infty} F_k$, where $\operatorname{int} \overline{F}_k = \varnothing$, $k \in \mathbb{N}$. Moreover, Baire proved that if $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is separately continuous, then $\mathcal{S}_{\mathcal{C}}(f)$ is an $\mathcal{F}_\sigma$–set (6) whose projections are of the first Baire category. Conversely, if $S \subset [0,1] \times [0,1]$ is an $\mathcal{F}_\sigma$–set whose projections are of the first Baire category, then there exists a separately continuous function $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ with $\mathcal{S}_{\mathcal{C}}(f) = S$ (cf. [Ker 1943], [Mas-Mik 2000]).

It is natural to ask whether the above results may be generalized to the case of separately continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \geq 3$, i.e. those functions f for

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(1) Augustin Cauchy (1789–1857) — French mathematician and physicist.
(2) Carl Johannes Thomae (1840–1921) — German mathematician.
(3) Eduard Heine (1821–1881) — German mathematician.
(4) Giuseppe Peano (1858–1932) — Italian mathematician.
(5) René-Louis Baire (1874–1932) — French mathematician.
(6) That is, a countable union of closed sets.

which $f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \in \mathcal{C}(\mathbb{R})$ for arbitrary $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$. H. Lebesgue (7) proved ([Leb 1905]) that every such a function is of the $(n-1)$ *Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty}$ of functions of the $(n-2)$ Baire class such that $f_k \longrightarrow f$ pointwise on $\mathbb{R}^n$. In particular, every separately continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Borel measurable. Moreover, H. Lebesgue proved that the above result is exact, i.e. for $n \geq 3$ there exists a separately continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that is not of the $(n-2)$ Baire class.

It is clear that one may formulate similar problems substituting the class $\mathcal{C}$ of continuous functions by other classes $\mathcal{F}$, e.g.:

- $\mathcal{F} = \mathcal{C}^k =$ the class of $\mathcal{C}^k$*–functions*, $k \in \mathbb{N} \cup \{\infty, \omega\}$, where $\mathcal{C}^\omega$ means the class of *real analytic functions*,
- $\mathcal{F} = \mathcal{H} =$ the class of *harmonic functions*,
- $\mathcal{F} = \mathcal{SH} =$ the class of *subharmonic functions* (in this case we allow that $f : D \times G \longrightarrow [-\infty, +\infty)$).

Thus our more general problem is the following one.

(S-$\mathcal{F}$) We are given two domains $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ and a function

$$f : D \times G \longrightarrow \mathbb{R}$$

that is *separately of class* $\mathcal{F}$ on $D \times G$, i.e.:

- $f(a, \cdot) \in \mathcal{F}(G)$ for arbitrary $a \in D$,
- $f(\cdot, b) \in \mathcal{F}(D)$ for arbitrary $b \in G$.

We ask whether $f \in \mathcal{F}(D \times G)$.

Moreover, in the case where the answer is negative, one may study the set $\mathcal{S}_{\mathcal{F}}(f)$ of all points $(a, b) \in D \times G$ such that $f \notin \mathcal{F}(U)$ for every neighborhood U of (a, b).

Observe that the Peano function (*) is separately real analytic. Consequently, our problem has the negative solution for $\mathcal{F} = \mathcal{C}^k$ with arbitrary $k \in \mathbb{N} \cup \{\infty, \omega\}$ and, therefore, one may be interested in the structure of $\mathcal{S}_{\mathcal{C}^k}(f)$. The structure of $\mathcal{S}_{\mathcal{C}^\omega}(f)$ was completely characterized in [Ray 1988], [Sic 1990] (8), and [Bło 1992] (cf. Theorem 4.6.2).

Surprisingly, in the case of harmonic functions the answer is positive — *every separately harmonic function is harmonic* — cf. [Lel 1961] (9) (Theorem 4.5.1).

In the case of subharmonic functions the answer is once again negative — cf. [Wie-Zei 1991].

Analogous problems may be formulated in the case where $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ are domains and $f : D \times G \longrightarrow \mathbb{C}$ is a function that is separately of class $\mathcal{F}$ with:

- $\mathcal{F} = \mathcal{O} =$ the class of all *holomorphic functions*,
- $\mathcal{F} = \mathcal{M} =$ the class of all *meromorphic functions*,
- $\mathcal{F} = \mathcal{PSH} =$ the class of all *plurisubharmonic functions* (in this case we allow that $f : D \times G \longrightarrow [-\infty, +\infty)$).

(7) Henri Lebesgue (1875–1941) — French mathematician.
(8) Józef Siciak (1931–) — Polish mathematician.
(9) Pierre Lelong (1912–) — French mathematician.

In the case of holomorphic functions the answer is positive — *every separately holomorphic function is holomorphic* (Theorem 2.1.5) — this is the famous *Hartogs theorem* (10) (cf. [Har 1906]). In the sequel we will be mostly concentrated on the holomorphic case. We would like to point out that investigations of the separately holomorphic functions begun at 1899 ([Osg 1899]) (11), that is almost at the same time as first Baire's results on separately continuous functions ([Bai 1899]).

Since the answer to the main question (S-$\mathcal{O}$) is positive, we may consider the following strengthened problem.

(S-$\mathcal{O}_\mathrm{H}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty set $B \subset G$, and a function $f : D \times G \longrightarrow \mathbb{C}$ such that:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$ (only in B),

we ask whether $f \in \mathcal{O}(D \times G)$.

The problem has a long history that began with [Huk 1930] (12) (Theorem 2.2.2) and has been continued in [Ter 1967], [Ter 1972] (13) — Theorems 4.1.1 and 4.1.4. Terada was the first who used the pluripotential theory — the newest tool at that tme. Roughly speaking, the final result says that the answer is positive iff the set B is not *pluripolar* (i.e. B is not thin from the point of view of the pluricomplex potential theory — cf. Definition 3.3.18).

The problem (S-$\mathcal{O}_\mathrm{H}$) leads to the following general question.

(S-$\mathcal{O}_\mathrm{C}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty sets $A \subset D$, $B \subset G$, and a function $f : (A \times G) \cup (D \times B) \longrightarrow \mathbb{C}$ such that:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,

we ask whether f may be holomorphically extended to an open (independent of f) neighborhood of the *cross* $\boldsymbol{X} := (A \times G) \cup (D \times B)$ (note that (S-$\mathcal{O}_\mathrm{H}$) is just the case where $A = D$). Investigations of (S-$\mathcal{O}_\mathrm{C}$) began with [Ber 1912] (14) and have been continued for instance in [Ber 1912], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981a], [Shi 1989], [Ngu-Sic 1991], [Ngu-Zer 1991], [Ngu-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002] in which it has been completely solved — Theorem 4.3.1. Roughly speaking, if the sets A, B are not pluripolar and *regular* (i.e. every point of A (resp. B) is a density point of A (resp. B) in the sense of the pluricomplex potential theory), then there exists a universal neighborhood $\widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$ such that every function f separately holomorphic on $\boldsymbol{X}$ extends holomorphically to $\widehat{\boldsymbol{X}}$.

The results extend (in a non-trivial way) to *N–fold crosses*

$$\boldsymbol{X} := \bigcup_{j=1}^{N} A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N$$

(10) Friedrich Hartogs (1874–1943) — German mathematician.
(11) William Osgood (1864–1943) — American mathematician.
(12) Masuo Hukuhara (1905–) — Japanese mathematician.
(13) Toshiaki Terada (1941–) — Japanese mathematician.
(14) Sergei Natanovich Bernstein (1880 — 1968)) — Russian mathematician.

with $A_j \subset D_j \subset \mathbb{C}^{n_j}$, $j = 1, \dots, N$, and *separately holomorphic functions*, i.e. functions $f : \boldsymbol{X} \longrightarrow \mathbb{C}$ such that $f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N) \in \mathcal{O}(D_j)$ for all $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$) — Theorem 4.3.1.

So far our separately holomorphic functions $f : \boldsymbol{X} \longrightarrow \mathbb{C}$ had no singularities on $\boldsymbol{X}$. The fundamental paper by E.M. Chirka and A. Sadullaev ([Chi-Sad 1988]) and next some applications to mathematical tomography ([Ökt 1998], [Ökt 1999]) showed that the following problems seems to be important.

(S-$\mathcal{O}_{\mathrm{S}}$) Let $A \subset D \subset \mathbb{C}^p$, $B \subset G \subset \mathbb{C}^q$ be as in (S-$\mathcal{O}_{\mathrm{C}}$), let

$$M \subset \boldsymbol{X} := (A \times G) \cup (D \times B),$$

and let $f : \boldsymbol{X} \setminus M \longrightarrow \mathbb{C}$ be a *separately holomorphic* on $\boldsymbol{X} \setminus M$, i.e.:

- $f(a, \cdot)$ is holomorphic in $\{w \in G : (a, w) \notin M\}$ for every $a \in A$,
- $f(\cdot, b)$ is holomorphic in $\{z \in D : (z, b) \notin M\}$ for every $b \in B$.

We ask whether there exist a universal open neighborhood $\widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$ and a relatively closed set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ (both independent of f) such that f extends holomorphically to $\widehat{\boldsymbol{X}} \setminus \widehat{M}$.

Observe that the case where $M = \varnothing$ reduces to (S-$\mathcal{O}_{\mathrm{C}}$). The problem generalizes to N–fold crosses and to separately meromorphic functions. A solution of (S-$\mathcal{O}_{\mathrm{S}}$) has been found in a series of papers [Sic 2001], [Jar-Pfl 2001a], [Jar-Pfl 2001b], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2003c], [Jar-Pfl 2007] — Chapters 5 and 7.

Analogous problems may be also stated for separately meromorphic functions, for example:

(S-$\mathcal{M}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a “thin” (in a certain sense) relatively closed set $S \subset D \times G$, and a function $f : D \times G \setminus S \longrightarrow \mathbb{C}$ that is *separately meromorphic* on $D \times G$, i.e.:

- $f(a, \cdot)$ extends meromorphically to G for every $a \in D$ with $\{a\} \times G \not\subset S$,
- $f(\cdot, b)$ extends meromorphically to D for every $b \in B$ with $D \times \{b\} \not\subset S$,

we ask under which assumptions on S the function f extends meromorphically to $D \times G$.

The problem generalizes in a natural way to crosses and N–fold crosses (also with singularities), cf. e.g. [Kaz 1978a], [Kaz 1978b], [Kaz 1984], [Shi 1989], [Shi 1991], [Jar-Pfl 2003c] — Chapter 6.

Similar questions as above may be formulated for a *boundary cross*. To be more precise:

(S-$\mathcal{O}_{\mathrm{B}}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty sets $A \subset \partial D$, $B \subset \partial G$, and a function $f : (A \times (G \cup B)) \cup ((D \cup A) \times B) \longrightarrow \mathbb{C}$ such that:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,
- $f(a, b) = \lim\limits_{D \ni z \to a} f(z, b) = \lim\limits_{G \ni w \to b} f(a, w)$, $(a, b) \in A \times B$, where the limits

are taken in a certain sense (e.g. non-tangential), we ask whether f may be holomorphically extended to a function $\widehat{f}$ on an open (independent of f) subset $\widehat{\boldsymbol{X}}$ of

$D \times G$ such that $\boldsymbol{X} \subset \overline{\widehat{\boldsymbol{X}}}$ and $f(a,b) = \lim\limits_{\widehat{\boldsymbol{X}} \ni (z,w) \to (a,b)} \widehat{f}(z,w)$, $(a,b) \in \boldsymbol{X}$ (where the limit has to be specialized) — Chapter 9.

Another possible generalization of the problem of holomorphicity of separately holomorphic functions is to consider non-linear fibers. Let us illustrate the main idea by the following particular case (cf. [Chi 2006]).

(S-$\mathcal{O}_{\mathrm{F}}$) Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, $\mathit{\Omega} \subset D \times \mathbb{C}^q$ be domains and let

$$D \times G \ni (z,w) \longmapsto (z, \varphi(z,w)) \in \mathit{\Omega}$$

be a homeomorphic mapping such that $\varphi(\cdot, w)$ is holomorphic for every $w \in G$. Suppose that $f : \mathit{\Omega} \longrightarrow \mathbb{C}$ is such that:

- $f(a,\cdot)$ is holomorphic on the fiber domain $\varphi(\{a\} \times G)$ for every $a \in D$,
- $f(\cdot, \varphi(\cdot, b)) \in \mathcal{O}(D)$ every $b \in G$.

We ask whether $f \in \mathcal{O}(\mathit{\Omega})$. The answer is positive — Chapter 10. Note that the classical Hartogs theorem is just the case where $\varphi(z,w) := w$ and $\mathit{\Omega} := D \times G$.

All above problems may be also formulated in the category of Riemann domains over $\mathbb{C}^n$ and/or complex manifolds.

The graph below represents interrelations between different parts of the book. We hope that it may help the reader to find an optimal path through the text.

[Road map of the book. Will be completed. . . . .]

Chapter 2
Classical results

2.1 Osgood and Hartogs theorems (1899 – 1906)

Definition 2.1.1. Let $\varnothing \neq \Omega \subset \mathbb{C}^n$ be open. We say that a function $f : \Omega \longrightarrow \mathbb{C}$ is *separately holomorphic* on Ω ($f \in \mathcal{O}_s(\Omega)$) if for any $a = (a_1, \dots, a_n) \in \Omega$ and $j \in \{1, \dots, n\}$, the function $\zeta \longmapsto f(a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots, a_n)$ is holomorphic in a neighborhood of $\zeta = a_j$ (as a function of one complex variable).

Clearly, $\mathcal{O}(\Omega) \subset \mathcal{O}_s(\Omega)$. At the end of the 19^{th} century, due to the Cauchy integral representation, the following equivalence was well known.

Theorem 2.1.2. *Let $\Omega \subset \mathbb{C}^n$ be open and let $f : \Omega \longrightarrow \mathbb{C}$. Then the following conditions are equivalent:*
(i) *f is complex differentiable at any point of Ω;*
(ii) *$f \in \mathcal{O}(\Omega)$;*
(iii) *$f \in \mathcal{O}_s(\Omega) \cap \mathcal{C}(\Omega)$.*

Thus $\mathcal{O}(\Omega) = \mathcal{O}_s(\Omega) \cap \mathcal{C}(\Omega)$. The first result dealing with separately holomorphic functions without the continuity assumption was the following one.

Theorem 2.1.3 (Osgood). (a) [Osg 1899] *If $f \in \mathcal{O}_s(\Omega)$ is locally bounded, then f is continuous. Consequently, by Theorem 2.1.2(iii), $\mathcal{O}(\Omega) = \{f \in \mathcal{O}_s(\Omega) : f$ is locally bounded$\}$.*

(b) [Osg 1900] *Suppose that $n = p + q$ and $f : \Omega \longrightarrow \mathbb{C}$ is such that for every $(a, b) \in \Omega \subset \mathbb{C}^p \times \mathbb{C}^q$ the functions $z \longmapsto f(z, b)$ and $w \longmapsto f(a, w)$ are holomorphic in neighborhoods of a and b, respectively (e.g. $n = 2$, $p = q = 1$, $f \in \mathcal{O}_s(\Omega)$). Then the set $\mathcal{S}_{\mathcal{O}}(f)$ is nowhere dense in Ω.*

Recall that $\mathcal{S}_{\mathcal{O}}(f)$ denotes the set of all points $a \in \Omega$ such that $f \notin \mathcal{O}(U)$ for every neighborhood U of a. It is clear that $\mathcal{S}_{\mathcal{O}}(f)$ is relatively closed in Ω.

Define

$$\begin{gathered}
\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \\
\mathbb{P}(a, r) = \mathbb{P}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\|_\infty < r\}, \quad a \in \mathbb{C}^n,\ r > 0, \\
\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0, r), \\
K(a, r) := \mathbb{P}_1(a, r), \quad K(r) := \mathbb{P}_1(r), \quad \mathbb{D} := K(1), \quad \mathbb{T} = \partial\mathbb{D}.
\end{gathered}$$

Proof. (a) Nowadays a standard proof of (a) is based on the Schwarz lemma. If

$|f(z)| \le C$ for $z \in \mathbb{P}(a,r) \subset \Omega$, then

$$\begin{aligned}|f(z)-f(a)| &\le |f(z_1,z_2,z_3,\dots,z_n)-f(a_1,z_2,z_3,\dots,z_n)|\\ &\quad+|f(a_1,z_2,z_3,\dots,z_n)-f(a_1,a_2,z_3,\dots,z_n)|+\dots\\ &\quad+|f(a_1,\dots,a_{n-1},z_n)-f(a_1,\dots,a_{n-1},a_n)|\\ &\le \frac{2C}{r}(|z_1-a_1|+|z_2-a_2|+\dots+|z_n-a_n|),\end{aligned}$$

and consequently f is continuous at a.

Another proof, based on the Montel theorem, may be done by induction on n. Suppose the result is true for $n-1$ and let $f \in \mathcal{O}_s(\Omega)$ be locally bounded. Take a polydisc $\mathbb{P}(a,r) \Subset \Omega$. Write $z=(z',z_n) \in \mathbb{C}^{n-1}\times\mathbb{C}$. By the inductive assumption, $f(\cdot,z_n) \in \mathcal{O}(\mathbb{P}_{n-1}(a',r))$ for all $z_n \in K(a_n,r)$. Take a sequence $\mathbb{P}(a,r) \ni a^k \longrightarrow a$ such that $f(a^k) \longrightarrow \alpha \in \mathbb{C}$. We like to show that $\alpha = f(a)$. By the Montel theorem (applied to the sequence $(f(\cdot,a_n^k))_{k=1}^\infty \subset \mathcal{O}(\mathbb{P}_{n-1}(a',r))$), there exists a subsequence $(k_s)_{s=1}^\infty$ such that $f(\cdot,a_n^{k_s}) \longrightarrow g$ locally uniformly in $\mathbb{P}_{n-1}(a',r)$. Since $f(z',\cdot) \in \mathcal{C}(K(a_n,r))$ for all $z' \in \mathbb{P}_{n-1}(a',r)$, we must have $g = f(\cdot,a)$. Thus $\alpha = \lim_{s\to+\infty} f((a^{k_s})',a_n^{k_s})) = g(a') = f(a)$.

(b) follows from a Baire argument. Let $\mathbb{P}_p(a,r)\times\mathbb{P}_q(b,r) \Subset \Omega$ be arbitrary. Define

$$A_k := \{z \in \mathbb{P}_p(a,r) : \forall_{w\in\mathbb{P}_q(b,r)} : |f(z,w)| \le k\}, \quad k \in \mathbb{N}.$$

Then A_k is closed in $\mathbb{P}_p(a,r)$ and $\mathbb{P}_p(a,r) = \bigcup_{k=1}^\infty A_k$. Hence, by Baire's theorem, there exists a k_0 such that A_{k_0} has a non-empty interior. Thus f is bounded on a non-empty open set $U = \mathbb{P}_p(c,\delta)\times\mathbb{P}_q(b,r) \subset \mathbb{P}_p(a,r)\times\mathbb{P}_q(b,r)$. By (a), $f \in \mathcal{O}(U)$. Hence $U \subset \Omega \setminus \mathcal{S}_{\mathcal{O}}(f)$. □

W. Osgood also observed that the proof of Theorem 2.1.3(b) shows that in order to prove that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for arbitrary open set $\Omega \subset \mathbb{C}^n$, it suffices to check the following lemma, which is nowadays called the *Hartogs lemma*.

Lemma 2.1.4 (Hartogs lemma). *Let $f : K(r)\times\mathbb{P}_m(r) \longrightarrow \mathbb{C}$ be such that:*
- *$f(a,\cdot) \in \mathcal{O}(\mathbb{P}_m(r))$ for every $a \in K(r)$,*
- *$f \in \mathcal{O}(K(r)\times\mathbb{P}_m(\delta))$ for some $0<\delta<r$.*

Then $f \in \mathcal{O}(K(r)\times\mathbb{P}_m(r))$.

Proof that Lemma 2.1.4 implies that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$. We use induction on n. For $n=1$ the theorem is trivial. Suppose that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for arbitrary open set $\Omega \subset \mathbb{C}^{n-1}$. Fix $\Omega \subset \mathbb{C}^n = \mathbb{C}\times\mathbb{C}^{n-1}$ and $f \in \mathcal{O}_s(\Omega)$. It is sufficient to show that f is holomorphic in a neighborhood of an arbitrary point $(z_0,w_0) \in \Omega$. Let $\mathbb{P}_n((z_0,w_0),2r) \subset \Omega$, and let

$$A_k := \{w \in \mathbb{P}_{n-1}(w_0,r) : \forall_{z\in K(z_0,r)} : |f(z,w)| \le k\}, \quad k \in \mathbb{N}.$$

Clearly $A_k \subset A_{k+1}$. Since $f(z,\cdot) \in \mathcal{C}(\mathbb{P}_{n-1}(w_0,2r))$ for arbitrary $z \in K(z_0,2r)$ (by the inductive assumption), the sets A_k are closed in $\mathbb{P}_{n-1}(w_0,r)$. Moreover, $\bigcup_{k\in\mathbb{N}} A_k = \mathbb{P}_{n-1}(w_0,r)$. Using Baire's property we conclude that $\operatorname{int} A_{k_0} \neq \varnothing$

for some k_0. Let $\mathbb{P}_{n-1}(\xi_0,\delta) \subset A_{k_0}$. In particular, by Theorem 2.1.3(a), $f \in \mathcal{O}(K(z_0,r) \times \mathbb{P}_{n-1}(\xi_0,\delta))$. Now we apply Lemma 2.1.4 (with $m := n-1$) to the function $K(r) \times \mathbb{P}_{n-1}(r) \ni (z,w) \longrightarrow f(z_0+z,\xi_0+w)$, and we conclude that $f \in \mathcal{O}(\mathbb{P}_n((z_0,\xi_0),r))$.It remains to observe that $(z_0,w_0) \in \mathbb{P}_n((z_0,\xi_0),r)$. □

The main step, based on the above remark by Osgood, was done by Hartogs in [Har 1906].

Theorem 2.1.5 (Hartogs theorem). *The Hartogs lemma (Lemma 2.1.4) is true. Consequently, $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for arbitrary open set $\Omega \subset \mathbb{C}^n$.*

Nowadays there exist various proofs of the Hartogs lemma. We present below two of them:

- Leja's proof [Lej 1950] (1), based on the *Leja's polynomial lemma* (Lemma 2.1.6),
- Koseki's proof [Kos 1966] (2), based on an elementary version of the Hartogs lemma (Lemma 2.1.7).

We like to point out that both proofs are based on classical complex analysis and are independent of the Hartogs lemma for plurisubharmonic functions (cf. Proposition 3.3.13).

2.1.1 Leja's proof of the Hartogs lemma

Let $\mathcal{P}(\mathbb{C}^n)$ denote the space of all complex polynomials of n complex variables.

Lemma 2.1.6 (Leja's polynomial lemma, cf. [Lej 1933a], [Lej 1933b]). *Let $K \subset \mathbb{C}$ be a compact set such that*

$$\inf\{\operatorname{diam} S : S \textit{ is a connected component of } K\} > 0$$

(e.g. K is a continuum (3)*) and let $\mathcal{F} \subset \mathcal{P}(\mathbb{C})$ be such that*

$$\forall_{z\in K} : \sup_{p\in\mathcal{F}} |p(z)| < +\infty,$$

i.e. $\mathcal{F}$ is pointwise bounded on K. Then

$$\forall_{a\in K}\ \forall_{\omega>1}\ \exists_{M=M(K,a,\omega,\mathcal{F})>0}\ \exists_{\eta=\eta(K,a,\omega)>0} : \sup_{p\in\mathcal{F}} \sup_{z\in K(a,\eta)} |p(z)| \le M\omega^{\deg p},$$

or equivalently,

$$\forall_{\omega>1}\ \exists_{M=M(K,\omega,\mathcal{F})>0}\ \exists_{\substack{\Omega=\Omega(K,\omega)\\ K\subset\Omega\ -\ open}} : \sup_{p\in\mathcal{F}} \sup_{z\in\Omega} |p(z)| \le M\omega^{\deg p}.$$

(1) Franciszek Leja (1885 — 1979) — Polish mathematician.
(2) Ken'iti Koseki (1917–1980) — Japanese mathematician.
(3) That is, a compact connected set having more than one point.

Notice that η and Ω are independent of $\mathcal{F}$.

Proof. Let $r > 0$ be such that $\operatorname{diam} S \geq 2r$ for every connected component S of K.

Step 1^0: Let $A \subset [0, r]$ be a closed set with $m := \mathcal{L}^1(A) > 0$. Then for every $d \in \mathbb{N}$ there exist $t_0, \dots, t_d \in A$, $0 \leq t_0 < \dots < t_d \leq r$, such that

$$t_k - t_j \geq \frac{k^2 - j^2}{d^2} m, \quad j, k = 0, \dots, d,\ j < k.$$

Proof of Step 1^0. Let $t_0 = \min A$, $s_0 := t_0 + \frac{1^2}{d^2} m$. Then $A_1 := A \setminus [0, s_0)$. Observe that A_1 is closed and non-empty (indeed, if $A \subset [t_0, s_0)$, then $m = \mathcal{L}^1(A) < s_0 - t_0 = \frac{1^2}{d^2} m \leq m$ — a contradiction).

Put $t_1 := \min A_1$, $s_1 := t_1 + \frac{2^2 - 1^2}{d^2} m$. Then $t_1 - t_0 \geq s_0 - t_0 = \frac{1^2 - 0^2}{d^2} m$. Let $A_2 := A \setminus [0, s_1)$; A_2 is again non-empty (if $A \subset [t_0, s_0) \cup [t_1, s_1)$), then $m < s_0 - t_0 + s_1 - t_1 = \frac{2^2}{d^2} m \leq m$ — a contradiction). Let $t_2 := \min A_2$. Then $t_2 - t_1 \geq s_1 - t_1 = \frac{2^2 - 1^2}{d^2} m$.

We continue and we get $t_0, \dots, t_{d-1} := \min A_{d-1}$, $s_{d-1} := t_{d-1} + \frac{d^2 - (d-1)^2}{d^2} m$, $A_d := A \setminus [0, s_{d-1})$. Suppose that $A_d = \varnothing$. Then $A \subset [t_0, s_0) \cup \dots [t_{d-1}, s_{d-1})$ and hence $m < s_0 - t_0 + \dots + s_{d-1} - t_{d-1} = \frac{d^2}{d^2} m = m$ — a contradiction. Thus $t_d := \min A_d$ is well defined and $t_d - t_{d-1} \geq s_{d-1} - t_{d-1} = \frac{d^2 - (d-1)^2}{d^2} m$. $\square$

Step 2^0: For any $B \subset K$ and $a \in K$, let

$$\pi_a(B) := \{t \in [0, r] : B \cap \partial K(a, t) \neq \varnothing\}.$$

Observe that if B is closed, then so is $\pi_a(B)$.

Step 3^0: Let

$$I(\alpha) := \exp\Big(\int_0^1 \log \frac{\alpha^2 + x^2}{x^2} dx\Big), \quad \alpha \geq 0.$$

Observe that $\log I(\alpha) = \log(1 + \alpha^2) + 2\alpha \arctan(1/\alpha)$. In particular, $\log I(\alpha) \leq (\pi + \alpha)\alpha$.

Step 4^0: *From now on p will denote an arbitrary polynomial from the family $\mathcal{F}$ and $d := \deg p$.*

For any closed set $B \subset K$, $a \in K$ and $\eta > 0$ we have

$$|p(z)| \leq \|p\|_B I^d(\alpha), \quad z \in \overline{K}(a, \eta),$$

where

$$\alpha^2 := \frac{\eta + r - \mathcal{L}^1(A)}{\mathcal{L}^1(A)}, \quad A := \pi_a(B).$$

In particular, if $B = K$, then $A = [0, r]$ and, consequently,

$$|p(z)| \leq \|p\|_K I^d(\sqrt{\eta/r}), \quad z \in K^{(\eta)},$$

where $K^{(\eta)} := \bigcup_{a \in K} \overline{\mathbb{P}}_n(a, \eta)$; notice that $K^{(\eta)}$ is also compact.

Proof of Step 4^0. The case where $\mathcal{L}^1(A) = 0$ is obvious. Assume that $m := \mathcal{L}^1(A) > 0$. Let $t_0, \dots, t_d$ be as in Step 1^0. Take arbitrary $z_k \in B \cap \partial K(a, t_k)$, $k = 0, \dots, d$. Let $T_d := \{z_0, \dots, z_d\}$,

$$L^{(k)}(z, T_d) := \prod_{\substack{j=0 \\ j \neq k}}^{d} \frac{z - z_j}{z_k - z_j}, \quad k = 0, \dots, d.$$

If $z \in \overline{K}(a, \eta)$, then $|z - z_j| \leq |z - a| + |z_j - a| \leq \eta + t_j$. Moreover, since $r - t_j \geq t_d - t_j \geq \frac{d^2 - j^2}{d^2} m$, we get $t_j \leq r - m + \frac{j^2}{d^2} m$, and, consequently, $|z - z_j| \leq (\alpha^2 + \frac{j^2}{d^2}) m$, $j = 0, \dots, d$. On the other hand, $|z_k - z_j| \geq |t_k - t_j| \geq \frac{|k^2 - j^2|}{d^2} m$. Thus, if $z \in \overline{K}(a, \eta)$, then

$$|L^{(k)}(z, T_d)| \leq \prod_{\substack{j=0 \\ j \neq k}}^{d} \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{|k^2 - j^2|}{d^2}} \leq 2 \prod_{j=1}^{d} \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{j^2}{d^2}} \leq 2 I^d(\alpha),$$

where the last inequality follows from the following inequality

$$\frac{1}{d} \sum_{j=1}^{d} \log \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{j^2}{d^2}} \leq \int_0^1 \log \frac{\alpha^2 + x^2}{x^2} dx = \log I(\alpha).$$

Then, for $z \in \overline{K}(a, \eta)$, we have

$$|p(z)|^s = \Big| \sum_{k=0}^{sd} p^s(z_k) L^{(k)}(z, T_{sd}) \Big| \leq \|p\|_B^s (sd + 1) 2 I^{sd}(\alpha),$$

which implies that

$$|p(z)| \leq \|p\|_B \sqrt[s]{2(sd + 1)} I^d(\alpha).$$

It remains to let $s \longrightarrow +\infty$. $\square$

Step 5^0: Let $(K_s)_{s=1}^{\infty}$ be a sequence of compact subsets of K such that $K_s \subset K_{s+1}$, $K = \bigcup_{s=1}^{\infty} K_s$. Then for every $\eta > 0$ there exists a sequence $(m_s(\eta))_{s=1}^{\infty} \subset [0, r]$, $m_s(\eta) \longrightarrow r$ such that

$$|p(z)| \leq \|p\|_{K_s} \Big(I(\sqrt{\eta / r}) I(\alpha_s) \Big)^d, \quad z \in K^{(\eta)},$$

where

$$\alpha_s^2 := \frac{\eta + r - m_s(\eta)}{m_s(\eta)}, \quad s \in \mathbb{N}.$$

Proof of Step 5^0. Take $a_1, \dots, a_N \in K$ such that $K \subset \bigcup_{k=1}^{N} \overline{K}(a_k, \eta) =: L$. Let $A_{k,s} := \pi_{a_k}(K_s)$. Then $A_{k,s} \nearrow [0, r]$ when $s \nearrow +\infty$, which implies that

$\mathcal{L}^1(A_{k,s}) \nearrow r$ when $s \nearrow +\infty$, $k = 1, \dots, N$. Put $m_s(\eta) := \min_{k=1,\dots,N} \mathcal{L}^1(A_{k,s})$. It is clear that $m_s(\eta) \longrightarrow r$. By Step 4^0 we know that

$$|p(z)| \le \|p\|_{K_s} I^d(\alpha_{k,s}) \le \|p\|_{K_s} I^d(\alpha_s), \quad z \in \overline{K}(a_k, \eta),$$

where

$$\alpha_{k,s}^2 := \frac{\eta + r - \mathcal{L}^1(A_{k,s})}{\mathcal{L}^1(A_{k,s})}.$$

Hence,

$$|p(z)| \le \|p\|_{K_s} I^d(\alpha_s), \quad z \in L,$$

and, finally, by Step 4^0, we get the required inequality. □

Step 6^0: Let

$$K_s := \{z \in K : \forall_{p \in \mathcal{F}} : |p(z)| \le s\}, \quad s \in \mathbb{N}.$$

Observe that K_s is compact, $K_s \subset K_{s+1}$, and $K = \bigcup_{s=1}^\infty K_s$ (because $\mathcal{F}$ is pointwise bounded on K). Let $F_{s,d} = \{z_{s,0}, \dots, z_{s,d}\}$ be the *d–th system of Fekete points for K_s*, i.e. $F_{s,d}$ realizes the maximum of the continuous function

$$K_s^{d+1} \ni (z_0, \dots, z_d) \longmapsto \prod_{0 \le j < k \le d} |z_j - z_k|.$$

Put

$$L^{(k)}(z, F_{s,d}) := \prod_{\substack{j=0 \\ j \ne k}}^{d} \frac{z - z_{s,j}}{z_{s,k} - z_{s,j}}, \quad k = 0, \dots, d.$$

Observe that $|L^{(k)}(z, F_{s,d})| \le 1$ for $z \in K_s$. Hence, by Step 5^0, we get

$$|p(z)| = \Big| \sum_{k=0}^{d} p(z_{s,k}) L^{(k)}(z, F_{s,d}) \Big| \le s(d+1) \Big(I(\sqrt{\eta/r}) I(\alpha_s) \Big)^d, \quad z \in K^{(\eta)},\ s \in \mathbb{N}.$$

Step 6^0: We move to the main proof. Fix an $\omega > 1$. Let $d_0 = d_0(\omega) \in \mathbb{N}$ be such that $\sqrt[d]{d+1} \le \sqrt[3]{\omega}$ for $d \ge d_0$. Let $\eta = \eta(r, \omega) > 0$ be so small that $I(\sqrt{\eta/r}) < \sqrt[3]{\omega}$. Finally, let $s_0 = s_0(r, \omega) \in \mathbb{N}$ be such $I(\alpha_s) \le \sqrt[3]{\omega}$ for $s \ge s_0$. In view of Step 5^0, if $d \ge d_0$, then

$$|p(z)| \le s(d+1) \Big(I(\sqrt{\eta/r}) I(\alpha_s) \Big)^d \le s\omega^d, \quad z \in K^{(\eta)},\ s \ge s_0$$

It remains to find an estimate for $d < d_0$. Let $S := \{z_0, \dots, z_{d_0}\} \subset K$ be an arbitrary set of $d_0 + 1$ distinct points. Put

$$M_0 := \max_{k=0,\dots,d_0} \sup_{p \in \mathcal{F}} |p(z_k)| < +\infty.$$

Then, for $d \le d_0$ we get

$$|p(z)| = \Big|\sum_{k=0}^{d_0} f(z_k)L^{(k)}(z,S)\Big| \le M_0 \sum_{k=0}^{d_0} \max_{z\in K^{(\eta)}} |L^{(k)}(z,S)| = M \le M\omega^d,$$
$$z \in K^{(\eta)}. \qquad \square$$

Proof of Lemma 2.1.4 via Leja's polynomial lemma. Let $n := 1+m$. Observe that it is sufficient to show that $f \in \mathcal{O}(\mathbb{P}_n(r'))$ for arbitrary $0 < r' < r$. Thus we may assume that $|f| \le C < +\infty$ in $K(r) \times \mathbb{P}_m(\delta)$ and $f(z,\cdot)$ is bounded for any $z \in \mathbb{P}_m(r)$. We have

$$f(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha, \quad z \in K(r),\ w \in \mathbb{P}_m(r),$$

where

$$f_\alpha(z) = \frac{1}{\alpha!}(D^\alpha f(z,\cdot))(0) = \frac{1}{\alpha!}(D^{(0,\alpha)}f)(z,0), \quad z \in K(r),\ \alpha \in \mathbb{Z}_+^m.$$

The last equality follows from the fact that $f \in \mathcal{O}(K(r) \times \mathbb{P}_m(\delta))$. In particular, $f_\alpha \in \mathcal{O}(K(r))$ for arbitrary α. Moreover, by the Cauchy inequalities, we obtain

$$|f_\alpha(z)| \le C/\delta^{|\alpha|}, \quad z \in K(r),\ \alpha \in \mathbb{Z}_+^m.$$

Applying once more the Cauchy inequalities (for the function $f(z,\cdot)$), we have

$$|f_\alpha(z)| \le \frac{\|f(z,\cdot)\|_{\mathbb{P}_m(r)}}{r^{|\alpha|}}, \quad z \in K(r),\ \alpha \in \mathbb{Z}_+^m,$$

where for a function $\varphi : A \longrightarrow \mathbb{C}$ we set $\|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}$. Consequently,

$$\limsup_{|\alpha|\to+\infty} |f_\alpha(z)|^{1/|\alpha|} \le 1/r, \quad z \in K(r).$$

Our aim is to show that the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha$ converges locally normally in $K(r) \times \mathbb{P}_m(r)$.

Take an arbitrary $\theta \in (0,1)$ and let $\omega > 1$ be such that $\theta_0 := \omega^2\theta < 1$. Fix a point $a \in K(r)$ and $0 < \rho < r - |a|$. Let $0 < \rho_0 < \rho$ be so small that $r\rho_0 \le \omega\delta\rho$. Write

$$f_\alpha(z) = \sum_{k=0}^{\infty} f_{\alpha,k}(z-a)^k, \quad z \in K(a, r-|a|),$$
$$p_\alpha(z) := \sum_{k=0}^{|\alpha|} f_{\alpha,k}(z-a)^k, \quad \mathcal{F} := \{(r/\omega)^{|\alpha|}p_\alpha : \alpha \in \mathbb{Z}_+^m\}.$$

In view of (2.1.1), the Cauchy inequalities imply that

$$|f_{\alpha,k}| \le \frac{C}{\delta^{|\alpha|}\rho^k}.$$

Consequently, in view of (2.1.2), if $z \in \overline{K}(a, \rho_0)$, then

$$|p_\alpha(z)| \le |f_\alpha(z)| + \sum_{k=|\alpha|+1}^{\infty} |f_{\alpha,k}(z-a)^k| \le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + \frac{C}{\delta^{|\alpha|}}\Big(\frac{\rho_0}{\rho}\Big)^{|\alpha|+1}\frac{1}{1-\frac{\rho_0}{\rho}}$$
$$\le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + \frac{C}{1-\frac{\rho_0}{\rho}}\frac{\rho_0}{\rho}\Big(\frac{\rho_0}{\delta\rho}\Big)^{|\alpha|} \le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + C_1\Big(\frac{\omega}{r}\Big)^{|\alpha|}.$$

Hence, the family $\mathcal{F}$ is pointwise bounded on $\overline{K}(a, \rho_0)$. By Leja's polynomial lemma there exist $0 < \eta \le \rho_0$ and $M > 0$ such that

$$\Big(\frac{r}{\omega}\Big)^{|\alpha|}|p_\alpha(z)| \le M\omega^{|\alpha|}, \quad z \in K(a,\eta),\ \alpha \in \mathbb{Z}_+^m.$$

Finally, for $(z, w) \in K(a, \eta) \times \mathbb{P}_m(\theta r)$ we get

$$|f_\alpha(z)w^\alpha| \le \Big(|p_\alpha(z)| + C_1\Big(\frac{\omega}{r}\Big)^{|\alpha|}\Big)(\theta r)^{|\alpha|} \le M(\omega^2\theta)^{|\alpha|} + C_1(\omega\theta)^{|\alpha|} \le (M + C_1)\theta_0^{|\alpha|},$$

which implies that the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha$ converges normally in $K(a, \eta) \times \mathbb{P}_m(\theta r)$. □

2.1.2 Koseki's proof of the Hartogs lemma

The main ingredient of Koseki's proof is the following lemma.

Lemma 2.1.7 (Koseki's lemma, cf. [Kos 1966]). *Let $\Omega \subset \mathbb{C}$ be open, $\varphi_\nu \in \mathcal{O}(\Omega)$, $p_\nu > 0$, $\nu \ge 1$. Assume that the sequence $(|\varphi_\nu|^{p_\nu})_{\nu=1}^\infty$ is locally uniformly bounded in Ω and*

$$\limsup_{\nu\to+\infty} |\varphi_\nu(z)|^{p_\nu} \le c, \quad z \in \Omega.$$

Then for any $K \Subset \Omega$ and $\varepsilon > 0$ there exists a ν_0 such that

$$|\varphi_\nu(z)|^{p_\nu} \le c + \varepsilon, \quad z \in K,\ \nu \ge \nu_0.$$

Proof. The result is local — it is sufficient to show that for any $\varepsilon > 0$ and $a \in \Omega$ there exist a disc $K(a, \eta) \subset \Omega$ and ν_0 such that

$$|\varphi_\nu(z)|^{p_\nu} \le c + \varepsilon, \quad z \in K(a,\eta),\ \nu \ge \nu_0.$$

We may assume that $\Omega = K(2)$, $a = 0$. Let $C > 0$ be such that $|\varphi_\nu|^{p_\nu} \le C$ in $\mathbb{D}$ for arbitrary ν. We may also assume that $\varphi_\nu \not\equiv 0$, $\nu \ge 1$. Write $\varphi_\nu = B_\nu\psi_\nu$ in $\mathbb{D}$, where B_ν is a finite Blaschke product and ψ_ν has no zeros in $\mathbb{D}$. Let $\chi_\nu \in \mathcal{O}(\mathbb{D})$ be a branch of $\psi_\nu^{p_\nu}$ in $\mathbb{D}$. Given arbitrary $\zeta \in \partial\mathbb{D}$, we have

$$\limsup_{\mathbb{D}\ni z\to\zeta} |\chi_\nu(z)| = \limsup_{\mathbb{D}\ni z\to\zeta} |\psi_\nu(z)|^{p_\nu} = \limsup_{\mathbb{D}\ni z\to\zeta} |\varphi_\nu(z)|^{p_\nu} \le C,$$

and so $|\chi_\nu| \le C$ in $\mathbb{D}$, $\nu \in \mathbb{N}$. In particular, the family $(\chi_\nu)_{\nu=1}^\infty$ is equicontinuous in $\mathbb{D}$. Fix an $\varepsilon > 0$ and let $0 < \eta < 1$ be such that $|\chi_\nu(z) - \chi_\nu(0)| \le \varepsilon/2$ for $z \in \overline{K}(\eta)$ and $\nu \ge 1$. Then

$$|\varphi_\nu(z)|^{p_\nu} \le |\psi_\nu(z)|^{p_\nu} = |\chi_\nu(z)| \le \varepsilon/2 + |\chi_\nu(0)|, \quad z \in \overline{K}(\eta),\ \nu \ge 1.$$

It remains to estimate $\chi_\nu(0)$. Observe that

$$|\chi_\nu(0)| \le \frac{1}{2\pi}\int_0^{2\pi} |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta, \quad \nu \ge 1.$$

Let

$$A_k := \{\theta \in [0, 2\pi] : |\varphi_\nu(e^{i\theta})|^{p_\nu} \le c + \varepsilon/4,\ \nu \ge k\}.$$

The sets A_k are closed, $A_k \subset A_{k+1}$, and $\bigcup_{k\in\mathbb{N}} A_k = [0, 2\pi]$. For $\nu \ge k$ we have

$$\begin{aligned}|\chi_\nu(0)| \le \frac{1}{2\pi}\Big(\int_{A_k} |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta + \int_{[0,2\pi]\setminus A_k} |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta\Big)\\ \le \frac{1}{2\pi}\Big((c+\varepsilon/4)\mathcal{L}^1(A_k) + C(2\pi - \mathcal{L}^1(A_k))\Big) \longrightarrow c + \varepsilon/4,\end{aligned}$$

where $\mathcal{L}^1$ denotes the Lebesgue measure in $\mathbb{R}$. Hence $|\chi_\nu(0)| \le c + \varepsilon/2$ for $\nu \gg 1$. □

Proof of Lemma 2.1.4 via Koseki's lemma. We begin as in the proof based on Leja's polynomial lemma:

$$f(z, w) = \sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z) w^\alpha, \quad z \in K(r),\ w \in \mathbb{P}_m(r),$$

where

$$f_\alpha \in \mathcal{O}(K(r)), \quad |f_\alpha(z)| \le C/\delta^{|\alpha|}, \quad z \in K(r),\ \alpha \in \mathbb{Z}_+^m, \tag{2.1.1}$$

$$\limsup_{|\alpha|\to+\infty} |f_\alpha(z)|^{1/|\alpha|} \le 1/r, \quad z \in K(r). \tag{2.1.2}$$

Write $\mathbb{Z}_+^m = \{\alpha_1, \alpha_2, \dots\}$ so that $|\alpha_\nu| \le |\alpha_{\nu+1}|$, $\nu = 1, 2, \dots$. Let $\Omega := K(r)$, $\varphi_\nu := f_{\alpha_\nu}$, $p_\nu := 1/|\alpha_\nu|$. Fix a $\theta \in (0, 1)$ and let $\varepsilon > 0$ be such that $(1 + r\varepsilon)\theta < 1$. Applying Lemma 2.1.7 to $K := \overline{K}(\theta r)$, we obtain $|\varphi_\nu(z)|^{p_\nu} \le 1/r + \varepsilon$ for $z \in \overline{K}(\theta r)$ and $\nu \ge \nu_0$. This means that

$$|f_\alpha(z)| \le (1/r + \varepsilon)^{|\alpha|}, \quad z \in K(\theta r),\ |\alpha| \gg 1.$$

Hence

$$|f_\alpha(z) w^\alpha| \le ((1 + r\varepsilon)\theta)^{|\alpha|}, \quad z \in K(\theta r),\ w \in \mathbb{P}_m(\theta r),\ |\alpha| \gg 1.$$

Consequently, the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z) w^\alpha$ is convergent normally in $\mathbb{P}_n(\theta r)$, which implies that $f \in \mathcal{O}(\mathbb{P}_n(\theta r))$. Since θ was arbitrary, we conclude that $f \in \mathcal{O}(\mathbb{P}_n(r))$. □

2.1.3 Generalized Hartogs lemma. Counterexamples

Lemma 2.1.4 may be easily generalized (Exercise) to the following form.

Lemma 2.1.8 (Hartogs lemma). *Let $U \subset \Omega \subset D \times \mathbb{C}^q$ be domains such that for every $a \in D$ the fiber $U_a := \{w \in \mathbb{C}^q : (a, w) \in U\}$ is non-empty and Ω_a is connected. Let $f : \Omega \longrightarrow \mathbb{C}$ be such that:*
- $f(a, \cdot) \in \mathcal{O}(\Omega_a)$, $a \in D$,
- $f \in \mathcal{O}(U)$.

Then $f \in \mathcal{O}(\Omega)$.

Lemma 2.1.4 is not true without the assumption that $f \in \mathcal{O}(K(r) \times \mathbb{P}_m(\delta))$ for some $0 < \delta < r$ (even if f satisfies some additional regularity conditions) — cf. for instance [Har 1906], [Lej 1950], [Fuk 1983].

Example 2.1.9 ([Lej 1950]). We construct a function $f : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ such that:
- $f(a, \cdot) \in \mathcal{O}(\mathbb{C})$ for every $a \in \mathbb{C}$,
- $f \in \mathcal{O}((\mathbb{C} \setminus \mathbb{R}_-) \times \mathbb{C})$, where $\mathbb{R}_- := \{x \in \mathbb{R} : x \leq 0\}$,

but f is unbounded near $(0, 0)$ (in particular, f is not holomorphic near $(0, 0)$).

Let

$$L_k := \bigcup_{x \in \mathbb{R}_-} K(x, 1/k) \subset \mathbb{C},$$

$$A_k := \overline{K}(k) \setminus L_k, \quad B_k := \overline{K}(k) \cap (\overline{L}_{k+1} \setminus L_{k+2}), \quad C_k := \overline{K}(k) \cap \overline{L}_{k+3}, \quad k \in \mathbb{N}.$$

By the Runge theorem, for each $k \in \mathbb{N}$, there exists a polynomial $P_k \in \mathcal{P}(\mathbb{C})$ such that

$$|P_k(z)| \leq 1/k^k, \quad z \in A_k \cup C_k, \qquad |P_k(z)| \geq k^k, \quad z \in B_k.$$

Let

$$f(z, w) := \sum_{k=1}^{\infty} P_k(z) w^k, \quad (z, w) \in \mathbb{C}^2.$$

Observe that f is well defined because for any $z \in \mathbb{C}$ there exists a $k_0(z) \in \mathbb{N}$ such that $z \in A_k \cup C_k$ for any $k \geq k_0(z)$ and therefore

$$|P_k(z) w^k| \leq (|w|/k)^k, \quad k \geq k_0(z).$$

In particular, $f(z, \cdot) \in \mathcal{O}(\mathbb{C})$ for any $z \in \mathbb{C}$. Moreover, for any $z_0 \in \mathbb{C} \setminus \mathbb{R}_-$ there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that $K(z_0, r_0) \subset A_k$ for $k \geq k_0$. Hence

$$|P_k(z) w^k| \leq (|w|/k)^k, \quad (z, w) \in K(z_0, r_0) \times \mathbb{C}, \ k \geq k_0,$$

and consequently, by the Weierstrass theorem, $f \in \mathcal{O}((\mathbb{C} \setminus \mathbb{R}_-) \times \mathbb{C})$.

Suppose that f is bounded in a neighborhood of $(0, 0)$, i.e. $|f(z, w)| \leq C$, $(z, w) \in \mathbb{P}_2(r)$. Then, by the Cauchy inequalities, we get

$$|P_k(z)| \leq C/r^k, \quad k \in \mathbb{N}, \ z \in K(r).$$

Consequently, taking $z \in B_k \cap K(r)$ with $k \gg 1$, we get

$$k^k \le |P_k(z)| \le C/r^k, \quad k \gg 1;$$

a contradiction.

Example 2.1.10 ([Fuk 1983]). We construct a function $f : \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{C}$ such that:

- $f(a, \cdot) \in \mathcal{O}(\mathbb{D})$, $a \in \mathbb{C}$,
- for any $w_0 \in \mathbb{D}_*$ the function $f(\cdot, w_0)$ is unbounded near 0 (in particular, not holomorphic near $(0, w_0)$).

Let

$$A_k := \Big\{x + iy \in \overline{K}(k) : x \le \frac{1}{2^k} - \frac{1}{2^{k+2}} \text{ or } x \ge \frac{1}{2^k} + \frac{1}{2^{k+2}}\Big\}, \quad \Omega_k := \operatorname{int} A_k,$$
$$B_k := \Big\{x + iy \in \overline{K}(k) : \frac{1}{2^k} - \frac{1}{2^{k+3}} \le x \le \frac{1}{2^k} + \frac{1}{2^{k+3}}\Big\}, \quad U_k := \operatorname{int} B_k.$$

By the Runge approximation theorem for each k there exists a polynomial $P_k \in \mathcal{P}(\mathbb{C})$ such that $|P_k| \le 1/2^k$ on A_k and $|P_k| \ge 2^{k^2}$ on B_k. Observe that:

- $\mathbb{C} = \bigcup_{k=1}^\infty \bigcap_{s=k}^\infty \Omega_s$,
- $1/2^k \in U_k \cap \bigcap_{s=k+1}^\infty \Omega_s$, $k \in \mathbb{N}$,
- $1/2^s \in A_k$ for $s \ne k$.

Define

$$f(z, w) := \sum_{k=1}^\infty P_k(z) w^k, \quad (z, w) \in \mathbb{C} \times \mathbb{D}.$$

Then:

- if $a \in \bigcap_{k=k_0}^\infty \Omega_k$, then $|P_k(a) w^k| \le 1/2^k$ for $k \ge k_0$, and hence $f(a, \cdot) \in \mathcal{O}(\mathbb{D})$;
- if $w_0 \in \mathbb{D}_*$ and $1/2^{k_0} < |w_0|$, then

$$\begin{aligned} |f(1/2^{kk_0}, w_0)| &\ge |P_{kk_0}(1/2^{kk_0}) w_0^{kk_0}| - \sum_{s \in \mathbb{N},\ s \ne kk_0} |P_s(1/2^{kk_0}) w_0^s| \\ &\ge 2^{(kk_0)^2} |w_0|^{kk_0} - \sum_{s \in \mathbb{N},\ s \ne kk_0} 1/2^s \ge (2^{k_0}|w_0|)^{kk_0} - 1 \underset{k \to +\infty}{\longrightarrow} +\infty. \end{aligned}$$

2.2 Hukuhara and Shimoda theorems (1930–1957)

Theorem 2.1.5 and Lemma 2.1.8 suggest the following problem, nowadays called the *Hukuhara problem.*

(S-$\mathcal{O}_{\mathrm{H}}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty set $B \subset G$, and a function $f : D \times G \longrightarrow \mathbb{C}$ that is *separately holomorphic* in the following sense:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,

we ask whether $f \in \mathcal{O}(D \times G)$.

In the above situation we write $f \in \mathcal{O}_s(\boldsymbol{X})$ with $\boldsymbol{X} := (D \times G) \cup (D \times B)$. Notice that from the set-theoretic point of view the set $\boldsymbol{X}$ is nothing else as the Cartesian product $D \times G$ which is, of course, independent of B. Writing $\boldsymbol{X} = (D \times G) \cup (D \times B)$ we point out the role played by the set B.

Remark 2.2.1. (a) Theorem 2.1.5 and Lemma 2.1.8 guaranties that the answer is positive (i.e. $\mathcal{O}_s(\boldsymbol{X}) = \mathcal{O}(D \times G)$) whenever B is open.

(b) Observe that the answer must be negative if B is too "thin". For example, if $B := g^{-1}(0)$, where $g \in \mathcal{O}(G)$, $g \not\equiv 0$, then for *arbitrary* function $\varphi : D \longrightarrow \mathbb{C}$, the function $f(z,w) := \varphi(z)g(w)$, $(z,w) \in D \times G$, belongs to $\mathcal{O}_s(\boldsymbol{X})$ (and, of course, may be not holomorphic on $D \times G$).

The next step in the development started in 1930 with the paper by M. Hukuhara [Huk 1930].

Theorem 2.2.2 (Hukuhara). *If $p = q = 1$ and B has an accumulation point in G, then every locally bounded function $f \in \mathcal{O}_s(\boldsymbol{X})$ is holomorphic on $D \times G$.*

Below (Theorem 2.2.4) we present a more general result (cf. [Ter 1972]) whose proof uses the same ideas as the original proof by Hukuhara.

Definition 2.2.3. We say that a set $B \subset \mathbb{C}^q$ is an *identity set at a point* $b_0 \in \overline{B}$ if for any open connected neighborhood U of b_0 and $f \in \mathcal{O}(U)$, if $f = 0$ on $B \cap U$, then $f \equiv 0$ on U.

Observe that if $q = 1$ and $B \subset G$ has an accumulation point $b_0 \in G$, then B is an identity set at b_0 in the sense of the above definition.

Theorem 2.2.4. *For arbitrary p and q, if B is an identity set at a point $b_0 \in G$, then every locally bounded function $f \in \mathcal{O}_s(\boldsymbol{X})$ is holomorphic on $D \times G$.*

The following notion will be very useful in the sequel.

Definition 2.2.5. Let Ω be a topological space (e.g. an open set in $\mathbb{C}^n$). We say that a sequence $(\Omega_k)_{k=1}^\infty$ of open subsets of Ω is an *exhaustion sequence for Ω* if $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$, $k \in \mathbb{N}$, and $\Omega = \bigcup_{k=1}^\infty \Omega_k$.

In the case where Ω is connected we will always assume that each Ω_k is also connected.

Proof. Let $(D_k)_{k=1}^\infty$ and $(G_k)_{k=1}^\infty$ be exhaustion sequences for D and G, respectively, with $b_0 \in G_1$. It suffices to prove that f is holomorphic on each $D_k \times G_k$. Thus, we may additionally assume that f is bounded.

Observe that f must be continuous. Indeed (cf. the proof of Theorem 2.1.3(a)), let

$$D \times G \ni (z_k, w_k) \longrightarrow (z_0, w_0) \in D \times G$$

and $f(z_k, w_k) \longrightarrow \alpha \in \mathbb{C}$. By a Montel argument we may assume that $f(z_k, \cdot) \longrightarrow g$ locally uniformly in G with $g \in \mathcal{O}(G)$. In particular, $f(z_k, w_k) \longrightarrow g(w_0) = \alpha$. Recall that if $b \in B$, then $f(\cdot, b) \in \mathcal{O}(D)$. Hence, $f(z_k, b) \longrightarrow f(z_0, b) = g(b)$,

$b \in B$. Since B is an identity set, we conclude that $f(z_0, \cdot) \equiv g$. Thus $\alpha = g(w_0) = f(z_0, w_0)$.

Fix an arbitrary polydisc $P = \mathbb{P}(a, r) \Subset D$ and define

$$\widetilde{f}(z, w) := \frac{1}{(2\pi i)^p} \int_{\partial_0 P} \frac{f(\zeta, w)}{z - \zeta} d\zeta, \quad (z, w) \in P \times G,$$

where $\partial_0 \mathbb{P}(a, r) := \partial K(a_1, r) \times \cdots \times \partial K(a_p, r)$. Then $\widetilde{f} \in \mathcal{O}_s(P \times G) \cap \mathcal{C}(P \times G)$ and so $\widetilde{f} \in \mathcal{O}(P \times G)$. Moreover, by the Cauchy integral formula, $\widetilde{f}(z, b) = f(z, b)$ for $(z, b) \in P \times B$. Since B is an identity set, we conclude that $\widetilde{f} = f$ in $P \times G$, which finishes the proof. □

It took another 30 years before I. Shimoda came back to the Hukuhara problem. He proved in [Shi 1957] an analogous result to the one of Osgood (Theorem 2.1.3(b)).

Theorem 2.2.6 (Shimoda (4)). *If $p = q = 1$ and B has an accumulation point in G, then for every function $f \in \mathcal{O}_s(\boldsymbol{X})$ the set $\mathcal{S}_{\mathcal{O}}(f)$ is nowhere dense.*

Below we present a more general result (cf. [Ter 1972]) whose proof goes along the same ideas as the original proof by Shimoda.

Theorem 2.2.7. *For arbitrary p and q, if B is an identity set at a point $b_0 \in G$, then for every function $f \in \mathcal{O}_s(\boldsymbol{X})$ the set $\Omega_0 := D \times G \setminus \mathcal{S}_{\mathcal{O}}(f)$ is dense in $D \times G$. Moreover, $\Omega_0 = U_0 \times G$, where U_0 is an open dense subset of D.*

Proof. First observe that if $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, r) \subset \Omega_0$ and $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, R) \subset D \times G$, then Lemma 2.1.8 implies that $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, R) \subset \Omega_0$. Consequently, Ω_0 must be of the form $\Omega_0 = U_0 \times G$.

Take an arbitrary polydisc $P = \mathbb{P}(a, r) \subset D$ and a point $b \in G$. Let $G_0 \Subset G$ be a subdomain of G such that $b, b_0 \in G_0$. Define

$$A_k := \{z \in P : \forall_{w \in G_0} : \ |f(z, w)| \le k\}, \quad k \in \mathbb{N}.$$

Then obviously $A_k \subset A_{k+1}$, $k \in \mathbb{N}$, and $P = \bigcup_{k=1}^{\infty} A_k$. Moreover, each A_k is closed in P. Indeed, let $A_k \ni z_s \longrightarrow z_0 \in P$. Using a Montel argument, we may assume that $f(z_s, \cdot) \longrightarrow g$ locally uniformly in G_0 with $g \in \mathcal{O}(G_0)$, $|g| \le k$. Since B is an identity set at b_0, we conclude that $g = f(z_0, \cdot)$.

Now, a Baire argument implies that there exists a k_0 such that $U := \operatorname{int} A_{k_0} \neq \varnothing$. Consequently, by Theorem 2.2.4, $f \in \mathcal{O}(U \times G_0)$. □

(4) Isae Shimoda (1916–) — Japanese mathematician.

Chapter 3
Prerequisities

For the reader's convenience we decided to collect in the present chapter various auxiliary results. Most of them may be found (with proofs) in [Jar-Pfl 2000]. Therefore, all the proofs which may be found in [Jar-Pfl 2000] will be skipped. Some of the results presented below will be very specialized — the reader should consult the *Road map of the book* at the end of Introduction to see where a given item will be really needed. We recommend to follow the graph from the *Road map of the book*.

3.1 Extension of holomorphic functions

Riemann domains appear in a very natural way while discussing problems related to holomorphic continuation. There exists an example of a bounded domain $D \subset \mathbb{C}^2$ such that every function $f \in \mathcal{O}(D)$ extends beyond D, but there is no domain $\widehat{D} \subset \mathbb{C}^2$ such that $D \subset \widehat{D}$ and each function $f \in \mathcal{O}(D)$ extends holomorphically to $\widehat{D}$ (cf. [Sha 1976]).

3.1.1 Riemann regions

See [Jar-Pfl 2000], § 1.1.

Definition 3.1.1. A pair (X, p) is called a *Riemann region over* $\mathbb{C}^n$ $(^1)$ (shortly $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$) if:

- X is a topological Hausdorff space,
- $p : X \longrightarrow \mathbb{C}^n$ is locally homeomorphic, i.e. each point $a \in X$ has an open neighborhood U such that $p(U)$ is open in $\mathbb{C}^n$ and $p|_U : U \longrightarrow p(U)$ is homeomorphic.

The mapping p is called the *projection*. For $z \in p(X)$ the set $p^{-1}(z)$ is called the *stalk* over z. A subset $A \subset X$ is said to be *univalent* if $p|_A : A \longrightarrow p(A)$ is homeomorphic.

If X is connected, then we say that (X, p) is a *Riemann domain over* $\mathbb{C}^n$ ($(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$).

If X is σ–compact, i.e. $X = \bigcup_{\nu=1}^{\infty} K_\nu$, where each K_ν is compact, then we say that (X, p) is *countable at infinity* ($(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$).

$(^1)$ Georg Riemann (1826–1866) — German mathematician.

We say that a Riemann region $(X,p)\in\mathfrak{R}(\mathbb{C}^n)$ is *relatively compact* ($(X,p)\in\mathfrak{R}_b(\mathbb{C}^n)$) if there exists $(X',p')\in\mathfrak{R}(\mathbb{C}^n)$ such that X is a relatively compact open set in X' and $p=p'|_X$,

Remark 3.1.2. (a) If $\Omega\subset\mathbb{C}^n$ is an open set, then $(\Omega,\mathrm{id})\in\mathfrak{R}_\infty(\mathbb{C}^n)$. This is the standard identification of open sets in $\mathbb{C}^n$ with Riemann regions.

(b) If $(X,p)\in\mathfrak{R}(\mathbb{C}^n)$, then p is an open mapping. In particular, the set $p(X)$ is open in $\mathbb{C}^n$. For any $a\in p(X)$ the stalk $p^{-1}(a)$ is a discrete subset of X.

(c) If $(X,p)\in\mathfrak{R}(\mathbb{C}^n)$, then the family $(U,p|_U)_U$, where U runs over all univalent open subsets of X, introduces on X an atlas of an n–dimensional complex manifold.

(d) If $(X,p)\in\mathfrak{R}(\mathbb{C}^n)$, $(Y,q)\in\mathfrak{R}(\mathbb{C}^m)$, then $(X\times Y,p\times q)\in\mathfrak{R}(\mathbb{C}^{n+m})$, where $(p\times q)(x,y):=(p(x),q(y))$.

(e) Let $(X,p)\in\mathfrak{R}_c(\mathbb{C}^n)$ and let Y be an open univalent subset such that $p(Y)=p(X)$. Then $Y=X$.

(f) Every Riemann domain is metrizable.

(g) $\mathfrak{R}_c(\mathbb{C}^n)\subset\mathfrak{R}_\infty(\mathbb{C}^n)$. Consequently, a Riemann region is countable at infinity iff it has an at most countable number of connected components.

Let $(X,p)\in\mathfrak{R}(\mathbb{C}^n)$. For $a\in X$ and $0<r\le+\infty$, we introduce on X the notion of a *polydisc centered at a of radius r* as an open univalent neighborhood $\widehat{\mathbb{P}}(a,r)=\widehat{\mathbb{P}}_X(a,r)$ of a such that $p(\widehat{\mathbb{P}}_X(a,r))=\mathbb{P}_n(p(a),r)$, where $\mathbb{P}_n(p(a),+\infty):=\mathbb{C}^n$. Notice that $\widehat{\mathbb{P}}_X(a,r)$ exists for small $r>0$. We define:

- the *distance to the boundary* $d_X: X\longrightarrow(0,+\infty]$:

$$d_X(a):=\sup\{r\in(0,+\infty]:\widehat{\mathbb{P}}_X(a,r)\text{ exists}\},\quad a\in X;$$

- the *maximal polydisc* centered at a point $a\in X$: $\widehat{\mathbb{P}}_X(a)=\widehat{\mathbb{P}}_X(a,d_X(a))$;
- $p_a:=p|_{\widehat{\mathbb{P}}_X(a)}$,
- $d_X(A):=\inf\{d_X(a):a\in A\}$, $A\subset X$;
- $A^{(r)}:=\overline{\bigcup_{x\in A}\widehat{\mathbb{P}}_X(x,r)}$, $0<r<d_X(A)$;
- $X_\infty:=\{a\in X: d_X(a)=+\infty\}$.

Remark 3.1.3. (a) The set X_∞ is the union of all connected components $Y\subset X$ such that $p|_Y:Y\longrightarrow\mathbb{C}^n$ is homeomorphic (cf. Remark 3.1.2(e)). Moreover,

$$|d_X(x)-d_X(a)|\le\|p(x)-p(a)\|_\infty,\quad a\in X\setminus X_\infty,\ x\in\widehat{\mathbb{P}}_X(a).$$

In particular, the function d_X is continuous.

(b) If $K\subset X$ is compact, then set $K^{(r)}$ is compact for any $0<r<d_X(K)$.

(c) If K is compact and univalent, then $K^{(r)}$ is univalent for small $r>0$.

For $z, \xi \in \mathbb{C}^n$ and $0 < r \le +\infty$ let $\Delta_\xi(z,r) := z + K(r)\xi$, where $\Delta_\xi(z,+\infty) := z + \mathbb{C}\xi$. For a point $a \in X$, $0 < r \le +\infty$, and $\xi \in \mathbb{C}^n$, we introduce on X the notion of a *disc in direction* ξ *centered at* a *of radius* r as a univalent set $\widehat{\Delta}_\xi(a,r)$ containing a such that $p(\widehat{\Delta}_\xi(a,r)) = \Delta_\xi(p(a),r)$. Observe that $\widehat{\Delta}_\xi(a,r)$ exists for small $r > 0$. Note that $\widehat{\Delta}_0(a,r) = \{a\}$ for every $r > 0$. We define the *distance to the boundary in direction* ξ:

$$\delta_{X,\xi} : X \longrightarrow (0,+\infty], \quad \delta_{X,\xi}(a) := \sup\{r > 0 : \widehat{\Delta}_\xi(a,r) \text{ exists}\}, \quad a \in X.$$

Remark 3.1.4. (a) The function

$$X \times \mathbb{C}^n \ni (x,\xi) \longrightarrow \delta_{X,\xi}(x) \in (0,+\infty]$$

is lower semicontinuous.

(b) The polydisc $\widehat{\mathbb{P}}_X(a,r)$ exists iff the disc $\widehat{\Delta}_\xi(a,r)$ exists for any ξ with $\|\xi\|_\infty = 1$. Moreover,

$$\widehat{\mathbb{P}}_X(a,r) = \bigcup_{\substack{\xi\in\mathbb{C}^n \\ \|\xi\|_\infty = 1}} \widehat{\Delta}_\xi(a,r), \quad d_X = \inf\{\delta_{X,\xi} : \xi \in \mathbb{C}^n,\ \|\xi\|_\infty = 1\}.$$

For $f : X \longrightarrow \mathbb{C}$ and $a \in X$, we define the *formal derivatives of* f *at* a

$$\frac{\partial f}{\partial z_j}(a) := \frac{\partial(f\circ p_a^{-1})}{\partial z_j}(p(a)), \quad \frac{\partial f}{\partial \overline{z}_j}(a) := \frac{\partial(f\circ p_a^{-1})}{\partial \overline{z}_j}(p(a)), \quad j = 1,\dots,n,$$

provided that the right hand sides exist, where $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$ on the right hand side are taken in the classical sense. If f is of class $\mathcal{C}^k$ in an open neighborhood of a and $\alpha, \beta \in \mathbb{Z}_+^n$ are such that $|\alpha| + |\beta| \le k$, then we may define the derivatives

$$D^{\alpha,\beta} f(a) := \Big(\frac{\partial}{\partial z_1}\Big)^{\alpha_1} \circ \dots \circ \Big(\frac{\partial}{\partial z_n}\Big)^{\alpha_n} \circ \Big(\frac{\partial}{\partial \overline{z}_1}\Big)^{\beta_1} \circ \dots \circ \Big(\frac{\partial}{\partial \overline{z}_n}\Big)^{\beta_n} f(a),$$
$$D^\alpha f(a) := D^{\alpha,0} f(a).$$

3.1.2 Holomorphic functions on Riemann regions

See [Jar-Pfl 2000], § 1.1.

Definition 3.1.5. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$. A function $f : X \longrightarrow \mathbb{C}$ is said to be *holomorphic* ($f \in \mathcal{O}(X)$) if for each open univalent subset $U \subset X$ the function $f \circ (p|_U)^{-1}$ is holomorphic in the standard sense on the open set $p(U) \subset \mathbb{C}^n$.

If $(Y,q) \in \mathfrak{R}(\mathbb{C}^m)$, then a continuous mapping $F : X \longrightarrow Y$ is said to be *holomorphic* ($F \in \mathcal{O}(X,Y)$) if $q \circ F \in \mathcal{O}(X,\mathbb{C}^m)$.

For $f \in \mathcal{O}(X)$ and $a \in X$, we define the *Taylor series* of f at a:

$$T_a f(z) := \sum_{\alpha \in \mathbb{Z}_+^n} \frac{D^\alpha f(a)}{\nu!}(z - p(a))^\alpha = T_{p(a)}(f \circ p_a^{-1})(z), \quad z \in \mathbb{C}^n,$$

and its *radius of convergence* $T_a f$,

$$d(T_a f) := \sup\{r > 0 : T_a f(z) \text{ is convergent for } z \in \mathbb{P}(p(a), r)\}.$$

Notice that $d(T_a f) \geq d_X(a)$ and $f(x) = T_a f(p(x))$ for $x \in \widehat{\mathbb{P}}_X(a)$. Moreover,

$$\frac{1}{d(T_a f)} = \limsup_{k \to +\infty} \Big(\max_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| = k}} \frac{1}{\alpha!} |D^\alpha f(a)| \Big)^{1/k}.$$

Proposition 3.1.6 (Identity principle). *Let* $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$, $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, F, $G \in \mathcal{O}(X, Y)$, *and assume that* $\operatorname{int}\{x \in X : F(x) = G(x)\} \neq \varnothing$. *Then* $F \equiv G$ *on* X.

3.1.3 Lebesgue measure on Riemann regions

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. A set $A \subset X$ is called *(Lebesgue) measurable* if for any open univalent set $U \subset X$ the set $p(A \cap U)$ is Lebesgue measurable in $\mathbb{C}^n$ (in the classical sense). Then:

- any Borel subset of X is measurable,
- a set $A \subset X$ is measurable iff any point $a \in X$ has an open univalent neighborhood U such that $p(A \cap U)$ is Lebesgue measurable in the classical sense.

Since X is countable at infinity we may write $X = \bigcup_{j=1}^\infty U_j$, where each U_j is open and univalent. Put $B_1 := U_1$, $B_j := U_j \setminus (U_1 \cup \cdots \cup U_{j-1})$, $j \in \mathbb{N}_2$. For any measurable set $A \subset X$ put

$$\mathcal{L}^X(A) := \sum_{j=1}^\infty \mathcal{L}^{2n}(p(A \cap B_j)),$$

where $\mathcal{L}^{2n}$ denotes the standard Lebesgue measure in $\mathbb{C}^n$. One can prove that $\mathcal{L}^X$ is a regular measure which is independent of the choice of the sequence $(U_j)_{j=1}^\infty$. It is called the *Lebesgue measure on* X. If $f : A \longrightarrow [0, +\infty]$ is a measurable function, then

$$\int_A f d\mathcal{L}^X = \sum_{j=1}^\infty \int_{p(A \cap B_j)} f \circ (p|_{U_j})^{-1} d\mathcal{L}^{2n}.$$

3.1.4 Sheaf of I–germs of holomorphic functions

See [Jar-Pfl 2000], Example 1.6.6.

Let I be an arbitrary non-empty set of indices. For $a \in \mathbb{C}^n$ define

$$\widetilde{\mathcal{O}}_a^I := \{(U, \boldsymbol{f}) : U \text{ is an open neighborhood of } a,\ \boldsymbol{f} = (f_i)_{i \in I} \subset \mathcal{O}(U)\},$$

For $(U, \boldsymbol{f})$, $(V, \boldsymbol{g}) \in \widetilde{\mathcal{O}}_a^I$ we define an equivalence relation

$$(U, \boldsymbol{f}) \overset{a}{\simeq} (V, \boldsymbol{g}) :\Longleftrightarrow \exists_{W-\text{ neighborhood of } a} :\ W \subset U \cap V,\ f_i|_W = g_i|_W,\ i \in I.$$

Put

$$\mathcal{O}_a^I := \widetilde{\mathcal{O}}_a^I / \overset{a}{\simeq}\ .$$

The class $\widehat{\boldsymbol{f}}_a := [(U, \boldsymbol{f})]_{\overset{a}{\simeq}}$ is called the *I–germ* of $\boldsymbol{f}$ at a. Notice that the *value of $\widehat{\boldsymbol{f}}_a$ at a* understood as

$$\widehat{\boldsymbol{f}}_a(a) := (f_i(a))_{i \in I}$$

is well defined.

Let $\mathcal{R}_a^I$ be the ring of all families $(S_i)_{i \in I}$ of power series centered at a that are convergent in a common (independent of $i \in I$) neighborhood of a, which may depend on the family $(S_i)_{i \in I}$ (i.e. $\inf\{d(S_i) : i \in I\} > 0$). Then the mapping

$$\mathcal{O}_a^I \ni \widehat{\boldsymbol{f}}_a \longrightarrow (T_a f_i)_{i \in I} \in \mathcal{R}_a^I \tag{3.1.1}$$

is an isomorphism. This gives an equivalent description of $\mathcal{O}_a^I$, which also introduces on $\mathcal{O}_a^I$ a structure of a commutative ring with the unit element — the *ring of I–germs of holomorphic functions at a*. Put

$$\mathcal{O}^I := \bigvee_{a \in \mathbb{C}^n} \mathcal{O}_a^I$$

and let $\pi^I : \mathcal{O}^I \longrightarrow \mathbb{C}^n$ be given by the formula $\pi^I(\widehat{\boldsymbol{f}}_a) := a$.

For $\widehat{\boldsymbol{f}}_a = [(U, \boldsymbol{f})]_{\overset{a}{\simeq}}$ put

$$\mathbb{V}(\widehat{\boldsymbol{f}}_a, U) := \{[(U, \boldsymbol{f})]_{\overset{b}{\simeq}} : b \in U\}.$$

One may easily check that:

- the system $\{\mathbb{V}(\widehat{\boldsymbol{f}}_a, U) : \widehat{\boldsymbol{f}}_a \in \mathcal{O}^I,\ (U, \boldsymbol{f}) \in \widehat{\boldsymbol{f}}_a\}$ is a neighborhood basis of a Hausdorff topology on $\mathcal{O}^I$ such that $\pi^I|_{\mathbb{V}(\widehat{\boldsymbol{f}}_a, U)} : \mathbb{V}(\widehat{\boldsymbol{f}}_a, U) \longrightarrow U$ is homeomorphic.

Thus $(\mathcal{O}^I, \pi^I) \in \mathfrak{R}(\mathbb{C}^n)$. It is called the *sheaf of I–germs of holomorphic functions in $\mathbb{C}^n$*. One can easily prove that

$$d_{\mathcal{O}^I}(\widehat{\boldsymbol{f}}_a) = \inf\{d(T_a f_i) : i \in I\}.$$

For $i_0 \in I$ define $\mathbb{F}_{i_0} : \mathcal{O}^I \longrightarrow \mathbb{C}$, $\mathbb{F}_{i_0}(\widehat{\boldsymbol{f}}_a) := f_{i_0}(a)$. Then

$$\mathbb{F}_{i_0} \circ (\pi^I|_{\mathbb{V}(\widehat{\boldsymbol{f}}_a, U)})^{-1} = f_{i_0} \text{ on } U.$$

This shows that $\mathbb{F}_{i_0} \in \mathcal{O}(\mathcal{O}^I)$.

3.1.5 Holomorphic extension of Riemann regions

See [Jar-Pfl 2000], § 1.4.

Definition 3.1.7. Let (X,p), $(Y,q) \in \mathfrak{R}(\mathbb{C}^n)$. A continuous mapping $\varphi : X \longrightarrow Y$ is said to be a *morphism* if $q \circ \varphi = p$.

If $\varphi : (X,p) \longrightarrow (Y,q)$ is a morphism such that φ is bijective and $\varphi^{-1} : Y \longrightarrow X$ is also a morphism, then we say that φ is an *isomorphism*.

Observe that if Ω_1, $\Omega_2 \subset \mathbb{C}^n$ are open and $\varphi : (\Omega_1, \mathrm{id}) \longrightarrow (\Omega_2, \mathrm{id})$ is a morphism, then $\Omega_1 \subset \Omega_2$ and φ is the inclusion operator.

Remark 3.1.8. Let $\varphi : (X,p) \longrightarrow (Y,q)$ be a morphism.

(a) If $\psi : (X,p) \longrightarrow (Y,q)$ is a morphism with $\varphi(a) = \psi(a)$ for some $a \in X$, then $\varphi = \psi$ on the connected component of X that contains a.

(b) φ is locally biholomorphic. In particular, φ is an open mapping.

(c) φ is an isomorphism iff φ is bijective.

(d) If $A \subset X$ is univalent, then $\varphi(A)$ is univalent. In particular:
- $\varphi(\widehat{\mathbb{P}}_X(a,r)) = \widehat{\mathbb{P}}_Y(\varphi(a),r)$, $a \in X$, $0 < r \le d_X(a)$,
- $d_Y \circ \varphi \ge d_X$,
- if φ is an isomorphism, then $d_Y \circ \varphi = d_X$.

(e) If every connected component of Y intersects $\varphi(X)$ and $d_Y \circ \varphi = d_X$, then $\varphi(X) = Y$.

(f) The mapping
$$\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X), \quad \varphi^*(g) := g \circ \varphi,$$
is injective iff every connected component of Y intersects $\varphi(X)$.

(g) $T_{\varphi(a)}g = T_a(g \circ \varphi)$, $g \in \mathcal{O}(Y)$, $a \in X$. In particular, $d(T_a f) \ge d_Y(\varphi(a))$ for any $a \in X$ and $f \in \varphi^*(\mathcal{O}(Y))$.

Definition 3.1.9. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \ne \mathcal{F} \subset \mathcal{O}(X)$. We say that a morphism $\varphi : (X,p) \longrightarrow (Y,q)$ is an *$\mathcal{F}$–extension* if φ^* is injective and $\mathcal{F} \subset \varphi^*(\mathcal{O}(Y))$, i.e. for each $f \in \mathcal{F}$ there exists exactly one $g =: f^\varphi \in \mathcal{O}(Y)$ such that $g \circ \varphi = f$. Put

$$\mathcal{F}^\varphi := \{f^\varphi : f \in \mathcal{F}\} = \{g \in \mathcal{O}(Y) : g \circ \varphi \in \mathcal{F}\}.$$

Notice that if X is connected, then Y must be connected.

If $\mathcal{F} = \mathcal{O}(X)$, then we say that $\varphi : (X,p) \longrightarrow (Y,q)$ is a *holomorphic extension*.

Remark 3.1.10. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. We define a morphism (cf. § 3.1.4 with $I := \mathcal{F}$):

$$\varphi = \varphi_{\mathcal{F}} : (X,p) \longrightarrow (\mathcal{O}^{(\mathcal{F})}, \pi^{(\mathcal{F})}),$$
$$\varphi(x) := [(\mathbb{P}_n(p(x), d_X(x)), (f \circ p_x^{-1})_{f\in\mathcal{F}})]_{\underset{\simeq}{p(x)}}, \quad x \in X.$$

After the identification (3.1.1), the mapping φ may be written as

$$\varphi(x) := (T_x f)_{f\in\mathcal{F}}, \quad x \in X.$$

Then φ is a morphism and $\mathbb{F}_f \circ \varphi = f$ for any $f \in \mathcal{F}$. Consequently, if $\widehat{X}$ denotes the union of those connected component of $\mathcal{O}^{(\mathcal{F})}$ that intersect $\varphi(X)$ and $\widehat{p} := \pi^{(\mathcal{F})}|_{\widehat{X}}$, then

$$\varphi : (X,p) \longrightarrow (\widehat{X}, \widehat{p})$$

is an $\mathcal{F}$–extension.

Remark 3.1.11. Let $\varphi : (X,p) \longrightarrow (Y,q)$ be a holomorphic extension. Then $f(X) = f^{\varphi}(Y)$ for every $f \in \mathcal{O}(X)$.

3.1.6 Regions of existence

See [Jar-Pfl 2000], § 1.7.

Definition 3.1.12. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. We say that (X,p) is an *$\mathcal{F}$–region of existence* if

$$d_X(a) = \inf\{d(T_a f) : f \in \mathcal{F}\}, \quad a \in X;$$

equivalently, for any $r > d_X(a)$ there exists an $f \in \mathcal{F}$ such that $d(T_a f) < r$.

If $\mathcal{F} = \{f\}$, then we say that (X,p) is a *region of existence of f*.

If $\mathcal{F} = \mathcal{O}(X)$, then we say that (X,p) is a *region of existence*.

If X is connected, then we say that (X,p) is an *$\mathcal{F}$–domain of existence*, *domain of existence of f*, and *domain of existence*, respectively.

Remark 3.1.13. (a) (X,p) is an $\mathcal{F}$–region of existence iff for any $\mathcal{F}$–extension

$$\varphi : (X,p) \longrightarrow (Y,q)$$

we have $d_Y \circ \varphi \equiv d_X$ (i.e. φ is surjective — cf. Remark 3.1.8(e)).

(b) If $(X,p) = (\Omega, \mathrm{id})$, where Ω is an open set in $\mathbb{C}^n$, then (Ω, id) is an $\mathcal{F}$–region of existence iff there are no domains Ω_0, $\widetilde{\Omega} \subset \mathbb{C}^n$ with $\varnothing \neq \Omega_0 \subset \Omega \cap \widetilde{\Omega}$, $\widetilde{\Omega} \not\subset \Omega$, such that for each $f \in \mathcal{F}$ there exists an $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ with $\widetilde{f} = f$ on Ω_0.

(c) (X,p) is an $\mathcal{F}$–region of existence iff there exists a dense subset $A \subset X$ such that $d_X(a) = \inf\{d(T_a f) : f \in \mathcal{F}\}$, $a \in A$.

3.1.7 Maximal holomorphic extensions

See [Jar-Pfl 2000], § 1.8.

Definition 3.1.14. An $\mathcal{F}$–extension $\varphi : (X, p) \longrightarrow (\widehat{X}, \widehat{p})$ is called *maximal* if for any $\mathcal{F}$–extension $\psi : (X, p) \longrightarrow (Y, q)$ there exists a morphism $\sigma : (Y, q) \longrightarrow (\widehat{X}, \widehat{p})$ such that $\sigma \circ \psi = \varphi$. The maximal $\mathcal{F}$–extension is uniquely determined up to an isomorphism. In the above situation we say that $\varphi : (X, p) \longrightarrow (\widehat{X}, \widehat{p})$ is the *$\mathcal{F}$–envelope of holomorphy of* (X, p). If $\mathcal{F} = \mathcal{O}(X)$, then we simply say that $\varphi : (X, p) \longrightarrow (\widehat{X}, \widehat{p})$ is the *envelope of holomorphy of* (X, p).

We say that (X, p) is an *$\mathcal{F}$–region of holomorphy* if for every $\mathcal{F}$–extension

$$\varphi : (X, p) \longrightarrow (Y, q)$$

the mapping φ is an isomorphism.

If (X, p) is an $\mathcal{O}(X)$–region of holomorphy, then we say that (X, p) is a *region of holomorphy.* If X is connected, then we say that (X, p) is an *$\mathcal{F}$–domain of holomorphy* and *domain of holomorphy*, respectively.

Remark 3.1.15. If $\varphi : (X, p) \longrightarrow (\widehat{X}, \widehat{p})$ is the maximal $\mathcal{F}$–extension, then $(\widehat{X}, \widehat{p})$ is an $\mathcal{F}^{\varphi}$–region of holomorphy.

Theorem 3.1.16 (Thullen theorem (2)). *Let (X, p) be a Riemann region over $\mathbb{C}^n$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. Then (X, p) has an $\mathcal{F}$–envelope of holomorphy.*

Proof. It suffices to prove that the $\mathcal{F}$–extension

$$\varphi_{\mathcal{F}} : (X, p) \longrightarrow (\widehat{X}, \widehat{p}),$$

constructed in Remark 3.1.10, is maximal. Let $\psi : (X, p) \longrightarrow (Y, q)$ be another $\mathcal{F}$–extension. By the same method as in Remark 3.1.10 we construct a morphism

$$\varphi_{\mathcal{F}^{\psi}} : (Y, q) \longrightarrow (\mathcal{O}^{(\mathcal{F}^{\psi})}, \pi^{(\mathcal{F}^{\psi})}).$$

Observe that $(\mathcal{O}^{(\mathcal{F}^{\psi})}, \pi^{(\mathcal{F}^{\psi})}) \simeq (\mathcal{O}^{(\mathcal{F})}, \pi^{(\mathcal{F})})$. Moreover, $\varphi_{\mathcal{F}^{\psi}} \circ \psi = \varphi_{\mathcal{F}}$. Consequently, $\varphi_{\mathcal{F}^{\psi}}(Y) \subset \widehat{X}$ (up to an isomorphism). $\square$

Definition 3.1.17. We say that $\mathcal{F}$ *separates points* in X if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exist $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$.

We say that $\mathcal{F}$ *weakly separates points* in X if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $p(x_1) = p(x_2)$ there exist $f \in \mathcal{F}$ and $\alpha \in \mathbb{Z}_+^n$ such that $D^\alpha f(x_1) \neq D^\alpha f(x_2)$, i.e. the exists an $f \in \mathcal{F}$ such that $T_{x_1} f \neq T_{x_2} f$.

We say that $\mathcal{F}$ is *d–stable* if: $f \in \mathcal{F} \Longrightarrow D^\alpha f \in \mathcal{F}$, $\alpha \in \mathbb{Z}_+^n$.

Observe that in $\mathcal{F}$ is d–stable and $p \in \mathcal{F}^n$, then $\mathcal{F}$ separates points in X iff $\mathcal{F}$ weakly separates points in X.

If (X, p) is univalent, then every family $\mathcal{F}$ weakly separates points in X.

(2) Peter Thullen (1907–1996) — German mathematician.

Remark 3.1.18. The morphism $\varphi_{\mathcal{F}}$ is injective iff $\mathcal{F}$ weakly separates points in X. Recall that $d_{\widehat{X}}(\varphi_{\mathcal{F}}(x)) = \inf\{d(T_x f) : f \in \mathcal{F}\}$, $x \in X$.

Proposition 3.1.19. *Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$, $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. Then the following conditions are equivalent:*

(i) *(X,p) is an $\mathcal{F}$–region of holomorphy;*

(ii) *$\mathcal{F}$ weakly separates points in X and (X,p) is an $\mathcal{F}$–region of existence;*

(iii) *there exists a dense subset $A \subset X$ with $A = p^{-1}(p(A))$ such that:*

- *for any x', $x'' \in A$ with $x' \neq x''$ and $p(x') = p(x'')$ there exists an $f \in \mathcal{F}$ such that $T_{x'}f \neq T_{x''}f$,*
- *$d_X(x) = \inf\{d(T_x f) : f \in \mathcal{F}\}$, $x \in A$.*

Proposition 3.1.20. *Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) *(X,p) is a region of holomorphy;*

(ii) *$\mathfrak{N}(\mathcal{O}(X)) := \{f \in \mathcal{O}(X) : (X,p)$ is an $\{f\}$–domain of existence$\} \neq \varnothing$;*

(iii) *$\mathfrak{N}(\mathcal{O}(X))$ is of the second Baire category in $\mathcal{O}(X)$.*

Remark 3.1.21. The above result remains true if we substitute $\mathcal{O}(X)$ by a *natural Fréchet space* (3) $\mathcal{F}$, i.e. a vector space $\mathcal{F} \subset \mathcal{O}(X)$ endowed with a structure of a Fréchet space such that if $f_k \longrightarrow f$ in $\mathcal{F}$, then $f_k \longrightarrow f$ locally uniformly in X. For example: $\mathcal{F} = \mathcal{H}^\infty(X) :=$ the space of all bounded holomorphic functions on X with the topology of uniform convergence.

3.1.8 Singular sets

See [Jar-Pfl 2000], § 3.4.

Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$, let M be a closed subset of X satisfying the following condition

$$\text{for any domain } D \subset X \text{ the set } D \setminus M \text{ is connected and dense in } D, \qquad (3.1.2)$$

and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X \setminus M)$.

Notice that:

- $\operatorname{int} M = \varnothing$;
- every pluripolar set (cf. Definition 3.3.18) satisfies (3.1.2);
- consequently, every thin set (cf. Definition 3.1.26) satisfies (3.1.2);
- in particular, every analytic set of dimension $\leq n-1$ satisfies (3.1.2).

Definition 3.1.22. We say that a point $a \in M$ is *non-singular with respect to $\mathcal{F}$* ($a \in M_{ns,\mathcal{F}}$) if there exists an open neighborhood U of a such that for each $f \in \mathcal{F}$ there exists a function $\widetilde{f} \in \mathcal{O}(U)$ with $\widetilde{f} = f$ on $U \setminus M$.

If $a \in M_{s,\mathcal{F}} := M \setminus M_{ns,\mathcal{F}}$, then we say that a is *singular with respect to $\mathcal{F}$*. If $M_{ns,\mathcal{F}} = \varnothing$, i.e. $M_{s,\mathcal{F}} = M$, then we say that M *is singular with respect to $\mathcal{F}$*. If $\mathcal{F} = \mathcal{O}(X \setminus M)$, then we simply say that M *is singular* and we skip the index $_{\mathcal{F}}$.

(3) René Fréchet (1878–1973) — French mathematician.

Remark 3.1.23. Notice the difference between the notion of “the singular analytic subset M of X” and “the singular points $\operatorname{Sing}(M)$ of an analytic subset M of X”. Recall that if M is an analytic subset of X, then a point $a \in M$ is called *regular* ($a \in \operatorname{Reg}(M)$) if there exists an open neighborhood U of a such that $M \cap U$ is a complex manifold. If $a \in \operatorname{Sing}(M) := M \setminus \operatorname{Reg}(M)$, then we say that a is *singular* — cf. [Chi 1989], § 2.3.

Remark 3.1.24. (a) The set $M_{s,\mathcal{F}}$ is closed in M and satisfies (3.1.2).

(b) Each function $f \in \mathcal{F}$ has a holomorphic extension $\widetilde{f} \in \mathcal{O}(X \setminus M_{s,\mathcal{F}})$.

(c) $M_{s,\mathcal{F}} = (M_{s,\mathcal{F}})_{s,\widetilde{\mathcal{F}}}$, where $\widetilde{\mathcal{F}} := \{\widetilde{f} : f \in \mathcal{F}\}$, i.e. $M_{s,\mathcal{F}}$ is singular with respect to $\widetilde{\mathcal{F}}$.

(d) $M_{s,\mathcal{F}} \cap U = (M \cap U)_{s,\mathcal{F}|_{U\setminus M}}$ for every open set $U \subset X$.

(e) If M is an analytic subset of X, then $\{a \in M : \dim_a M \le n-2\} \subset M_{ns}$ (cf. [Chi 1989], Appendix I). In other words, if $M \neq \varnothing$ is singular, then M is of pure codimension one.

Proposition 3.1.25. *Let $M \subset X$ be an analytic subset of pure dimension $(n-1)$, and let $M = \bigcup_{i\in I} M_i$ be the decomposition of M into irreducible components (cf.* [Chi 1989]*, Section 5.4). Then $M_{s,\mathcal{F}} = \bigcup_{i: M_i \subset M_{s,\mathcal{F}}} M_i$. In particular, the set $M_{s,\mathcal{F}}$ is also analytic.*

Definition 3.1.26. A set $M \subset X$ is *thin in* X if for any $a \in X$ there exist a connected neighborhood $U \subset X$ of a and a holomorphic function $\varphi \in \mathcal{O}(U)$, $\varphi \not\equiv 0$, such that $P \cap U \subset \varphi^{-1}(0)$. Note that every thin set is pluripolar (cf. Definition 3.3.18).

Proposition 3.1.27. *If M is a closed thin set, then $M_{s,\mathcal{F}}$ is analytic.*

3.2 Holomorphic convexity

See [Jar-Pfl 2000], § 1.10.

Definition 3.2.1. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$. For a compact set $K \Subset X$ put

$$\widehat{K}^{\mathcal{O}(X)} := \{x \in X : \forall_{f \in \mathcal{O}(X)} :\ |f(x)| \le \|f\|_K\}.$$

The set $\widehat{K}^{\mathcal{O}(X)}$ is called the *holomorphic hull* of K. We say that K is *holomorphically convex* if $K = \widehat{K}^{\mathcal{O}(X)}$. We say that (X,p) is *holomorphically convex* if $\widehat{K}^{\mathcal{O}(X)}$ is compact for every compact $K \Subset X$.

Proposition 3.2.2. *Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then X is holomorphically convex iff there exists a sequence $(K_j)_{j=1}^\infty$ of holomorphically convex compact sets such that $K_j \subset \operatorname{int} K_{j+1}$, $j \in \mathbb{N}$, and $X = \bigcup_{j=1}^\infty K_j$.*

Theorem 3.2.3 (Cartan–Thullen theorem [4]). *Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) (X,p) *is a region of holomorphy;*

(ii) $\mathcal{O}(X)$ *separates points in* X *and* $d_X(\widehat{K}^{\mathcal{O}(X)}) = d_X(K)$ *for every compact* $K \Subset X$*;*

(iii) $\mathcal{O}(X)$ *separates points in* X *and* $d_X(\widehat{K}^{\mathcal{O}(X)}) > 0$ *for every compact* $K \Subset X$*;*

(iv) $\mathcal{O}(X)$ *separates points in* X *and for any set* $A \subset X$ *with* $d_X(A) = 0$ *there exists an* $f \in \mathcal{O}(X)$ *such that* $\sup_A |f| = +\infty$*;*

(v) $\mathcal{O}(X)$ *separates points in* X *and* X *is holomorphically convex;*

(vi) $\mathcal{O}(X)$ *separates points in* X *and for any infinite set* $A \subset X$ *with no limit points in* X *there exists an* $f \in \mathcal{F}$ *such that* $\sup_A |f| = +\infty$.

Notice that in fact, if X is holomorphically convex, then $\mathcal{O}(X)$ separates points in X — cf. Theorem 3.5.9.

Definition 3.2.4. Any $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ satisfying (vi) is called a *Riemann–Stein region over* $\mathbb{C}^n$.

3.3 Plurisubharmonic functions

See [Jar-Pfl 2000], § 2.1.

Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ (notice that in fact the majority of results remains true for arbitrary $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$).

Definition 3.3.1. For $u : X \longrightarrow \mathbb{R}_{-\infty} := [-\infty, +\infty)$, $a \in X$, and $\xi \in \mathbb{C}^n$, we put

$$\lambda \overset{u_{a,\xi}}{\longmapsto} (u \circ p_a^{-1})(p(a) + \lambda\xi).$$

A function $u : X \longrightarrow \mathbb{R}_{-\infty}$ is called *plurisubharmonic (psh)* in X ($u \in \mathcal{PSH}(X)$) if:

- u is upper semicontinuous on X,
- for every $a \in X$ and $\xi \in \mathbb{C}^n$ the function $u_{a,\xi}$ is subharmonic in a neighborhood of zero (as a function of one complex variable).

Notice that the above definition has a local character. Consequently, whenever we are interested in local properties of psh functions, we may assume that $(X,p) = (D, \mathrm{id})$, where D is a domain in $\mathbb{C}^n$.

We say that a function $u : X \longrightarrow \mathbb{R}_+$ is *logarithmically plurisubharmonic (log–psh)* if $\log u \in \mathcal{PSH}(X)$.

For $I \subset \mathbb{R}_{-\infty}$ we put $\mathcal{PSH}(X, I) := \{u \in \mathcal{PSH}(X) : u(X) \subset I\}$.

Remark 3.3.2. (a) For an upper semicontinuous function $u : X \longrightarrow \mathbb{R}_{-\infty}$ the following conditions are equivalent:

[4] Henri Cartan (1904–2008) — French mathematician.

(i) $u \in \mathcal{PSH}(X)$;

(ii) $\forall_{a\in X}\ \forall_{\xi\in\mathbb{C}^n:\|\xi\|_\infty=1}\ \exists_{0<R\le d_X(a)}$:

$$u(a) = u_{a,\xi}(0) \le \frac{1}{2\pi}\int_0^{2\pi} u_{a,\xi}(re^{i\theta})d\theta, \quad 0<r<R;$$

(iii) $\forall_{a\in X}\ \forall_{\xi\in\mathbb{C}^n:\|\xi\|_\infty=1}\ \exists_{0<R\le d_X(a)}$:

$$u(a) \le \frac{1}{\pi r^2}\int_{K(r)} u_{a,\xi}(\zeta)d\mathcal{L}^2(\zeta), \quad 0<r<R;$$

(iv) $\forall_{a\in X}\ \forall_{\xi\in\mathbb{C}^n:\|\xi\|_\infty=1}\ \exists_{0<R\le d_X(a)}\ \forall_{0<r<R}\ \forall_{f\in\mathcal{P}(\mathbb{C})}$: if $u_{a,\xi} \le \operatorname{Re} f$ on $\partial K(r)$, then $u(a) \le \operatorname{Re} f(0)$ (where $\mathcal{P}(\mathbb{C})$ stands for the space of all complex polynomials of one complex variable);

(v) $\forall_{a\in X}\ \forall_{\xi\in\mathbb{C}^n:\|\xi\|_\infty=1}\ \exists_{0<R\le d_X(a)}\ \forall_{0<r<R}\ \forall_{h\in\mathcal{H}(K(r))\cap\mathcal{C}(\overline{K}(r))}$: if $u_{a,\xi} \le h$ on $\partial K(r)$, then $u(a) \le h(0)$ (where $\mathcal{H}(\Omega)$ stands for the space of all functions harmonic in Ω);

(vi) for any $a \in X$ and $\xi \in \mathbb{C}^n$ the function

$$K(\delta_{X,\xi}(a)) \ni \lambda \longmapsto (u \circ (p|_{\widehat{\Delta}(a,\xi)})^{-1})(p(a)+\lambda\xi)$$

is subharmonic;

(vii) $u \circ (p|_U)^{-1} \in \mathcal{PSH}(p(U))$ for any univalent open set $U \subset X$.

(b) $\mathcal{PSH}(X) + \mathcal{PSH}(X) = \mathcal{PSH}(X), \quad \mathbb{R}_{>0}\cdot\mathcal{PSH}(X) = \mathcal{PSH}(X)$.

(c) $|f|$ is log–psh on X for any $f \in \mathcal{O}(X)$.

(d) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ and $u_\nu \searrow u$ pointwise on X, then $u \in \mathcal{PSH}(X)$.

In particular, if $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X,[-\infty,0])$, then $\sum_{\nu=1}^\infty u_\nu \in \mathcal{PSH}(X)$.

(e) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ and $u_\nu \longrightarrow u$ locally uniformly in X, then $u \in \mathcal{PSH}(X)$.

(f) If $u_1,\dots,u_N \in \mathcal{PSH}(X)$, then $\max\{u_1,\dots,u_N\} \in \mathcal{PSH}(X)$ (cf. Proposition 3.3.11).

(g) (Liouville type theorem) If $u \in \mathcal{PSH}(\mathbb{C}^n)$ and $\sup_{\mathbb{C}^n} u < +\infty$, then $u \equiv \text{const}$.

(h) Let $I \subset \mathbb{R}$ be an open interval and let $\varphi : I \longrightarrow \mathbb{R}$ be convex and increasing. Then $\varphi \circ u \in \mathcal{PSH}(X)$ for every $u \in \mathcal{PSH}(X,I)$. Consequently:

If $u \in \mathcal{PSH}(X)$, then $e^u \in \mathcal{PSH}(X)$ (in particular, any log–psh function is psh).

If $u \in \mathcal{PSH}(X,\mathbb{R}_+)$, then $u^p \in \mathcal{PSH}(X)$ for every $p \ge 1$.

(i) If u_1, u_2 are log–psh, then $u_1 + u_2$ is log–psh.

(j) (Maximum principle) If X is connected, $u \in \mathcal{PSH}(X)$, and $u \le u(a)$ for some $a \in X$, then $u \equiv u(a)$. Consequently, if $Y \Subset X$ is a domain, $u \in \mathcal{PSH}(Y)$, and $u \not\equiv$ const, then

$$u(x) < \sup\{\limsup_{Y \ni y \to \zeta} u(y) : \zeta \in \partial Y\}, \quad x \in Y.$$

Let $\Omega \subset \mathbb{C}^n$ be open and let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. We define the *Levi form of u at a* ([5])

$$\mathcal{L}u(a;\xi) := \sum_{j,k=1} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a)\xi_j\overline{\xi}_k, \quad a \in \Omega,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n.$$

Observe that

$$\mathcal{L}u(a;\xi) = \frac{\partial^2 u_{a,\xi}}{\partial\lambda\partial\overline{\lambda}}(0).$$

Consequently, we have the following

Proposition 3.3.3. *Let $u \in \mathcal{C}^2(X, \mathbb{R})$. Then*

$$u \in \mathcal{PSH}(X) \iff \forall_{a\in\Omega,\ \xi\in\mathbb{C}^n} : \mathcal{L}u(a;\xi) \ge 0.$$

Remark 3.3.4. Let $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, $F \in \mathcal{O}(Y, X)$, $u \in \mathcal{C}^2(X, \mathbb{R})$. Then

$$\mathcal{L}(u \circ F)(b;\eta) = \mathcal{L}u(F(b); (p \circ F)'(b)(\eta)), \quad b \in Y,\ \eta \in \mathbb{C}^m.$$

Consequently, if $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^2(X, \mathbb{R})$, then $u \circ F \in \mathcal{PSH}(Y)$ — cf. Proposition 3.3.16.

Definition 3.3.5. We say that a function $u \in \mathcal{C}^2(X, \mathbb{R})$ is *strictly plurisubharmonic* if

$$\forall_{a\in\Omega,\ \xi\in(\mathbb{C}^n)_*} : \mathcal{L}u(a;\xi) > 0.$$

Proposition 3.3.6. *Let $Y \subset X$ be open, $v \in \mathcal{PSH}(Y)$, $u \in \mathcal{PSH}(X)$. Assume that*

$$\limsup_{Y \ni y \to \zeta} v(y) \le u(\zeta), \quad \zeta \in \partial Y.$$

Put

$$\widetilde{u}(x) := \begin{cases} \max\{v(x), u(x)\}, & x \in Y \\ u(x), & x \in X \setminus Y \end{cases}.$$

Then $\widetilde{u} \in \mathcal{PSH}(X)$.

([5]) Eugenio Elia Levi (1883–1917) — Italian mathematician.

To simplify notation we will use the following abbreviations:

$$e^z := (e^{z_1}, \dots, e^{z_n}), \quad z \cdot w := (z_1 w_1, \dots, z_n w_n),$$
$$z = (z_1, \dots, z_n),\ w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0} := (\mathbb{R}_{>0})^n$. If

$$\partial_0 \mathbb{P}(a, \boldsymbol{r}) \xrightarrow{u} \mathbb{R}_{-\infty}$$

is bounded from above and measurable, i.e. the function

$$[0, 2\pi)^n \ni \theta \longmapsto u(a + \boldsymbol{r} \cdot e^{i\theta})$$

is Lebesgue measurable, then we define

$$\boldsymbol{P}(u; a, \boldsymbol{r}; z) := \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \Big(\prod_{j=1}^n \frac{r_j^2 - |z_j - a_j|^2}{|r_j e^{i\theta_j} - (z_j - a_j)|^2} \Big) u(a + \boldsymbol{r} \cdot e^{i\theta}) d\mathcal{L}^n(\theta),$$
$$z = (z_1, \dots, z_n) \in \mathbb{P}(a, \boldsymbol{r}),$$

$$\boldsymbol{J}(u; a, \boldsymbol{r}) := \boldsymbol{P}(u; a; \boldsymbol{r}; a) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} u(a + \boldsymbol{r} \cdot e^{i\theta}) d\mathcal{L}^n(\theta).$$

If $u : \mathbb{P}(a, \boldsymbol{r}) \longrightarrow \mathbb{R}_{-\infty}$ is bounded from above and measurable, then we define

$$\boldsymbol{A}(u; a, \boldsymbol{r}) := \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{\mathbb{P}(a,\boldsymbol{r})} u \, d\mathcal{L}^{2n} = \frac{1}{\mathcal{L}^{2n}(\mathbb{P}(a, \boldsymbol{r}))} \int_{\mathbb{P}(a,\boldsymbol{r})} u \, d\mathcal{L}^{2n}.$$

Proposition 3.3.7. *Let $\Omega \subset \mathbb{C}^n$ be open, $u \in \mathcal{PSH}(\Omega)$, $a \in \Omega$. Then*

$$\boldsymbol{J}(u; a, \boldsymbol{r}) \searrow u(a), \quad \boldsymbol{A}(u; a, \boldsymbol{r}) \searrow u(a) \quad \textit{when } \boldsymbol{r} \searrow 0.$$

Proposition 3.3.8. *Let $u_1, u_2 \in \mathcal{PSH}(X)$. If $u_1 \le u_2$ almost everywhere in X, then $u_1 \le u_2$ everywhere.*

Proposition 3.3.9. *Let $\Omega \subset \mathbb{C}^n$ be open, $u \in \mathcal{PSH}(\Omega)$, $\mathbb{P}(a, \boldsymbol{r}) \Subset \Omega$, $\boldsymbol{r} \in \mathbb{R}^n_{>0}$. Then*

$$\begin{aligned} &u(z) \le \boldsymbol{P}(u; a, \boldsymbol{r}; z), \quad z \in \mathbb{P}(a, \boldsymbol{r}), \\ &u(a) \le \boldsymbol{J}(u; a, \boldsymbol{r}), \\ &u(a) \le \boldsymbol{A}(u; a, \boldsymbol{r}). \end{aligned}$$

Proposition 3.3.10. *If X is connected, $u \in \mathcal{PSH}(X)$, and $u \not\equiv -\infty$, then u is locally integrable; in particular, the set $u^{-1}(-\infty)$ is of zero measure.*

Proposition 3.3.11. *If a family $(u_i)_{i \in I} \subset \mathcal{PSH}(X)$ is locally bounded from above, then the function*

$$u := (\sup_{i \in I} u_i)^*$$

is psh in X.

Here v^* denotes the *upper regularization* of v, $v^*(x) := \limsup_{y\to x} v(y)$, $x \in X$.

Proposition 3.3.12. *If a sequence* $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ *is locally bounded from above, then the function*

$$u := (\limsup_{\nu\to\infty} u_\nu)^*$$

is psh on X.

Proposition 3.3.13 (Hartogs lemma). *Let* $(u_k)_{k=1}^\infty \subset \mathcal{PSH}(X)$ *be a sequence locally bounded from above. Assume that for some* $m \in \mathbb{R}$

$$\limsup_{k\to+\infty} u_k \le m.$$

Then for every compact subset $K \subset X$ *and for every* $\varepsilon > 0$*, there exists a* k_0 *such that*

$$\max_K u_k \le m + \varepsilon, \quad k \ge k_0.$$

Definition 3.3.14 (Regularization). Let

$$\Phi(z_1, \dots, z_n) := \Psi(z_1) \cdot \dots \cdot \Psi(z_n), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

where $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ is such that:

$$\operatorname{supp} \Psi = \overline{\mathbb{D}}, \quad \Psi(z) = \Psi(|z|),\ z \in \mathbb{C}, \quad \int \Psi d\mathcal{L}^2 = 1.$$

Put

$$\Phi_\varepsilon(z) := \frac{1}{\varepsilon^{2n}}\Phi(\frac{z}{\varepsilon}), \quad z \in \mathbb{C}^n,\ \varepsilon > 0.$$

Let

$$X_\varepsilon := \{x \in X : d_X(x) > \varepsilon\}, \quad \varepsilon > 0.$$

For every function $u \in L^1(X, \text{loc})$, define

$$\begin{aligned} u_\varepsilon(x) :&= \int_{\widehat{\mathbb{P}}_X(x)} u(y)\Phi_\varepsilon(p(x) - p(y))d\mathcal{L}^X(y) \\ &= \int_{\mathbb{D}^n} (u \circ p_x^{-1})(p(x) + \varepsilon w)\Phi(w)d\mathcal{L}^{2n}(w), \quad x \in X_\varepsilon. \end{aligned}$$

The function u_ε is called the *ε–regularization* of u. Observe that for $a \in X_\varepsilon$ and $x \in \widehat{\mathbb{P}}_X(a, d_X(a) - \varepsilon)$ we get

$$\begin{aligned} u_\varepsilon(x) &= \int_{\widehat{\mathbb{P}}_X(a)} u(y)\Phi_\varepsilon(p(x) - p(y))d\mathcal{L}^X(y) \\ &= \int_{\mathbb{D}^n} (u \circ p_a^{-1})(p(x) + \varepsilon w)\Phi(w)d\mathcal{L}^{2n}(w). \end{aligned} \tag{†}$$

Proposition 3.3.15. *If $u \in \mathcal{PSH}(X)$, $u \not\equiv -\infty$, then $u_\varepsilon \in \mathcal{PSH}(X_\varepsilon) \cap \mathcal{C}^\infty(X_\varepsilon)$ and $u_\varepsilon \searrow u$ pointwise in X when $\varepsilon \searrow 0$.*

Proposition 3.3.16. *Let $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, $F \in \mathcal{O}(Y, X)$. Then $u \circ F \in \mathcal{PSH}(Y)$ for any $u \in \mathcal{PSH}(X)$.*

Corollary 3.3.17. *Let $u : X \longrightarrow \mathbb{R}_{-\infty}$ be upper semicontinuous. Then u is psh on X iff for any analytic disc $\varphi : \mathbb{D} \longrightarrow X$ the function $u \circ \varphi$ is subharmonic in $\mathbb{D}$.*

Definition 3.3.18. A set $M \subset X$ is called *(locally) pluripolar* ($M \in \mathcal{PLP}$) if any point $a \in M$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a \subset v_a^{-1}(-\infty)$. For $A \subset X$ put

$$\mathcal{PLP}(A) := \{P \in \mathcal{PLP}(X) : P \subset A\}.$$

By Proposition 3.3.10, if M is pluripolar, then $\mathcal{L}^X(M) = 0$. It is clear that any thin set (cf. Definition 3.1.26) is pluripolar.

Proposition 3.3.19. (a) *Let $(u_i)_{i \in I} \subset \mathcal{PSH}(X)$ be locally bounded from above. Put $u := \sup_{i \in I} u_i$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is of zero measure.*

(b) *Let $(u_\nu)_{\nu \in \mathbb{N}} \subset \mathcal{PSH}(X)$ be a sequence locally bounded from above. Put $u := \limsup_{\nu \to +\infty} u_\nu$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is of zero measure.*

Notice that in fact the set $\{x \in X : u(x) < u^*(x)\}$ is pluripolar — cf. Theorem 3.3.29.

Theorem* 3.3.20 (Josefson theorem; cf. [Jos 1978]). *If $M \subset \mathbb{C}^n$ is pluripolar, then there exists a $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

Proposition 3.3.21. *Let $M_j \subset \mathbb{C}^n$ be pluripolar, $j \in \mathbb{N}$. Then $M := \bigcup_{j=1}^\infty M_j$ is pluripolar.*

Theorem 3.3.22. *Let $M \subset X$ be pluripolar. Then there exists a $v \in \mathcal{PSH}(X)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

Proof. We may assume that X is connected. Let $X = \bigcup_{k=1}^\infty U_k$ be an open covering by univalent sets (cf. Remark 3.1.2(g)). Then each set $A_k := p(M \cap U_k)$ is pluripolar, and consequently, by Proposition 3.3.21, the set $A := \bigcup_{k=1}^\infty A_k$ is pluripolar. Hence, by the Josefson theorem (Theorem 3.3.20), there exists a $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $u = -\infty$ on A. By Proposition 3.3.10, $u|_{p(X)} \not\equiv -\infty$. Now, we only need to put $v := u \circ p$. Then $v \in \mathcal{PSH}(X)$ (Proposition 3.3.16), $v \not\equiv -\infty$, and $v = -\infty$ on M. □

Remark 3.3.23. [Will be completed [El 1980]. . . . .]

Proposition 3.3.24. *Let $M_j \subset X$ be pluripolar, $j \in \mathbb{N}$. Then $M := \bigcup_{j=1}^\infty M_j$ is pluripolar.*

Proposition 3.3.25 (Removable singularities of psh functions). *Let M be a closed pluripolar subset of X.*

(a) *Let $u \in \mathcal{PSH}(X \setminus M)$ be locally bounded from above in X* ([6]). *Define*

$$\widetilde{u}(z) := \limsup_{X \setminus M \ni w \to z} u(w), \quad z \in X$$

(notice that $\widetilde{u}$ is well-defined). Then $\widetilde{u} \in \mathcal{PSH}(X)$.

(b) *For every function $u \in \mathcal{PSH}(X)$ we have*

$$u(z) = \limsup_{X \setminus M \ni w \to z} u(w), \quad z \in X.$$

(c) *The set $X \setminus M$ is connected.*

Corollary 3.3.26. *Let M be a closed pluripolar subset of X. Let $f \in \mathcal{O}(X \setminus M)$ be locally bounded in X. Then f extends holomorphically to X.*

Proposition 3.3.27. *Let $(Y, q) \in \mathfrak{R}_\infty(\mathbb{C}^m)$.*

(a) *If $A \subset X \times Y$ is pluripolar, then*

$$P := \{z \in X : A_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \in \mathcal{PLP}(X),$$

where

$$A_{(z,\cdot)} := \{w \in Y : (z, w) \in A\}.$$

(b) *If $A \subset X \times Y$ is thin, then*

$$P := \{z \in X : A_{(z,\cdot)} \textit{ is not thin in } Y\} \in \mathcal{PLP}(X).$$

(c) *Let $Q \subset X \times Y$ be such that $Q_{(a,\cdot)} \in \mathcal{PLP}(Y)$, $a \in X$. Let $C \subset X \times Y$ be such that*

$$\{z \in X : C_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \notin \mathcal{PLP}(X)$$

(e.g. $C = C' \times C'' \subset X \times Y$, where $C' \notin \mathcal{PLP}(X)$, $C'' \notin \mathcal{PLP}(Y)$). Then $C \setminus Q \notin \mathcal{PLP}(X \times Y)$.

Proof. We may assume that $X = D$ and $Y = G$ are domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively.

(a) Let $v \in \mathcal{PSH}(D \times G)$, $v \not\equiv -\infty$, be such that $A \subset v^{-1}(-\infty)$ (Theorem 3.3.20). Fix a compact $K \Subset G$ with $\operatorname{int} K \neq \varnothing$. Define

$$u(z) := \sup\{v(z, w) : w \in K\}, \quad z \in D.$$

Then $u \in \mathcal{PSH}(D)$ and $u \not\equiv -\infty$. If $z \in P$, then $A_{(z,\cdot)} \notin \mathcal{PLP}$. Hence, $v(z, \cdot) \equiv -\infty$, and consequently, $u(z) = -\infty$. Thus $A \subset u^{-1}(-\infty)$.

(b) Using the definition of a thin sets and Lindelöf theorem, we get

$$A \subset \bigcup_{k=1}^{\infty} \{(z, w) \in U_k \times V_k : \varphi_k(z, w) = 0\},$$

([6]) That is every point $a \in X$ has a neighborhood V_a such that u is bounded from above in $V_a \setminus M$.

where $U_k \times V_k \subset D \times G$ is connected, $\varphi_k \in \mathcal{O}(U_k \times V_k)$, and $\varphi_k \not\equiv 0$. Observe that for any k the set

$$P_k := \{z \in U_k : \varphi_k(z, \cdot) \equiv 0\} = \bigcap_{w \in V_k} \{z \in U_k : \varphi_k(z, w) = 0\}$$

is analytic in U_k. Hence the set $P_0 := \bigcup_{k=1}^{\infty} P_k$ is pluripolar. If $a \notin P_0$, then

$$A_{(a,\cdot)} \subset \bigcup_{k \in \mathbb{N}:\, a \in U_k} \{w \in V_k : \varphi_k(a, w) = 0\},$$

and consequently, the set $A_{(a,\cdot)}$ is thin.

(c) Suppose that $C \setminus Q$ is pluripolar. Then, by (a), there exists a pluripolar set $P \subset D$ such that the fiber $(C \setminus Q)_{(a,\cdot)}$ is pluripolar, $a \in D \setminus P$. Consequently, the fiber $C_{(a,\cdot)}$ is pluripolar, $a \in D \setminus P$; a contradiction. $\square$

Exercise 3.3.28. The set P in Proposition 3.3.27(b) need not be thin. Complete the following example. Let $X = Y := \mathbb{D}$,

$$A := (\{0\} \times \{|w| = 1/4\}) \cup \bigcup_{k \in \mathbb{N}_2} \{1/k\} \times \{|w| = 1 - 1/k\}.$$

Then $P = \{0\} \cup \{1/k : k \in \mathbb{N},\ k \geq 2\}$.

Theorem* 3.3.29 (Bedford–Taylor theorem; cf. [Kli 1991], Th. 4.7.6). (a) *Assume that a family $(u_i)_{i \in I} \subset \mathcal{PSH}(X)$ is locally bounded from above. Put $u := \sup_{i \in I} u_i$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is pluripolar.*

(b) *Assume that a sequence $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{PSH}(X)$ is locally bounded from above. Put $u := \limsup_{\nu \to +\infty} u_\nu$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is pluripolar.*

3.4 Relative extremal function

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$, $A \subset X$.

Definition 3.4.1. The *relative extremal function of A with respect to X* is defined as the upper semicontinuous regularization $h^*_{A,X}$ of the function

$$h_{A,X} := \sup\{u :\ u \in \mathcal{PSH}(X),\ u \leq 1,\ u|_A \leq 0\}.$$

For an open set $Y \subset X$ we put $h_{A,Y} := h_{A \cap Y, Y}$, $h^*_{A,Y} := h^*_{A \cap Y, Y}$.

Put $\omega_{A,X} := \lim_{k \to +\infty} h^*_{A,X_k}$, where $(X_k)_{k=1}^{\infty}$ is an exhaustion sequence for X (cf. Definition 2.2.5). The function $\omega_{A,X}$ is called the *generalized relative extremal function of A in X*. One can easily check that the definition is independent of the exhausting sequence $(X_k)_{k=1}^{\infty}$.

Proposition 3.4.2. (a) *If Y is a connected component of X, then $h_{A,X} = h_{A,Y}$ and $\omega_{A,X} = \omega_{A,Y}$ on Y.*

(b) $h^*_{A,X} \in \mathcal{PSH}(X)$ *(cf. Proposition 3.3.11), $\omega_{A,X} \in \mathcal{PSH}(X)$ (cf. Remark 3.3.2(d)). In particular, by the maximum principle (cf. Proposition 3.3.2(j)), if X is connected and $h^*_{A,X} \not\equiv 1$, then $h^*_{A,X}(z) < 1$, $z \in X$.*

(c) *If $Y_1 \subset Y_2 \subset X$ are open, $A_1 \subset Y_1$, and $A_1 \subset A_2 \subset Y_2$, then $h_{A_2,Y_2} \le h_{A_1,Y_1}$ (and so $h^*_{A_2,Y_2} \le h^*_{A_1,Y_1}$) on Y_1. In particular, $h^*_{A,X} \le \omega_{A,X}$.*

(d) *There exists a $P \in \mathcal{PLP}(A)$ such that $h^*_{A,X} = 0$ on $A \setminus P$ (cf. Theorem 3.3.29(a)) and hence, by* (b), *$h^*_{A,X} \le h_{A\setminus P,X}$. Consequently, by Proposition 3.3.24, there exists a $P \in \mathcal{PLP}(A)$ such that $\omega_{A,X} = 0$ on $A \setminus P$ and hence, by* (b,c), *$h^*_{A,X} \le \omega_{A,X} \le h_{A\setminus P,X} \le h^*_{A\setminus P,X}$.*

Proposition 3.4.3. *If $A \notin \mathcal{PLP}$, then $\omega_{A,\mathbb{C}^n} \equiv 0$ (and so $h^*_{A,\mathbb{C}^n} \equiv 0$).*

Proof. Let $u := \omega_{A,\mathbb{C}^n}$. Then $u \in \mathcal{PSH}(\mathbb{C}^n)$ and $u \le 1$. Thus $u \equiv$ const (cf. Remark 3.3.2(g)). Since $A \notin \mathcal{PLP}$, Proposition 3.4.2(d) implies that there exists an $a \in A$ such that $u(a) = 0$. $\square$

Definition 3.4.4. We say that a set $A \subset X$ is *pluriregular at a point* $a \in \overline{A}$ if $h^*_{A,U}(a) = 0$ for any open neighborhood U of a. Observe that A is pluriregular at a iff there exists a basis $\mathcal{U}(a)$ of neighborhoods of a such that $h^*_{A,U}(a) = 0$ for every $U \in \mathcal{U}(a)$. Define

$$A^* := \{a \in \overline{A} : A \text{ is pluriregular at } a\}.$$

We say that A is *locally pluriregular* if $A \neq \varnothing$ and A is pluriregular at every point $a \in A$, i.e. $\varnothing \neq A \subset A^*$. Observe that any non-empty open set is locally pluriregular.

Remark 3.4.5. $\omega_{A,X} = 0$ on A^*. Consequently:

(a) $h^*_{A,X} \le \omega_{A,X} \le h_{A^*,X} \le h^*_{A^*,X}$,

(b) if A is locally pluriregular, then $\omega_{A,X} = h^*_{A,X}$.

Proposition 3.4.6. $A \setminus A^* \in \mathcal{PLP}$.

Proof. We may assume that X is connected. Let $(U_k)_{k=1}^\infty$ be a basis of the topology of X. Put $P_k := \{z \in U_k : h_{A,U_k}(z) < h^*_{A,U_k}(z)\}$, $P := \bigcup_{k=1}^\infty P_k$. Then $P \in \mathcal{PLP}$ (cf. Theorem 3.3.29(a) and Proposition 3.3.24). If $a \in (A\setminus P)\cap U_k$, then $h^*_{A,U_k}(a) = h_{A,U_k}(a) = 0$, $k \in \mathbb{N}$. Thus $A \setminus A^* \subset P$. $\square$

Proposition 3.4.7 ([Ale-Hec 2004]). *Assume that X is connected. The following conditions are equivalent:*

(i) *for every $A \subset X$ we have $\omega_{A,X} \equiv h^*_{A,X}$;*

(ii) *for any $A \subset X$ and $P \in \mathcal{PLP}(X)$ we have $h^*_{A\cup P,X} \equiv h^*_{A,X}$;*

(iii) *for every $P \in \mathcal{PLP}(X)$ we have $h^*_{P,X} \equiv 1$;*

(iv) *for every $P \in \mathcal{PLP}(X)$ there exists a $v \in \mathcal{PSH}(X)$, $v \not\equiv -\infty$, $v \le 0$, such that $P \subset v^{-1}(-\infty)$;*

(v) *for every $A \subset X$ we have $h^*_{A,X} \equiv h^*_{A^*,X}$.*

Moreover:

- *by Theorem 3.3.22, condition* (iv) *(and, consequently, each other condition) is always satisfied if* (X, p) *is relatively compact (cf. Definition 3.1.1);*
- *conditions* (ii), (iii), (iv) *are also equivalent if we fix a pluripolar set* $P \subset X$.

Proof. (ii) $\Longrightarrow$ (iii): $h^*_{P,X} \equiv h^*_{\varnothing,X} \equiv 1$.

(iii) $\Longrightarrow$ (iv): By Proposition 3.3.19(a) there exists an $a \in X$ such that $h_{P,X}(a) = 1$. Take a sequence $(u_k)_{k=1}^{\infty} \subset \mathcal{PSH}(X)$ with $u_k \le 1$, $u_k \le 0$ on P, and $u_k(a) \ge 1 - 1/2^k$, $k \in \mathbb{N}$. Define $v := \sum_{k=1}^{\infty}(u_k - 1)$. Then $v \in \mathcal{PSH}(X)$, $v \le 0$, $P \subset v^{-1}(-\infty)$, and $v(a) \ge -1$.

(iv) $\Longrightarrow$ (ii): Let $u \in \mathcal{PSH}(X)$, $u \le 1$ on X, $u \le 0$ on A. Then, for every $\varepsilon > 0$, we get $u + \varepsilon v \le 1$ on X and $u + \varepsilon v \le 0$ on $A \cup P$. Thus $u + \varepsilon v \le h_{A\cup P,X}$ and hence $u + \varepsilon v \le h^*_{A\cup P,X}$. Thus $u \le h^*_{A\cup P,X}$ on $X \setminus v^{-1}(-\infty)$. Consequently, by Proposition 3.3.8, $u \le h^*_{A\cup P,X}$ and, finally, $h^*_{A,X} \le h^*_{A\cup P,X}$.

(ii) $\Longrightarrow$ (i): By Proposition 3.4.2(d) there exists a set $P \in \mathcal{PLP}(A)$ such that $h^*_{A,X} \le \omega_{A,X} \le h^*_{A\setminus P,X} \overset{\text{(ii)}}{=} h^*_{A,X}$.

(i) $\Longrightarrow$ (ii): Using the fact that (iv) is always satisfied for relatively compact open sets and the implication (iv) $\Longrightarrow$ (ii), for every exhaustion sequence $(X_k)_{k=1}^{\infty}$, we have

$$h^*_{A\cup P,X} = \omega_{A\cup P,X} = \lim_{k\to+\infty} h^*_{A\cup P,X_k} = \lim_{k\to+\infty} h^*_{A,X_k} = \omega_{A,X} = h^*_{A,X}.$$

(v) $\Longrightarrow$ (iii): $h^*_{P,X} = h^*_{P^*,X} = h^*_{\varnothing,X} \equiv 1$.

(ii) $\Longrightarrow$ (v): The inequality "$\le$" follows from Remark 3.4.5. By Proposition 3.4.6(b) and (ii) we get $h^*_{A,X} = h^*_{A\cap A^*,X} \ge h^*_{A^*,X}$. $\square$

Proposition 3.4.8. *Let* $G \subset \mathbb{C}^{n-k}$ *be an arbitrary domain and let* $A \subset \mathbb{C}^k \times G =: D$. *Then* $h_{A,D}(z,w) = h_{B,G}(w)$, $(z,w) \in D$, *where*

$$B := \operatorname{pr}_G(A) = \{w \in G : \exists_{z\in\mathbb{C}^k} :\ (z,w) \in A\}.$$

Proof. It is clear that $h_{B,G}(w) \le h_{A,D}(z,w)$. Conversely, if $u \in \mathcal{PSH}(D)$, $u \le 1$ on D and $u \le 0$ on A, then $u(z,w) = v(w)$ with $v \in \mathcal{PSH}(G)$. Obviously, $v \le 1$ on G and $v \le 0$ on B. hence $v \le h_{B,G}$. $\square$

Proposition 3.4.9 ([Ale-Hec 2004]). *Let* $G \subset \mathbb{C}^{n-1}$ *be an arbitrary bounded domain, let* $B \subset \mathbb{C}$ *be polar, and let* $C \subset G$, $C \notin \mathcal{PLP}$. *Put* $D := \mathbb{C} \times G \subset \mathbb{C}^n$, $A := B \times C$. *Then* $h^*_{A,D}(z) < \omega_{A,D}(z) = 1$, $z \in D$.

Proof. By Propositions 3.4.7 and 3.4.8, $h^*_{A,D}(z,w) = h^*_{C,G}(w) < 1$, $(z,w) \in D$. Since $A \in \mathcal{PLP}$, Proposition 3.4.7 implies that for very exhaustion sequence $(D_k)_{k=1}^{\infty}$ we have $\omega_{A,D} = \lim_{k\to+\infty} h^*_{A,D_k} = 1$. $\square$

Theorem 3.4.10 (Product property; [Edi-Pol 1997], [Edi 2002]). *Let* $D_j \subset \mathbb{C}^{n_j}$ *be a domain,* $A_j \subset D_j$, $j = 1, 2$. *Assume that* A_1, A_2 *are open or* A_1, A_2 *are compact. Then*

$$h_{A_1\times A_2,D_1\times D_2}(z_1,z_2) = \max\{h_{A_1,D_1}(z_1),\ h_{A_2,D_2}(z_2)\}, \quad (z_1,z_2) \in D_1 \times D_2.$$

Moreover, if D_1, D_2 are bounded, then for arbitrary subsets $A_1 \subset D_1$, $A_2 \subset D_2$ we have

$$h^*_{A_1 \times A_2, D_1 \times D_2}(z_1, z_2) = \max\{h^*_{A_1,D_1}(z_1),\ h^*_{A_2,D_2}(z_2)\}, \quad (z_1, z_2) \in D_1 \times D_2.$$

Proposition 3.4.11. (a) $\omega_{A,X} = h^*_{A^*,X}$.

(b) *If $X \in \mathfrak{R}_b(\mathbb{C}^n)$ (cf. Definition 3.1.1), then $h^*_{A\cup P,X} \equiv h^*_{A,X}$ for any $A \subset X$ and $P \in \mathcal{PLP}(X)$.*

(c) *If $X \in \mathfrak{R}_b(\mathbb{C}^n)$, then a set $P \subset X$ is pluripolar iff $h^*_{P,X} \equiv 1$.*

(d) *$\omega_{A\cup P,X} \equiv \omega_{A,X}$ for arbitrary $A \subset X$ and $P \in \mathcal{PLP}(X)$.*

(e) *If $P \in \mathcal{PLP}$, then $(A \setminus P)^* = A^*$. In particular, if A is locally pluriregular, then $A \setminus P$ is locally pluriregular.*

(f) *$A \cap A^*$ is locally pluriregular.*

(g) *If $A \subset X$, $B \subset Y$ are locally pluriregular, then $A \times B \subset X \times Y$ is locally pluriregular.*

(h) *Let $P \subset A \times B \subset X \times Y$. Assume that B is locally pluriregular and for any $a \in A$ the fiber $P_{(a,\cdot)}$ is pluripolar (we do not assume that P is pluripolar). Then for any open set $V \times W \subset D \times G$ we have $\omega_{(A\times B)\setminus P,\, V\times W} = \omega_{A\times B,\, V\times W}$. In particular, if A is also locally pluriregular, then $A \times B \setminus P$ is locally pluriregular (cf. (g, e)).*

Proof. (a) The inequality "$\leq$" follows from Remark 3.4.5. Take an arbitrary exhaustion sequence $(X_k)_{k=1}^{\infty}$. By Proposition 3.4.7(v) (applied to the relatively compact open sets X_k, $k \in \mathbb{N}$), we get

$$\omega_{A,X} = \lim_{k\to+\infty} h^*_{A,X_k} = \lim_{k\to+\infty} h^*_{A^*,X_k} \geq h^*_{A^*,X}.$$

(b) and (c) follow directly from Proposition 3.4.7.

(d) follows from (b).

(e) We only need to show that $A^* \subset (A \setminus P)^*$. Fix a point $a \in A^*$. If U is an open bounded neighborhood of a, then, using (b), we have $h^*_{A\setminus P,U}(a) = h^*_{A,U}(a) = 0$. It remains to observe that $a \in \overline{A \setminus P}$ (otherwise, a has a bounded neighborhood U such that $U \cap (A \setminus P) = \varnothing$, which implies that $1 = h_{A\setminus P,U}(a) = h^*_{A,U}(a) = 0$; a contradiction).

(f) follows from (e) and Proposition 3.4.6(b).

(g) Take $(a, b) \in A \times B$ and let U, V be arbitrary univalent relatively compact neighborhoods of a and b, respectively. By Theorem 3.4.10, we get

$$h^*_{A\times B,U\times V}(a, b) = \max\{h^*_{A,U}(a),\ h^*_{B,V}(b)\} = 0.$$

(h) Take an open set $V \times W \subset D \times G$. We may assume that V and W are relatively compact. Take a $u \in \mathcal{PSH}(V \times W)$, $u \leq 1$, with $u \leq 0$ on $(A \times B) \cap (V \times W) \setminus P$. Then for any $a \in A \cap V$ we have $u(a, \cdot) \leq 0$ on $(B \cap W) \setminus P_{(a,\cdot)}$. Hence $u(a, b) \leq h^*_{B\setminus P_{(a,\cdot)},W}(b) = h^*_{B,W}(b) = 0$, $b \in B \cap W$. □

Proposition 3.4.12. *Let $X_k \nearrow X \Subset Y$ and let $A_k \subset X_k$, $A_k \nearrow A$. Then $h^*_{A_k,X_k} \searrow h^*_{A,X}$.*

Proof. Let $u_k := h_{A_k,X_k}$. Obviously, $u_{k+1} \le u_k$. Let $v := \lim_{k\to+\infty} u_k^*$. Then $v \in \mathcal{PSH}(X)$ and $h^*_{A,X} \le v \le 1$. Put $P_k := \{z \in X_k : u_k(z) < u_k^*(z)\}$. Then $P_k \in \mathcal{PLP}$, $k \in \mathbb{N}$ (cf. Theorem 3.3.29(a)), and hence $P := \bigcup_{k=1}^{\infty} P_k \in \mathcal{PLP}$. Observe that $v = \lim_{k\to+\infty} u_k \le 0$ on $A \setminus P$. Consequently, by Proposition 3.4.11(b), $v \le h^*_{A\setminus P,X} = h^*_{A,X}$. □

Proposition 3.4.13. (a) *If $A \subset X \in \mathfrak{R}_b(\mathbb{C}^n)$, $A \notin \mathcal{PLP}$, $0 < \mu < 1$, and*

$$\Omega_\mu := \{z \in X : h^*_{A,X}(z) < \mu\},$$

then $A \cap S \notin \mathcal{PLP}$ for any connected component S of Ω_μ (in particular, $A \cap S \neq \varnothing$).

(b) *If A is locally pluriregular, then $h^*_{A,\Omega_\mu} = (1/\mu)h^*_{A,X}$ on Ω_μ.*

Proof. (a) By Proposition 3.4.11(c) we have to prove that $h^*_{A,\Omega_\mu}(z) < 1$, $z \in \Omega_\mu$. Let $B := \{z \in A : h^*_{A,X}(z) = 0\}$. Then $B \subset A \cap \Omega_\mu$ and therefore it suffices to show that $h^*_{B,\Omega_\mu}(z) < 1$ for any $z \in \Omega_\mu$. Observe that

$$A \setminus B \subset \{z \in X : h_{A,X}(z) < h^*_{A,X}(z)\}.$$

Consequently, the set $A \setminus B$ is pluripolar and hence, by Proposition 3.4.11(b), $h^*_{A,X} = h^*_{B,X}$. Put

$$P := \{z \in X : h_{B,X}(z) < h^*_{B,X}(z)\} \cup \{z \in \Omega_\mu : h_{B,\Omega_\mu}(z) < h^*_{B,\Omega_\mu}(z)\};$$

P is pluripolar. Define

$$u := \begin{cases} \max\{h^*_{B,X},\ \mu h^*_{B,\Omega_\mu}\} & \text{on } \Omega_\mu \\ h^*_{B,X} & \text{on } X \setminus \Omega_\mu \end{cases}.$$

Then $u \in \mathcal{PSH}(X)$ (cf. Proposition 3.3.6), $u \le 1$ on X, and $u = 0$ on $B \setminus P$. Thus $u \le h^*_{B\setminus P,X} = h^*_{B,X}$ and, finally, $h^*_{B,\Omega_\mu} \le (1/\mu)u \le (1/\mu)h^*_{B,X} < 1$ in Ω_μ.

(b) If A is locally pluriregular, then $B = A$ and hence $h^*_{A,\Omega_\mu} \le (1/\mu)h^*_{A,X}$ in Ω_μ. The converse inequality is obvious. □

Proposition 3.4.14. *Let D_j be Riemann a domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. Put*

$$\widehat{\boldsymbol{X}} := \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^{N} h^*_{A_j,D_j}(z_j) < 1\Big\}.$$

Then

$$h^*_{A_1\times\dots\times A_N,\widehat{\boldsymbol{X}}}(z) = \sum_{j=1}^{N} h^*_{A_j,D_j}(z_j), \quad z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}.$$

Proof. The inequality "$\geq$" is obvious. To get the opposite inequality we proceed by induction on $N \geq 2$.

Let $N = 2$ (cf. [Sic 1981a]): Put $u := h^*_{A_1 \times A_2, \widehat{\boldsymbol{X}}} \in \mathcal{PSH}(\widehat{\boldsymbol{X}})$ and fix a point $(a_1, a_2) \in \widehat{\boldsymbol{X}}$. If $a_1 \in A_1$, then $u(a_1, \cdot) \in \mathcal{PSH}(D_2)$, $u(a_1, \cdot) \leq 1$, and $u(a_1, \cdot) \leq 0$ on A_2. Therefore,

$$u(a_1, \cdot) \leq h^*_{A_2,D_2} = h^*_{A_1,D_1}(a_1) + h^*_{A_2,D_2} \quad \text{on } D_2.$$

In particular, $u(a_1, a_2) \leq h^*_{A_1,D_1}(a_1) + h^*_{A_2,D_2}(a_2)$. The same argument works if $a_2 \in A_2$. If $a_1 \notin A_1$, then $h^*_{A_1,D_1}(a_1) + h^*_{A_2,D_2}(a_2) < 1$ and hence

$$\mu := 1 - h^*_{A_1,D_1}(a_1) \in (0, 1].$$

Put

$$(D_2)_\mu := \{z_2 \in D_2 : h^*_{A_2,D_2}(z_2) < \mu\}.$$

It is clear that $A_2 \subset (D_2)_\mu \ni a_2$. Put

$$v := \frac{1}{\mu}\Big(u(a_1, \cdot) - h^*_{A_1,D_1}(a_1)\Big) \in \mathcal{PSH}((D_2)_\mu).$$

Then $v \leq 1$ and $v \leq 0$ on A_2. Therefore, by Proposition 3.4.13(b),

$$v \leq h^*_{A_2,(D_2)_\mu}(a_2) = \frac{1}{\mu} h^*_{A_2,D_2}(a_2) \quad \text{on } (D_2)_\mu.$$

Consequently, $u(a_1, a_2) \leq h^*_{A_1,D_1}(a_1) + h^*_{A_2,D_2}(a_2)$, which finishes the proof for $N = 2$.

Now, assume that the formula is true for $N - 1 \geq 2$. Put

$$\widehat{\boldsymbol{Y}} := \{(z_1, \dots, z_{N-1}) \in D_1 \times \dots \times D_{N-1} : \sum_{j=1}^{N-1} h^*_{A_j,D_j}(z_j) < 1\}.$$

By the inductive hypothesis, we conclude that

$$h^*_{A_1 \times \dots \times A_{N-1}, \widehat{\boldsymbol{Y}}}(z') = \sum_{j=1}^{N-1} h^*_{A_j,D_j}(z_j), \quad z' = (z_1, \dots, z_{N-1}) \in \widehat{\boldsymbol{Y}}.$$

Now we apply the case $N = 2$ to the following situation:

$$\widehat{\boldsymbol{Z}} := \{(z', z_N) \in \widehat{\boldsymbol{Y}} \times D_N : h^*_{A_1 \times \dots \times A_{N-1}, \widehat{\boldsymbol{Y}}}(z') + h^*_{A_N,D_N}(z_N) < 1\}.$$

So

$$h^*_{A_1 \times \dots \times A_{N-1}, \widehat{\boldsymbol{Y}}}(z') + h^*_{A_N,D_N}(z_N) = h^*_{A_1 \times \dots \times A_N, \widehat{\boldsymbol{Z}}}(z), \quad z = (z', z_N) \in \widehat{\boldsymbol{Z}}.$$

It remains to observe that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{Y}}$. $\square$

Definition 3.4.15. We say that an Riemann region (X, p) is *hyperconvex* if there exists a function $u \in \mathcal{PSH}(X, \mathbb{R}_-)$ such that

$$\{x \in X : u(x) < t\} \Subset X, \quad t < 0.$$

Theorem* 3.4.16 ([Bed-Tay 1976], [Bed 1981]). *Let* $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. *There exists a* Monge–Ampère operator

$$\mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc}) \ni u \longmapsto (dd^c u)^n \in \mathfrak{K}(X),$$

where $\mathfrak{K}(X)$ *denotes the space of all non-negative Borel measure on* X*, such that:*

(a) *if* $u \in \mathcal{PSH}(X) \cap \mathcal{C}^2(X, \mathbb{R})$*, then* $(dd^c u)^n = 4^n n! \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right]_{j,k=1,\dots,n} \mathcal{L}^X$,

(b) *if* $\mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc}) \ni u_\nu \searrow u \in \mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc})$*, then we get* $(dd^c u_\nu)^n \longrightarrow (dd^c u)^n$ *in the weak sense.*

Definition 3.4.17. The measure $\mu_{A,X} := (dd^c h^*_{A,X})^n$ is called the *equilibrium measure for* A.

Theorem* 3.4.18 ([Bed 1981], [Zer 1986], [Kli 1991], [Ale-Zer 2001]). *Let* $X \Subset Y$ *be a hyperconvex open set and let* $K \Subset X$ *be compact.*

(a) $\mu_{K,X}(X \setminus K) = 0$.

(b) *Let* $P \subset K$ *be such that* $\mu_{K,X}(P) = 0$. *Then* $h^*_{K \setminus P, X} \equiv h^*_{K,X}$.

3.5 Pseudoconvexity

See [Jar-Pfl 2000], § 2.2.

Definition 3.5.1. Let $\mathcal{S} \subset \mathcal{PSH}(X)$. For a compact set $K \subset X$ we put

$$\widetilde{K}^{\mathcal{S}} := \{x \in X : \forall_{u \in \mathcal{S}} : u(x) \le \sup_K u\}.$$

We say that a Riemann region $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is *pseudoconvex* if for any compact set $K \subset X$ the set $\widetilde{K}^{\mathcal{PSH}(X)}$ is relatively compact.

Remark 3.5.2. (a) $\widetilde{K}^{\mathcal{PSH}(X)} \subset \widehat{K}^{\mathcal{O}(X)}$. Consequently, if $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is holomorphically convex, then (X, p) is pseudoconvex.

(b) If (X, p) is hyperconvex (Definition 3.4.15), then (X, p) is pseudoconvex.

3.5.1 Smooth regions

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ and let $\Omega \Subset X$ be open.

Definition 3.5.3. We say that $\partial\Omega$ is *smooth of class* $\mathcal{C}^k$ (or $\mathcal{C}^k$*–smooth*) *at a point* $a \in \partial\Omega$ if there exist an open neighborhood U of a and a function $u \in \mathcal{C}^k(U, \mathbb{R})$ such that

$$\Omega \cap U = \{x \in U : u(x) < 0\}, \quad U \setminus \overline{\Omega} = \{x \in U : u(x) > 0\},$$
$$\operatorname{grad} u(x) \neq 0, \quad x \in U \cap \partial\Omega,$$

where

$$\operatorname{grad} u(x) := \Big(\frac{\partial u}{\partial \overline{z}_1}(x), \dots, \frac{\partial u}{\partial \overline{z}_n}(x)\Big).$$

Here $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, where $u \in \mathcal{C}^\omega$ means that u is real analytic. The function u is called a *local defining function for* Ω *at* a. We say that Ω is $\mathcal{C}^k$*–smooth* if $\partial\Omega$ is $\mathcal{C}^k$–smooth at any point $a \in \partial\Omega$. Put

$$T_x^{\mathbb{C}}(\partial\Omega) := \Big\{\xi \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial u}{\partial z_j}(x)\xi_j = 0\Big\}, \quad x \in U \cap \partial\Omega.$$

The space $T_x^{\mathbb{C}}(\partial\Omega)$ is called the *complex tangent space to* $\partial\Omega$ *at* x. The definition of $T_x^{\mathbb{C}}(\partial\Omega)$ is independent of u. If $n = 1$, then $T_x^{\mathbb{C}}(\partial\Omega) = \{0\}$.

We say that $\partial\Omega$ is *strongly pseudoconvex at a point* $a \in \partial\Omega$ if there exist an open neighborhood U of a and a local defining function $u \in \mathcal{C}^2(U, \mathbb{R})$ such that

$$\mathcal{L}u(x;\xi) > 0, \quad x \in U \cap \partial\Omega,\ \xi \in T_x^{\mathbb{C}}(\partial\Omega) \setminus \{0\}.$$

The definition is independent of u. We say that Ω is *strongly pseudoconvex* if $\partial\Omega$ is strongly pseudoconvex at any point $a \in \partial\Omega$. If $n = 1$, then any $\mathcal{C}^2$–smooth open set $\Omega \Subset X$ is strongly pseudoconvex.

3.5.2 Pseudoconvexity in terms of the boundary distance

Proposition 3.5.4. *Let* $(X, p) \in \mathfrak{R}(\mathbb{C})$*. Then* $-\log d_X \in \mathcal{SH}(X)$*.*

Theorem 3.5.5. *Let* $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$*. Then the following conditions are equivalent:*

(i) *for any compact* $K \subset X$ *the set* $\widetilde{K}^{\mathcal{PSH}(X)\cap\mathcal{C}^\infty(X)}$ *is compact;*

(ii) (X, p) *is pseudoconvex;*

(iii) *for any* $\xi \in \mathbb{C}^n$ *the function* $-\log\delta_{X,\xi}$ *is psh on* X*;*

(iv) $-\log d_X \in \mathcal{PSH}(X)$*;*

(v) *there exists an* exhaustion function $u \in \mathcal{PSH}(X) \cap \mathcal{C}(X)$*, i.e. for any* $t \in \mathbb{R}$ *the set* $\{x \in X : u(x) < t\}$ *is relatively compact;*

(vi) *there exists an exhaustion function* $u \in \mathcal{PSH}(X)$*;*

(vii) *there exists a strictly psh exhaustion function* $u \in \mathcal{C}^\infty(X)$ *(cf. Definition 3.3.5).*

3.5.3 Basic properties of pseudoconvex domains

Theorem 3.5.6. *Let $(X,p) \in \mathfrak{R}_c(\mathbb{C}^n)$, $(Y,q) \in \mathfrak{R}_c(\mathbb{C}^m)$.*

(a) *If $X = \bigcup_{\nu\in\mathbb{N}} X_\nu$, where X_ν is a pseudoconvex open subset of X with $X_\nu \subset X_{\nu+1}$, $\nu \in \mathbb{N}$, then X is pseudoconvex.*

(b) *If $Y = \operatorname{int}\bigcap_{\nu\in\mathbb{N}} X_\nu$, where X_ν is a pseudoconvex open subset of X, $\nu \in \mathbb{N}$, then Y is pseudoconvex.*

(c) *If $(X_j,p_j) \in \mathfrak{R}_c(\mathbb{C}^{n_j})$ is pseudoconvex, $j = 1,\dots,N$, then $X_1 \times \cdots \times X_N$ is pseudoconvex.*

(d) *Any Riemann domain over $\mathbb{C}$ is pseudoconvex.*

(e) *If X is pseudoconvex and $u \in \mathcal{PSH}(X)$, then $Y := \{x \in X : u(x) < 0\}$ is pseudoconvex.*

(f) *If X is pseudoconvex and $Y \subset X$ is an open set such that for any point $a \in \partial Y$ there exists an open neighborhood U_a such that $Y \cap U_a$ is pseudoconvex, then Y is pseudoconvex.*

(g) *If X is pseudoconvex and M is an analytic subset of X of pure dimension $(n-1)$, then $X \setminus M$ is pseudoconvex.*

(h) *If $Z \subset X \times Y$ is pseudoconvex, then for any $y_0 \in Y$ the fiber $Z_{y_0} := \{x \in X : (x,y_0) \in Z\}$ is a pseudoconvex open subset of X.*

(i) *If X is pseudoconvex, $f : X \longrightarrow Y$ be holomorphic, and $Z \subset Y$ is open pseudoconvex, then $f^{-1}(Z)$ is pseudoconvex.*

3.5.4 Smooth pseudoconvex domains

So far, pseudoconvex domains were characterized by the plurisubharmonicity of the function $-\log d_X$. In the case of smooth open subsets $\Omega \Subset X$ we can say more, namely:

Theorem 3.5.7. *Let $(X,p) \in \mathfrak{R}_c(\mathbb{C}^n)$ and let $\Omega \Subset X$ be a $\mathcal{C}^2$–smooth open set. Then $(\Omega, p|_\Omega)$ is pseudoconvex iff any local defining function $u \in \mathcal{C}^2(U,\mathbb{R})$ satisfies the following* Levi condition

$$\mathcal{L}u(x;\xi) \geq 0, \quad x \in U \cap \partial\Omega,\ \xi \in T_x^{\mathbb{C}}(\partial\Omega).$$

Theorem 3.5.8. *Let $\Omega \Subset X$ be strongly pseudoconvex.*

(a) *If Ω is $\mathcal{C}^k$–smooth ($k \geq 2$), then there exist an open neighborhood U of $\overline{\Omega}$ and a strictly psh defining function $u \in \mathcal{C}^k(U,\mathbb{R})$.*

In particular, any strongly pseudoconvex open set is hyperconvex.

(b) *For any open neighborhood U of $\overline{\Omega}$ there exists a strongly pseudoconvex $\mathcal{C}^\infty$–smooth open set Ω' such that $\overline{\Omega} \subset \Omega' \subset U$. Consequently, every function $f \in \mathcal{O}(\Omega)$ may be approximated locally uniformy in Ω by functions holomorphic in a neighborhood of $\overline{\Omega}$ (cf.* [Jar-Pfl 2000], *Proposition 2.7.7).*

3.5.5 Levi problem

In view of Remark 3.5.2(a) it is natural to ask whether any pseudoconvex Riemann region is a region of holomorphy. This is the famous *Levi Problem.* The problem, formulated by E.E. Levi in 1910 , was solved by Oka only in 1942 for $n = 2$ and in 1954 by Oka, Norguet, and Bremermann for $n > 2$ ([7]) ([8]) ([9]). [BIOGRAPHICAL DATA OF NORGUET. WILL BE COMPLETED. . . . .]

Theorem 3.5.9 (Solution of the Levi problem). *Let* $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$*. Then the following conditions are equivalent:*
(i) (X, p) *is a region of holomorphy;*
(ii) $\mathcal{O}(X)$ *separates points in* X *and* (X, p) *is holomorphically convex;*
(iii) (X, p) *is holomorphically convex;*
(iv) (X, p) *is pseudoconvex.*

Proposition 3.5.10. *If* $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$ *is a domain of holomorphy, then every* $u \in \mathcal{PSH}(X)$ *is a* Hartogs plurisubharmonic function*, i.e. there exists a sequence* $(f_k)_{k=1}^\infty \subset \mathcal{O}(X)$ *such that:*
- *the sequence* $(|f_k|^{1/k})_{k=1}^\infty$ *is locally bounded in* X*,*
- $u = v^*$*, where* $v := \limsup_{k\to+\infty}(1/k)\log|f_k|$.

Proof. The Hartogs domain

$$Y := \{(z, w) \in X \times \mathbb{C} : |w| < e^{-u(z)}\}$$

is a domain of holomorphy (cf. Theorem 3.5.6(e)). Let $f \in \mathcal{O}(Y)$ be non-continuable beyond Y. Write f in form of the Hartogs series

$$f(z, w) = \sum_{k=0}^{\infty} f_k(z)w^k, \quad (z, w) \in Y,$$

where $f_k \in \mathcal{O}(X)$, $k \in \mathbb{N}$. Obviously $v := \limsup_{k\to+\infty}(1/k)\log|f_k| \le u$. In particular, $v^* \le u$ and, by the Hartogs lemma (Proposition 3.3.13), the sequence $(|f_k|^{1/k})_{k=1}^\infty$ is locally bounded in X. Suppose that $v^*(a) < u(a)$. Then $v(z) \le v^*(z) < -\log R < u(a)$, $z \in \widehat{\mathbb{P}}_X(a, r) \Subset X$. Thus $f(z, \cdot)$ extends holomorphically to $K(R)$ for every $z \in \widehat{\mathbb{P}}_X(a, r)$. Consequently, by the Hartogs lemma (Lemma 2.1.4), the function f extends holomorphically to $\widehat{\mathbb{P}}_X(a, r) \times K(R)$. Since f is non-continuable, we conclude that $R \le e^{-u(a)}$; a contradiction. □

3.6 The Grauert boundary of a Riemann domain

See [Jar-Pfl 2000], § 1.5.

([7]) Kiyoshi Oka (1901–1978) — Japanese mathematician.
([8]) François Norguet (1929–) — French mathematician.
([9]) Hans–Joachim Bremermann (1926–1996) — German mathematician.

Let (X, p), $(Y, q) \in \mathfrak{R}_c(\mathbb{C}^n)$ and let $\varphi : X \longrightarrow Y$ be a morphism. Our aim is to define an abstract boundary $\overset{=\varphi}{\partial} X$ of X with respect to the morphism φ. The idea of such an abstract boundary is due to H. Grauert [10].

In the case where $(X, p) = (G, \mathrm{id})$ (G is a domain in $\mathbb{C}^n$), $(Y, q) = (\mathbb{C}^n, \mathrm{id})$, $\varphi = \mathrm{id}$, the abstract boundary $\overset{=\mathrm{id}}{\partial} G$ coincides with the set of, so-called, prime ends of G.

For $a \in X$ let $\mathfrak{B}_c(a)$ denote the family of all open connected neighborhoods U of a.

Definition 3.6.1. We say that a filter basis $\mathfrak{a}$ of subdomains of X is a *φ–boundary point of X* [11] if:

- $\mathfrak{a}$ has no accumulation points in X,
- there exists a point $y_0 \in Y$ such that $\lim \varphi(\mathfrak{a}) = y_0$,
- for any $V \in \mathfrak{B}_c(y_0)$ there exists exactly one connected component $U =: \mathfrak{C}(\mathfrak{a}, V)$ of $\varphi^{-1}(V)$ such that $U \in \mathfrak{a}$,
- for any $U \in \mathfrak{a}$ there exists a $V \in \mathfrak{B}_c(y_0)$ such that $U = \mathfrak{C}(\mathfrak{a}, V)$.

Let $\overset{=\varphi}{\partial} X$ denote the set of all φ–boundary points of X. We put

$$\overset{=\varphi}{X} := X \cup \overset{=\varphi}{\partial} X$$

and we extend φ to $\overline{\overline{\varphi}} : \overset{=\varphi}{X} \longrightarrow Y$ by putting $\overline{\overline{\varphi}}(\mathfrak{a}) := y_0$ if $\mathfrak{a}$ and y_0 are as above. Moreover, we put $\overset{=\varphi}{p} := q \circ \overline{\overline{\varphi}}$. We endow $\overset{=\varphi}{X}$ with a Hausdorff topology which coincides with the initial topology on X and is such that the mapping $\overline{\overline{\varphi}}$ is

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[10] Hans Grauert (1930–) — German mathematician.

[11] We say that a non-empty family $\mathfrak{F}$ of subsets of a topological space X is a *filter* if:

- $A \in \mathfrak{F},\ A \subset B \Longrightarrow B \in \mathfrak{F}$,
- $A_1, A_2 \in \mathfrak{F} \Longrightarrow A_1 \cap A_2 \in \mathfrak{F}$,
- $\varnothing \notin \mathfrak{F}$.

A non-empty family $\mathfrak{P}$ of non-empty subsets of X is said to be a *filter basis* if:

- $\forall_{A_1, A_2 \in \mathfrak{P}}\ \exists_{A \in \mathfrak{P}} : A \subset A_1 \cap A_2$.

It is clear that for each filter basis $\mathfrak{P}$ the family $\mathfrak{F}_{\mathfrak{P}} := \{A \subset X : \exists_{B \in \mathfrak{P}} : B \subset A\}$ is a filter.

We say that a filter $\mathfrak{F}$ is *convergent* to a point $a \in X$ if each neighborhood of a belongs to $\mathfrak{F}$. We shortly write $a \in \lim \mathfrak{F}$.

We say that a filter basis $\mathfrak{P}$ is convergent to a if $a \in \lim \mathfrak{F}_{\mathfrak{P}}$ (equivalently, each neighborhood of a contains an element of $\mathfrak{P}$); we put $\lim \mathfrak{P} := \lim \mathfrak{F}_{\mathfrak{P}}$.

We say that a is an *accumulation point* of a filter $\mathfrak{F}$ (resp. filter basis $\mathfrak{P}$) if $a \in \overline{A}$ for any $A \in \mathfrak{F}$ (resp. $A \in \mathfrak{P}$).

Let us recall a few elementary properties of filters:

- If $\mathfrak{F} \subset \mathfrak{F}'$ are filters and if a is an accumulation point of $\mathfrak{F}'$, then a is an accumulation point of $\mathfrak{F}$.
- If $a \in \lim \mathfrak{F}$, then $a \in \lim \mathfrak{F}'$ for any filter $\mathfrak{F}' \supset \mathfrak{F}$.
- If a is an accumulation point of $\mathfrak{F}$, then there exists a filter $\mathfrak{F}' \supset \mathfrak{F}$ such that $a \in \lim \mathfrak{F}'$.
- $a \in \overline{A}$ iff there exists a filter basis $\mathfrak{P}$ consisting of subsets of A such that $a \in \lim \mathfrak{P}$.
- Let Y be another topological space and let $\varphi : X \longrightarrow Y$. Then φ is continuous iff for any filter basis $\mathfrak{P}$ in X the filter basis $\varphi(\mathfrak{P}) := \{\varphi(A) : A \in \mathfrak{P}\}$ satisfies the relation: $\varphi(\lim \mathfrak{P}) \subset \lim \varphi(\mathfrak{P})$.
- X is Hausdorff iff any filter in X converges to at most one point. If X is a Hausdorff space and $\lim \mathfrak{F} = \{a\}$, then we write $\lim \mathfrak{F} = a$.

continuous: by an open neighborhood of a point $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ we mean any set of the form

$$\widehat{U}_{\mathfrak{a}} := U \cup \{\mathfrak{b} \in \overset{=\varphi}{\partial} X : \ U \text{ belongs to the filter generated by } \mathfrak{b}\},$$

where $U \in \mathfrak{a}$.

Proposition 3.6.2. *For any* $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ *and for any neighborhood* $\widehat{U}_{\mathfrak{a}} \subset \overset{=\varphi}{X}$ *there exists a neighborhood* $\widehat{W}_{\mathfrak{a}} \subset \widehat{U}_{\mathfrak{a}}$ *such that* $d_X = d_U$ *on* W*. In particular,*

$$\lim_{X \ni y \to \mathfrak{a}} d_X(y) = 0.$$

Let $\mathfrak{K}(A)$ denote the family of all relatively closed pluripolar subsets of A.

Proposition 3.6.3. (a) *Assume that* $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ *is such that there exists a neighborhood* $U \subset \overset{=\varphi}{X}$ *of* $\mathfrak{a}$ *with the following properties:*

- $V := \overline{\overline{\varphi}}(U)$ *is open in* Y,
- $P := \overline{\overline{\varphi}}(U \cap \overset{=\varphi}{\partial} X) \in \mathfrak{K}(V)$,
- $\varphi : U \setminus \overset{=\varphi}{\partial} X \longrightarrow V \setminus P$ *is biholomorphic.*

Then the mapping $\overline{\overline{\varphi}}|_U : U \longrightarrow V$ *is homeomorphic.*

(b) *Let* Σ *denote the set of all points* $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ *which satisfies the above conditions. Put*

$$\overset{*\varphi}{X} := X \cup \Sigma.$$

Then:

- $(\overset{*\varphi}{X}, \overset{=\varphi}{p}|_{\overset{*\varphi}{X}})$ *is a Riemann domain over* $\mathbb{C}^n$,
- $\overline{\overline{\varphi}}|_{\overset{*\varphi}{X}} : (\overset{*\varphi}{X}, \overset{=\varphi}{p}|_{\overset{*\varphi}{X}}) \longrightarrow (Y, q)$ *is a morphism,*
- $\Sigma \in \mathfrak{K}(\overset{*\varphi}{X})$.

The following proposition shows that $\overset{*\varphi}{X}$ is in some sense maximal.

Proposition 3.6.4. *Suppose that* $W \subset X$ *is an open subset such that:*

- $\varphi(W) = V \setminus P$, *where* V *is an open subset of* Y *and* $P \in \mathfrak{K}(V)$,
- $\varphi : W \longrightarrow V \setminus P$ *is biholomorphic.*

Then there exists an open set $U \subset \overset{*\varphi}{X}$ *such that* $W \subset U$ *and* $\overline{\overline{\varphi}} : U \longrightarrow V$ *is biholomorphic.*

3.7 The Docquier–Grauert criteria

See [Jar-Pfl 2000], § 2.9.

The aim of this section is to localize the description of the pseudoconvexity. The main local criteria for the pseudoconvexity are contained in the following theorem. Let $(X,p) \in \mathfrak{R}_c(\mathbb{C}^n)$. Put

$$\varDelta := \mathbb{D}^{n-1} \times \overline{\mathbb{D}}, \quad \delta\varDelta := \mathbb{D}^{n-1} \times \partial\mathbb{D}.$$

Theorem 3.7.1 (Docquier–Grauert criteria). *The following conditions are equivalent:*

(p_0) *(X,p) is a Riemann–Stein domain;*

(p_1) *there exists a function $u \in \mathcal{PSH}(X)$ such that $\{x \in X : u(x) < t\} \Subset X$ for any $t \in \mathbb{R}$;*

(p_2) $\operatorname{int}(\widetilde{K}^{\mathcal{PSH}(X)}) \Subset X$ *for any compact $K \subset X$;*

(p_3) *$X = \bigcup_{\nu=1}^{\infty} \varOmega_\nu$, where $\varOmega_\nu$ is a is strongly pseudoconvex domain with real analytic boundary, and $\varOmega_\nu \Subset \varOmega_{\nu+1}$, $\nu \geq 1$;*

(p_4) *there exist a Riemann–Stein domain (Y,q) over $\mathbb{C}^n$ and a morphism*

$$\varphi : (X,p) \longrightarrow (Y,q)$$

such that any point $a \in \overset{=\varphi}{\partial} X$ has a neighborhood $U \subset \overset{=\varphi}{X}$ (cf. § 3.6) such that the region $(X \cap U, p)$ is holomorphically convex;

(p_5) (Kontinuitätssatz) *for any sequence of holomorphic mappings*

$$\psi_\nu : D_\nu \longrightarrow X,$$

where $D_\nu \subset \mathbb{C}$ is a neighborhood of $\overline{\mathbb{D}}$, $\nu \geq 1$, we have the following implication: if $\bigcup_{\nu=1}^{\infty} \psi_\nu(\partial\mathbb{D}) \Subset X$, then $\bigcup_{\nu=1}^{\infty} \psi_\nu(\overline{\mathbb{D}}) \Subset X$;

(p_6) *for any biholomorphic mapping $f : W \longrightarrow f(W) \subset X$, where $W \subset \mathbb{C}^n$ is a neighborhood of $\varDelta$, if $f(\delta\varDelta) \Subset X$, then $f(\varDelta) \Subset X$;*

(p_7) *there exist a Riemann–Stein domain (Y,q) over $\mathbb{C}^n$ and a morphism*

$$\varphi : (X,p) \longrightarrow (Y,q)$$

such that there is no continuous mapping $f : \overline{\mathbb{D}}^n \longrightarrow \overset{=\varphi}{X}$ with the following properties:

($\dagger_1$) $f(\delta\varDelta) \Subset X$,

($\dagger_2$) $f(\mathbb{D}^n) \subset X$,

($\dagger_3$) $f(\overline{\mathbb{D}}^n) \cap \overset{=\varphi}{\partial} X \neq \varnothing$,

($\dagger_4$) *$\overset{=}{\varphi} \circ f$ extends to a biholomorphic mapping in a neighborhood of $\overline{\mathbb{D}}^n$;*

(p_7^*) *there exist a Riemann–Stein domain (Y,q) over $\mathbb{C}^n$ and a morphism*

$$\varphi : (X,p) \longrightarrow (Y,q)$$

such that any point $a \in \overset{=\varphi}{\partial} X$ has an open neighborhood $U \subset \overset{=\varphi}{X}$ such that there is no continuous mapping $f : \overline{\mathbb{D}}^n \longrightarrow U$ with the above properties ($\dagger_1 - \dagger_4$).

Notice that the Docquier–Grauert criteria remain true in the case where (X,p) be a Riemann domain over an n–dimensional connected Stein manifold M.

3.8 Meromorphic functions

See [Jar-Pfl 2000], § 3.6.

Let $(X,p) \in \mathfrak{R}_c(\mathbb{C}^n)$.

Definition 3.8.1. A function $f : X \setminus S \longrightarrow \mathbb{C}$, where $S = S(f)$ is a closed subset of X with (3.1.2), is said to be *meromorphic on* X ($f \in \mathcal{M}(X)$) if:

(a) $f \in \mathcal{O}(X \setminus S)$ and S is singular for f in the sense of § 3.1.8,

(b) for any point $a \in S$ there exist an open connected neighborhood U of a and functions $\varphi, \psi \in \mathcal{O}(U)$, $\psi \not\equiv 0$, such that $\psi f = \varphi$ on $U \setminus S$. we say that (φ, ψ) is a local representation of f at a. Note that in view of (a) we must have $\psi(a) = 0$. Consequently, either $S = \varnothing$ or S is an $(n-1)$–dimensional set of pure codimension one.

The set $\mathcal{R}(f) := X \setminus S(f)$ is called the set of *regular* points of f.

We say that a point $a \in S$ is a *pole* of f ($a \in \mathcal{P}(f)$) if there exists a local representation (φ, ψ) of f at a such that $\varphi(a) \neq 0$.

We say that a point $a \in S$ is a *point of indeterminacy* of f ($a \in \mathcal{I}(f)$) if for every local representation (φ, ψ) of f at a we have $\varphi(a) = 0$.

Obviously, $S(f) = \mathcal{P}(f) \cup \mathcal{I}(f)$ and $\mathcal{P}(f) \cap \mathcal{I}(f) = \varnothing$. Moreover, $\mathcal{I}(f)$ is an analytic set of dimension $\leq n-2$. In particular, if $n = 1$, then $\mathcal{I}(f) = \varnothing$.

The theory of extension of holomorphic mappings developed in § 3.1 may be repeated word for word for meromorphic functions and leads to the following Thullen theorem (cf. Theorem 3.1.16).

Theorem 3.8.2 (Thullen theorem). *Let $\varnothing \neq \mathcal{F} \subset \mathcal{M}(X)$. Then (X,p) has an $\mathcal{F}$–envelope of meromorphy $\alpha : (X,p) \longrightarrow (\widetilde{X}, \widetilde{p})$ such that $(\widetilde{X}, \widetilde{p})$ is a Riemann–Stein domain. In particular, the envelope of meromorphy of (X,p) coincides with its envelope of holomorphy.*

Theorem 3.8.3. *Let $f \in \mathcal{M}(X)$. Then there exist $\varphi, \psi \in \mathcal{O}(X)$, $\psi \not\equiv 0$, such that $f = \varphi/\psi$.*

3.9 Sections of regions of holomorphy

This section is based on [Jar-Pfl 2005b].

Remark 3.9.1. (a) Let (X,p) be an $\mathcal{S}$–region of holomorphy and let $U \subset X$ be a univalent domain for which there exists a domain $V \supset p(U)$ such that for every $f \in \mathcal{S}$ there exists a function $F_f \in \mathcal{O}(V)$ such that $F_f = f \circ (p|_U)^{-1}$ on $p(U)$. Then there exists a univalent domain $W \supset U$ with $p(W) = V$.

Indeed, we only need to observe that we may assume that (X,p) coincides with $(\widehat{X}, \widehat{p})$ constructed in Remark 3.1.10.

(b) If $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is an $\mathcal{S}$–region of holomorphy, then there exists a finite or countable subfamily $\mathcal{S}_0 \subset \mathcal{S}$ such that (X,p) is an $\mathcal{S}_0$–region of holomorphy.

Indeed, we may assume that X is connected. The case where $(X,p) \simeq (\mathbb{C}^n, \mathrm{id})$ is trivial. Thus assume that $d_X(x) < +\infty$, $x \in X$. Let $A \subset X$ be a countable dense subset such that $A = p^{-1}(p(A))$. By proposition 3.1.19, for any $x \in A$ and $r > d_X(x)$ there exists an $f_{x,r} \in \mathcal{S}$ such that $d(T_x f_{x,r}) < r$, and for $x', x'' \in A$, with $x' \neq x''$ and $p(x') = p(x'')$, there exists an $f_{x',x''} \in \mathcal{S}$ such that $T_{x'} f_{x',x''} \neq T_{x''} f_{x',x''}$. Now, we may take

$$\mathcal{S}_0 := \{f_{x,r} : x \in A,\ \mathbb{Q} \ni r > d_X(x)\} \\ \cup \{f_{x',x''} : x', x'' \in A,\ x' \neq x'',\ p(x') = p(x'')\}.$$

Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$, $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^\ell$,

$$p = (u,v) : X \longrightarrow \mathbb{C}^k \times \mathbb{C}^\ell.$$

Put $\Omega := p(X)$, $\Omega_k := u(X)$, $\Omega^\ell := v(X)$. For $a \in \Omega_k$ define $X_a := u^{-1}(a)$, $p_a := v|_{X_a}$. Similarly, for $b \in \Omega^\ell$, put $X^b := v^{-1}(b)$, $p^b := u|_{X^b}$.

Remark 3.9.2. For every $a \in \Omega_k$, (X_a, p_a) is a Riemann region over $\mathbb{C}^\ell$. If (X,p) is countable at infinity, then so is (X_a, p_a).

Let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(X)$. For $a \in \Omega_k$ define $f_a := f|_{X_a}$, $\mathcal{S}_a := \{f_a : f \in \mathcal{S}\} \subset \mathcal{O}(X_a)$, and analogously, $f^b := f|_{X^b}$, $\mathcal{S}^b := \{f^b : f \in \mathcal{S}\} \subset \mathcal{O}(X^b)$, $b \in \Omega^\ell$.

Theorem 3.9.3 ([Jar-Pfl 2005b]). *Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ and let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(X)$. Assume that (X,p) is an $\mathcal{S}$–region of holomorphy. Then there exists a pluripolar set $P_k \subset \Omega_k$ such that for every $a \in \Omega_k \setminus P_k$, (X_a, p_a) is an $\mathcal{S}_a$–region of holomorphy.*

Proof. By Remark 3.9.1(b), we may assume that $\mathcal{S}$ is finite or countable.

Step 1. *There exists a pluripolar set $P \subset \Omega_k$ such that for any $a \in \Omega_k \setminus P$, (X_a, p_a) is an $\mathcal{S}_a$–region of existence.*

Define $R_{f,b}(x) := d(T_x f_{u(x)})$, $f \in \mathcal{S}$, $b \in \Omega^\ell$, $x \in X^b$. Recall that

$$1/R_{f,b}(x) = \limsup_{\nu \to +\infty} \Big(\max_{\beta \in \mathbb{Z}_+^\ell :\ |\beta| = \nu} \frac{1}{\beta!} |\Omega^{(0,\beta)} f(x)| \Big)^{1/\nu}, \quad x \in X^b.$$

Obviously, $R_{f,b}(x) \geq d_X(x)$, $x \in X^b$. By the Cauchy inequalities, we get

$$\frac{1}{\beta!} |\Omega^{(0,\beta)} f(x)| \leq \frac{\sup_{\widehat{\mathbb{P}}_X(x_0,r)} |f|}{r^{|\beta|}}, \quad 0 < r < d_X(x_0),\ x \in \widehat{\mathbb{P}}_X(x_0, r/2),\ \beta \in \mathbb{Z}_+^\ell.$$

Consequently, the function $-\log(R_{f,b})_*$ (where $_*$ denotes the lower semicontinuous regularization on X^b) is plurisubharmonic on X^b. Put

$$P_{f,b} := u(\{x \in X^b : (R_{f,b})_*(x) < R_{f,b}(x)\}) \subset \Omega_k.$$

It is known that $P_{f,b}$ is pluripolar (cf. Theorem 3.3.29(b)). Put

$$R_b := \inf_{f \in \mathcal{S}} R_{f,b}, \quad \widehat{R}_b := \inf_{f \in \mathcal{S}} (R_{f,b})_*.$$

Observe that $-\log(\widehat{R}_b)_*$ is plurisubharmonic on X^b. Put

$$P_b := u(\{x \in X^b : (\widehat{R}_b)_*(x) < \widehat{R}_b(x)\}) \subset \Omega_k.$$

The set P_b is also pluripolar (cf. Theorem 3.3.29(a)). Now let $B \subset \Omega^\ell$ be a dense countable set. Define

$$P := \Big(\bigcup_{f\in\mathcal{S},\, b\in B} P_{f,b}\Big) \cup \Big(\bigcup_{b\in B} P_b\Big) \subset \Omega_k.$$

Then P is pluripolar.

Take an $a \in \Omega_k \setminus P$ and suppose that X_a is not an $\mathcal{S}_a$–region of existence. Then there exist a point $x_0 \in X_a$ and a number $r > d_{X_a}(x_0)$ such that $b := v(x_0) \in B$ and $R_b(x_0) > r$. Since $a \notin P$, we have

$$(\widehat{R}_b)_*(x_0) = \widehat{R}_b(x_0) = \inf_{f\in\mathcal{S}} (R_{f,b})_* = \inf_{f\in\mathcal{S}} R_{f,b} = R_b(x_0) > r.$$

In particular, there exists $0 < \varepsilon < d_X(x_0)$ such that $(\widehat{R}_b)_*(x) > r$, $x \in \widehat{\mathbb{P}}_{X^b}(x_0)$. Since,

$$R_b(x) = \inf_{f\in\mathcal{S}} R_{f,b}(x) \geq \inf_{f\in\mathcal{S}} (R_{f,b})_*(x) = \widehat{R}_b(x) \geq (\widehat{R}_b)_*(x),$$

we conclude that $R_b(x) > r$, $x \in \widehat{\mathbb{P}}_{X^b}(x_0)$. Put $U := \widehat{\mathbb{P}}_X(x_0, \varepsilon)$. Hence, by the classical Hartogs lemma (cf. Lemma 2.1.4), for every $f \in \mathcal{S}$, the function $f \circ (p|_U)^{-1}$ extends holomorphically to $V := \mathbb{P}(a, \varepsilon) \times \mathbb{P}(b, r)$. Since (X, p) is an $\mathcal{S}$–domain of holomorphy, by Remark 3.9.1(a), there exists a univalent domain $W \subset X$, $U \subset W$, such that $p(W) = V$. In particular, $d_{X_a}(x_0) \geq r$; a contradiction.

Step 2. *There exists a pluripolar set $P \subset \Omega_k$ such that for any $a \in \Omega_k \setminus P$ the family $\mathcal{S}_a$ weakly separates points in X_a.*

Take $a \in \Omega_k$, $x', x'' \in X_a$ with $x' \neq x''$ and $p_a(x') = p_a(x'') =: b$. Since $\mathcal{S}$ weakly separates points in X, there exists an $f \in \mathcal{S}$ such that $T_{x'}f \neq T_{x''}f$. Put $r := \min\{d(T_{x'}f), d(T_{x''}f)\}$ and let

$$P_{a,x',x''} := \bigcap_{w\in\mathbb{P}(b,r)} \{z \in \mathbb{P}(a, r) : T_{x'}f(z, w) = T_{x''}f(z, w)\}.$$

Then $P_{a,x',x''} \subsetneq \mathbb{P}(a, r)$ is an analytic subset. For any $z \in \mathbb{P}(a, r) \setminus P_{a,x',x''}$ we have $T_{x'}f(z, \cdot) \not\equiv T_{x''}f(z, \cdot)$ on $\mathbb{P}(b, r)$.

Take a countable dense set $A \subset \Omega_k$. For any $a \in A$ let $B_a \subset X_a$ be a countable dense subset such that $p_a^{-1}(p_a(B_a)) = B_a$. Then

$$P := \bigcup_{\substack{a\in A,\, x',x''\in B_a \\ x'\neq x'',\, p_a(x')=p_a(x'')}} P_{a,x',x''}$$

is a pluripolar set.

Fix $a_0 \in \Omega_k \setminus P$, $x_0', x_0'' \in X_{a_0}$, with $x_0' \neq x_0''$ and $p_{a_0}(x_0') = p_{a_0}(x_0'') =: b_0$. Put $r := \min\{d_X(x_0'), d_X(x_0'')\}$. Let $a \in A \cap \mathbb{P}(a_0, r/2)$ and $x', x'' \in B_a$ be such that

$x' \in \mathbb{P}_X(x'_0, r/2)$, $x'' \in \mathbb{P}_X(x''_0, r/2)$, $p_a(x') = p_a(x'')$. Since $a_0 \notin P$, we conclude that $T_{x'}f(a_0,\cdot) \not\equiv T_{x''}f(a_0,\cdot)$ on $\mathbb{P}(b_0, r/2)$. Consequently, $T_{x'_0}f(a_0,\cdot) \not\equiv T_{x''_0}f(a_0,\cdot)$ on $\mathbb{P}(b_0, r/2)$, which implies that $T_{x'_0}f_{a_0} \neq T_{x''_0}f_{a_0}$. $\square$

Corollary 3.9.4. *Let $D \subset \mathbb{C}^k \times \mathbb{C}^\ell$ be a domain, let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(D)$ and let $A \subset \mathrm{pr}_{\mathbb{C}^k}(D)$. Assume that for any $a \in A$ we are given a domain $G(a) \supset D_a$ in $\mathbb{C}^\ell$ such that:*

- *for any $f \in \mathcal{S}$, the function $f(a,\cdot)$ extends to an $\widehat{f}_a \in \mathcal{O}(G(a))$,*
- *the domain $G(a)$ is a $\{\widehat{f}_a : f \in \mathcal{S}\}$–domain of holomorphy.*

Let (X,p) be the $\mathcal{S}$–envelope of holomorphy of D. Then there exists a pluripolar set $P \subset A$ such that for every $a \in A \setminus P$ we have $(X_a, p_a) \simeq (G(a), \mathrm{id})$.

Proposition 3.9.5. *Let $D \subset \mathbb{C}^p$ be a domain, let $(G, \pi_G) \in \mathfrak{R}_c(\mathbb{C}^q)$, and let $\Omega \subset D \times G$ be a Riemann domain of holomorphy over $\mathbb{C}^p \times \mathbb{C}^q$ (Ω is considered with the projection $\pi_\Omega := \mathrm{id}_D \times \pi_G$). Let $M \subset \Omega$ be a relatively closed pluripolar set that is singular with respect to a family $\mathcal{S} \subset \mathcal{O}(\Omega \setminus M)$. Then there exists a pluripolar set $P \subset D$ such that for any $a \in D \setminus P$, the fiber $M_{(a,\cdot)}$ is singular with respect to the family $\mathcal{S}^a := \{f(a,\cdot) : f \in \mathcal{S}\} \subset \mathcal{O}(\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)})$.*

Proof. Observe that $\Omega \setminus M$ is a domain of holomorphy with respect to the family $\mathcal{F}_0 := \mathcal{F} \cup \mathcal{O}(\Omega)$. By Theorem 3.9.3, there exists a pluripolar set $P \subset D$ such that for any $a \in D \setminus P$, the fiber $\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)}$ is a domain of holomorphy with respect to the family $(\mathcal{F}_0)^a$. In particular, for any $a \in D \setminus P$, the fiber $M_{(a,\cdot)}$ is singular with respect to $\mathcal{F}^a$. $\square$

Lemma 3.9.6. *Let $D \subset \mathbb{C}^k$, $G_0 \subset G \subset \mathbb{C}^\ell$ be domains of holomorphy and let $A \subset D$. Assume that for every $a \in A$ we are given a relatively closed pluripolar set $M(a) \subset G$. Let $\mathcal{S}$ denote the set of all functions $f \in \mathcal{O}(D \times G_0)$ such that for every $a \in A$, the function $f(a,\cdot)$ extends to an $\widehat{f}_a \in \mathcal{O}(G \setminus M(a))$. Assume that for every $a \in A$ the set $M(a)$ is singular with respect to the family $\{\widehat{f}_a : f \in \mathcal{S}\}$. Then there exists a pluripolar set $P \subset A$ such that if we put $A_0 := A \setminus P$, then the set*

$$M(A_0) := \bigcup_{a \in A_0} \{a\} \times M(a)$$

is relatively closed in $A_0 \times G$.

Proof. First observe that every function from $\mathcal{O}(G)$ may be regarded as an element of $\mathcal{S}$, which implies that for every $a \in A$ the domain $G(a) := G \setminus M(a)$ is a $\{\widehat{f}_a : f \in \mathcal{S}\}$–domain of holomorphy.

Let (X,p) be the $\mathcal{S}$–envelope of holomorphy of $D \times G_0$. Since D and G are domains of holomorphy, we may assume that $p(X) \subset D \times G$.

By Corollary 3.9.4, there exists a pluripolar set $P \subset A$ such that for every $a \in A_0 := A \setminus P$ we have $(X_a, p_a) \simeq (G(a), \mathrm{id})$. Thus p is injective on the set $B := p^{-1}(A_0 \times G)$ and $p(B) = \bigcup_{a \in A_0} \{a\} \times G(a) = (A_0 \times G) \setminus M(A_0)$. Hence $p(B) = p(X) \cap (A_0 \times G)$ and, consequently, $p(B)$ is relatively open in $A_0 \times G$. $\square$

Chapter 4

Classical cross theorem

4.1 Terada theorem (1967 – 1972)

Recall the Hukuhara problem (§ 2.2):

(S-$\mathcal{O}_{\mathrm{H}}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty set $B \subset G$, and a function $f \in \mathcal{O}_s(\boldsymbol{X})$, where $\boldsymbol{X} := (D \times G) \cup (D \times B)$, we ask whether $f \in \mathcal{O}(D \times G)$. After Theorems 2.2.4 and 2.2.7, the next important step was the one by T. Terada ([Ter 1967]) who finally was able to answer the question raised by Hukuhara. We are going to present a "modern" proof of Terada's theorem, based on the notion of relative extremal function — cf. § 3.4.

Theorem 4.1.1 (Terada). *If $B \notin \mathcal{PLP}$, then $\mathcal{O}_s(\boldsymbol{X}) = \mathcal{O}(D \times G)$.*

Proof. Fix an $f \in \mathcal{O}_s(\boldsymbol{X})$. By Theorem 2.2.7, $f \in \mathcal{O}(U_0 \times G)$, where U_0 is an open dense subset of D. To prove that $U_0 = D$ we only need to show that if $\mathbb{P}(a, r) \Subset U_0$ and $\mathbb{P}(a, R) \subset D$ for $0 < r < R$, then $\mathbb{P}(a, R) \subset U_0$. Write

$$f(z, w) = \sum_{\alpha \in \mathbb{Z}_+^p} f_\alpha(w)(z - a)^\alpha, \quad (z, w) \in \mathbb{P}(a, r) \times G, \tag{4.1.1}$$

where $f_\alpha \in \mathcal{O}(G)$, $\alpha \in \mathbb{Z}_+^p$. Moreover, by the Cauchy inequalities, for every compact $K \Subset G$, we get

$$\|f_\alpha\|_K \le \frac{\|f\|_{\mathbb{P}(a,r) \times K}}{r^{|\alpha|}}, \quad \alpha \in \mathbb{Z}_+^p.$$

Consequently, the sequence of log-psh functions

$$u_k := \Big(\sum_{|\alpha| = k} |f_\alpha| \Big)^{1/k}, \quad k \in \mathbb{N},$$

is locally bounded in G. Define $u := \limsup_{k \to +\infty} u_k$. Notice that $\log u^* \in \mathcal{PSH}(G)$ — cf. Proposition 3.3.12. We know that $u \le 1/r$ on G and $u \le 1/R$ on B. We may assume that G is bounded and B is locally pluriregular (cf. Proposition 3.4.6(b)). By Theorem 3.3.29(a) and Proposition 3.4.11(b),

$$\frac{\log u^* + \log r}{\log R/r} + 1 \le h^*_{B,G}$$

(cf. Definition 3.4.1). Hence

$$u(w) < \frac{1}{R}\Big(\frac{R}{r}\Big)^\mu, \quad w \in G_\mu := \{w \in G : h^*_{B,G}(w) < \mu\}, \quad 0 < \mu < 1. \tag{4.1.2}$$

Recall that $B \subset G_\mu$ (cf. Remark 3.4.5) and every connected component of G_μ intersects B (cf. Proposition 3.4.13).

Inequality (4.1.2) and the Hartogs lemma for plurisubharmonic functions (cf. Proposition 3.3.13) imply that the series (4.1.1) converges locally uniformly to a (holomorphic) function $\widetilde{f}$ in the open set

$$\bigcup_{0<\mu<1} \mathbb{P}(a, R(r/R)^\mu) \times G_\mu.$$

A standard argument shows that $\widetilde{f} = f$ and, finally $\mathbb{P}(a,R) \subset U_0$. $\square$

Exercise 4.1.2. Simplify the proof of Theorem 4.1.1 under additional assumption that B is of positive Lebesgue measure.

Exercise 4.1.3. Prove the Hartogs, Hukuhara, Shimoda, and Terada theorems in the case where D and G are Riemann domains over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively.

Theorem 4.1.4 ([Ter 1972]). *Assume that $D \subset \mathbb{C}^p$ is a domain such that:*

(†) *there exist a sequence $(\Omega_k)_{k=1}^\infty$ of open subsets of D and a sequence $(z_k)_{k=1}^\infty \subset D$ for which:*

(*) $D = \bigcup_{k=1}^\infty \bigcap_{s=k}^\infty \Omega_s,\ z_k \in (\bigcap_{s=k+1}^\infty \Omega_s) \setminus \Omega_k,\ z_k \longrightarrow z_0 \in D,$

(**) $\forall_{k\in\mathbb{N}}\ \forall_{M>0}\ \forall_{\varepsilon>0}\ \exists_{\varphi\in\mathcal{O}(D)} :\ |\varphi(z_k)| \geq M,\ |\varphi| \leq \varepsilon$ *on* Ω_k.

Let $G \subset \mathbb{C}^q$ be a domain of holomorphy and let $B \subset G$ be an $\mathcal{F}_\sigma$ pluripolar set. Then $\mathcal{O}_s(\boldsymbol{X}) \subsetneq \mathcal{O}(D \times G)$.

Remark 4.1.5. (a) $D := \mathbb{D}$ satisfies (†).

Indeed, if A_k, $k \in \mathbb{N}$, are as in Example 2.1.10, then we take $\Omega_k := \operatorname{int}(A_k \cap \mathbb{D})$, $z_k := 1/2^k$, $z_0 := 0$. Condition (**) may be checked using Runge's theorem (like in Example 2.1.10).

(b) Taking $\Omega_k \times \mathbb{D}^{p-1}$, we easily conclude that $\mathbb{D}^p$ also satisfies (†). Thus every polydisc $\mathbb{P}(a,r)$ satisfies (†) with arbitrary $z_0 \in \mathbb{P}(a,r)$ (use a biholomorphic mapping $\Phi : \mathbb{P}(a,r) \longrightarrow \mathbb{D}^n$ with $\Phi(z_0) = 0$).

(c) Consequently, every bounded domain $D \subset \mathbb{C}^p$ satisfies (†) with arbitrary $z_0 \in D$.

(d) Assumption that G is a domain of holomorphy is unessential — we may always substitute G by its envelope of holomorphy (which may be a Riemann domain over $\mathbb{C}^q$).

(e) $\boxed{?}$ We do not know whether the assumption that $B \in \mathcal{F}_\sigma$ is essential $\boxed{?}$

Proof of Theorem 4.1.4. Let $u \in \mathcal{PSH}(\mathbb{C}^q)$, $u \not\equiv -\infty$, be such that $B \subset u^{-1}(-\infty)$ (cf. Theorem 3.3.20). Since G is a domain of holomorphy, $u = v^*$, where

$$v = \limsup_{m\to+\infty} \frac{1}{m} \log |g_m|$$

$(g_m)_{m=1}^\infty \subset \mathcal{O}(G)$ is such that the sequence $(|g_m|^{1/m})_{m=1}^\infty$ is locally bounded in G (cf. Proposition 3.5.10). Let $w_0 \in G$ be such that $u(w_0) = v(w_0) > -\infty$

(cf. Propositions 3.3.10, 3.3.19). Fix a sequence $(G_k)_{k=1}^\infty$ of subdomains of G such that $w_0 \in G_k \Subset G_{k+1} \Subset G$, $G = \bigcup_{k=1}^\infty G_k$ and $B = \bigcup_{k=1}^\infty B_k$, where B_k is compact, $\varnothing \neq B_k \subset B \cap G_k$, $B_k \subset B_{k+1}$.

Let $c_k := \sup_{G_{k+1}} u$. Observe that

$$\limsup_{m\to+\infty} \frac{1}{m} \log |e^{-mc_k} g_m(w)| = v - c_k \le u - c_k \begin{cases} < 0 & \text{on } G_{k+1} \\ = -\infty & \text{on } B_k \end{cases}.$$

Put $Q_k := c_k - u(w_0) > 0$. Using the Hartogs lemma for plurisubharmonic functions (cf. Proposition 3.3.13), for every $n_k > 0$, we choose an $m_k \in \mathbb{N}$ such that with $\psi_k := e^{-m_k c_k} g_{m_k} \in \mathcal{O}(G)$ we have:

(1) $|\psi_k| \le 1$ on G_k,

(2) $|\psi_k(w_0)| \ge e^{-2m_k Q_k}$,

(3) $|\psi_k| \le e^{-m_k n_k Q_k}$ on B_k.

Take an arbitrary exhaustion sequence $(D_k)_{k=1}^\infty$ for D. Using (**), we construct inductively $M_k > 0$, $\varphi_k \in \mathcal{O}(D)$, and $n_k > 0$ such that:

(4) $M_k e^{-m_k Q_k} \ge k + 1 + \sum_{s=1}^{k-1} |\varphi_s(z_k)\psi_s(w_0)|$ (we choose $M_k > 0$),

(5) $|\varphi_k(z_k)| \ge M_k$, $|\varphi_k| \le 1/2^k$ on Ω_k (we choose φ_k),

(6) $|\varphi_k| e^{-m_k n_k Q_k} \le 1/2^k$ on D_k (we choose $n_k > 0$).

Define

$$f(z,w) := \sum_{k=1}^\infty \varphi_k(z)\psi_k(w), \quad (z,w) \in D \times G.$$

Take an arbitrary $a \in D$, say $a \in \Omega_k$ for $k \ge k_0$. Then we get

$$|\varphi_k(a)\psi_k(w)| \overset{(1),(5)}{\le} \frac{1}{2^k}, \quad w \in G_k,\ k \ge k_0.$$

Hence $f(a,\cdot) \in \mathcal{O}(G)$.

Take an arbitrary $b \in B$, say $b \in B_k$ for $k \ge k_0$. Then we get

$$|\varphi_k(z)\psi_k(b)| \overset{(3)}{\le} |\varphi_k(z)| e^{-m_k n_k Q_k} \overset{(6)}{\le} \frac{1}{2^k}, \quad z \in D_k,\ k \ge k_0.$$

Thus $f(\cdot,b) \in \mathcal{O}(D)$. Consequently, $f \in \mathcal{O}_s(\boldsymbol{X})$, $\boldsymbol{X} := (D \times G) \cup (D \times B)$. To prove that $f \notin \mathcal{O}(D \times G)$ it suffices to show that $|f(z_k, w_0)| \ge k$, $k \in \mathbb{N}$. We have

$$\begin{aligned} |f(z_k,w_0)| &\ge |\varphi_k(z_k)\psi_k(w_0)| - \sum_{s=1}^{k-1} |\varphi_s(z_k)\psi_s(w_0)| - \sum_{s=k+1}^\infty |\varphi_s(z_k)\psi_s(w_0)| \\ &\overset{(1),(2),(4),(5)}{\ge} M_k e^{-2m_k Q_k} - (M_k e^{-2m_k Q_k} - k - 1) - \sum_{s=k+1}^\infty \frac{1}{2^s} \ge k. \qquad \square \end{aligned}$$

4.2 Crosses

After Terada's results (cf. Theorems 4.1.1, 4.1.4) it was clear that the next step should be a solution of the following general problem (cf. Chapter 1).

(S-$\mathcal{O}_{\mathrm{C}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty sets $A \subset D$, $B \subset G$. Define the *cross*

$$\boldsymbol{X} = \boldsymbol{K}(A, B; D, G) := (A \times G) \cup (D \times B).$$

We say that a function $f : \boldsymbol{X} \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{X}$ ($f \in \mathcal{O}_s(\boldsymbol{X})$) if:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$.

We ask whether there exists an open neighborhood $\widehat{\boldsymbol{X}} \subset D \times G$ of $\boldsymbol{X}$ such that every function $f \in \mathcal{O}_s(\boldsymbol{X})$ extends holomorphically to $\widehat{\boldsymbol{X}}$. Observe that the Hukuhara problem was just the case where $A = D$ and $\widehat{\boldsymbol{X}} = D \times G$. Notice once again that different crosses may have the same geometric image.

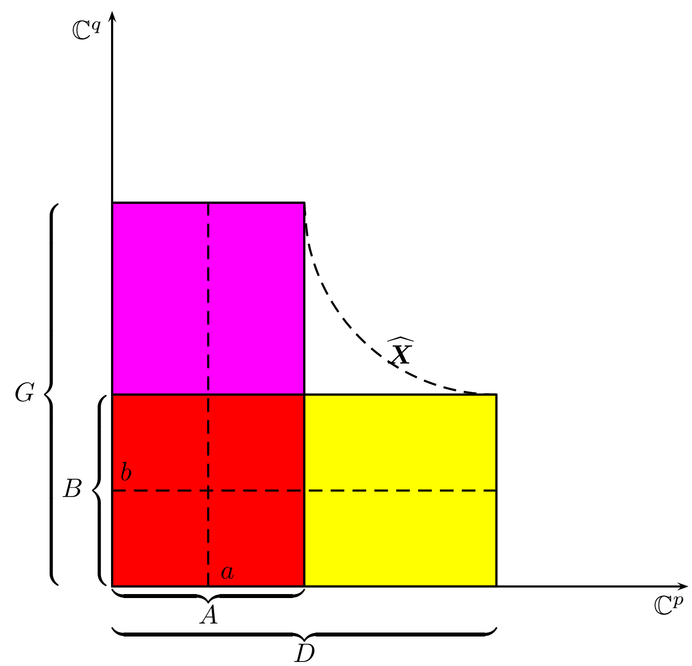

Figure 4.2.1. $\boldsymbol{X} = (A \times G) \cup (D \times B) \subset \widehat{\boldsymbol{X}}$.

Remark 4.2.1. To get an insight into the problem consider the following elementary situation.

Recall that a domain $\Omega \subset \mathbb{C}^n$ is a *Reinhardt domain* if for every $a = (a_1, \dots, a_n) \in \Omega$ the set

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = |a_j|,\ j = 1, \dots, n\}$$

is contained in Ω (cf. [Jar-Pfl 2008], Definition 1.5.2). A domain $\Omega \subset \mathbb{C}^n$ is a*complete Reinhardt domain* if for every $a = (a_1, \dots, a_n) \in \Omega$ the set

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| \le |a_j|,\ j = 1, \dots, n\}$$

is contained in Ω (cf. [Jar-Pfl 2008], Definition 1.3.8) A Reinhardt (1) domain $\Omega \subset \mathbb{C}^n$ with $0 \in \Omega$ is a domain of holomorphy iff Ω is *logarithmically convex*, i.e. the set $\log \Omega$ is convex (cf. [Jar-Pfl 2008], Theorem 1.11.13), where

$$\log \Omega := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in \Omega.$$

Moreover, if $\Omega \subset \mathbb{C}^n$ is a Reinhardt domain with $0 \in \Omega$, then its envelope of holomorphy is a complete Reinhardt $\widehat{\Omega}$ with $\log \widehat{\Omega} = \operatorname{conv}(\log \Omega)$, where $\operatorname{conv}(A)$ denotes the convex hull of A (cf. [Jar-Pfl 2008], § 1.12).

Assume that each of the sets $A \subset D \subset \mathbb{C}^p$, $B \subset G \subset \mathbb{C}^q$ is complete Reinhardt domains of holomorphy. Then $\Omega := (A \times G) \cup (D \times B)$ is a Reinhardt domain in $\mathbb{C}^{p+q}$ with $0 \in \Omega$. Observe that, by the Hartogs lemma (cf. Lemma 2.1.4), $\mathcal{O}_s(\boldsymbol{X}) = \mathcal{O}(\Omega)$. Consequently, every function $f \in \mathcal{O}_s(\boldsymbol{X})$ extends holomorphically to the envelope $\widehat{\Omega}$ of holomorphy of Ω, which satisfies $\log \widehat{\Omega} = \operatorname{conv}(\log \Omega)$. Note that $\log \Omega = (\log A \times \log G) \cup (\log D \times \log B)$, where each of the sets $\log A \subset \log D \subset \mathbb{R}^p$, $\log B \subset \log G \subset \mathbb{R}^q$ is a convex domain. Thus

$$\begin{aligned} \log \widehat{\Omega} = \{t(a', b') + (1-t)(a'', b'') :& \\ &(a', b') \in \log A \times \log G,\ (a'', b'') \in \log D \times \log B,\ t \in [0,1]\}. \end{aligned}$$

[Will be completed. . . . .]

Finally,

$$\widehat{\Omega} = \{(z, w) \in D \times G : h^*_{A,D}(z) + h^*_{B,G}(w) < 1\}.$$

In the general situation, in view of the above remark, we put

$$\widehat{\boldsymbol{X}} = \widehat{\boldsymbol{K}}(A, B; D, G) := \{(z, w) \in D \times G : \omega_{A,D}(z) + \omega_{B,G}(w) < 1\}.$$

4.2.1 N–fold crosses

The extension problem (S-$\mathcal{O}_{\mathrm{C}}$) may be generalized to more complicated objects.

(1) Karl August Reinhardt (1895–1941) — German mathematician.

Definition 4.2.2. Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $\varnothing \neq A_j \subset D_j$, $j = 1, \dots, N$, $N \geq 2$. Let

$$A'_j := A_1 \times \dots \times A_{j-1},\ j = 2, \dots, N, \quad A''_j := A_{j+1} \times \dots \times A_N,\ j = 1, \dots, N-1.$$

Similarly, for $a = (a_1, \dots, a_N) \in A_1 \times \dots \times A_N$, we write $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$. Define the *$N$–fold cross*

$$\boldsymbol{X} = \boldsymbol{K}(A_1, \dots, A_N; D_1, \dots, D_N) = \boldsymbol{K}((A_j, D_j)_{j=1}^N) := \bigcup_{j=1}^{N} (A'_j \times D_j \times A''_j),$$

where $A'_1 \times D_1 \times A''_1 := D_1 \times A''_1$ and $A'_N \times D_N \times A''_N := A'_N \times D_N$.

Define the *center of the cross* $\boldsymbol{X}$

$$\boldsymbol{c}(\boldsymbol{X}) := A_1 \times \dots \times A_N.$$

Observe that the geometric image of $\boldsymbol{X}$ may be the same for different systems $(A_j, D_j)_{j=1}^N$. For example, if $A_j = D_j$, $j = 1, \dots, N-1$, then the geometric image of $\boldsymbol{X}$ is just the Cartesian product $D_1 \times \dots \times D_N$ independently of A_N.

We say that a function $f : \boldsymbol{X} \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{X}$ ($f \in \mathcal{O}_s(\boldsymbol{X})$) if for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$, the function

$$D_j \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic in D_j.

(S-$\mathcal{O}_{\mathrm{C}}$) We ask whether there exists an open neighborhood $\widehat{\boldsymbol{X}} \subset D_1 \times \dots \times D_N$ of $\boldsymbol{X}$ such that every $f \in \mathcal{O}_s(\boldsymbol{X})$ extends holomorphically to $\widehat{\boldsymbol{X}}$.

Define

$$\begin{aligned}\widehat{\boldsymbol{X}} = \widehat{\boldsymbol{K}}(A_1, \dots, A_N; D_1, \dots, D_N) = \widehat{\boldsymbol{K}}((A_j, D_j)_{j=1}^N) :& \\ = \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^{N} \omega_{A_j, D_j}(z_j) < 1\Big\}&.\end{aligned}$$

Exercise 4.2.3. Prove the following properties of N–fold crosses.

(a) $\boldsymbol{X}$ is connected.

(b) If $A_1, \dots, A_N \notin \mathcal{PLP}$, then $\boldsymbol{X} \notin \mathcal{PLP}$.

(c) If $A_1, \dots, A_N$ are locally pluriregular, then $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$ (use Remark 3.4.5).

(d) If $D_1, \dots, D_N$ are domains of holomorphy, then $\widehat{\boldsymbol{X}}$ is a region of holomorphy (use Theorem 3.5.6(e)).

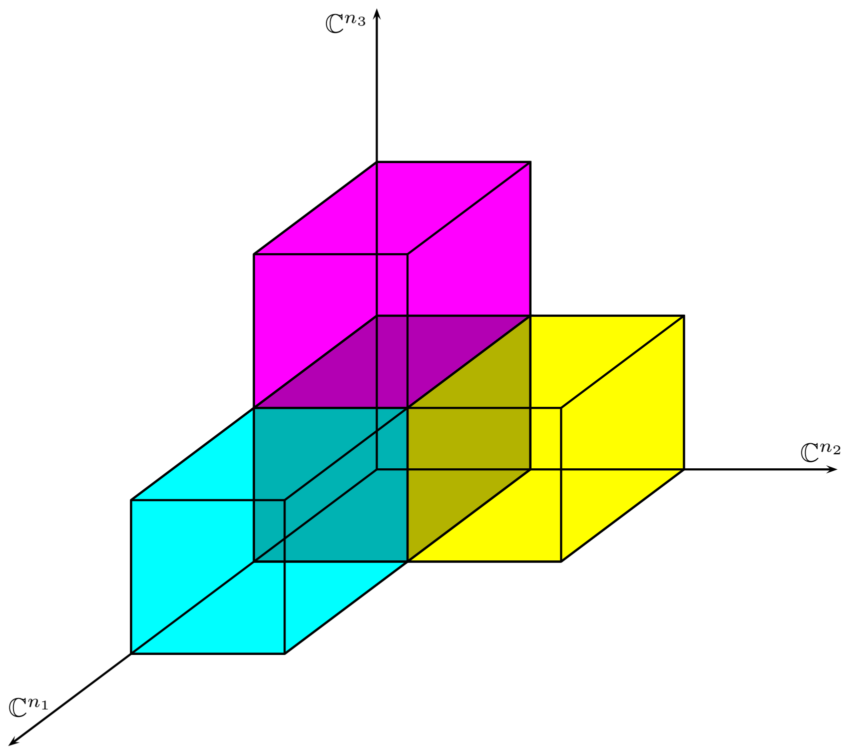

Figure 4.2.2. $\boldsymbol{X} = (D_1 \times A_2 \times A_3) \cup (A_1 \times D_2 \times A_3) \cup (A_1 \times A_2 \times D_3)$.

(e) If $(D_{j,k})_{k=1}^{\infty}$ is a sequence of subdomains of D_j with $D_{j,k} \subset D_{j,k+1}$, $A_{j,k} := D_{j,k} \cap A_j \neq \varnothing$, $k \in \mathbb{N}$, and $D_j = \bigcup_{k=1}^{\infty} D_{j,k}$, $j = 1, \dots, N$, then

$$\widehat{\boldsymbol{K}}((A_j, D_{j,k})_{j=1}^{N}) \nearrow \widehat{\boldsymbol{X}}.$$

(f) If $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$ (cf. Definition 3.1.1), $(D_{j,k})_{k=1}^{\infty}$ is a sequence of subdomains of D_j such that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, $j = 1, \dots, N$, then

$$\widehat{\boldsymbol{K}}((A_{j,k}, D_{j,k})_{j=1}^{N}) \nearrow \widehat{\boldsymbol{X}}$$

(use Proposition 3.4.12).

(g) If $A_1, \dots, A_N$ are locally pluriregular, then $\widehat{\boldsymbol{X}}$ is connected.

Hint: We may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$. It suffices to show that every point $a = (a_1, \dots, a_N) \in \widehat{\boldsymbol{X}}$ may be connected in $\widehat{\boldsymbol{X}}$ with a point from $\boldsymbol{c}(\boldsymbol{X})$. Put $\varepsilon := \sum_{j=1}^{N-1} h^*_{A_j,D_j}(a_j)$. If $\varepsilon = 0$, then $\{(a_1, \dots, a_{N-1})\} \times D_N \subset \widehat{\boldsymbol{X}}$. If $\varepsilon > 0$, then by Proposition 3.4.13, the connected component S

of the open set $\{z_N \in D_N : h^*_{A_N,D_N}(z_N) < 1-\varepsilon\}$ that contains a_N, intersects A_N. Consequently, a may be connected inside of $\widehat{\boldsymbol{X}}$ with $(a_1,\dots,a_{N-1},b_N)$, where $b_N \in A_N$. Repeating the above argument, we easily show that a may be connected inside of $\widehat{\boldsymbol{X}}$ with a point $b \in \boldsymbol{c}(\boldsymbol{X})$.

(h) If $P_j \in \mathcal{PLP}(D_j)$, $j = 1,\dots,N$, then

$$\widehat{\boldsymbol{K}}((A_j \setminus P_j, D_j)_{j=1}^N) = \widehat{\boldsymbol{X}}$$

(use Proposition 3.4.11(d)). In particular,

$$\widehat{\boldsymbol{K}}((A_j \cap A_j^*, D_j)_{j=1}^N) = \widehat{\boldsymbol{X}},$$

where A_j^* is given in Definition 3.4.4, $j = 1,\dots,N$ (use Proposition 3.4.6).

(i) If $A_1,\dots,A_N$ are locally pluriregular and

$$\widehat{\boldsymbol{Y}} := \widehat{\boldsymbol{K}}((A_j, D_j)_{j=1}^{N-1}) \subset D_1 \times \dots \times D_{N-1},$$

then

$$\widehat{\boldsymbol{K}}(A'_N, A_N; \widehat{\boldsymbol{Y}}, D_N) = \widehat{\boldsymbol{X}}$$

(use Proposition 3.4.14).

(j) Assume that $B_j \subset A_j$, $B_j \notin \mathcal{PLP}$, $j = 1,\dots,N$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be such that $f = 0$ on $B_1 \times \dots \times B_N$. Then $f = 0$ on $\boldsymbol{X}$.

Hint: It suffices to show that $f = 0$ on $\boldsymbol{c}(\boldsymbol{X})$. Fix a point $(a_1,\dots,a_N) \in \boldsymbol{c}(\boldsymbol{X})$. We know that that for any $b_j \in B_j$, $j = 1,\dots,N-1$, we have $f(b_1,\dots,b_{N-1},\cdot) = 0$ on B_N. Since $B_N \notin \mathcal{PLP}$, we conclude that $f(b_1, \dots, b_{N-1},\cdot) = 0$ on D_j and, therefore, $f(b_1,\dots,b_{N-1},a_N) = 0$. Now we repeat the same procedure with respect to the $(N-1)$-th variables: $f(b_1, \dots, b_{N-2},\cdot,a_N) = 0$ on B_{N-1} and hence $f(b_1,\dots,b_{N-2},a_{N-1},a_N) = 0$. Finite induction finishes the proof.

4.3 Main cross theorem (1969 — 2001)

The problem of holomorphic continuation of separately holomorphic functions defined on N–fold crosses has been investigated in several paper, e.g. [Ber 1912], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981a], [Shi 1989], [Ngu-Sic 1991], [Ngu-Zer 1991], [Ngu-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002] and has led to the following result. The breaking point of the proof was made in [Zah 1976].

Theorem 4.3.1. *Assume that D_j is a Riemann domain over $\mathbb{C}^{n_j}$ such that $\mathcal{O}(D_j)$ separates points in D_j (cf. Definition 3.1.17) and $A_j \subset D_j$ is locally pluriregular, $j = 1,\dots,N$. Put $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$. Then*

(*) *there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{X}} |f|$.*

A proof will be presented in § 4.8.

Remark 4.3.2. (a) Let $(X,p) \in \mathfrak{R}_c(\mathbb{R}^n)$ be such that $\mathcal{O}(X)$ separates points in X. Let $\alpha : (X,p) \longrightarrow (\widehat{X},\widehat{p})$ be a maximal holomorphic extension (cf. Theorem 3.1.16). Then α is injective (Remark 3.1.18). Thus we may assume that X is a subdomain of $\widehat{X}$, $p = \widehat{p}|_X$, $\alpha = \mathrm{id}$.

(b) Let $\widehat{D}_j$ denote the envelope of holomorphy of D_j with D_j being a subdomain of $\widehat{D}_j$ (cf. (a)). Then for every function $g \in \mathcal{O}(D_j)$ there exists (exactly one) extension $\widehat{g} = \boldsymbol{\mathcal{E}}_j(g) \in \mathcal{O}(\widehat{D}_j)$ with $\widetilde{g} = g$ on D_j and $\sup_{\widehat{D}_j} |\widehat{g}| = \sup_{D_j} |g|$, $j = 1, \dots, N$ (cf. Remark 3.1.11).

Put $\boldsymbol{Y} = \boldsymbol{K}((A_j, \widehat{D}_j)_{j=1}^N)$. Since $\omega_{A_j,\widehat{D}_j} \le \omega_{A_j,D_j}$ on D_j, $j = 1, \dots, N$, we get $\widehat{\boldsymbol{X}} \subset \widehat{\boldsymbol{Y}}$. For $f \in \mathcal{O}_s(\boldsymbol{X})$ define $g : \boldsymbol{Y} \longrightarrow \mathbb{C}$,

$$g(a_j', z_j, a_j'') := \boldsymbol{\mathcal{E}}_j(f(a_j', \cdot, a_j''))(z_j), \quad (a_j', z_j, a_j'') \in A_j' \times \widehat{D}_j \times A_j'', \quad j = 1, \dots, N.$$

It is clear that $g \in \mathcal{O}_s(\boldsymbol{Y})$ and $\sup_{\boldsymbol{Y}} |g| = \sup_{\boldsymbol{X}} |f|$. Suppose that g extends to an $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ with $\sup_{\widehat{\boldsymbol{Y}}} |\widehat{g}| = \sup_{\boldsymbol{Y}} |g|$. Then $\widehat{f} := \widehat{g}|_{\widehat{\boldsymbol{X}}}$ gives the extension to $\widehat{\boldsymbol{X}}$ with $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{X}} |f|$. Consequently,

- *in the extension problem described in Theorem 4.3.1 we may always assume that $\widehat{D}_j = D_j$, i.e. D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$.*

Sometimes the assumption that $A_1, \dots, A_N$ are locally pluriregular is too restrictive and it is better to consider the following equivalent form of Theorem 4.3.1.

Theorem 4.3.3. *Let D_j be as in Theorem 4.3.1 and let $A_j \subset D_j$ be non-pluripolar, $j = 1, \dots, N$. Put*

$$\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N), \quad \boldsymbol{Y} := \boldsymbol{K}((A_j \cap A_j^*, D_j)_{j=1}^N)$$

(recall that $\widehat{\boldsymbol{Y}} = \widehat{\boldsymbol{X}}$ — cf. Exercise 4.2.3(h)). Let $f \in \mathcal{O}_s(\boldsymbol{X})$. Then
(**) *there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{Y}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| \le \sup_{\boldsymbol{X}} |f|$.*

It is obvious that Theorem 4.3.3 $\Longrightarrow$ Theorem 4.3.1. Conversely, since $A_j \cap A_j^*$ is locally pluriregular (cf. Proposition 3.4.11(f)), $j = 1, \dots, N$, Theorem 4.3.1 implies that there exists an $f \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{Y}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{Y}} |f| \le \sup_{\boldsymbol{X}} |f|$.

Remark 4.3.4. Let D_j, A_j, $j = 1, \dots, N$, $\boldsymbol{X}$, and $\widehat{\boldsymbol{X}}$ be as in Theorem 4.3.1.

In the case where $N = 2$ we will always write $D := D_1$, $p := n_1$, $G := D_2$, $q := n_2$, $A := A_1$, $B := A_2$.

We present below procedures which allow us to prove Theorems 4.3.1 and 4.3.3 under some additional useful assumptions.

(P1) Let $(D_{j,k})_{k=1}^\infty$ be an exhaustion sequence for D_j with $A_{j,k} := A_j \cap D_{j,k} \neq \varnothing$ (observe that $A_{j,k}$ is locally pluriregular), $k \in \mathbb{N}$, $j = 1, \dots, N$. Put $\boldsymbol{X}_k := \boldsymbol{K}((A_{j,k}, D_{j,k})_{j=1}^N)$. Notice that $\boldsymbol{X}_k \nearrow \boldsymbol{X}$, $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$ (cf. Exercise 4.2.3(e)).

Obviously, if $f \in \mathcal{O}_s(\boldsymbol{X})$, then $f|_{\boldsymbol{X}_k} \in \mathcal{O}_s(\boldsymbol{X}_k)$, $k \in \mathbb{N}$. Suppose that for each k there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\widehat{f}_k = f$ on $\boldsymbol{X}_k$ and $\sup_{\widehat{\boldsymbol{X}}_k} |\widehat{f}_k| \leq \sup_{\boldsymbol{X}_k} |f|$. Then (*) is true.

Indeed, since $\widehat{f}_{k+1} = \widehat{f}_k$ on the non-pluripolar set $\boldsymbol{X}_k$, we conclude that $\widehat{f}_{k+1} = \widehat{f}_k$ in the domain $\widehat{\boldsymbol{X}}_k$. Thus, we obtain an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ with $\widehat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{k\in\mathbb{N}} \sup_{\widehat{\boldsymbol{X}}_k} |\widehat{f}_k| \leq \sup_{k\in\mathbb{N}} \sup_{\boldsymbol{X}_k} |f| = \sup_{\boldsymbol{X}} |f|$.

In particular:

- (†) *we may always assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$,*
- *if $D_1, \dots, D_N$ are Riemann domains of holomorphy, then we may assume that they are strongly pseudoconvex with real analytic boundaries* (cf. § 3.5.1),
- *we may always assume that $f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N) \in \mathcal{O}(\overline{D}_j)$ for every $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$, $j = 1, \dots, N$.*

(P2) Assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$ (as in (†)). Let $A_{j,k} \nearrow A_j$, $j = 1, \dots, N$. We assume that each set $A_{j,k}$ is non-pluripolar. Put

$$\boldsymbol{X}_k := \boldsymbol{K}((A_{j,k}, D_j)_{j=1}^N), \quad \boldsymbol{Y}_k := \boldsymbol{K}((A_{j,k} \cap A_{j,k}^*, D_j)_{j=1}^N).$$

Observe that $\boldsymbol{Y}_k \subset \boldsymbol{X}_k \nearrow \boldsymbol{X}$ and $\widehat{\boldsymbol{Y}}_k = \widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$ (cf. Proposition 3.4.6 and Exercise 4.2.3(f)). Suppose that (**) holds for each k, i.e. there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\widehat{f}_k = f$ on $\boldsymbol{Y}_k$ and $\sup_{\widehat{\boldsymbol{X}}_k} |\widehat{f}_k| \leq \sup_{\boldsymbol{X}_k} |f| \leq \sup_{\boldsymbol{X}} |f|$. Then (*) is true.

Indeed, since $\boldsymbol{Y}_k \notin \mathcal{PLP}$, we get $\widehat{f}_{k+1} = \widehat{f}_k$ on $\widehat{\boldsymbol{X}}_k$. Thus we get a function $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| \leq \sup_{\boldsymbol{X}} |f|$ and $\widehat{f} = f$ on each $\boldsymbol{Y}_k$. It remains to use Exercise 4.2.3(j) to show that $\widehat{f} = f$ on every $\boldsymbol{X}_k$ and hence $\widehat{f} = f$ on $\boldsymbol{X}$.

In particular,

- *if $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$ (as in (†)), then we may always assume that $A_j \Subset D_j$, $j = 1, \dots, N$.*

(P3) *We may additionally assume that $N = 2$.*

Indeed, suppose that the result is true for $N = 2$. We proceed by induction on $N \geq 2$. Suppose that the theorem is true for $N - 1 \geq 2$. Put $\boldsymbol{Y} := \boldsymbol{K}((A_j, D_j)_{j=1}^{N-1})$, $\boldsymbol{Z} := \boldsymbol{K}(A_N', A_N; \widehat{\boldsymbol{Y}}, D_N)$. Observe that if $z_N \in A_N$, then $f(\cdot, z_N) \in \mathcal{O}_s(\boldsymbol{Y})$. By inductive assumption there exists an $\widehat{f}_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ with $\widehat{f}_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{Y}$ and $\sup_{\widehat{\boldsymbol{Y}}} |\widehat{f}_{z_N}| \leq \sup_{\boldsymbol{Y}} |f(\cdot, z_N)| \leq \sup_{\boldsymbol{X}} |f|$. Define $g : \boldsymbol{Z} \longrightarrow \mathbb{C}$,

$$g(z', z_N) := \begin{cases} \widehat{f}_{z_N}(z'), & \text{if } (z', z_N) \in \widehat{\boldsymbol{Y}} \times A_N \\ f(z', z_N), & \text{if } (z', z_N) \in A_N' \times D_N \end{cases}.$$

Obviously, g is well-defined and $g \in \mathcal{O}_s(\boldsymbol{Z})$. It is clear that holomorphic functions on $\widehat{\boldsymbol{Y}}$ separate points and A_N' is locally pluriregular. Using the case $N = 2$, we find an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}})$ with $\widehat{f} = g$ on $\boldsymbol{Z}$ and $\sup_{\widehat{\boldsymbol{Z}}} |\widehat{f}| \leq \sup_{\boldsymbol{Z}} |g| \leq \sup_{\boldsymbol{X}} |f|$. It remains to recall that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$ (cf. Exercise 4.2.3(i)).

Observe that the above proof shows that if the main theorem is true for $N = 2$ and bounded functions $f \in \mathcal{O}_s(\boldsymbol{X})$, then it holds for arbitrary N and bounded separately holomorphic functions.

(P4) *If $N = 2$, then we may additionally assume that f is bounded.*

Indeed, we already know (by (P1)) that we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, 2$ (as in (†)) and that for arbitrary $(a_1, a_2) \in A_1 \times A_2$ we have $f(a_1, \cdot) \in \mathcal{O}(\overline{D}_2)$, $f(\cdot, a_2) \in \mathcal{O}(\overline{D}_1)$. Define

$$A_{1,k} := \{z_1 \in A_1 : |f(z_1, \cdot)| \le k \text{ on } D_2\},$$
$$A_{2,k} := \{z_2 \in A_2 : |f(\cdot, z_2)| \le k \text{ on } D_1\}, \quad k \in \mathbb{N}.$$

Observe that $A_{j,k} \nearrow A_j$. We may assume that $A_{j,k} \notin \mathcal{PLP}$, $k \gg 1$, $j = 1, 2$. Observe that $|f| \le k$ on $\boldsymbol{K}(A_{1,k}, A_{2,k}; D_1, D_2)$, $k \in \mathbb{N}$. Now we only need to use (P2).

(P5) Assume that $N = 2$ and let $f \in \mathcal{O}_s(\boldsymbol{X})$ be bounded.

(a) If B is an identity set, then $f|_{A \times G}$ is continuous.

(b) If B is an identity set and $A \Subset D$, then f extends to an $\widetilde{f} \in \mathcal{O}_s(\boldsymbol{Z})$, where $\boldsymbol{Z} := \boldsymbol{K}(\overline{A}, B; D, G)$.

(c) If A, B are identity sets, then f is continuous on $\boldsymbol{X}$.

(d) If $A \Subset D$, $B \Subset G$ are identity sets, then f extends to a continuous function $\widetilde{f} \in \mathcal{O}_s(\boldsymbol{Z})$, with $\boldsymbol{Z} := \boldsymbol{K}(\overline{A}, \overline{B}; D, G)$.

Indeed, for the proof of (a) let $A \times G \ni (z_s, w_s) \longrightarrow (z_0, w_0) \in A \times G$, $f(z_s, w_s) \longrightarrow \alpha$. The sequence of holomorphic functions $(f(z_s, \cdot))_{s=1}^{\infty}$ is bounded. Consequently, by the Montel theorem, we may assume that $f(z_s, \cdot) \longrightarrow g$ locally uniformly in G with $g \in \mathcal{O}(G)$. In particular, $f(z_s, w_s) \longrightarrow g(w_0) = \alpha$. On the other hand, $f(z_s, w) \longrightarrow f(z_0, w)$ for $w \in B$. Hence, $g = f(z_0, \cdot)$ on B and, finally, $g = f(z_0, \cdot)$ on G, which gives (a).

(b) Let $A \ni z_k \longrightarrow z_0 \in \overline{A} \subset D$. By a Montel argument there exists a subsequence $(k_s)_{s=1}^{\infty}$ such that $f(z_{k_s}, \cdot) \longrightarrow g^{z_0}$ locally uniformly in G. Observe that $g^{z_0}(w) = f(z_0, w)$, $w \in B$. Consequently, g^{z_0} does not depend neither on the subsequences and nor the sequence $(z_k)_{k=1}^{\infty} \subset A$ with $z_k \longrightarrow z_0$. Define

$$\widetilde{f}(z, w) := \begin{cases} f(z, w), & \text{on } \boldsymbol{X}, \\ g^z(w), & \text{on } \overline{A} \times G \end{cases}.$$

Then $\widetilde{f}$ is well defined and separately holomorphic on $\boldsymbol{Y}$.

(c) In view of (a), to prove that f is continuous on $\boldsymbol{X}$, we only need to consider the case where

$$A \times G \ni (z_s, w_s) \longrightarrow (z_0, w_0) \in (D \times B) \setminus (A \times G),$$

$f(z_s, w_s) \longrightarrow \alpha$. Analogously as in the first part of the proof, we may assume that $f(z_s, \cdot) \longrightarrow g$ locally uniformly in G. Hence $g(w_s) \longrightarrow g(w_0) = \alpha$. Moreover, $g(w) = f(z_0, w)$ on $w \in B$. In particular, $g(w_0) = f(z_0, w_0)$, which finishes the proof.

(d) follows from (b).

(P6) Assume that $N = 2$, D, G are relatively compact (as (†)), and $A \Subset D$, $B \Subset G$ are non-pluripolar. Put

$$\boldsymbol{Y} := \boldsymbol{K}(A \cap A^*, B \cap B^*; D, G), \quad \boldsymbol{Z} := \boldsymbol{K}(\overline{A}, \overline{B}; D, G),$$
$$\boldsymbol{W} := \boldsymbol{K}(\overline{A} \cap (\overline{A})^*, \overline{B} \cap (\overline{B})^*; D, G).$$

Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be bounded. We know by (P5) that f extends a continuous $\widetilde{f} \in \mathcal{O}_s(\boldsymbol{Z})$. Suppose that (**) holds for $\boldsymbol{Z}$, i.e. there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}})$ such that $\widehat{f} = \widetilde{f}$ on $\boldsymbol{W}$ and $\sup_{\widehat{\boldsymbol{Z}}} |\widehat{f}| \leq \sup_{\boldsymbol{Z}} |\widetilde{f}|$. Observe that $\boldsymbol{Y} \subset \boldsymbol{Z}$ and $\widehat{\boldsymbol{X}} = \widehat{\boldsymbol{Y}} \subset \widehat{\boldsymbol{W}} = \widehat{\boldsymbol{Z}}$. Thus $\widehat{f}|_{\widehat{\boldsymbol{X}}}$ solves (**) for $\boldsymbol{X}$.

(P7) *Summarizing, to prove Theorem 4.3.1 in its full generality, it suffices to prove Theorem 4.3.3 under the following additional assumptions:*
- $N = 2$,
- *D, G are strongly pseudoconvex domains with real analytic boundaries,*
- *A, B are compact non-pluripolar,*
- $f(a, \cdot) \in \mathcal{O}(\overline{G})$, $a \in A$, $f(\cdot, b) \in \mathcal{O}(\overline{D})$, $b \in B$,
- $|f| \leq 1$ *on* $\boldsymbol{X}$ *(and* f *is continuous on* $\boldsymbol{X}$*).*

4.4 Siciak's approach

The aim of this section is present some of Siciak's results from the paper [Sic 1969a]. J. Siciak was the one who initiated modern theory of separately holomorphic functions on crosses. To be historically correct, one should mention that already in 1911 Bernstein (cf. [Ber 1912]) discussed the following general 2-fold cross situation: $n_1 = n_2 = 1$, $D_1 = D_2 =$ an ellipse with foci $1, -1$, $A_1 = A_2 = [-1, 1]$, $f \in \mathcal{O}_s(\boldsymbol{K}(A_1, A_2; D_1 D_2))$ bounded. It seems that this result has been not recognized for a long time up to a paper by Akhiezer and Ronkin (cf. [Akh-Ron 1973], see also [Ron 1977]).

Observe that in the following results the domains $D_1, \dots, D_N$ and sets A_1, $\dots$, A_N satisfy very restrictive assumptions, much more restrictive than those considered in Remark 4.3.4. Since not all reduction procedures (from Remark 4.3.4) preserve these special additional assumptions, we can apply only some of them.

Theorem 4.4.1. *Let $D \subset \mathbb{C}^p$ be a domain and let $G_1, \dots, G_q \subset \mathbb{C}$ be simply connected domains symmetric with respect to the real axis $\mathbb{R}$. Assume that $A \subset D$ is locally pluriregular and $B_j = [a_j, b_j] \subset G_j \cap \mathbb{R}$, $a_j < b_j$, $j = 1, \dots, q$. Put $\boldsymbol{X} := \boldsymbol{K}(A, B_1, \dots, B_q; D, G_1, \dots, G_q)$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be bounded on $\boldsymbol{X}$. Then there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{X}} |f|$. If A is additionally compact, then the result remains true for locally bounded $f \in \mathcal{O}_s(\boldsymbol{X})$.*

Notice that the assumptions that f is bounded or locally bounded are, in fact, superfluous (cf. Theorem 4.3.1).

We need some auxiliary results, whose proofs may be found e.g. in [Gol 1983].

Lemma 4.4.2. *Let $D \subset \mathbb{H}^+ := \{x + iy : x \in \mathbb{R},\ y > 0\}$ be a simply connected domain and let $L \subset \mathbb{R} \cap \partial D$ be an open interval. Assume that $g : D \longrightarrow \mathbb{H}^+$ is a biholomorphic mapping. Then g extends to a continuous injective mapping $\widetilde{g} : D \cup L \longrightarrow \overline{\mathbb{H}}^+$ with $\widetilde{g}(L) \subset \mathbb{R}$.*

Lemma 4.4.3. *For every $-\infty \le c < -1 < 1 < d \le +\infty$ there exist $0 < \rho \le +\infty$ and a biholomorphic mapping $h : \mathbb{H}^+ \longrightarrow \mathfrak{R}$ with*

$$\mathfrak{R} = \mathfrak{R}(\rho) := \{u + iv : u \in (0, \rho),\ v \in (0, \pi)\},$$

such that $\widetilde{h}(c) = \rho + i\pi$, $\widetilde{h}(-1) = i\pi$, $\widetilde{h}(1) = 0$, $\widetilde{h}(d) = \rho$, where $\widetilde{h}$ denotes the extension of h to a homeomorphic mapping $\widetilde{h} : \overline{\mathbb{H}}^+ \longrightarrow \overline{\mathfrak{R}}$ (which exists by the Carathéodory theorem).

Corollary 4.4.4. *Let $D \subset \mathbb{H}^+$ be a simply connected domain such that $(c, d) \subset \mathbb{R} \cap \partial D$ with $-\infty \le c < -1 < 1 < d \le +\infty$. Then there exist $0 < \rho \le +\infty$ and a biholomorphic mapping $g : \mathbb{H}^+ \longrightarrow \mathfrak{R}$ with $\mathfrak{R} = \mathfrak{R}(\rho)$ that extends to a continuous injective mapping $\widetilde{g} : D \cup (c, d) \longrightarrow \overline{\mathfrak{R}}$ such that $\widetilde{g}((c, -1]) = (\rho + i\pi, i\pi]$, $\widetilde{g}(-1) = i\pi$, $\widetilde{g}([-1, 1]) = [i\pi, 0]$, $\widetilde{g}(1) = 0$, $\widetilde{g}([1, d)) = [0, \rho)$.*

Lemma 4.4.5. *Let $D \subset \mathbb{C}$ be a simply connected domain symmetric with respect to a line L. Let $[a, b] \subset L \cap D$, $a \ne b$. Then there exist uniquely determined $R \in (1, +\infty]$ and*

$$g : D \longrightarrow \mathfrak{E} := \{w \in \mathbb{C} : |w + \sqrt{w^2 - 1}| < R\}, \quad g \text{ biholomorphic},$$

such that $g([a, b]) = [-1, 1]$, $g(a) = -1$, $g(b) = 1$, and the branch of $\sqrt{w^2 - 1}$ is chosen so that $\sqrt{x^2 - 1} > 0$ for $x \in (1, +\infty)$.

Proof. Let

$$g_1(z) := \frac{2}{b - a}\Big(z - \frac{a + b}{2}\Big), \quad z \in \mathbb{C}.$$

Then g_1 maps biholomorphically D onto the simply connected domain $D_1 := g_1(D)$ that is symmetric with respect to the real axis, $g_1([a, b]) = [-1, 1]$, $g_1(a) = -1$, $g_1(b) = 1$. If $D_1 = \mathbb{C}$, then we put $R := +\infty$, $g := g_1$.

Assume that $D_1 \ne \mathbb{C}$. Let $(c, d) := D_1 \cap \mathbb{R}$ (observe that $D_1 \cap \mathbb{R}$ must be connected because D_1 is symmetric and simply connected — Exercise). Put

$$D_1^+ := \{z \in D_1 : \operatorname{Im} z > 0\};$$

observe that D_1^+ is a simply connected domain. Then there exist $\rho \in (0, +\infty]$ and a biholomorphic mapping

$$g_2 : D_1^+ \longrightarrow D_2 := \{u + iv : u \in (0, \rho),\ v \in (0, \pi)\}$$

such that $\widetilde{g}_2((c, -1]) = (\rho + i\pi, i\pi]$, $\widetilde{g}_2(-1) = i\pi$, $\widetilde{g}_2([-1, 1]) = [i\pi, 0]$, $\widetilde{g}_2(1) = 0$, $\widetilde{g}_2([1, d)) = [0, \rho)$ (Corollary 4.4.4). The mapping $g_3 := \exp$ maps biholomorphically D_2 onto the domain

$$D_3 := \{w \in \mathbb{C} : 1 < |w| < R := e^\rho,\ \operatorname{Im} w > 0\},$$

$g_3([\rho + i\pi, i\pi] = [-R, -1]$, $g_3([i\pi, 0]) = C^+ := \{\zeta \in \mathbb{T} : \operatorname{Im}\zeta \geq 0\}$, $g_3([0, \rho]) = [1, R]$.

Next, the Zhukovski mapping $g_4(z) := \frac{1}{2}(z + 1/z)$ maps D_3 onto the domain $\mathcal{E}^+ := \{w \in \mathcal{E} : \operatorname{Im} w > 0\}$, $g_4([-R, -1]) = [-\frac{1}{2}(R + 1/R), -1]$, $g_4(C^+) = [-1, 1]$, $g_4([1, R]) = [1, \frac{1}{2}(R + 1/R]$.

Let $g_5 := g_4 \circ g_3 \circ g_2 : D_1^+ \longrightarrow \mathcal{E}^+$,

$$g_6(w) := \begin{cases} g_5(w), & w \in D_1^+ \\ g_4 \circ g_3 \circ \widetilde{g}_2(w), & w \in (c, d) \\ \overline{g_5(\overline{w})}, & \overline{w} \in D_1^+ \end{cases}.$$

Finally, $g := g_6 \circ g_1$ satisfies all the required properties.

It remains to prove that R and g are uniquely determined. Suppose that $h : D \longrightarrow \mathcal{E}' := \{w \in \mathbb{C} : |w + \sqrt{w^2 - 1}| < R'\}$ is another biholomorphic mapping with the above properties. Then the biholomorphic mapping $f := g_4^{-1} \circ h \circ g^{-1} \circ g_4 : \mathbb{A}(1, R) \longrightarrow \mathbb{A}(1, R')$ with $f(\pm 1) = \pm 1$, where

$$\mathbb{A}(r_-, r_+) := \{z \in \mathbb{C} : r_- < |z| < r_+\}.$$

Consequently, $R' = R$ and $g \equiv h$. $\square$

Corollary 4.4.6. *Assume that D is a simply connected domain symmetric with respect to the real line $\mathbb{R}$ and $[a, b] \subset D \cap \mathbb{R}$, $a < b$. Let R and g be as in Lemma 4.4.5. Then the function $\Phi(z) := g(z) + \sqrt{g^2(z) - 1}$, $z \in D \setminus [a, b]$, is the unique biholomorphic mapping of $D \setminus [a, b]$ onto $\mathbb{A}(1, R)$ such that $\Phi(a) = -1$, $\Phi(b) = 1$. Moreover:*

- $\overline{\Phi(\overline{z})} = \Phi(z)$, $z \in D \setminus [a, b]$,
- *the limits* $\Phi(x + i0) := \lim_{y \to 0+} \Phi(x + iy)$ *and* $\Phi(x - i0) := \lim_{y \to 0-} \Phi(x + iy)$ *exist for* $x \in [a, b]$,
- $\Phi(x + i0) = \overline{\Phi(x - i0)} = 1/\Phi(x - i0)$, $x \in [a, b]$,
- *the functions* $(a, b) \ni x \longmapsto \Phi(x + i0)$ *and* $(a, b) \ni x \longmapsto \Phi(x - i0)$ *are real analytic,*
- *the function* $m(x) := g'(x)/\sqrt{1 - g^2(x)} = i\Phi'(x - i0)/\Phi(x - i0)$, $x \in (a, b)$, *is Riemann integrable and* $\int_a^b m(x)dx = \arcsin g|_a^b = \pi$.

Put

$$\Phi_k(z) := \frac{1}{2} \begin{cases} \Phi^k(z) + \Phi^{-k}(z), & z \in D \setminus [a, b] \\ \Phi^k(x - i0) + \Phi^{-k}(x - i0), & z = x \in [a, b] \end{cases}, \quad k \in \mathbb{Z}_+$$

Then:

- $\Phi_k \in \mathcal{O}(D)$,
- $|\Phi_k| \leq |\Phi|^k$,
- $|\Phi_k| \leq 1$ *on* $[a, b]$.

Corollary 4.4.7. *Let D, a, b, R, and Φ be as in Corollary 4.4.6. Then*

$$\omega_{[a,b],D} = h^*_{[a,b],D} = \frac{\log|\Phi|}{\log R}.$$

Proof. Let $u := \frac{\log |\Phi|}{\log R}$. It clear that $u \in \mathcal{H}(D \setminus [a,b])$ and $0 < u < 1$ on $D \setminus [a,b]$. Moreover, u is continuous on D and $u = 0$ on $[a,b]$. Hence $u \in \mathcal{SH}(D)$ and, consequently, $h_{[a,b],D} \geq u$. Applying the maximum principle to the subharmonic function $h^*_{[a,b],D} - u$ on $D \setminus [a,b]$, gives the converse inequality. □

Lemma 4.4.8. *Let D, a, b, R, m, Φ, and $(\Phi_k)_{k=1}^{\infty}$ be as in Corollary 4.4.6. Let $f \in \mathcal{O}(D)$. Then*

$$f(z) = \sum_{k=0}^{\infty} c_k \Phi_k(z), \quad z \in D, \tag{4.4.1}$$

where

$$c_k := \frac{2^{\operatorname{sgn} k}}{\pi} \int_a^b m(x) f(x) \Phi_k(x) dx, \quad k \in \mathbb{Z}_+.$$

Moreover:

- *the series converges locally uniformly in D,*
- *if $|f| \leq M$, then $|c_k| \leq 2M/R^k$, $k \in \mathbb{N}$.*

Proof. Applying the Laurent expansion to the function $F := f \circ \Phi^{-1}$ in the annulus $\mathbb{A}(1,R)$ gives

$$F(w) = a_0 + \sum_{k=1}^{\infty} (a_k w^k + a_{-k} w^{-k}), \quad 1 < |w| < R,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|w|=r} \frac{F(\zeta)}{\zeta^{k+1}} d\zeta = \frac{1}{2\pi i} \int_{|\Phi(z)|=r} \frac{f(z)}{\Phi^{k+1}(z)} \Phi'(z) dz, \quad 1 < r < R,\ k \in \mathbb{Z}_+.$$

Since F is continuous for $1 \leq |w| < R$, we have

$$\begin{aligned} a_k &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F(\zeta)}{\zeta^{k+1}} d\zeta \\ &= \frac{1}{2\pi i} \int_a^b \frac{f(x)}{\Phi^{k+1}(x - i0)} \Phi'(x - i0) dx - \frac{1}{2\pi i} \int_a^b \frac{f(x)}{\Phi^{k+1}(x + i0)} \Phi'(x + i0) dx \\ &= \frac{1}{2\pi} \int_a^b m(x) f(x) \Big(\Phi^k(x - i0) + \Phi^{-k}(x - i0) \Big) dx = \frac{1}{\pi} \int_a^b m(x) f(x) \Phi_k(x) dx. \end{aligned}$$

In particular, $c_0 = a_0$, $c_k = 2a_k = 2a_{-k}$, $k \in \mathbb{N}$. Consequently, we get (4.4.1) on $D \setminus [a,b]$, the series being locally uniformly convergent in $D \setminus [a,b]$. It remains to observe that series (4.4.1) is uniformly convergent in $\{z \in D : |\Phi(z)| \leq \theta R\}$ for arbitrary $0 < \theta < 1$. In fact, if $|\Phi(z)| \leq \theta R$, then

$$|c_k \Phi_k(z)| \leq \frac{2M}{R^k} |\Phi(z)|^k \leq 2M\theta^k, \quad k \in \mathbb{N}.$$

□

Proof of Theorem 4.4.1. First observe that, using the same procedures as in Remark 4.3.4, we may reduce the proof to the case where D is bounded, $A \Subset D$, and $q = 1$. In the case where A is compact we easily reduce the proof to the case where f is bounded, $|f| \le M$ on $\boldsymbol{X}$. In particular, f is continuous on $\boldsymbol{X}$ (cf. Remark 4.3.4(P5)).

Write $G := G_1$, $[a,b] := [a_1, b_1]$, $F := [a,b]$. Let R, g be associated to $(G,[a,b])$ as in Lemma 4.4.5 and let m, Φ, $(\Phi_k)_{k=1}^{\infty}$ be associated by Corollary 4.4.6. Define

$$c_k(z) := \frac{2^{\operatorname{sgn} k}}{\pi} \int_a^b m(t) f(z,t) \Phi_k(t)\,dt, \quad z \in D,\ k \in \mathbb{Z}_+.$$

We have (cf. Lemma 4.4.8):

$$f(z,w) = \sum_{k=0}^{\infty} c_k(z)\Phi_k(w), \quad (z,w) \in A \times G,$$

$$c_k \in \mathcal{O}(D), \quad |c_k| \le 2M, \quad |c_k(z)| \le \frac{2M}{R^k}, \quad z \in A,\ k \in \mathbb{Z}_+.$$

Hence

$$|c_k| \le 2MR^{k(h^*_{A,D}-1)} \text{ on } D, \quad k \in \mathbb{N}.$$

For $0 < \theta < 1$ define

$$\Omega_\theta := \Big\{(z,w) \in D \times G : h^*_{A,D} + h^*_{B,G}(w) = h^*_{A,D} + \frac{\log|\Phi(w)|}{\log R} < 1 + \frac{\log\theta}{\log R}\Big\}.$$

Observe that $\Omega_\theta \nearrow \widehat{\boldsymbol{X}}$ when $\theta \nearrow 1$. For $(z,w) \in \Omega_\theta$ we get

$$|c_k(z)\Phi_k(w)| \le 2MR^{k(h^*_{A,D}(z)-1)}|\Phi(w)|^k \le 2M(R^{h^*_{A,D}(z)-1}|\Phi(w)|)^k \le 2M\theta^k, \quad k \in \mathbb{N}.$$

Thus, the series is uniformly convergent on Ω_θ and its sum $\widehat{f}$ satisfies the inequality

$$\sup_{\Omega_\theta} |\widehat{f}| \le \frac{2M}{1-\theta}.$$

Using the same argument for the function f^m instead of f we conclude that

$$\sup_{\Omega_\theta} |\widehat{f}^m| \le \frac{2M^m}{1-\theta},$$

which gives

$$\sup_{\Omega_\theta} |\widehat{f}| \le M\Big(\frac{2}{1-\theta}\Big)^{1/m}.$$

Letting $m \longrightarrow +\infty$ leads to the conclusion that $|\widehat{f}| \le M$ on Ω_θ. $\square$

Theorem 4.4.9. *Let $D_1, \dots, D_n \subset \mathbb{C}$ be simply connected domains symmetric with respect to the real axis $\mathbb{R}$ and let $A_j = [a_j, b_j] \subset D_j \cap \mathbb{R}$, $a_j < b_j$, $j = 1, \dots, n$. Put $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^n)$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$. Then there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| \le \sup_{\boldsymbol{X}} |f|$.*

We need some auxiliary results.

Theorem 4.4.10 (Leja's polynomial lemma). *Let $K_1, \dots, K_n \subset \mathbb{C}$ be continua, $K := K_1 \times \dots \times K_n \subset \mathbb{C}^n$, and let $\mathcal{F} \subset \mathcal{P}(\mathbb{C}^n)$ be such that*

$$\forall_{z \in K} : \sup_{p \in \mathcal{F}} |p(z)| < +\infty,$$

i.e. $\mathcal{F}$ is pointwise bounded on K. Then

$$\forall_{a \in K}\ \forall_{\omega > 1}\ \exists_{M = M(K,a,\omega,\mathcal{F}) > 0}\ \exists_{\eta = \eta(K,a,\omega) > 0} : \sup_{p \in \mathcal{F}} \sup_{z \in \mathbb{P}(a,\eta)} |p(z)| \le M\omega^{\deg p},$$

equivalently,

$$\forall_{\omega > 1}\ \exists_{M = M(K,\omega,\mathcal{F}) > 0}\ \exists_{\substack{\Omega = \Omega(K,\omega) \\ K \subset \Omega\ -\ open}} : \sup_{p \in \mathcal{F}} \sup_{z \in \Omega} |p(z)| \le M\omega^{\deg p}.$$

Notice that η is independent of $\mathcal{F}$.

Proof. The case $n = 1$ is covered by Lemma 2.1.6. Assume that the result is true for $(n-1)$ variables and consider the case of n variables. Take a point $a' \in K' := K_1 \times \dots \times K_{n-1}$ and put $\mathcal{F}_{a'} := \{p(a', \cdot) : p \in \mathcal{F}\}$. By Lemma 2.1.6 there exist an open neighborhood $\Omega_n := \bigcup_{\zeta \in K_n} K(\zeta, \eta(K_n, \zeta, \sqrt{\omega}))$ of K_n (Ω_n does not depend on a') and a constant $M(a')$ such that

$$|p(a', z_n)| \le M(a')\sqrt{\omega}^{\deg p}, \quad z_n \in \Omega_n,\ f \in \mathcal{F}.$$

Now, let $\mathcal{F}' := \{\sqrt{\omega}^{-\deg p} p(\cdot, z_n) : z_n \in \Omega_n\}$. Observe that $\mathcal{F}'$ is pointwise bounded on K'. Thus, by the inductive assumption, there exist an open neighborhood Ω' of K' and a constant M such that

$$\sqrt{\omega}^{-\deg p} |p(z', z_n)| \le M\sqrt{\omega}^{\deg p}, \quad z' \in \Omega',\ z_n \in \Omega_n,\ p \in \mathcal{F}. \qquad \square$$

Theorem 4.4.11. *Let K be a compact subset of an open set $\Omega \subset \mathbb{C}^n$. Assume that for every point $a \in A$ there exist continua $K_1, \dots, K_n \subset \mathbb{C}$ such that $a \in K_1 \times \dots \times K_n \subset K$. Let a sequence $(f_\alpha)_{\alpha \in \mathbb{Z}_+^m} \subset \mathcal{O}(\Omega)$ be locally uniformly bounded in Ω and such that*

$$\limsup_{|\alpha| \to +\infty} (|f_\alpha(z)| R^\alpha)^{1/|\alpha|} \le 1, \quad z \in K,$$

where $R \in \mathbb{R}_{>0}^n$. Then for every $\omega > 1$ there exist a constant $M = M(\omega) > 0$ and an open neighborhood Ω_ω of K, $\Omega_\omega \subset \Omega$, such that

$$|f_\alpha(z)| R^\alpha \le M\omega^{|\alpha|}, \quad z \in \Omega_\omega,\ \alpha \in \mathbb{Z}_+^m.$$

Proof. Take an arbitrary $\omega > 1$. It suffices to show that for every point $a \in K$ there exist $M, \eta > 0$ such that

$$|f_\alpha(z)|R^\alpha \le M\omega^{|\alpha|}, \quad z \in \mathbb{B}(a,\eta),\ \alpha \in \mathbb{Z}_+^m.$$

Fix a point $a \in K$ and let

$$\mathbb{B}(a,r_0) \subset \Omega, \quad 0 < \rho < r < r_0, \quad \rho/r \le \omega/\max\{R_1,\dots,R_m\}.$$

Put

$$M_1 := \sup_{\alpha\in\mathbb{Z}_+^m} \max_{z\in\overline{\mathbb{B}}(a,r)} |f_\alpha(z)| < +\infty.$$

Write

$$f_\alpha(z) = \sum_{k=0}^{\infty} f_{\alpha,k}(z-a), \quad z \in \mathbb{B}(a,r_0),$$

where $f_{\alpha,k}$ is a homogeneous polynomial of degree k. Put

$$p_\alpha(z) := \sum_{k=0}^{|\alpha|} f_{\alpha,k}(z-a), \quad \mathcal{F} := \{(R/\omega)^\alpha p_\alpha : \alpha \in \mathbb{Z}_+^m\}.$$

The Cauchy inequalities imply that

$$|f_{\alpha,k}| \le \frac{M_1}{r^k}$$

Consequently, if $z \in \overline{\mathbb{B}}(a,\rho)$, then

$$\begin{aligned}|p_\alpha(z)| &\le |f_\alpha(z)| + \sum_{k=0}^{|\alpha|+1} |f_{\alpha,k}(z-a)| \le M_2(z)\frac{\omega^{|\alpha|}}{R^\alpha} + M_1 \sum_{k=|\alpha|+1}^{\infty} \Big(\frac{\rho}{r}\Big)^k \\ &\le M_2(z)\frac{\omega^{|\alpha|}}{R^\alpha} + M_1\Big(\frac{\rho}{r}\Big)^{|\alpha|+1}\frac{1}{1-\frac{\rho}{r}} \le M_2(z)\frac{\omega^{|\alpha|}}{R^\alpha} + M_3\Big(\frac{\rho}{r}\Big)^{|\alpha|} \le M_4(z)\frac{\omega^{|\alpha|}}{R^\alpha}.\end{aligned}$$

Hence, the family $\mathcal{F}$ is pointwise bounded on $\overline{\mathbb{B}}(a,\rho)$. By Leja's polynomial lemma (Theorem 4.4.10) there exist $0 < \eta \le \rho$ and $M > 0$ such that

$$(R/\omega)^\alpha |p_\alpha(z)| \le M\omega^{|\alpha|}, \quad z \in \mathbb{B}(a,\eta),\ \alpha \in \mathbb{Z}_+^m.$$

Finally, for $z \in \mathbb{B}(a,\eta)$, we get

$$R^\alpha|f_\alpha(z)| \le R^\alpha|p_\alpha(z)| + M_3\omega^{|\alpha|} \le M\omega^{2|\alpha|} + M_3\omega^{|\alpha|} \le (M+M_3)\omega^{2|\alpha|}. \qquad \square$$

Proof of Theorem 4.4.9. We use induction on n. The case $n = 1$ is trivial ($\boldsymbol{X} = \boldsymbol{X} = \widehat{\boldsymbol{X}} = D$). Suppose that the result is true for $n-1$. Fix an $f \in \mathcal{O}_s(\boldsymbol{X})$. In view of Theorem 4.4.1 we only need to show that f is locally bounded on $\boldsymbol{X}$, i.e. for any subdomains $D_j^0 \Subset D_j$ with $A_j \subset D_j^0$, $j = 1, \dots, n$, the function f is bounded on $\boldsymbol{K}((A_j, D_j^0)_{j=1}^n)$. Fix an $j_0 \in \{1, \dots, n\}$. We are going to show that f is bounded on $A'_{j_0} \times D^0_{j_0} \times A''_{j_0}$. We may assume that $j_0 = n$.

We want to prove that f is bounded on $A'_n \times D^0_n$. Put

$$\boldsymbol{Y} := \boldsymbol{K}((A_j, D_j)_{j=1}^{n-1}), \quad \boldsymbol{Z} := \boldsymbol{K}(A'_n, A_n; \widehat{\boldsymbol{Y}}, D_n).$$

Recall that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$. In view of the inductive assumption, for every $z_n \in A_n$, the function $f(\cdot, z_n)$ extends to a function $\widehat{f}_{z_n}$ holomorphic on $\widehat{\boldsymbol{Y}}$ with $\widehat{f}_{z_n} = f(\cdot, z_n)$ on Y. Define $g : Z \longrightarrow \mathbb{C}$,

$$g(z', z_n) := \begin{cases} f(z', z_n), & (z', z_n) \in A'_n \times D_n \\ \widehat{f}_{z_n}(z'), & (z', z_n) \in \widehat{\boldsymbol{Y}} \times A_n \end{cases}.$$

Then $g \in \mathcal{O}_s(\boldsymbol{Z})$. For $0 < \varepsilon < 1$ put

$$\widehat{\boldsymbol{Y}}_\varepsilon := \Big\{(z_1, \dots, z_{n-1}) \in \widehat{\boldsymbol{Y}} : \sum_{j=1}^{n-1} h^*_{A_j, D_j}(z_j) < 1 - \varepsilon\Big\}.$$

Using a Baire argument, we show that there exist a constant $C > 0$ and a non-trivial interval $[a'_n, b'_n] \subset [a_n, b_n]$ such that $|g(z', z_n)| \le C$ on $\widehat{\boldsymbol{Y}}_\varepsilon \times [a'_n, b'_n]$ and $g|_{\widehat{\boldsymbol{Y}}_\varepsilon \times [a'_n, b'_n]}$ is continuous (cf. Remark 4.3.4(d, f)). Let R, g, m, Φ, $(\Phi_k)_{k=1}^\infty$ be associated to $(D_n, [a'_n, b'_n])$. Define

$$c_k(z') := \frac{2^{\operatorname{sgn} k}}{\pi} \int_{a'_n}^{b'_n} m_q(t) g(z', t) \Phi_k(t) dt, \quad z' \in \widehat{\boldsymbol{Y}}_\varepsilon,\ k \in \mathbb{Z}_+.$$

We have

$$g(z', z_n) = \sum_{k=0}^\infty c_k(z') \Phi_k(z_n), \quad (z', z_n) \in A'_n \times D_n,$$

$$c_k \in \mathcal{O}(\widehat{\boldsymbol{Y}}_\varepsilon), \quad |c_k| \le 2C, \quad |c_k(z')| \le \frac{2 \max_{t \in [a'_n, b'_n]} |g(z', t)|}{R^k},\ z' \in A'_n,\ k \in \mathbb{Z}_+.$$

Hence, by Theorem 4.4.10,

$$|c_k| \le \frac{C(\varepsilon)}{(R_q e^{-\varepsilon})^k} \text{ on } A'_n, \quad k \in \mathbb{Z}_+.$$

Finally, if

$$D_{n,\varepsilon} := \{z_n \in D_n : |\Phi(z_n)| \le R e^{-2\varepsilon}\},$$

then

$$|g(z', z_n)| \le C(\varepsilon) \sum_{k=0}^\infty \frac{(R e^{-2\varepsilon})^k}{(R e^{-\varepsilon})^k} = \frac{C(\varepsilon)}{1 - e^{-\varepsilon}}, \quad (z', z_n) \in A'_n \times D_{n,\varepsilon}.$$

In remains to observe that for sufficiently small ε we get $D^0_n \subset D_{n,\varepsilon}$. $\square$

4.5 Browder and Lelong theorems

Theorem 4.5.1 ([Bro 1961], [Lel 1961]). (a) *If $\Omega \subset \mathbb{R}^n \simeq \mathbb{R}^n + i0 \subset \mathbb{C}^n$ is open, then*

$$\mathcal{A}(\Omega) = \{f \in \mathcal{A}_s(\Omega) : \forall_{a\in\Omega}\ \exists_{r>0}\ \forall_{x\in\mathbb{R}^n\cap\mathbb{P}(a,r)\subset\Omega}\ \forall_{j\in\{1,\dots,n\}} : \\ f(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n) \in \mathcal{O}(K(a_j,r))\} =: \mathcal{L}_\Omega,$$

where $\mathcal{A}(\Omega)$ denotes the space of all real analytic functions $f : \Omega \longrightarrow \mathbb{R}$.

(b) *If $\Omega \subset \mathbb{R}^{n_1}\times\dots\times\mathbb{R}^{n_N}$, then $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega) = \mathcal{H}(\Omega)$, where $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega)$ denotes the space of all functions $f : \Omega \longrightarrow \mathbb{R}$ such that for every $(a_1,\dots,a_N) \in \Omega$, the function $x_j \longmapsto f(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_N)$ is harmonic in a neighborhood of a_j (as a function of n_j variables), $j = 1,\dots,N$.*

Notice that the function

$$f(x,y) := \begin{cases} xye^{-\frac{1}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is separately analytic and of class $\mathcal{C}^\infty(\mathbb{R}^2)$, but not analytic (near $(0,0)$) (Exercise).

Proof. (a) Fix an $f \in \mathcal{L}_\Omega$ and an $a = (a_1,\dots,a_n) \in \Omega$. Let r be as in the definition of $\mathcal{L}_\Omega$. Take an arbitrary $0 < s < r$ and put

$$\boldsymbol{X} := \boldsymbol{K}(([a_j - s, a_j + s], K(a_j,r))_{j=1}^n).$$

Directly from the definition of $\mathcal{L}_\Omega$ it follows that f extends to an $\widetilde{f} \in \mathcal{O}_s(\boldsymbol{X})$. Now Theorem 4.4.9 implies that $\widetilde{f}$ extends holomorphically to $\widehat{\boldsymbol{X}}$, which is a $\mathbb{C}^n$–neighborhood of a. In particular, f is real analytic in an $\mathbb{R}^n$–neighborhood of a.

(b) It suffices to show that $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega) \subset \mathcal{L}_\Omega$. In fact, we only need to observe that if $f \in \mathcal{H}(\overline{\mathbb{B}}(r)\cap\mathbb{R}^n)$, then f extends holomorphically to $\mathbb{P}(r/\sqrt{n})$.

Indeed, it is well known that f may be represented by its real Taylor series $f(x) = \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha$, $x \in \mathbb{B}(r)\cap\mathbb{R}^n$ that is convergent locally uniformly in $\mathbb{B}(r)\cap\mathbb{R}^n$. Consequently, the complex series $\sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha z^\alpha$ is locally convergent in $\mathbb{P}(r/\sqrt{n})$. $\square$

4.6 p–separately analytic functions

Definition 4.6.1. Let $\Omega \subset \mathbb{R}^{n_1}\times\dots\times\mathbb{R}^{n_N}$ ($N \geq 2$) be open, let $f : \Omega \longrightarrow \mathbb{R}$, and let $1 \leq p \leq N-1$. We say that f is p–separately analytic in Ω ($f \in$

$\mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$), if for any $a = (a_1, \dots, a_N) \in \Omega$ and $1 \le i_1 < \dots < i_p \le N$ the function

$$(x_{i_1}, \dots, x_{i_p}) \longmapsto f(a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_p-1}, x_{i_p}, a_{i_p+1}, \dots, a_N)$$

is analytic in an open neighborhood of $(a_{i_1}, \dots, a_{i_p})$.

Observe that $\mathcal{A}_{(n_1,\dots,n_N),1}(\Omega) = \mathcal{A}_{(n_1,\dots,n_N)}(\Omega) =$: the space of all functions $f : \Omega \longrightarrow \mathbb{R}$ such that for any $(a_1, \dots, a_N) \in \Omega$ and $j \in \{1, \dots, N\}$, the function $x_j \longmapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_N)$ is analytic in a neighborhood of a_j (as a function of n_j variables).

Theorem 4.6.2. *Let $\Omega \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}$ be open, $f : \Omega \longrightarrow \mathbb{R}$, $1 \le p \le N - 1$, and let*

$$S = \mathcal{S}_{\mathcal{A}}(f) := \{a \in \Omega : \forall_{a \in U \subset \Omega} : \ f \notin \mathcal{A}(U)\}.$$

(a) *If $f \in \mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$, then:*

(*) $\operatorname{pr}_{\mathbb{R}^{n_{j_1}} \times \dots \times \mathbb{R}^{n_{j_{N-p}}}}(S) \in \mathcal{PLP}(\mathbb{C}^{n_{j_1}} \times \dots \times \mathbb{C}^{n_{j_{N-p}}})$ *for all* $1 \le j_1 < \dots < j_{N-p} \le N$.

(b) *For every relatively closed set $S \subset \Omega$ with (*), there exists a function $f \in \mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$ such that $S = \mathcal{S}_{\mathcal{A}}(f)$.*

In the case where $N = 2$, $n_1 = n_2 = 1$, the result was proved in [Ray 1989], [Ray 1990]. In the general case, part (a) with $p \ge N/2$ and part (b) with an arbitrary p were proved in [Sic 1990]. Finally, part (a) with arbitrary p was proved in [Bło 1992].

Proof. [A SKETCH OF THE PROOF. WILL BE COMPLETED. . . .] □

4.7 Separate subharmonicity

See [Jar-Pfl 2000], § 2.1. Let $\Omega \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}$ be open. A function $u : \Omega \longrightarrow \mathbb{R}_{-\infty}$ is said to *separately subharmonic* ($u \in \mathcal{SH}_{(n_1,\dots,n_N)}(\Omega)$) if for every $(a_1, \dots, a_N) \in \Omega$, the function $x_j \longmapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_N)$ is subharmonic in a neighborhood of a_j (as a function of n_j variables), $j = 1, \dots, N$.

In view of Theorem 4.5.1(b), one could conjecture every separately subharmonic functions is subharmonic.

In the case where $\Omega \subset \mathbb{C}^n$ is open, one could at least conjecture that a function $u : \Omega \longrightarrow \mathbb{R}_{-\infty}$ is plurisubharmonic iff every $a \in X$ and $\xi \in \mathbb{C}^n$ the function $u_{a,\xi}$ is subharmonic in a neighborhood of zero, i.e. iff u is subharmonic on complex affine lines through Ω. Observe that every such a function is of class $\mathcal{SH}_{(2,\dots,2)}(\Omega)$. The above conjecture has been formulated by P. Lelong, who proved ([Lel 1945]) that the answer is positive if we additionally assume that u is locally bounded from above in Ω. $\boxed{?}$ The general answer is still not known $\boxed{?}$

It is known that if $\Omega \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}$ is open and $u \in \mathcal{SH}_{(n_1,\dots,n_N)}(\Omega)$ is such that for every point $a \in \Omega$ there exist an open neighborhood $U \subset \Omega$, a number

$0 < r \le +\infty$, and a function $v \in L^r(\Omega)$ such that $u \le v$ in U, then $u \in \mathcal{SH}(\Omega)$ ([Rii 1989]). The case $r = +\infty$ is due to Avanissian [Ava 1961]. The case $r = 1$ is due to Arsove [Ars 1966].

In particular, if $\Omega \subset \mathbb{C}^n$ is open and $u : \Omega \longrightarrow \mathbb{R}_{-\infty}$ is subharmonic on every complex affine line and such that for every point $a \in \Omega$ there exist an open neighborhood $U \subset \Omega$, a number $0 < r \le +\infty$, and a function $v \in L^r(\Omega)$ such that $u \le v$ in U, then $u \in \mathcal{PSH}(\Omega)$.

The example constructed in [Wie-Zei 1991] shows that in general the answer is negative and we have $\mathcal{SH}_{(n_1,\dots,n_N)}(\Omega) \not\subset \mathcal{SH}(\Omega)$. More precisely, there exists a function $u : \mathbb{C}^2 \longrightarrow \mathbb{R}_+$ such that for every $(z_0, w_0) \in \mathbb{C}^2$ the functions $u(z_0, \cdot)$ and $u(\cdot, w_0)$ are $\mathcal{C}^\infty$ subharmonic, but $u \notin \mathcal{SH}(\mathbb{C}^2)$.

Indeed, let

$$\widetilde{u}_k(z) := \begin{cases} k^k \operatorname{Re}(-iz^k), & \text{if } 0 < \operatorname{Arg} z < \pi/k \\ 0, & \text{otherwise} \end{cases}, \quad z \in \mathbb{C},$$

$$u_k := \widetilde{u}_k * \Phi_{1/k^3}\Big(z - \frac{1}{k^2}\exp(\frac{\pi i}{2k})\Big), \quad z \in \mathbb{C},$$

$$u(z,w) := \sum_{k=1}^{\infty} u_k(z) u_k(w), \quad (z,w) \in \mathbb{C}^2,$$

where $(\Phi_\varepsilon)_{\varepsilon>0}$ are regularization functions as in Definition 3.3.14. Observe that:

- if $z = re^{i\varphi}$, then $\operatorname{Re}(-iz^k) = r^k \sin(k\varphi)$;
- consequently, $\widetilde{u}_k$ is a non-negative continuous function;
- $\widetilde{u}_k$ is subharmonic on $\Delta_k := \{0 < \operatorname{Arg} z < \pi/k\}$ and, consequently, on $\mathbb{C}$;
- u_k is a non-negative subharmonic function on $\mathbb{C}$ (cf. Proposition 3.3.15);
- $\operatorname{supp} u_k \subset \Delta_k$ for $k \gg 1$;
- for every $z_0 \in \mathbb{C}$ we have $u_k(z_0) = 0$ for $k \gg 1$;
- consequently, $u(z_0, \cdot)$ is a well defined non-negative $\mathcal{C}^\infty$ subharmonic function on $\mathbb{C}$;
- $\sqrt{u(\frac{2}{k}\exp(\frac{\pi i}{2k}), \frac{2}{k}\exp(\frac{\pi i}{2k}))} \ge u_k(\frac{2}{k}\exp(\frac{\pi i}{2k})) \ge \widetilde{u}_k(\frac{2}{k}\exp(\frac{\pi i}{2k}) - \frac{1}{k^2}\exp(\frac{\pi i}{2k})) = \widetilde{u}_k((\frac{2}{k} - \frac{1}{k^2})\exp(\frac{\pi i}{2k})) = k^k \operatorname{Re}(-i(\frac{2}{k} - \frac{1}{k^2})^k (\exp(\frac{\pi i}{2k}))^k) = (2 - \frac{1}{k})^k \longrightarrow +\infty.$

4.8 Proof of the cross theorem

We need some auxiliary results. We begin with the following general theorem from functional analysis.

Theorem 4.8.1 ([Mit 1961], see also [Jar-Pfl 2000], Lemma 3.5.9). *Let $\mathcal{H}_0, \mathcal{H}_1$ be separable Hilbert spaces with $\dim \mathcal{H}_0 = \dim \mathcal{H}_1 = \infty$, and let $T : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ be a linear injective compact operator* (2) *such that $T(\mathcal{H}_0)$ is dense in $\mathcal{H}_1$. Then there exists an orthogonal basis $(b_k)_{k\in\mathbb{N}} \subset \mathcal{H}_0$ such that:*

(2) Recall that a linear operator $T : X \longrightarrow Y$, where X and Y are locally convex topological vector spaces, is *compact* if for any bounded set $B \subset X$ the set $T(B)$ is relatively compact in Y.

- *$(T(b_k))_{k=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}_1$,*
- *$\|b_k\|_{\mathcal{H}_0} =: \nu_k \nearrow +\infty$ when $k \nearrow +\infty$.*

Moreover, if there exist a locally convex nuclear space $\boldsymbol{\mathcal{V}}$ and linear continuous operators $\mathcal{H}_0 \xrightarrow{T_1} \boldsymbol{\mathcal{V}} \xrightarrow{T_2} \mathcal{H}_1$ such that $T = T_2 \circ T_1$, then the above basis $(b_k)_{k\in\mathbb{N}}$ may be chosen in such a way that the series $\sum_{k=1}^{\infty} \nu_k^{-\varepsilon}$ is convergent for any $\varepsilon > 0$.

In the case of the cross theorem the above general result implies the following fundamental theorem.

Theorem 4.8.2 ([Zah 1976], [Zer 1982], [Zer 1986], [Ngu-Zer 1991], [Zer 1991], [Ale-Zer 2001], [Zer 2002], see also [Jar-Pfl 2000], Lemma 3.5.10). *Let $\Omega \Subset X$ be a strongly pseudoconvex open set on a Riemann region (X,p) over $\mathbb{C}^n$ and let $A \subset \Omega$ be compact and such such that $A \cap S$ is non-pluripolar for every connected component S of Ω. Put*

- *$\mathcal{H}_0 := L_h^2(\Omega)$ ([3]),*
- *$\mathcal{H}_1 := \mathrm{cl}_{L^2(A,\mu_{A,\Omega})}(L_h^2(\Omega)|_A)$ = the closure of $L_h^2(\Omega)|_A$ in $L^2(A,\mu_{A,\Omega})$,*

where $\mu_{A,\Omega}$ is the equilibrium measure for A (cf. Definition 3.4.17).
Then the linear operators

$$\mathcal{H}_0 \ni f \xrightarrow{T_1} f \in \mathcal{O}(\Omega), \quad \mathcal{O}(\Omega) \ni f \xrightarrow{T_2} f|_A \in \mathcal{H}_1$$

are well defined, injective, and continuous ([4]). Moreover, T_1 is compact. In particular, the operator $\mathcal{H}_0 \ni f \overset{T:=T_2\circ T_1}{\longrightarrow} f|_A \in \mathcal{H}_1$ is compact.

Let $(b_k)_{k=1}^{\infty} \subset \mathcal{H}_0$, $(\nu_k)_{k=1}^{\infty}$ be as in Theorem 4.8.1. Then for any $\alpha \in (0,1)$ and for any compact

$$K \subset \{z \in \Omega : h_{A,\Omega}^*(z) < \alpha\}$$

there exists a constant $C = C(\alpha, K) > 0$ such that

$$\|b_k\|_K \le C\nu_k^{\alpha}, \quad k \in \mathbb{N}. \tag{4.8.1}$$

Remark 4.8.3. An independent proof of Theorem 4.8.2 has been given by A. Zeriahi in [Zer 2002].

Proof of Theorem 4.3.3. To simplify the proof we assume additionally that $D_1, \dots, D_N$ are domains of holomorphy.

We already know (cf. Remark 4.3.4(P7)) that we may assume that $N = 2$, D, G are strongly pseudoconvex domains with real analytic boundaries, $A \Subset D$, $B \Subset G$ are compact and non-pluripolar, $f(a,\cdot) \in \mathcal{O}(\overline{G})$, $a \in A$, $f(\cdot,b) \in \mathcal{O}(\overline{D})$, $b \in B$, $|f| \le 1$ on $\boldsymbol{X}$, and f is continuous on $\boldsymbol{X}$.

Let $\mu := \mu_{A,D}$, $\mathcal{H}_0 := L_h^2(D)$, $\mathcal{H}_1 :=$ the closure of $\mathcal{H}_0|_A$ in $L^2(A,\mu)$, and let $(b_k)_{k=1}^{\infty}$ be the basis from Theorem 4.8.2; $\nu_k := \|b_k\|_{\mathcal{H}_0}$, $k \in \mathbb{N}$. For any $w \in B$ we have $f(\cdot,w) \in \mathcal{H}_0$ and $f(\cdot,w)|_A \in \mathcal{H}_1$. Hence

$$f(\cdot,w) = \sum_{k=1}^{\infty} c_k(w) b_k,$$

([3]) $L_H^2(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$; observe that $L_h^2(\Omega)$ is a complex Hilbert space with the scalar product given by the formula $(f,g) \longmapsto \int_\Omega f\overline{g}\,d\mathcal{L}^{\Omega}$.

([4]) Recall that $\mathcal{O}(\Omega)$ is a nuclear space.

where

$$c_k(w) = \frac{1}{\nu_k^2}\int_D f(z,w)\overline{b}_k(z)d\mathcal{L}^{2n}(z) = \int_A f(z,w)\overline{b}_k(z)d\mu(z), \quad k \in \mathbb{N};$$

cf. Theorem 4.8.1. The series is convergent in $L_h^2(D)$ (in particular, locally uniformly in D) and in $L^2(A,\mu)$. Since f is continuous, the formula

$$\widehat{c}_k(w) := \int_A f(z,w)\overline{b}_k(z)d\mu(z), \quad w \in G,$$

defines a holomorphic function on G, $k \in \mathbb{N}$. We are going to prove that the series

$$\sum_{k=1}^{\infty} \widehat{c}_k(w)b_k(z)$$

converges locally uniformly in $\widehat{\boldsymbol{X}}$.

Take a compact $K \times L \subset \widehat{\boldsymbol{X}}$ and let $\alpha > \max_K h^*_{A,D}$, $\beta > \max_L h^*_{B,G}$ be such that $\alpha + \beta < 1$. First, we will prove that there exists a constant $C'(L,\beta) > 0$ such that

$$\|\widehat{c}_k\|_L \le C'(L,\beta)\nu_k^{\beta-1}, \quad k \in \mathbb{N}. \tag{4.8.2}$$

Suppose for a moment that (4.8.2) is true. Then, using Theorems 4.8.2 and 4.8.1, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\|\widehat{c}_k\|_L\|b_k\|_K &\le \sum_{k=1}^{\infty} C'(L,\beta)\nu_k^{\beta-1}C(K,\alpha)\nu_k^{\alpha} \\ &= C'(L,\beta)C(K,\alpha)\sum_{k=1}^{\infty}\nu_k^{\alpha+\beta-1} =: M(K,L) < +\infty,\end{aligned}$$

which gives the normal convergence on $K \times L$. Let

$$\widehat{f}(z,w) := \sum_{k=1}^{\infty}\widehat{c}_k(w)b_k(z), \quad (z,w) \in \widehat{\boldsymbol{X}};$$

obviously $\widehat{f}$ is holomorphic. Recall that $\widehat{f} = f$ on $D \times B$. Hence $\widehat{f} = f$ on $\boldsymbol{Y} := \boldsymbol{K}(A \cap A^*, B \cap B^*; D, G) \subset \boldsymbol{X} \cap \widehat{\boldsymbol{X}}$.

Moreover, if K, L are as above, then

$$\sup_{K\times L}|\widehat{f}| \le M(K,L) < +\infty.$$

Taking f^m instead of f, we conclude that

$$\sup_{K\times L}|\widehat{f^m}| \le M(K,L) < +\infty, \quad m \in \mathbb{N},$$

which implies that $|\widehat{f}| \le 1$.

We move to the proof of (4.8.2). By the Hölder inequality, we get

$$|\widehat{c}_k(w)| \le \sqrt{\mu(A)}, \quad w \in G,\ k \in \mathbb{N}$$

(recall that $|f| \le 1$). On the other hand, if $w \in B$, then

$$|\widehat{c}_k(w)| = |c_k(w)| = \Big| \frac{1}{\nu_k^2} \int_D f(z,w)\overline{b}_k(z) d\mathcal{L}^{2n}(z) \Big| \le \frac{1}{\nu_k} \sqrt{\mathcal{L}^{2n}(D)}.$$

For $k \in \mathbb{N}$ such that $\nu_k > 1$, let

$$u_k := \frac{\log |\widehat{c}_k|}{\log \nu_k}.$$

The sequence $(u_k)_{k=1}^\infty$ is bounded from above in G, $u := \limsup_{k\to+\infty} u_k \le 0$, and $u \le -1$ on B. Let $P := \{w \in G : u(w) < u^*(w)\}$; $P \in \mathcal{PLP}$. Thus $u^* \in \mathcal{PSH}(G)$, $u^* \le 0$, and $u^* = u \le -1$ on $B \setminus P$. Consequently, $1 + u^* \le h^*_{B\setminus P,G} = h^*_{B,G}$ (cf. Proposition 3.4.11(d)). Hence $1 + u^* < \beta$ on L. Now, by the Hartogs lemma, $u_k < \beta - 1$ on L for $k \gg 1$, which implies (4.8.2). $\square$

4.9 Cross theorem for generalized crosses

Definition 4.9.1. Let $D_j \in \mathfrak{R}_c(\mathbb{C}^{n_j})$, let $\varnothing \ne A_j \subset D_j$, and let $\Sigma_j \subset A'_j \times A''_j$, $j = 1, \dots, N$. We define a *generalized N-fold cross*

$$\begin{aligned}\boldsymbol{T} &:= \boldsymbol{GK}(A_1, \dots, A_N; D_1, \dots, D_N; \Sigma_1, \dots, \Sigma_N) = \boldsymbol{GK}((A_j, D_j, \Sigma_j)_{j=1}^N) := \\ &= \bigcup_{j=1}^N \Big\{ (a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j \Big\}\end{aligned}$$

and its *center*

$$\boldsymbol{c}(\boldsymbol{T}) := \boldsymbol{T} \cap (A_1 \times \dots \times A_N) = (A_1 \times \dots \times A_N) \setminus \varDelta_0,$$

where

$$\varDelta_0 := \bigcap_{j=1}^N \Big\{ (a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \in \Sigma_j \Big\}.$$

We say that a function $f : \boldsymbol{T} \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{T}$ ($f \in \mathcal{O}_s(\boldsymbol{T})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the function

$$D_j \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic in D_j.

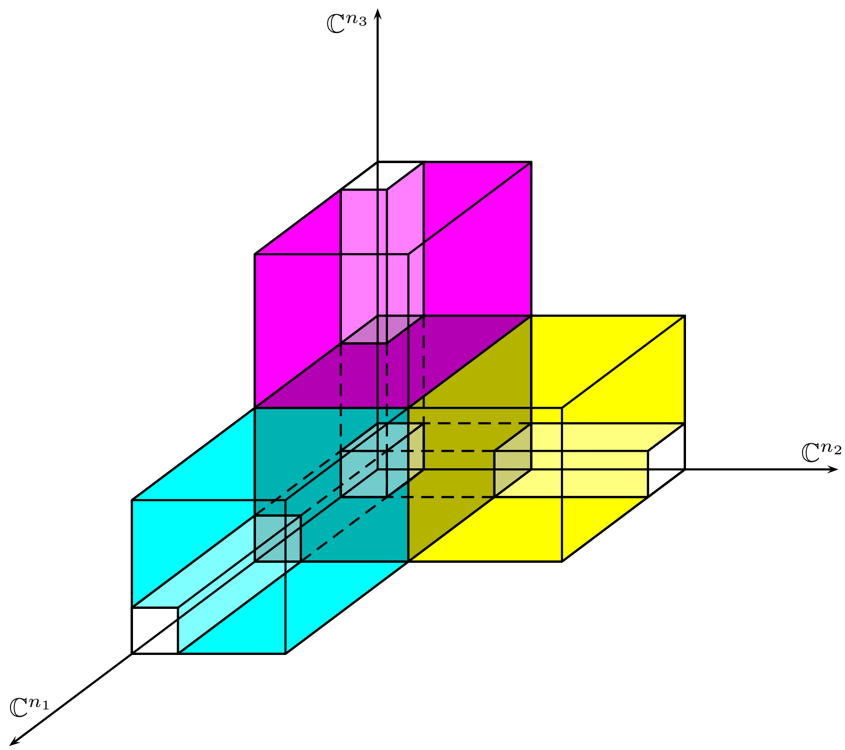

Figure 4.9.1. Generalized 3–fold cross.

Observe that:

- $\boldsymbol{GK}((A_j, D_j, \varnothing)_{j=1}^N) = \boldsymbol{K}((A_j, D_j)_{j=1}^N)$;
- if $N = 2$, then $\boldsymbol{GK}(A_1, A_2; D_1, D_2; \Sigma_1, \Sigma_2) = \boldsymbol{K}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2)$;

roughly speaking, generalized 2–fold crosses are nothing new in comparison with the standard 2–fold crosses; for $N \geq 3$ generalized N–fold crosses are geometrically different than the standard ones — for instance, this makes the theory of extension with singularities for $N \geq 3$ essentially more difficult — cf. Chapters 5, 7;

- if one of the sets $\Sigma_1, \dots, \Sigma_N$ is pluripolar, then $\mathit{\Delta}_0 \in \mathcal{PLP}$.

Theorem 4.9.2 (Extension theorem for generalized crosses). *Assume that D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A'_j \times A''_j$ is pluripolar, $j = 1, \dots, N$, $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$, and $\boldsymbol{T} := \boldsymbol{GK}((A_j, D_j, \Sigma_j)_{j=1}^N)$. Then for every $f \in \mathcal{O}_s(\boldsymbol{T})$ such that*

(*) *for any $j \in \{1, \dots, N\}$ and $b_j \in D_j$, the function*

$$A'_j \times A''_j \setminus \Sigma_j \ni (z'_j, z''_j) \longmapsto f(z'_j, b_j, z''_j)$$

is continuous,

there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{T}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{T}} |f|$.

Remark 4.9.3. (a) The assumption (*) is obviously a necessary condition for a function $f \in \mathcal{O}_s(\boldsymbol{T})$ to be holomorphically extendible to $\widehat{\boldsymbol{X}}$. $\boxed{?}$ We do not know whether the theorem remains true without (*) $\boxed{?}$

(b) If $N = 2$, then every generalized 2–fold cross is a "standard" 2–fold cross and $\widehat{\boldsymbol{X}} = \widehat{\boldsymbol{T}}$. Consequently, the result follows from Theorem 4.3.1 for arbitrary $f \in \mathcal{O}_s(\boldsymbol{T})$ (without the assumption (*)).

(c) The case where $\Sigma_1 = \cdots = \Sigma_N = \varnothing$ follows immediately from Theorem 4.3.1 (without (*)).

Proof of Theorem 4.9.2. We apply induction on N. As we already observed the result is true for $N = 2$. Assume that the result is true for $N - 1 \geq 2$. Take an $f \in \mathcal{O}_s(\boldsymbol{T})$ with (*). Let

$$Q := \{z_N \in A_N : \exists_{j \in \{1,\dots,N-1\}} : (\Sigma_j)_{(\cdot,z_N)} \notin \mathcal{PLP}\}.$$

Then, by Proposition 3.3.27, $Q \in \mathcal{PLP}$. Take a $z_N \in A_N \setminus Q$ and define

$$\boldsymbol{T}(z_N) := \boldsymbol{GK}((A_j, D_j, (\Sigma_j)_{(\cdot,z_N)})_{j=1}^{N-1}), \quad \boldsymbol{Y} := \boldsymbol{K}((A_j, D_j)_{j=1}^{N-1}).$$

Put $\widetilde{A}''_j := A_{j+1} \times \cdots \times A_{N-1}$, $\widetilde{a}''_j := (a_{j+1}, \dots, a_{N-1})$, $j = 1, \dots, N-1$. Observe that

$$\boldsymbol{T}_{(\cdot,z_N)} = \boldsymbol{T}(z_N) \cup (A'_N \setminus \Sigma_N).$$

It is clear that $f(\cdot, z_N) \in \mathcal{O}_s(\boldsymbol{T}(z_N))$. Moreover, the function $f(\cdot, z_N)$ satisfies (*) on $\boldsymbol{T}(z_N)$. Indeed, let $j \in \{1, \dots, N-1\}$, $b_j \in D_j$. Then the continuity of the function

$$A'_j \times \widetilde{A}''_j \setminus (\Sigma_j)_{(\cdot,z_N)} \ni (z'_j, \widetilde{z}''_j) \longmapsto f(z'_j, b_j, \widetilde{z}''_j, z_N)$$

follows directly from the condition (*) for the function f.

By the inductive assumption, there exists an $\widehat{f}_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ with $\widehat{f}_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{T}(z_N)$ and $\sup_{\widehat{\boldsymbol{Y}}} |\widehat{f}_{z_N}| = \sup_{\boldsymbol{T}(z_N)} |f(\cdot, z_N)|$. Consider the 2–fold cross

$$\boldsymbol{Z} := \widehat{\boldsymbol{K}}(A'_N \setminus \Sigma_N, A_N \setminus Q; \widehat{\boldsymbol{Y}}, D_N) = ((A'_N \setminus \Sigma_N) \times D_N) \cup (\widehat{\boldsymbol{Y}} \times (A_N \setminus Q)).$$

Observe that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$ (cf. Proposition 3.4.14). Let $g : \boldsymbol{Z} \longrightarrow \mathbb{C}$ be given by the formula

$$g(z', z_N) := \begin{cases} f(z', z_N), & \text{if} \quad (z', z_N) \in (A'_N \setminus \Sigma_N) \times D_N \\ \widehat{f}_{z_N}(z'), & \text{if} \quad (z', z_N) \in \widehat{\boldsymbol{Y}} \times (A_N \setminus Q) \end{cases}.$$

Observe that g is well-defined.

Indeed, let $(z', z_N) \in ((A'_N \setminus \Sigma_N) \times D_N) \cap (\widehat{\boldsymbol{Y}} \times (A_N \setminus Q))$. If $z' \in \boldsymbol{T}(z_N)$, then obviously $\widehat{f}_{z_N}(z') = f(z', z_N)$. Suppose that $z' \notin \boldsymbol{T}(z_N)$. Then $z' \in P(z_N)$, where

$$P(z_N) := \bigcap_{j=1}^{N-1} \Big\{(w'_j, w_j, \widetilde{w}''_j) \in A'_j \times A_j \times \widetilde{A}''_j : (w'_j, \widetilde{w}''_j) \in (\Sigma_j)_{(\cdot,z_N)}\Big\}.$$

In view of the definition of Q, the set $P(z_N)$ is pluripolar. Take a sequence

$$A'_N \setminus (\Sigma_N \cup P(z_N)) \ni z'^{\nu} \longrightarrow z'.$$

Then $z'^{\nu} \in \boldsymbol{T}(z_N)$. Thus $\widehat{f}_{z_N}(z'^{\nu}) = f(z'^{\nu}, z_N)$ and, by (*) (with $j := N$ and $b_N := z_N$), $\widehat{f}_{z_N}(z') = f(z', z_N)$.

It is clear that $g \in \mathcal{O}_s(\boldsymbol{Z})$ and $\sup_{\boldsymbol{Z}} |g| \le \sup_{\boldsymbol{T}} |f|$.

By Theorem 4.3.1, we get a holomorphic extension $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ with $\widehat{f} = g$ on $\boldsymbol{Z}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{Z}} |g| \le \sup_{\boldsymbol{T}} |f|$. It remains to show that $\widehat{f} = f$ on $\boldsymbol{T}$.

Take a point $a \in \boldsymbol{T}$. If $a \in (A'_N \setminus \Sigma_N) \times D_N \subset \boldsymbol{Z}$, then $\widehat{f}(a) = g(a) = f(a)$. In the remaining case we may assume that for instance $a = (a_1, a''_1) \in D_1 \times (A''_1 \setminus \Sigma_1)$.

Let $T_0 := \bigcup_{z_N \in A_N \setminus Q} \boldsymbol{T}(z_N) \times \{z_N\} \subset \widehat{\boldsymbol{Y}} \times (A_N \setminus Q) \subset \boldsymbol{Z}$. On the other hand $T_0 \subset \bigcup_{z_N \in A_N \setminus Q} \boldsymbol{T}_{(\cdot, z_N)} \times \{z_N\} \subset \boldsymbol{T}$. Observe that if $b = (b', b_N) \in T_0$, then $\widehat{f}(b) = g(b) = \widehat{f}_{b_N}(b') = f(b)$. Thus, we only need to show that there exists a sequence $(b^{\nu})_{\nu=1}^{\infty} \subset T_0 \cap (\{a_1\} \times (A''_1 \setminus \Sigma_1))$ with $b^{\nu} \longrightarrow a$ (and then use the continuity of $f(a_1, \cdot)$ on $A''_1 \setminus \Sigma_1$).

Since Q is pluripolar, we may find a sequence $b^{\nu}_N \longrightarrow a_N$ with $b^{\nu}_N \in A_N \setminus Q$. Let $P := \bigcup_{\nu=1}^{\infty} (\Sigma_1)_{(\cdot, b^{\nu}_N)}$. In view of the definition of Q, the set P is pluripolar. In particular, we may find a sequence $(b^{\nu}_2, \dots, b^{\nu}_{N-1}) \longrightarrow (a_2, \dots, a_{N-1})$ with $(b^{\nu}_2, \dots, b^{\nu}_{N-1}) \in (A_2 \times \dots \times A_{N-1}) \setminus P$. Put $b^{\nu} := (a_1, b^{\nu}_2, \dots, b^{\nu}_N)$. Then $b^{\nu} \longrightarrow a$ and obviously $b^{\nu} \in \boldsymbol{T}(b^{\nu}_N) \times \{b^{\nu}_N\} \subset T_0$. $\square$

Remark 4.9.4. In the context of Theorem 4.9.2, one may formulate the following general problem:

Assume that D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\varnothing \neq B_j \subset A'_j \times A''_j$, $j = 1, \dots, N$,

$$\boldsymbol{W} := \bigcup_{j=1}^{N} \Big\{ (a'_j, a_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \in B_j \Big\}.$$

We say that a function $f : \boldsymbol{W} \longrightarrow \mathbb{C}$ is *separately holomorphic* on $\boldsymbol{W}$ ($f \in \mathcal{O}_s(\boldsymbol{W})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in B_j$, the function

$$D_j \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic in D_j.

$\boxed{?}$ Given an $f \in \mathcal{O}_s(\boldsymbol{W})$, we look for conditions on $B_1, \dots, B_N$, and f, under which the exists an open neighborhood $\varOmega$ of $\boldsymbol{W}$ (independent of f) such that f extends holomorphically to $\varOmega$ $\boxed{?}$

It is clear that the configuration of the sets must be special. For example, if one of the branches $\{(a'_j, a_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \in B_j\}$ does not intersect the others, then the answer is definitively negative.

4.10 Chirka–Sadullaev theorem

Theorem 4.10.1 ([Chi-Sad 1988] ([5])). *Let $A \subset \mathbb{D}^p$ be locally pluriregular, let $\boldsymbol{X} := \boldsymbol{K}(A, \mathbb{D}; \mathbb{D}^p, \mathbb{C}) = (A \times \mathbb{C}) \cup \mathbb{D}^{p+1}$ and let $M \subset \boldsymbol{X}$ be a relatively closed set such that:*

- *$M \cap \mathbb{D}^{p+1} = \varnothing$,*
- *$M_{(a,\cdot)}$ is polar for every $a \in A$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}} = \mathbb{D}^p \times \mathbb{C}$ such that:

- *$\widehat{M} \cap \boldsymbol{X} \subset M$,*
- *for every $f \in \mathcal{O}(\mathbb{D}^{p+1})$ such that for every $a \in A$, the function $f(a, \cdot)$ extends holomorphically to $\mathbb{C} \setminus M_{(a,\cdot)}$, there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$,*
- *$\mathbb{D}^p \times \mathbb{C} \setminus \widehat{M}$ is a domain of holomorphy,*
- *if all the fibers $M_{(a,\cdot)}$, $a \in A$, are discrete, then $\widehat{M}$ is analytic.*

Proof. It is known (cf. [Chi-Sad 1988]) that each function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ has the univalent domain of existence $G_f \subset \mathbb{D}^p \times \mathbb{C}$. Let G denote the connected component of $\operatorname{int} \bigcap_{f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)} G_f$ that contains $\mathbb{D}^{p+1}$ and let $\widehat{M} := \mathbb{D}^p \times \mathbb{C} \setminus G$. It remains to show that $\widehat{M}$ is pluripolar. Take $(a, b) \in A \times \mathbb{C} \setminus M$. Since $M_{(a,\cdot)}$ is polar, there exists a curve $\gamma : [0, 1] \longrightarrow \mathbb{C} \setminus M_{(a,\cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = b$. Since M is relatively closed, there exists an $\varepsilon > 0$ so small that $\mathbb{P}(a, \varepsilon) \subset \mathbb{D}^p$ and

$$(\mathbb{P}(a, \varepsilon) \times (\gamma([0, 1]) + K(\varepsilon))) \cap M = \varnothing.$$

Put $V_b := \mathbb{D} \cup (\gamma([0, 1]) + K(\varepsilon))$ and consider the cross

$$\boldsymbol{Y} := \boldsymbol{K}(A \cap \mathbb{P}(a, \varepsilon), \mathbb{D}; \mathbb{P}(a, \varepsilon), V_b) \subset \boldsymbol{X} \setminus M.$$

Then $f|_{\boldsymbol{Y}} \in \mathcal{O}_s(\boldsymbol{Y})$ for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$. Consequently, by Theorem 4.3.1, we get $\widehat{\boldsymbol{Y}} \subset G$. In particular, $(a, b) \in \{a\} \times V_b \subset \widehat{\boldsymbol{Y}} \subset G$.

Thus $\widehat{M}_{(a,\cdot)} \subset M_{(a,\cdot)}$ for all $a \in A$, and therefore, by Lemma 5 from [Chi-Sad 1988], $\widehat{M}$ is pluripolar.

In the case where all the fibers $M_{(a,\cdot)}$, $a \in A$, are discrete, Lemma 8 from [Chi-Sad 1988] implies that $\widehat{M}$ is analytic.

[A MORE DETAILED PROOF. WILL BE COMPLETED. . . . . .] $\square$

4.11 Grauert–Remmert, Dloussky, and Chirka theorems

The following three extensions theorems with singularities, which are nowadays standard tools in complex analysis.

([5]) See also [Jar-Pfl 2001b].

Theorem 4.11.1. *Let G be a Riemann domain over $\mathbb{C}^n$ such that $\mathcal{O}(G)$ separates points in G and let $\widehat{G}$ be its envelope of holomorphy. We assume that G is a subdomain of $\widehat{G}$.*

(a) (Grauert–Remmert — [Gra-Rem 1956]) *Let $M \subset \widehat{G}$ be an analytic subset of codimension one. Then $\widehat{G} \setminus M$ is the envelope of holomorphy of $G \setminus M$.*

(b) (Dloussky — [Dlo 1977], see also [Por 2002]) *Let $M \subset G$ be a relatively closed thin subset. Then there exists an analytic subset $\widehat{M}$ of $\widehat{G}$ such that $\widehat{M} \cap G \subset M$ and $\widehat{G} \setminus \widehat{M}$ is the envelope of holomorphy of $G \setminus M$.*

(c) (Chirka — [Chi 1993]) *Let $M \subset G$ be a relatively closed pluripolar set. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{G}$ such that $\widehat{M} \cap G \subset M$ and $\widehat{G} \setminus \widehat{M}$ is the envelope of holomorphy of $G \setminus M$.*

Roughly speaking, the above results say that if $M \subset G$ is analytic (resp. pluripolar), then $\widehat{G \setminus M} = \widehat{G} \setminus \widehat{M}$ where $\widehat{M} \subset \widehat{G}$ is analytic (resp. pluripolar) and $\widehat{M} \cap G \subset M$. Observe that in general $\widehat{M} \cap G \subsetneq M$, e.g.:

- if M is analytic and $\dim M \leq n - 2$, then $\widehat{M} = \varnothing$ (cf. [Jar-Pfl 2008], Propositions 1.9.11, 1.9.14),
- if M is a compact pluripolar set, then $\widehat{M} = \varnothing$ (cf. [Jar-Pfl 2008], Theorem 1.9.1).

[A SKETCH OF PORTEN'S PROOF OF THE DLOUSSKY THEOREM. WILL BE COMPLETED. . . . .]

[A SKETCH OF THE PROOF OF THE CHIRKA THEOREM. WILL BE COMPLETED.]

Chapter 5

Cross theorem with singularities

5.1 Formulation of the extension problem with singularities

The notion of separately holomorphic functions on a cross $\boldsymbol{X}$ extends in a natural way to $\boldsymbol{X} \setminus M$.

Definition 5.1.1. Let $D_j \in \mathfrak{R}_c(\mathbb{C}^{n_j})$ and let $\varnothing \neq A_j \subset D_j$, $j = 1, \dots, N$. Put $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$. Let $M \subset \boldsymbol{X}$ be such that for any $a = (a_1, \dots, a_N) \in \boldsymbol{c}(\boldsymbol{X}) = A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the *fiber*

$$M_{(a'_j, \cdot, a''_j)} := \{z_j \in D_j : (a'_j, z_j, a''_j) \in M\}$$

is closed in D_j $(^1)$. We say that a function $f : \boldsymbol{X} \setminus M \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{X} \setminus M$ ($f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$) if for any $a \in \boldsymbol{c}(\boldsymbol{X})$ and $j \in \{1, \dots, N\}$, either $M_{(a'_j, \cdot, a''_j)} = D_j$ or $M_{(a'_j, \cdot, a''_j)} \subsetneq D_j$ and the function

$$D_j \setminus M_{(a'_j, \cdot, a''_j)} \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic.

In the context of Theorems 4.3.1 and 4.11.1, one may formulate the following *extension problem with singularities* (in its elementary version).

Definition 5.1.2. Let D_j, A_j, $j = 1, \dots, N$, be as in Theorem 4.3.1. Let M be an analytic subset of an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$ (resp. $M \subset \boldsymbol{X}$ be a relatively closed pluripolar set). We ask whether there exists an analytic subset (resp. a relatively closed pluripolar set) $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:

(A) $\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$ (resp. $\widehat{M} \cap T \subset M$ for a set $T \subset \boldsymbol{X}$ with $T \setminus M \notin \mathcal{PLP}$),

(B) if M is analytic in $\widehat{\boldsymbol{X}}$ (i.e. $U = \widehat{\boldsymbol{X}}$), then $\widehat{M}$ is the union of all one codimensional irreducible components of M,

(C) for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$ (resp. on $T \setminus M$),

(D) the set $\widehat{M}$ is singular with respect to the family $\widehat{\mathcal{F}} := \{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$.

Roughly speaking, $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is the envelope of holomorphy of $\boldsymbol{X} \setminus M$ with respect to $\mathcal{O}_s(\boldsymbol{X} \setminus M)$. Obviously, by Theorem 4.3.1, if $M = \varnothing$, then $\widehat{M} = \varnothing$.

$(^1)$ For example, M is relatively closed in $\boldsymbol{X}$. We do not exclude the cases where $M_{(a'_j, \cdot, a''_j)} = \varnothing$ or $M_{(a'_j, \cdot, a''_j)} = D_j$.

Remark 5.1.3. (a) Observe that Theorem 4.10.1 solves a special case of our extension problem with singularities ($D = \mathbb{D}^p$, $G := \mathbb{C}$, $B = \mathbb{D}$).

(b) We will see in Theorem 5.3.1 that in the case $N = 2$ the set T will be always of the form $T = \boldsymbol{K}(A', B'; D, G)$, where $A' \subset A$, $B' \subset B$ are such that $A \setminus A'$, $B \setminus B'$ are pluripolar and for any $(a, b) \in A' \times B'$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are pluripolar. Observe that in such a case the set $T \setminus M$ is automatically non-pluripolar (cf. Exercise 5.3.3(b)).

(c) Note that the function $\widehat{f}$ in (C) is uniquely determined because $T \setminus M \notin \mathcal{PLP}$.

(d) Observe that if $\widehat{M}$ satisfies (A), (B), (C), then we may always replace $\widehat{M}$ by $\widehat{M}_{s,\,\widehat{\mathcal{F}}}$ (cf. § 3.1.8). Thus, condition (D) is a consequence of (A), (B), (C).

(e) Since $\widehat{\boldsymbol{X}}$ is a domain of holomorphy and $\mathcal{O}(\widehat{\boldsymbol{X}})|_{\widehat{\boldsymbol{X}} \setminus \widehat{M}} \subset \widehat{\mathcal{F}}$, condition (D) is satisfied iff $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is an $\widehat{\mathcal{F}}$–domain of holomorphy. Notice that if $T = \boldsymbol{X}$, then $\widehat{\mathcal{F}} = \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$.

(f) Observe that in $\widehat{M}$ is analytic and non-empty, then $\widehat{M}$ must be of pure codimension one (cf. Proposition 3.1.25).

Our next aim to is prove that the above (and even some more general) extension problems with singularities have always solutions. This will be done in §§ 5.3, 5.4 and in Chapter 7.

First we like to clarify why we require $\widehat{M} \cap T \subset M$ and not simply $\widehat{M} \cap \boldsymbol{X} \subset M$ (like in the analytic case).

Remark 5.1.4. Let $N = 2$, $n_1 = n_2 = 1$, $D_1 = D_2 = \mathbb{C}$, $A_1 := \mathbb{D}$, $\boldsymbol{X} := \boldsymbol{K}(\mathbb{D}, A_2; \mathbb{C}, \mathbb{C})$. Note that $\widehat{\boldsymbol{X}} = \mathbb{C}^2$ (cf. Proposition 3.4.3). Assume that $M \subset \{0\} \times \mathbb{C}$ is a closed (pluripolar) set and suppose that $\widehat{M}$ is a solution of the above extension problem with singularities for which $T = \boldsymbol{X}$.

Put $\boldsymbol{Y} := \boldsymbol{K}(\mathbb{D}_*, A_2; \mathbb{C}_*, \mathbb{C}) \subset \boldsymbol{X} \setminus M \subset \mathbb{C}^2 \setminus \widehat{M}$. Then $\widehat{\boldsymbol{Y}} = \mathbb{C}_* \times \mathbb{C}$. If $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$, then $f|_Y \in \mathcal{O}_s(\boldsymbol{Y})$. Thus, every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\widetilde{f} \in \mathcal{O}(\mathbb{C}_* \times \mathbb{C})$ with $\widetilde{f} = f$ in $\boldsymbol{Y}$ and, consequently, $\widetilde{f} = \widehat{f}$ on $\mathbb{C}_* \times \mathbb{C} \setminus \widehat{M}$. Since $\mathbb{C}^2 \setminus \widehat{M}$ is a domain of holomorphy, we conclude that $\widehat{M} \subset \{0\} \times \mathbb{C}$. Consider the following two particular cases.

(a) Let $A_2 = \mathbb{D}$, $M := \{0\} \times \overline{\mathbb{D}}$. Let $f_0 : \boldsymbol{X} \setminus M \longrightarrow \mathbb{C}$,

$$f_0(z, w) := \begin{cases} 1/z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0,\ |w| > 1 \end{cases},$$

and observe that $f_0 \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$. Since f_0 extends to an $\widehat{f_0} \in \mathcal{O}(\mathbb{C}^2 \setminus \widehat{M})$ with $\widehat{f_0} = f_0$ on $\boldsymbol{X} \setminus M$, we conclude that $\widehat{f_0}(z, w) = 1/z$, $(z, w) \in (\widehat{\boldsymbol{X}} \setminus \widehat{M}) \cap (\mathbb{C}_* \times \mathbb{C})$. Hence $\{0\} \times \mathbb{C} \subset \widehat{M}$. Thus $\widehat{M} = \{0\} \times \mathbb{C}$. Consequently, $\widehat{M} \cap \boldsymbol{X} = \{0\} \times \mathbb{C} \not\subset M$; a contradiction.

(b) Let $A_2 := \{w \in \mathbb{C} : r < |w| < 1\}$, where $0 < r < 1$,

$$M := \{0\} \times \{w \in \mathbb{C} : |w| = r\}.$$

Now we look at the function $f_0 \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ defined by

$$f_0(z,w) := \begin{cases} w, & \text{if } z \neq 0 \text{ or } z = 0,\ |w| > r \\ 0, & \text{if } z = 0,\ |w| < r \end{cases}.$$

Obviously, $\widehat{f}_0(z,w) = w$, $(z,w) \in \mathbb{C}^2 \setminus \widehat{M}$. Since $\widehat{M} \cap \boldsymbol{X} \subset M$ and $\widehat{f}_0 = f_0$ on $\boldsymbol{X} \setminus M$, we get $w = \widehat{f}_0(0,w) = f_0(0,w) = 0$, $|w| < r$; a contradiction.

5.2 Öktem and Siciak theorems

The next step after Theorem 4.10.1 was done 10 years later by Öktem who studied the following range problem in the *mathematical tomography* (cf. [Ökt 1998], [Ökt 1999]).

For $\omega = (\cos\alpha, \sin\alpha)$ let $\omega^\perp := (-\sin\alpha, \cos\alpha)$. Define

$$\ell_{\omega,p} := \{x = (x_1, x_2) \in \mathbb{R}^2 : \langle x, \omega\rangle = x_1 \cos\alpha + x_2 \sin\alpha = p\}, \quad p \in \mathbb{R},$$

and let $\mathcal{L}_{\omega,p}$ be the Lebesgue measure on the line $\ell_{\omega,p}$. For $\mu \in \mathbb{R}_*$, the *exponential Radon transform* is given by the following mapping

$$\mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{R}) \ni h \xmapsto{R_\mu} R_\mu(h), \quad R_\mu(h) : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R},$$
$$R_\mu(h)(\omega, p) := \int_{\ell_{\omega,p}} h(x) e^{\mu\langle x, \omega^\perp\rangle} d\mathcal{L}_{\omega,p}(x).$$

The main problem is to recover h from $R_\mu(h)$ which is measured. So it is important to know the shape of the range of R_μ. For $g : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{C}$ let $\widehat{g} : \mathbb{T} \times \mathbb{C} \longrightarrow \mathbb{C}$ be the Fourier transform of g with respect to the second variable, i.e.

$$\widehat{g}(\omega, \zeta) := \int_{\mathbb{R}} g(\omega, p) e^{-i\zeta p} d\mathcal{L}^1(p).$$

Theorem 5.2.1 (Öktem (1998)). *Let $g : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{C}$ and $\mu \neq 0$. Then the following statements are equivalent:*

(i) *there is $h \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C})$ with $g = R_\mu(h)$;*

(ii) *$g \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}, \mathbb{C})$ and $\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it)$ whenever $\omega, \sigma \in \mathbb{T}$ and $t \in \mathbb{R}$ are such that $t\omega + \mu\omega^\perp = -t\sigma + \mu\sigma^\perp$.*

To prove this result Öktem used the following extension theorem with singularities.

Theorem 5.2.2 (Öktem (1998/1999)). *Let*

$$\boldsymbol{X} := \boldsymbol{K}(\mathbb{R}, \mathbb{R}; \mathbb{C}, \mathbb{C}) = (\mathbb{R} \times \mathbb{C}) \cup (\mathbb{C} \times \mathbb{R}), \quad M := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = z_2\}$$

(note that $\mathbb{C}^2 = \widehat{\boldsymbol{X}}$). Then for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\mathbb{C}^2 \setminus M)$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$.

Öktem's result (Theorem 5.2.2) was extended by J. Siciak in [Sic 2001].

Theorem 5.2.3 (Siciak (2001)). *Let $D_j = \mathbb{C}$, let $A_j \subset \mathbb{C}$ be locally regular, $j = 1, \dots, N$, and let $\boldsymbol{X} := \boldsymbol{K}((A_j, \mathbb{C})_{j=1}^N)$ (note that $\widehat{\boldsymbol{X}} = \mathbb{C}^N$). Let*

$$M := \{z \in \mathbb{C}^N : P(z) = 0\},$$

where P is a non-constant polynomial of N–complex variables. Then for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there is an $\widehat{f} \in \mathcal{O}(\mathbb{C}^N \setminus M)$ such that $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$.

The above theorem has been generalized in cf. [Jar-Pfl 2001a], [Jar-Pfl 2001b], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2008] to various cross theorems with analytic and pluripolar singularities, which will be presented in the next sections.

5.3 Extension theorems with singularities in the case where $N = 2$

We begin with the case $N = 2$. We should point out that this is case is essentially simpler than the case $N \geq 3$. Although in the present section we are interested in the case $N = 2$, some results (whose proofs for $N \geq 3$ are not essentially different from the case $N = 2$) will be presented for arbitrary N. Our aim is to prove the following theorem.

Theorem 5.3.1 (Extension theorem with singularities). *Let D and G be a Riemann domains of holomorphy over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively. Let $A \subset D$, $B \subset G$ be locally pluriregular, $\boldsymbol{X} := \boldsymbol{K}(A, B, ; D, G)$, and let $M \subset \boldsymbol{X}$ be a relatively closed set. Put*

$$A' = A'(M) := \{a \in A : M_{(a,\cdot)} \in \mathcal{PLP}\},$$
$$B' = B'(M) := \{b \in B : M_{(\cdot,b)} \in \mathcal{PLP}\}$$

and assume that $A \setminus A' \in \mathcal{PLP}$, $B \setminus B' \in \mathcal{PLP}$ (e.g. $M \in \mathcal{PLP}$ — cf. Proposition 3.3.27). Put $\boldsymbol{X}' := \boldsymbol{K}(A', B'; D, G)$. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:

- *$\widehat{M} \cap \boldsymbol{X}' \subset M$,*
- *for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{X}' \setminus M$,*
- *the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,*
- *if for any $(a, b) \in A' \times B'$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are thin in G and D, respectively, then $\widehat{M}$ is analytic,*
- *if $M \subset U$ be an analytic subset of an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$, then $\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$; moreover, if $U = \widehat{\boldsymbol{X}}$, then $\widehat{M}$ is the union of all irreducible one-codimensional components of M.*

The case where $G = \mathbb{C}^q$, and B is open was studied in [Chi-Sad 1988] (for $q = 1$) (2) and in [Kaz 1988] (for arbitrary q).

To simplify formulations we will use the following terminology related to Theorem 5.3.1:

- we say that we are dealing with the *pluripolar case* if $M \subset \boldsymbol{X}$ is a relatively closed such that $A'(M), B'(M) \in \mathcal{PLP}$;
- we say that we are dealing with the *analytic case* if M is an analytic subset of an open neighborhood U of $\boldsymbol{X}$.

Remark 5.3.2. (a) In the language of Definition 5.1.2 we have $T = \boldsymbol{X}'$.

(b) Observe that in the analytic case we have

$$A' = \{a \in A : M_{(a,\cdot)} \neq G\}, \quad B' = \{b \in B : M_{(\cdot,b)} \neq D\}.$$

In particular, $\boldsymbol{X} \setminus M = \boldsymbol{X}' \setminus M$ and if $\widehat{M} \cap \boldsymbol{X}' \subset M$, then $\widehat{M} \cap \boldsymbol{X} \subset M$. Thus, in the analytic case we may take $T = \boldsymbol{X}$.

(c) Suppose that the pluripolar case is already proved. Let M be an analytic subset of U. Then $M \cap \boldsymbol{X}$ is pluripolar and for all $(a, b) \in A' \times B'$, the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are analytic.

We may apply the pluripolar case to $M \cap \boldsymbol{X}$ and we get an analytic set $\widehat{M}$ such that:

- $\widehat{M} \cap \boldsymbol{X} \subset M$,
- for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$,
- the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$.

Thus in order to finish the proof in the analytic case we have to show that $\widehat{M} \cap U_0 \subset M$. $\boxed{?}$ Unfortunately, we do not know any elementary proof of this inclusion (having the inclusion $\widehat{M} \cap \boldsymbol{X} \subset M$) $\boxed{?}$ Consequently, the analytic case will be proved using different methods.

(d) *Suppose that the pluripolar case is already proved. Then, in order to prove the analytic case, we may assume that* $U = \widehat{\boldsymbol{X}}$.

Indeed, as we have observed in (c), we only need to show that $\widehat{M} \cap U_0 \subset M$, where $\widehat{M}$ constructed via the pluripolar case. Take arbitrary $(a, b) \in A \times B$ and domains of holomorphy $D' \Subset D$, $G' \Subset G$ with $(a, b) \in D' \times G'$. Since $(\{a\} \times G) \cup (D \times \{b\}) \subset \boldsymbol{X} \subset U$, there exists an $r > 0$ such that $\widehat{\mathbb{P}}(a, r) \subset D'$, $\widehat{\mathbb{P}}(b, r) \subset G'$, and $(\widehat{\mathbb{P}}(a, r) \times G') \cup (D' \times \widehat{\mathbb{P}}(b, r)) \subset U$ (3). Put

$$\begin{aligned}
\boldsymbol{Y} &:= \boldsymbol{K}(A \cap \widehat{\mathbb{P}}(a, r), B \cap \widehat{\mathbb{P}}(b, r); \widehat{\mathbb{P}}(a, r), G') \subset \boldsymbol{X}, \\
\boldsymbol{Z} &:= \boldsymbol{K}(A \cap \widehat{\mathbb{P}}(a, r), B \cap \widehat{\mathbb{P}}(b, r); D', \widehat{\mathbb{P}}(b, r)) \subset \boldsymbol{X}.
\end{aligned}$$

Observe that $\widehat{\boldsymbol{Y}} \subset \widehat{\mathbb{P}}(a, r) \times G' \subset U$. Consequently, $M \cap \boldsymbol{Y}$ satisfies the "global" assumptions (with respect to domains $\widehat{\mathbb{P}}(a, r)$, G' and test sets $A \cap \widehat{\mathbb{P}}(a, r)$, $B \cap$

(2) Cf. Theorem 4.10.1.

(3) Here and in the sequel, to simplify notation, we will write $\widehat{\mathbb{P}}(a, r)$ without specifying the Riemann domain in which the "polydisc" is contained — it will always follow from the context.

$\widehat{\mathbb{P}}(b,r)$). Hence $\widehat{M \cap \boldsymbol{Y}} \subset M$. It remains to show that $\widehat{M} \cap \widehat{\boldsymbol{Y}} \subset \widehat{M \cap \boldsymbol{Y}}$. Indeed, since $\mathcal{O}_s(\boldsymbol{X} \setminus M)|_{\boldsymbol{Y} \setminus M} \subset \mathcal{O}_s(\boldsymbol{Y} \setminus M)$, each function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ has an extension $\widehat{f}^{\boldsymbol{Y}} \in \mathcal{O}(\widehat{\boldsymbol{Y}} \setminus \widehat{M \cap \boldsymbol{Y}})$ with $\widehat{f}^{\boldsymbol{Y}} = f$ on $\boldsymbol{Y} \setminus M$. Since $\widehat{M}$ is singular, we must have $\widehat{M} \cap \widehat{\boldsymbol{Y}} \subset \widehat{M \cap \boldsymbol{Y}}$.

Repeating the same argument with respect to $\boldsymbol{Z}$ we conclude that $\widehat{M} \cap \widehat{\boldsymbol{Z}} \subset M$. Thus $\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$.

(e) Let $S \subset \widehat{\boldsymbol{X}}$ be an analytic set of pure codimension one. Then $S \cap \boldsymbol{X} \neq \varnothing$.

Indeed, suppose that $S \cap \boldsymbol{X} = \varnothing$. Since S is of pure codimension one, $\widehat{\boldsymbol{X}} \setminus S$ is a domain of holomorphy, and therefore, there exists a $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus S)$ such that $\widehat{\boldsymbol{X}} \setminus S$ is the domain of existence of g (cf. Proposition 3.1.20). Since $\boldsymbol{X} \subset \widehat{\boldsymbol{X}} \setminus S$, we conclude that $g|_{\boldsymbol{X}} \in \mathcal{O}_s(\boldsymbol{X})$. By Theorem 4.3.1 there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{g} = g$ on $\boldsymbol{X}$, and consequently, on $\widehat{\boldsymbol{X}} \setminus S$. Thus g extends holomorphically to $\widehat{\boldsymbol{X}}$; a contradiction.

(f) Let $M \subsetneq \widehat{\boldsymbol{X}}$ be an analytic set. Suppose that $\widehat{M} \subset \widehat{\boldsymbol{X}}$ is an analytic set such that:

- $\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$,
- every function $f \in \mathcal{F} := \mathcal{O}_s(\boldsymbol{X} \setminus M) \cap \mathcal{C}(\boldsymbol{X} \setminus M)$ extends to an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$,
- the set $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$.

Then $\widehat{M}$ is the union of all irreducible components of M of codimension one. *In particular, the last assertion of Theorem 5.3.1 follows from the others.*

Indeed, let $\widetilde{M}$ be the union of all irreducible components of M of codimension one. Consider two cases:

$\widetilde{M} \neq \varnothing$: Similarly as in (a), there exists a non-continuable function $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widetilde{M})$. Then $g|_{\boldsymbol{X} \setminus M} \in \mathcal{O}_s(\boldsymbol{X} \setminus M) \cap \mathcal{C}(\boldsymbol{X} \setminus M)$ and, therefore, there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{g} = g$ on $\boldsymbol{X} \setminus M$. Hence, $\widehat{g} = g$ on $\widehat{\boldsymbol{X}} \setminus (\widehat{M} \cup \widetilde{M})$. Since g is non-continuable, we conclude that $\widetilde{M} \subset \widehat{M}$. The set $\widehat{M}$, as a non-empty singular set, is also of pure codimension one. Since $\widehat{M} \cap U_0 \subset M$ and $S \cap U_0 \neq \varnothing$ for every irreducible component of $\widehat{M}$ (by (e)), we conclude (using the identity principle for analytic sets) that $\widehat{M} \subset M$ (cf. [Chi 1993], § 5.3). Consequently, $\widehat{M} \subset \widetilde{M}$

$\widetilde{M} = \varnothing$: It remains to exclude the situation when $\widehat{M} \neq \varnothing$. If $\widetilde{M} = \varnothing$, then the codimension of M is ≥ 2. If $\widehat{M} \neq \varnothing$ then the codimension of $\widehat{M}$ is 1. Since we have $\widehat{M} \subset M$ (as above), we get a contradiction.

(g) Our assumption that the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$, $(a,b) \in A' \times B'$, are pluripolar is in fact very weak and there is a lot of non-pluripolar sets $M \subset \boldsymbol{X}$ that fulfil this condition — cf. Remark 5.4.9.

Exercise 5.3.3. Let $\boldsymbol{T}$ be an N–fold generalized cross and let $M \subset \boldsymbol{c}(\boldsymbol{T})$. Prove the following statements.

(a) Let $A_1, \dots, A_N$ be locally pluriregular, let $\Sigma'_N \supset \Sigma_N$ be pluripolar, and assume that the fiber $M_{(a'_N,\cdot)}$ is pluripolar for every $a'_N \in A'_N \setminus \Sigma_N$. Then the set $((A'_N \setminus \Sigma'_N) \times A_N) \setminus M$ is dense in $\boldsymbol{c}(\boldsymbol{T}) \setminus M$.

Hint: Take an $a = (a', a_N) \in \boldsymbol{c}(\boldsymbol{T}) \setminus M$. Since A'_N is locally pluriregular and Σ'_N is pluripolar, there exists a sequence $A'_N \setminus \Sigma'_N \ni a'^k \longrightarrow a'$. The set $P := \bigcup_{k=1}^{\infty} M_{(a'^k,\cdot)}$ is pluripolar. In particular, there exists a sequence $A_N \setminus P \ni a_N^k \longrightarrow a_N$. Then $((A'_N \setminus \Sigma'_N) \times A_N) \setminus M \ni (a'^k, a_N^k) \longrightarrow a$.

(b) If $A_1, \dots, A_N \notin \mathcal{PLP}$, $\Sigma_1, \dots, \Sigma_N \in \mathcal{PLP}$, and $M_{(a'_j,\cdot,a''_j)} \in \mathcal{PLP}$ for each $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, then

$$\{(a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j,\ a_j \notin M_{(a'_j,\cdot,a''_j)}\}, \quad j = 1, \dots, N,$$

are not pluripolar and therefore, $\boldsymbol{c}(\boldsymbol{T}) \setminus M \notin \mathcal{PLP}$ (use Proposition 3.3.27(c)).

(c) If $A_1, \dots, A_N$ are locally pluriregular, $\Sigma_1, \dots, \Sigma_N \in \mathcal{PLP}$, and $M_{(a'_j,\cdot,a''_j)} \in \mathcal{PLP}$ for every $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, then the sets

$$\{(a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j,\ a_j \notin M_{(a'_j,\cdot,a''_j)}\}, \quad j = 1, \dots, N,$$

are locally pluriregular, and therefore, $\boldsymbol{c}(\boldsymbol{T}) \setminus M$ is locally pluriregular (use Proposition 3.4.11(h)).

The main "technical tool" in the proof of Theorem 5.3.1 is the following theorem.

Theorem 5.3.4 (Glueing theorem). *Let D, G be Riemann domains of holomorphy over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, let $A \subset D$, $B \subset G$ be locally pluriregular, $\boldsymbol{X} := \boldsymbol{K}(A, B; D, G)$, let $M \subset \boldsymbol{X}$ and let $A' \subset A$, $B' \subset B$ be such that:*

- *for any $(a, b) \in A \times B$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are relatively closed in G and D, respectively,*
- *$A \setminus A'$, $B \setminus B'$ are pluripolar,*
- *for any $(a, b) \in A' \times B'$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are pluripolar.*

In the analytic case we additionally assume that $M \subsetneq U$ is an analytic set in a connected open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$.

Fix a family $\varnothing \neq \mathcal{F} \subset \mathcal{O}_s(\boldsymbol{X} \setminus M)$.

Let $(D_k)_{k=1}^{\infty}$, $(G_k)_{k=1}^{\infty}$ be exhaustion sequences of Riemann domains of holomorphy for D and G, respectively, such that

$$\varnothing \neq A'_k := A' \cap D_k \subset A \cap D_k =: A_k,$$
$$\varnothing \neq B'_k := B' \cap G_k \subset B \cap G_k =: B_k.$$

Put $\varXi_k := A'_k \times B'_k \setminus M$ or $\varXi_k := A_k \times B_k$, $k \in \mathbb{N}$.

We assume that for each $k \in \mathbb{N}$, $(a, b) \in \varXi_k$, there exist:

- *polydiscs $\widehat{\mathbb{P}}_D(a, r_{k,a}) \subset D_k$, $\widehat{\mathbb{P}}_G(b, s_{k,b}) \subset G_k$,*
- *relatively closed pluripolar sets*

$$S_{k,a} \subset \widehat{\mathbb{P}}_D(a, r_{k,a}) \times G_k =: V_{k,a}, \quad S^{k,b} \subset D_k \times \widehat{\mathbb{P}}_G(b, s_{k,b}) =: V^{k,b},$$

such that:

- $$S_{k,a} \cap ((A' \cap \widehat{\mathbb{P}}_D(a, r_{k,a})) \times G_k) \subset M,$$
  $$S^{k,b} \cap (D_k \times (B' \cap \widehat{\mathbb{P}}_G(b, s_{k,b}))) \subset M,$$
- *for any* $f \in \mathcal{F}$ *there exist* $\widetilde{f}_{k,a} \in \mathcal{O}(V_{k,a} \setminus S_{k,a})$, $\widetilde{f}^{k,b} \in \mathcal{O}(V^{k,b} \setminus S^{k,b})$ *with*
  $$\widetilde{f}_{k,a} = f \quad \text{on } (A' \cap \widehat{\mathbb{P}}_D(a, r_{k,a})) \times G_k \setminus M,$$
  $$\widetilde{f}^{k,b} = f \quad \text{on } D_k \times (B' \cap \widehat{\mathbb{P}}_G(b, s_{k,b})) \setminus M,$$
- *in the analytic case we additionally assume that*
  $$V_{k,a} \cup V^{k,b} \subset U, \quad S_{k,a} \cup S^{k,b} \subset M.$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *such that:*
- $\widehat{M} \cap \boldsymbol{X}' \subset M$, *where* $\boldsymbol{X}' := \boldsymbol{K}(A', B'; D, G)$,
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *with* $\widehat{f} = f$ *on* $\boldsymbol{X}' \setminus M$,
- $\widehat{M}$ *is singular with respect to the family* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if all the sets* $S_{k,a}$, $S^{k,b}$, $(a,b) \in \Xi_k$, $k \in \mathbb{N}$, *are thin, then* $\widehat{M}$ *is analytic,*
- *in the analytic case we additionally have* $\widehat{M} \cap U_0 \subset M$ *for an open neighborhood* $U_0 \subset U$ *of* $\boldsymbol{X}'$; *moreover, if* $\Xi_k = A_k \times B_k$, *then* $\widehat{M} \cap U_0 \subset M$ *for an open neighborhood* $U_0 \subset U$ *of* $\boldsymbol{X}$ (4).

Proof. We may assume that for any $k \in \mathbb{N}$ and $(a,b) \in \Xi_k$
- $S_{k,a}$ is singular with respect to the family $\{\widetilde{f}_{k,a} : f \in \mathcal{F}\}$,
- $S^{k,b}$ is singular with respect to the family $\{\widetilde{f}^{k,b} : f \in \mathcal{F}\}$.

In particular, $S_{k,a}$ (resp. $S^{k,b}$) is thin iff it is analytic.

Fix a $k \in \mathbb{N}$ and define (details are explained below):

$$V_k := \bigcup_{(a,b) \in \Xi_k} V_{k,a} \cup V^{k,b}, \quad \widetilde{f}_k := \bigcup_{(a,b) \in \Xi_k} \widetilde{f}_{k,a} \cup \widetilde{f}^{k,b},$$
$$S_k := \bigcup_{(a,b) \in \Xi_k} S_{k,a} \cup S^{k,b} \subset V_k,$$
$$\boldsymbol{X}_k := \boldsymbol{K}(A_k, B_k; D_k, G_k), \quad \boldsymbol{X}'_k := \boldsymbol{K}(A'_k, B'_k; D_k, G_k).$$

Observe that $\boldsymbol{X}'_k \subset V_k$. Indeed, let $(z,w) \in \boldsymbol{X}'_k$, e.g. $z = a \in A'_k$, $w \in G_k$. Since $M_{(a,\cdot)}$ is pluripolar, there exists a $b \in B'_k \setminus M_{(a,\cdot)}$. Then $(a,b) \in A'_k \times B'_k \setminus M$ and $(z,w) \in \widehat{\mathbb{P}}_D(a, r_{k,a}) \times G_k = V_{k,a}$.

Notice that in the case $\Xi_k = A_k \times B_k$ we obviously have $\boldsymbol{X}_k \subset V_k$. Moreover, in the analytic case we get $V_k \subset U$.

Take an $f \in \mathcal{F}$. We want to glue the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k\}$ and the functions $\{\widetilde{f}_{k,a},\ \widetilde{f}^{k,b} : (a,b) \in \Xi_k\}$ to obtain a global holomorphic function $\widetilde{f}_k$ on $V_k \setminus S_k$.

(4) Note that this is the only place where the case $\Xi_k = A_k \times B_k$ plays an important role.

Let $(a,b) \in \Xi_k$. Observe $\widetilde{f}_{k,a} = f = \widetilde{f}^{k,b}$ on the non-pluripolar set

$$(A' \cap \widehat{\mathbb{P}}_D(a, r_{k,a})) \times (B' \cap \widehat{\mathbb{P}}_G(b, s_{k,b})) \setminus M$$

(cf. Exercise 5.3.3(b)). Hence

$$\widetilde{f}_{k,a} = \widetilde{f}^{k,b} \text{ on } \widehat{\mathbb{P}}_D(a, r_{k,a}) \times \widehat{\mathbb{P}}_G(b, s_{k,b}) \setminus (S_{k,a} \cup S^{k,b}).$$

Since $S_{k,a}$ and $S^{k,b}$ are singular, we conclude that

$$S_{k,a} \cap (\widehat{\mathbb{P}}_D(a, r_{k,a}) \times \widehat{\mathbb{P}}_G(b, s_{k,b})) = S^{k,b} \cap (\widehat{\mathbb{P}}_D(a, r_{k,a}) \times \widehat{\mathbb{P}}_G(b, s_{k,b})).$$

Now let $a', a'' \in A'_k$ be such that $C := \widehat{\mathbb{P}}_D(a', r_{k,a'}) \cap \widehat{\mathbb{P}}_D(a'', r_{k,a''}) \neq \varnothing$. Fix a $b \in B'_k \setminus (M_{(a',\cdot)} \cup M_{(a'',\cdot)})$. We know that

$$\widetilde{f}_{k,a'} = \widetilde{f}^{k,b} = \widetilde{f}_{k,a''} \text{ on } C \times \widehat{\mathbb{P}}_G(b, r_{k,b}) \setminus (S_{k,a'} \cup S^{k,b} \cup S_{k,a''}).$$

Hence, by the identity principle, we conclude that

$$\widetilde{f}_{k,a'} = \widetilde{f}_{k,a''} \text{ on } C \times G_k \setminus (S_{k,a'} \cup S_{k,a''})$$

and, moreover,

$$S_{k,a'} \cap (C \times G_k) = S_{k,a''} \cap (C \times G_k).$$

The same argument works for $b', b'' \in B'_k$.

Let U_k be the connected component of $V_k \cap \widehat{\boldsymbol{X}}'_k$ with $\boldsymbol{X}'_k \subset U_k$. Recall that $\widehat{\boldsymbol{X}}'_k = \widehat{\boldsymbol{X}}_k$ (cf. Exercise 4.2.3(h)). Observe that in the case where $\Xi_k = A_k \times B_k$ we have $\boldsymbol{X}_k \subset U_k$.

We have constructed a relatively closed pluripolar set $S_k \subset U_k$ such that:

- $S_k \cap \boldsymbol{X}'_k \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\widetilde{f}_k \in \mathcal{O}(U_k \setminus S_k)$ with $\widetilde{f}_k = f$ on $\boldsymbol{X}'_k \setminus M$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k\}$ are thin, then S_k is analytic,
- in the analytic case we have $S_k \subset M$.

Recall that $\boldsymbol{X}'_k \subset U_k \subset \widehat{\boldsymbol{X}}_k$. Observe that the envelope of holomorphy $\widehat{U}_k$ of U_k coincides with $\widehat{\boldsymbol{X}}_k$. In fact, let $h \in \mathcal{O}(U_k)$, then $h|_{\boldsymbol{X}'_k} \in \mathcal{O}_s(\boldsymbol{X}'_k)$. So, in virtue of Theorem 4.3.1, there exists an $\widehat{h} \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\widehat{h} = h$ on $\boldsymbol{X}'_k$. Hence $\widehat{h} = h$ on U_k.

Applying Theorem 4.11.1, we find a relatively closed pluripolar set $\widehat{M}_k \subset \widehat{\boldsymbol{X}}_k$ such that:

- $\widehat{M}_k \cap U_k \subset S_k$,
- for any $f \in \mathcal{F}$ there exists an function $\widehat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k)$ with $\widehat{f}_k = f_k$ on $U_k \setminus S_k$ (in particular, $\widehat{f}_k = f$ on $\boldsymbol{X}'_k \setminus M$),
- the set $\widehat{M}_k$ is singular with respect to the family $\{\widehat{f}_k : f \in \mathcal{F}\}$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k\}$ are analytic, then $\widehat{M}_k$ is analytic.

Recall that $\boldsymbol{X}_k \nearrow \boldsymbol{X}$ and $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$. Using again the singularity of the $\widehat{M}_k$'s, we get $\widehat{M}_{k+1} \cap \widehat{\boldsymbol{X}}_k = \widehat{M}_k$ and, consequently:

- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{\boldsymbol{X}}$ with $\widehat{M} \cap \boldsymbol{X}' \subset M$ (in the analytic case we have $\widehat{M} \cap \boldsymbol{X} \subset M$),
- for each $f \in \mathcal{F}$, the function $\widehat{f} := \bigcup_{k=1}^{\infty} \widehat{f}_k$ is holomorphic on $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ with $\widehat{f} = f$ on $\boldsymbol{X}' \setminus M$ (in the analytic case we have $\boldsymbol{X}' \setminus M = \boldsymbol{X} \setminus M$),
- $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \varXi_k,\ k \in \mathbb{N}\}$ are thin, then $\widehat{M}$ is analytic,
- in the analytic case, if $U_0 := \bigcup_{k=1}^{\infty} U_k$, then

$$\widehat{M} \cap U_0 = \bigcup_{k=1}^{\infty} \widehat{M}_k \cap U_k \subset \bigcup_{k=1}^{\infty} S_k \subset M. \qquad \square$$

First we use Theorem 5.3.4 to prove that the analytic case may be always reduced to the case where $U = \widehat{\boldsymbol{X}}$.

Lemma 5.3.5. *Suppose that Theorem 5.3.1 is true in the analytic case with $U = \widehat{\boldsymbol{X}}$. Then it is true in the analytic case with arbitrary U.*

Proof. We only need to check all the assumptions of Theorem 5.3.4 with $\varXi_k = A_k \times B_k$ (notice that this is the only place where this case will be used). Fix a $k \in \mathbb{N}$ and $(a,b) \in \varXi_k$. We are going to construct $r_{k,a}$, $S_{k,a}$, and $\widetilde{f}_{k,a}$ (the construction of $s_{k,b}$, $S^{k,b}$, and $\widetilde{f}^{k,b}$ is symmetric). Let $r > 0$ be such that $\widehat{\mathbb{P}}((a,b),r) \Subset D_k \times G_k$, $\widehat{\mathbb{P}}(a,r) \times G_{k+1} \subset U$. Consider the 2–fold cross

$$\boldsymbol{Y}_{k,(a,b)} := \boldsymbol{K}(A \cap \widehat{\mathbb{P}}(a,r), B_{k+1}; \widehat{\mathbb{P}}(a,r), G_{k+1}) \subset \boldsymbol{X}.$$

Observe that every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ belongs to $\mathcal{O}_s(\boldsymbol{Y}_{k,(a,b)} \setminus M)$. Thus we are in the special case and our assumptions imply that for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\widetilde{f}_{k,a} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{k,(a,b)} \setminus M)$ with $\widetilde{f}_{k,a} = f$ on $\boldsymbol{Y}_{k,(a,b)} \setminus M$. Note that $\{a\} \times G_{k+1} \subset \widehat{\boldsymbol{Y}}_{k,(a,b)}$. Let $r_{k,a} \in (0,r)$ be so small that $V_{k,a} := \mathbb{P}(a, r_{k,a}) \times G_k \subset \widehat{\boldsymbol{Y}}_{k,(a,b)}$. Then the triple $(r_{k,a},\ M \cap V_{k,a},\ \widetilde{f}_{k,a}|_{V_{k,a}})$ solves our problem. $\square$

Remark 5.3.6. Observe that in the analytic case with $U = \widehat{\boldsymbol{X}}$ we only need to prove that every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends holomorphically to an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus M)$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$.

Lemma 5.3.7. *In the analytic case with $U = \widehat{\boldsymbol{X}}$ it suffices to consider only the case where $M = g^{-1}(0)$ with $g \in \mathcal{O}(\widehat{\boldsymbol{X}})$, $g \not\equiv 0$, in particular, M is of pure codimension one.*

Proof. Since $\widehat{\boldsymbol{X}}$ is pseudoconvex, M may be written as

$$M = \{z \in \widehat{\boldsymbol{X}} : g_1(z) = \cdots = g_k(z) = 0\},$$

where $g_j \in \mathcal{O}(\widehat{\boldsymbol{X}})$, $g_j \not\equiv 0$, $j = 1, \dots, k$. Put $M_j := g_j^{-1}(0)$, $j = 1, \dots, k$.

Take an $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$. Observe that $f_j := f|_{\boldsymbol{X} \setminus M_j} \in \mathcal{O}_s(\boldsymbol{X} \setminus M_j)$. Suppose that for each j there exists an $\widehat{f}_j \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus M_j)$ such that $\widehat{f}_j = f$ on $\boldsymbol{X} \setminus M_j$.

Gluing the functions $(\widehat{f}_j)_{j=1}^k$, leads to an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus M)$ with $\widehat{f} = \widehat{f}_j$ on $\widehat{\boldsymbol{X}} \setminus M_j$, $j = 1, \dots, k$. Therefore, $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$. □

We move to the proof of Theorem 5.3.1. In the analytic case we assume that M is as in Lemma 5.3.7. The main idea is to apply Theorem 5.3.4 with $\varXi_k = A'_k \times B'_k \setminus M$. Thus in fact we have to check the following lemma.

Lemma 5.3.8. *For any $a \in A'$ and a domain of holomorphy $G' \Subset G$ with $B' \cap G' \neq \varnothing$ there exist an $r > 0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a, r) \times G' =: V \subset \widehat{\boldsymbol{X}}$ such that:*

- *$((A' \cap \widehat{\mathbb{P}}(a, r)) \times G') \cap S \subset M$,*
- *for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widetilde{f} \in \mathcal{O}(V \setminus S)$ such that $\widetilde{f} = f$ on $(A' \cap \widehat{\mathbb{P}}(a, r)) \times G' \setminus S$,*
- *if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S is analytic,*
- *in the analytic case we have $S \subset M$.*

First, we reduce the proof of Lemma 5.3.8 to a proof of the following lemma.

Lemma 5.3.9. *For any $(a, b) \in A' \times G$ and for any polydiscs $\widehat{\mathbb{P}}(a, r_0) \Subset D$, $\widehat{\mathbb{P}}(b, R_0) \Subset G$ with $R_0 > r_0$, if $M \cap \widehat{\mathbb{P}}((a, b), r_0) = \varnothing$, then for every $0 < R' < R_0$ there exist an $0 < r' < r_0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a, r') \times \widehat{\mathbb{P}}(b, R') =: V \subset \widehat{\boldsymbol{X}}$ such that:*

- *$((A' \cap \widehat{\mathbb{P}}(a, r')) \times \widehat{\mathbb{P}}(b, R')) \cap S \subset M$,*
- *for every function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$, where*

$$\begin{aligned}\boldsymbol{Y} &:= \boldsymbol{K}(A' \cap \widehat{\mathbb{P}}(a, r_0), \widehat{\mathbb{P}}(b, r_0); \widehat{\mathbb{P}}(a, r_0), \widehat{\mathbb{P}}(b, R_0)) \\ &= \widehat{\mathbb{P}}((a, b), r_0) \cup ((A' \cap \widehat{\mathbb{P}}(a, r_0)) \times \widehat{\mathbb{P}}(b, R_0)), \quad (^5)\end{aligned}$$

there exists an $\widetilde{h} \in \mathcal{O}(V \setminus S)$ such that $\widetilde{h} = h$ on $(A' \cap \widehat{\mathbb{P}}(a, r')) \times \widehat{\mathbb{P}}(b, R') \setminus M$,

- *if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S is analytic,*
- *in the analytic case we have $S \subset M$.*

Proof that Lemma 5.3.9 implies Lemma 5.3.8. Let a and G' be as in Lemma 5.3.8. Fix a domain $G'' \Subset G$ with $G' \Subset G''$. Let $\varOmega$ be the set of all $w \in G''$ such that there exist $r_w > 0$ with

$$\widehat{\mathbb{P}}((a, w), r_w) \subset \widehat{\boldsymbol{X}} \cap (\widehat{\mathbb{P}}(a, r) \times G'')),$$

and a relatively closed pluripolar set $S_w \subset \widehat{\mathbb{P}}((a, w), r_w)$ such that:

- $S_w \cap ((A' \cap \mathbb{P}(a, r_w)) \times \widehat{\mathbb{P}}(w, r_w)) \subset M$,
- every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\widetilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a, w), r_w) \setminus S_w)$ with $\widetilde{f}_w = f$ on $(A' \cap \mathbb{P}(a, r_w)) \times \widehat{\mathbb{P}}(w, r_w) \setminus M$,

---

(5) Notice that, by the Terada theorem (Theorem 4.1.1), the space $\mathcal{O}_s(\boldsymbol{Y} \setminus M)$ consist of all functions $h \in \mathcal{O}(\widehat{\mathbb{P}}((a, b), r_0)$ such that $h(z, \cdot)$ extends holomorphically to $\widehat{\mathbb{P}}(b, R_0)$ for every $z \in A' \cap \widehat{\mathbb{P}}(a, r_0)$.

- S_w is singular with respect to the family $\{\widetilde{f}_w : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S is analytic,
- in the analytic case we have $S_w \subset M$.

It is clear that Ω is open. Observe that $\Omega \neq \varnothing$. Indeed, since $B' \cap G' \setminus M_{(a,\cdot)} \neq \varnothing$, we find a point $w \in B' \cap G' \setminus M_{(a,\cdot)}$. Therefore there is a polydisc $\widehat{\mathbb{P}}((a,w),r') \subset \widehat{\boldsymbol{X}} \setminus M$. Put

$$\boldsymbol{Z} := \boldsymbol{K}(A' \cap \mathbb{P}(a,r'), B' \cap \widehat{\mathbb{P}}(w,r'); \widehat{\mathbb{P}}(a,r'), \widehat{\mathbb{P}}(w,r')).$$

Observe that for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ the function $f|_{\boldsymbol{Z}}$ belongs to $\mathcal{O}_s(\boldsymbol{Z})$. Let $0 < r_w < r'$ be such that $\widehat{\mathbb{P}}((a,w),r_w) \subset \widehat{\boldsymbol{Z}}$. By Theorem 4.3.1, for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widetilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a,w),r_w))$ with

$$\widetilde{f}_w = f \text{ on } \widehat{\mathbb{P}}((a,w),r_w) \cap \boldsymbol{Z} \supset (A' \cap \mathbb{P}(a,r_w)) \times \widehat{\mathbb{P}}(w,r_w).$$

Consequently, $w \in \Omega$.

Moreover, Ω is relatively closed in G''. Indeed, let c be an accumulation point of Ω in G'' and let $\widehat{\mathbb{P}}(c,3R) \subset G''$. Take a point $w \in \Omega \cap \widehat{\mathbb{P}}(c,R) \setminus M_{(a,\cdot)}$ and let $0 < \rho < \min\{r_w, 2R\}$ be such that $\widehat{\mathbb{P}}((a,w),\rho) \cap (M \cup S_w) = \varnothing$. Observe that $\widetilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a,w),\rho))$ and

$$\widetilde{f}_w(z,\cdot) = f(z,\cdot) \in \mathcal{O}(\widehat{\mathbb{P}}(w,\rho) \setminus M_{(z,\cdot)}), \quad z \in A' \cap \widehat{\mathbb{P}}(a,\rho).$$

Define

$$\begin{aligned}\boldsymbol{Y} &:= \boldsymbol{K}(A' \cap \widehat{\mathbb{P}}(a,\rho), \widehat{\mathbb{P}}(w,\rho); \widehat{\mathbb{P}}(a,\rho), \widehat{\mathbb{P}}(w,2R)) \\ &= \widehat{\mathbb{P}}((a,w),\rho) \cup ((A' \cap \widehat{\mathbb{P}}(a,\rho)) \times \widehat{\mathbb{P}}(w,2R))\end{aligned}$$

and put $\widetilde{\widetilde{f}}_w : \boldsymbol{Y} \setminus M \longrightarrow \mathbb{C}$,

$$\widetilde{\widetilde{f}}_w := \begin{cases} \widetilde{f}_w, & \text{on } \widehat{\mathbb{P}}((a,w),\rho) \\ f, & \text{on } (A' \cap \widehat{\mathbb{P}}(a,\rho)) \times \widehat{\mathbb{P}}(w,2R) \setminus M. \end{cases}$$

Then $\widetilde{\widetilde{f}}_w$ is well defined and $\widetilde{\widetilde{f}}_w \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$. Now, by Lemma 5.3.9 (with $b := w$, $r_0 := \rho$, $R_0 := 2R$, $R' := R$) there exist $0 < r' < \rho$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a,r') \times \widehat{\mathbb{P}}(w,R)$ such that:

- $S \cap ((A' \cap \widehat{\mathbb{P}}(a,r')) \times \widehat{\mathbb{P}}(w,R)) \subset M$,
- every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\widehat{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}(a,r') \times \widehat{\mathbb{P}}(w,R) \setminus S)$ with $\widehat{f}_w = \widetilde{\widetilde{f}}_w$ on $(A' \cap \mathbb{P}(a,r')) \times \widehat{\mathbb{P}}(w,R) \setminus M$,
- S is singular with respect to the family $\{\widehat{f}_w : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S is analytic,
- in the analytic case we have $S \subset M$.

Take an $r_c > 0$ so small that $\widehat{\mathbb{P}}((a,c),r_c) \subset \widehat{\mathbb{P}}(a,\rho') \times \widehat{\mathbb{P}}(w,R)$ and put

$$S_c := S \cap \widehat{\mathbb{P}}((a,c),r_c), \quad \widetilde{f}_c := \widehat{f}_w|_{\widehat{\mathbb{P}}((a,c),r_c) \setminus S}.$$

Obviously $\widetilde{f}_c = \widehat{f}_w = \widetilde{\widetilde{f}}_w = f$ on $(A' \cap \widehat{\mathbb{P}}(a, r_c)) \times \widehat{\mathbb{P}}(c, r_c) \setminus M$. Hence $c \in \Omega$.

Thus $\Omega = G''$. There exists a finite set $T \subset \overline{G}'$ such that

$$\overline{G}' \subset \bigcup_{w \in T} \widehat{\mathbb{P}}(w, r_w).$$

Define $r := \min\{r_w : w \in T\}$. Take $w', w'' \in T$ with

$$C := \widehat{\mathbb{P}}(w', r_{w'}) \cap \widehat{\mathbb{P}}(w'', r_{w''}) \neq \varnothing.$$

Then $\widetilde{f}_{w'} = f = \widetilde{f}_{w''}$ on $(A' \cap \widehat{\mathbb{P}}(a, r)) \times (\widehat{\mathbb{P}}(w', r_{w'}) \cap \widehat{\mathbb{P}}(w'', r_{w''})) \setminus M$. Consequently, $\widetilde{f}_{w'} = \widetilde{f}_{w''}$ on $\widehat{\mathbb{P}}(a, r) \times C \setminus (S_{w'} \cup S_{w''})$. Since $S_{w'}$ and $S_{w''}$ are singular, we conclude that they coincide on $\widehat{\mathbb{P}}(a, r) \times C$ and that the functions $\widetilde{f}_{w'}$ and $\widetilde{f}_{w''}$ glue together.

Thus we get a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a, r) \times G' =: V$ such that $S \cap ((A' \cap \widehat{\mathbb{P}}(a, r)) \times G') \subset M$ and any function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends holomorphically to an $\widetilde{f} \in \mathcal{O}(V \setminus S)$ with $\widetilde{f} = f$ on $(A' \cap \mathbb{P}(a, r)) \times G' \setminus M$. Moreover, in the analytic case we have $S \subset M$. □

In the next step we reduce the proof of Lemma 5.3.9 to a proof of the following lemma.

Lemma 5.3.10. *Let $A \subset \mathbb{P}(r_0) \subset \mathbb{C}^p$ be locally pluriregular and let M be a relatively closed subset of the cross $\boldsymbol{Z} := \boldsymbol{K}(A, K(r_0); \mathbb{P}(r_0), K(R_0))$ with $R_0 > r_0$ such that:*

- *the fiber $M_{(z,\cdot)}$ is polar for all $z \in A' \subset A$,*
- *$A \setminus A'$ is pluripolar,*
- *$M \cap (\mathbb{P}(r_0) \times K(r_0))$ is pluripolar,*
- *$B' := \{w \in K(r_0) : M_{(\cdot,w)} \in \mathcal{PLP}\}$ (note that $K(r_0) \setminus B' \in \mathcal{PLP}$),*
- *in the analytic case we have $M = g^{-1}(0)$ with $g \in \mathcal{O}(\widehat{\boldsymbol{Z}})$, $g \not\equiv 0$, and $A' := \{z \in A : M_{(z,\cdot)} \neq K(R_0)\}$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{Z}}$ such that:

- *$\widehat{M} \cap \boldsymbol{Z}' \subset M$, with $\boldsymbol{Z}' := \boldsymbol{K}(A', B'; \mathbb{P}(r_0), K(R_0))$,*
- *for every $f \in \mathcal{F} := \mathcal{O}(\mathbb{P}(r_0) \times K(r_0) \setminus M) \cap \mathcal{O}_s(\boldsymbol{Z} \setminus M)$ (6) there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{Z}' \setminus M$,*
- *if $M \cap (\mathbb{P}(r_0) \times K(r_0))$ is analytic and all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then $\widehat{M}$ is analytic,*
- *in the analytic case we have $\widehat{M} \subset M$.*

Proof that Lemma 5.3.10 implies Lemma 5.3.9. Consider the configuration like in Lemma 5.3.9. We may assume that $\widehat{\mathbb{P}}(a, r_0) = \mathbb{P}_p(r_0) \subset \mathbb{C}^p$, $\widehat{\mathbb{P}}(b, R_0) = \mathbb{P}_q(R_0) \subset$

(6) Observe that if $M \cap (\mathbb{P}(r_0) \times K(r_0)) \neq \varnothing$ and $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M)$, then f need not belong to $\mathcal{O}(\mathbb{P}(r_0) \times K(r_0) \setminus M)$. For example: take $p = 1$, $r_0 = 1$, assume additionally that $A \subsetneq \mathbb{D}$ is closed in $\mathbb{D}$, and let $M := A \times \{0\}$. Define $f : \boldsymbol{Z} \setminus M \to \mathbb{C}$, $f(z, w) := 0$ if $w \neq 0$, $f(z, 0) := 1$. Then $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M) \setminus \mathcal{O}(\mathbb{D}^2 \setminus M)$.

$\mathbb{C}^q$. Put

$$\begin{aligned}\boldsymbol{Y} := \boldsymbol{K}(A' \cap \mathbb{P}_p(r_0), \mathbb{P}_q(r_0); \mathbb{P}_p(r_0), \mathbb{P}_q(R_0)) \\ = \mathbb{P}_{p+q}(r_0) \cup ((A' \cap \mathbb{P}_p(r_0)) \times \mathbb{P}_q(R_0)).\end{aligned}$$

Let R'_0 be the supremum of all $0 < R' < R_0$ such that there exist an $r = r_{R'} \in (0, r_0)$, and a relatively closed pluripolar set $S = S_{R'} \subset V := \mathbb{P}_p(r) \times \mathbb{P}_q(R')$ for which:

- $S \cap ((A' \cap \mathbb{P}_p(r)) \times \mathbb{P}_q(R_0)) \subset M$,
- for any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ there exists an $\widetilde{h} = \widetilde{h}_{R'} \in \mathcal{O}(V \setminus S)$ such that $\widetilde{h} = h$ on $(A' \cap \mathbb{P}_p(r)) \times \mathbb{P}_q(R') \setminus M$,
- the set S is singular with respect to the family $\{\widetilde{h} : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$ (in particular, $S \cap \mathbb{P}_{p+q}(r_0) = \varnothing$),
- if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S is analytic,
- in the analytic case we have $S \subset M$.

It suffices to show that $R'_0 = R_0$. Suppose that $R'_0 < R_0$. Fix $R'_0 < R'' < R_0$ and choose $0 < R' < \widetilde{R}' < R'_0$ such that $\sqrt[q]{R'^{q-1}R''} > R'_0$. Let $r := r_{\widetilde{R}'}$, $S := S_{\widetilde{R}'}$, $\widetilde{h} := \widetilde{h}_{\widetilde{R}'}$. Fix an R''' with $R'_0 < R''' < \sqrt[q]{R'^{q-1}R''}$. Put

$$M_q := \Big(S \cap (\mathbb{P}_p(r) \times \overline{\mathbb{P}}_q(R'))\Big) \cup \Big(M \setminus (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))\Big).$$

Observe that:

- the set $M_q \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R')) = S \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))$ is pluripolar,
- $\widetilde{h} \in \mathcal{O}(\mathbb{P}_p(r) \times \mathbb{P}_q(R') \setminus M_q)$ for every $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$,
- $M_q \cap \boldsymbol{Y} \subset M$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then the set $M_q \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R')) = S \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))$ is analytic and all the fibers $(M_q)_{(z,\cdot)} \subset M_{(z,\cdot)}$, $z \in A' \cap \mathbb{P}_p(r)$, are thin,
- in the analytic case we have $M_q \subset M$.

Write $w = (w', w_q) \in \mathbb{C}^q = \mathbb{C}^{q-1} \times \mathbb{C}$. Let

$$C := \{(z, w') \in (A' \cap \mathbb{P}_p(r)) \times \mathbb{P}_{q-1}(R') : (M_q)_{(z,w',\cdot)} \text{ is polar}\}.$$

In the case where all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin we put

$$C := \{(z, w') \in (A' \cap \mathbb{P}_p(r)) \times \mathbb{P}_{q-1}(R') : (M_q)_{(z,w',\cdot)} \text{ is discrete}\}.$$

By Proposition 3.4.11(h), C is locally pluriregular. Observe that for every $c \in C$ and for every $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$, the function $\widetilde{h}(c, \cdot)$ is holomorphic in $K(R_0) \setminus (M_q)_{(c,\cdot)}$. Consequently, applying Lemma 5.3.10 to the cross

$$\boldsymbol{Z}_q := \boldsymbol{K}(C, K(R'); \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R'), K(R_0)),$$

we conclude that there exists a relatively closed pluripolar set $S_q \subset \widehat{\boldsymbol{Z}}_q$ such that:

- $S_q \cap \boldsymbol{Z}'_q \subset M_q$,
- any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a $\widetilde{h}_q \in \mathcal{O}(\widehat{\boldsymbol{Z}}_q \setminus S_q)$ with $\widetilde{h}_q = h$ on $\boldsymbol{Z}'_q \setminus M_q$,
- S_q is singular with respect to the family $\{\widetilde{h}_q : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A'$, are thin, then S_q is analytic,
- in the analytic case we have $S_q \subset M$.

Using the product property of the relative extremal function (Theorem 3.4.10), we get

$$\begin{aligned}\widehat{\boldsymbol{Z}}_q &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times K(R_0) : \\ &\qquad h^*_{C,\mathbb{P}_p(r)\times\mathbb{P}_{q-1}(R')}(z, w') + h^*_{K(R'),K(R_0)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times K(R_0) : \\ &\qquad h^*_{A'\times\mathbb{P}_{q-1}(R'),\mathbb{P}_p(r)\times\mathbb{P}_{q-1}(R')}(z, w') + h^*_{K(R'),K(R_0)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times K(R_0) : \\ &\qquad h^*_{A,\mathbb{P}_p(r)}(z) + h^*_{K(R'),K(R_0)}(w_q) < 1\}.\end{aligned}$$

Consequently, since $R'' < R_0$, we find an $r_q \in (0, r)$ such that

$$\mathbb{P}_p(r_q) \times \mathbb{P}_{q-1}(R') \times K(R'') \subset \widehat{\boldsymbol{Z}}_q.$$

Thus any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a function $\widetilde{h}_q$ on $(\mathbb{P}_p(r_q) \times \mathbb{P}_{q-1}(R') \times K(R'')) \setminus S_q$ and S_q is singular with respect to the family $\{\widetilde{h}_q : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$.

Repeating the above argument for the coordinates w_ν, $\nu = 1, \dots, q-1$, and gluing the obtained sets, we find an $r_* \in (0, r)$ and a relatively closed pluripolar set $S_0 := \bigcup_{j=1}^q S_j$ such that any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a function $\widetilde{h}_0 := \bigcup_{j=1}^q \widetilde{h}_j$ holomorphic in $\mathbb{P}_p(r_*) \times W \setminus S_0$, where

$$W := \bigcup_{j=1}^q \mathbb{P}_{j-1}(R') \times \mathbb{P}(R'') \times \mathbb{P}_{q-j}(R').$$

Observe that W is a complete Reinhardt domain in $\mathbb{C}^q$. Let $\widehat{W}$ denote the envelope of holomorphy of W (it is known that $\widehat{W}$ is a complete logarithmically convex Reinhardt domain in $\mathbb{C}^q$ (cf. Remark refRemReinhardt). Applying Theorem 4.11.1, we find a relatively closed pluripolar subset $\widehat{S}_0$ of $\mathbb{P}_p(r_*) \times \widehat{W}$ such that

- $\widehat{S}_0 \cap (\mathbb{P}_p(r_*) \times \widehat{W}) \subset S_0$,
- any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends to an $\widetilde{h} \in \mathcal{O}(\mathbb{P}_p(r_*) \times \widehat{W} \setminus \widehat{S}_0)$,
- $\widehat{S}_0$ is singular with respect to the family $\{\widetilde{h} : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$,
- if all the fibers $M_{(a,\cdot)}$, $a \in A'$, are thin, then $\widehat{S}_0$ is analytic,
- in the analytic case we have $\widehat{S}_0 \subset M$.

Since $\widehat{W}$ is logarithmically convex, we must have $\mathbb{P}_q(\sqrt[q]{R'^{q-1}R'_0}) \subset \widehat{W}$. Consequently, $\mathbb{P}_q(R''') \subset \widehat{W}$. Recall that $R''' > R'_0$. Let $0 < \rho < r_*$ be such that $\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''') \subset \mathbb{P}_p(r_*) \times \widehat{W}$. Put $r_{R'''} := \rho$, $S' = S_{R'''} := \widehat{S}_0 \cap (\mathbb{P}_p(\rho) \times \mathbb{P}_q(R'''))$. Then any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to $(\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''')) \setminus S'$.

To get a contradiction it remains show that $S' \cap ((A' \cap \mathbb{P}_p(\rho)) \times \mathbb{P}_q(R''')) \subset M$ (7). Take $(z, w) \in ((A' \cap \mathbb{P}_p(\rho)) \times \mathbb{P}_q(R''')) \setminus M$. Since $M_{(z,\cdot)}$ is pluripolar, there exists a curve $\gamma : [0, 1] \longrightarrow \mathbb{P}_q(R''') \setminus M_{(z,\cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = w$. We may assume that for small $\varepsilon > 0$ we have

$$\mathbb{P}_p(z, \varepsilon) \times (\gamma([0, 1]) + \mathbb{P}_q(\varepsilon)) \Subset (\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''')) \setminus M.$$

Put $V_w := \gamma([0, 1]) + \mathbb{P}_q(\varepsilon)$. Consider the cross

$$\boldsymbol{W} := \boldsymbol{K}(A \cap \mathbb{P}_p(z, \varepsilon), \mathbb{P}_q(\varepsilon); \mathbb{P}_p(z, \varepsilon), V_w).$$

Then $h \in \mathcal{O}_s(\boldsymbol{W})$ for any $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$. Consequently, by Theorem 4.3.1, $(z, w) \in \widehat{\boldsymbol{W}} \subset \mathbb{P}_p(r) \times \mathbb{P}_q(R''') \setminus S'$. □

Thus, it remains to prove Lemma 5.3.10 in the pluripolar and analytic cases.

5.3.1 Proof of Lemma 5.3.10 in the pluripolar case

Theorem 5.3.11. *Assume that:*

- $\mathbb{P}(r_0) \subset \mathbb{C}^p$, $K(R_0) \subset \mathbb{C}$, $R_0 > r_0$, $A \subset \mathbb{P}(r_0)$ *is locally pluriregular,*
- $\boldsymbol{Z} = \boldsymbol{K}(A, K(r_0); \mathbb{P}(r_0), K(R_0))$,
- $M \subset \boldsymbol{Z}$ *is a relatively closed set such that* $M \cap (\mathbb{P}(r_0) \cap K(r_0))$ *is pluripolar and* $M_{(a,\cdot)} \in \mathcal{PLP}$ *for every* $a \in A'$, *where* $A' \subset A$ *is such that* $A \setminus A' \in \mathcal{PLP}$.

Put

$$\mathcal{F} := \mathcal{O}_s(\boldsymbol{Z} \setminus M) \cap \mathcal{O}(\mathbb{P}(r_0) \times K(r_0) \setminus M).$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{Z}}$ *such that:*

- $\widehat{M} \cap \boldsymbol{Z}' \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}} \setminus \widehat{M})$ *with* $\widehat{f} = f$ *on* $\boldsymbol{Z}' \setminus M$,
- *the set* $\widehat{M}$ *is singular with respect to the family* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if all the fibers* $M_{(a,\cdot)}$, $a \in A'$, *are discrete, then* $\widehat{M}$ *is analytic.*

Proof. We are going to apply Theorem 5.3.4 (with $D := \mathbb{P}(r_0)$, $G := K(R_0)$, $B := K(r_0)$, $B' := \{b \in B : M_{(\cdot,b)} \in \mathcal{PLP}\}$). Keep all the notation from Theorem 5.3.4. Assume additionally that $B = K(r_0) \Subset G_k$ for every k. Take $(a, b) \in \Xi_k = A'_k \times B'_k \setminus M$.

The "horizontal" direction is simple: we take $s = s_{k,b} > 0$ such that $K(b, s) \subset K(r_0)$ and let $\widetilde{S}^{k,b} := M \cap (D_k \times K(b, s)) =: V^{k,b}$; $\widetilde{S}^{k,b}$ is relatively closed

(7) Note that in the analytic case we have obviously $S' \subset M$.

pluripolar. Let $S^{k,b}$ be the singular part of $\widetilde{S}^{k,b}$ with respect to the family $\{f|_{V^{k,b}\setminus M} : f \in \mathcal{F}\}$ and let $\widetilde{f}^{k,b}$ denote the extension of $f|_{V^{k,b}\setminus M}$ to $V^{k,b} \setminus S^{k,b}$.

The "vertical" direction is more complicated: we have to show that there exist an $r = r_{k,a} > 0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}_D(a,r) \times G_k =: V_{k,a}$ such that:

- $\widehat{\mathbb{P}}(a,r) \subset D_k$,
- $S \cap ((A \cap \mathbb{P}(a,r)) \times G_k) \subset M$,
- any function from $f \in \mathcal{F}$ extends holomorphically to an $\widetilde{f} \in \mathcal{O}(V_{k,a} \setminus S)$ with $\widetilde{f} = f$ on $(A' \cap \widehat{\mathbb{P}}(a,r)) \times G_k \setminus M$ (8).

For $c \in K(R_0)$, let $\rho = \rho_c > 0$ be such that $K(c,\rho) \Subset K(R_0)$ and $M_{(a,\cdot)} \cap \partial K(c,\rho) = \varnothing$ (cf. [Arm-Gar 2001], Th. 7.3.9). Take $\rho^- = \rho_c^- > 0$, $\rho^+ = \rho_c^+ > 0$ such that $\rho^- < \rho < \rho^+$, $K(c,\rho^+) \Subset K(R_0)$, and $M_{(a,\cdot)} \cap \overline{P} = \varnothing$, where $P = P_c := \mathbb{A}(c,\rho^-,\rho^+)$. Let $\gamma : [0,1] \longrightarrow G \setminus M_{(a,\cdot)}$ be a curve such that $\gamma(0) = 0$ and $\gamma(1) \in \partial K(c,\rho)$. There exists an $\varepsilon = \varepsilon_c > 0$ such that

$$\Big(\mathbb{P}(a,\varepsilon) \times ((\gamma([0,1]) + K(\varepsilon)) \cup P)\Big) \cap M = \varnothing.$$

Put $V = V_c := K(r_0) \cup (\gamma([0,1]) + K(\varepsilon)) \cup P$ and consider the cross

$$\boldsymbol{Y} = \boldsymbol{Y}_c := \boldsymbol{K}(A \cap \mathbb{P}(a,\varepsilon), K(r_0); \mathbb{P}(a,\varepsilon), V).$$

Then $f \in \mathcal{O}_s(\boldsymbol{Y})$ for any $f \in \mathcal{F}$. Consequently, by Theorem 4.3.1, any function from $\mathcal{F}$ extends holomorphically to $\widehat{\boldsymbol{Y}} \supset \{a\} \times V$. Shrinking P, ε and V, we may assume that any function $f \in \mathcal{F}$ extends to a function $\widetilde{f} = \widetilde{f}_c \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times W)$, where

$$W = W_c := K(r_0 - \varepsilon) \cup (\gamma([0,1]) + K(\varepsilon)) \cup P.$$

In particular, $\widetilde{f}$ is holomorphic in $\mathbb{P}(a,\varepsilon) \times P$, and therefore may be represented by the Hartogs–Laurent series

$$\begin{aligned}\widetilde{f}(z,w) = \sum_{\nu=0}^{\infty} \widetilde{f}_\nu(z)(w-c)^\nu + \sum_{\nu=1}^{\infty} \widetilde{f}_{-\nu}(z)(w-c)^{-\nu}&\\ =: \widetilde{f}^+(z,w) + \widetilde{f}^-(z,w), \quad (z,w) \in \mathbb{P}(a,\varepsilon) \times P,&\end{aligned}$$

where $\widetilde{f}^+ \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times \mathbb{P}(c,\rho^+))$ and $\widetilde{f}^- \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times (\mathbb{C} \setminus \overline{\mathbb{P}}(c,\rho^-)))$. Recall that for any $z \in A' \cap \mathbb{P}(a,\varepsilon)$ the function $\widetilde{f}(z,\cdot)$ extends holomorphically to $K(R_0) \setminus M_{(z,\cdot)}$. Consequently, for any $z \in A' \cap \mathbb{P}(a,\varepsilon)$ the function $\widetilde{f}^-(z,\cdot)$ extends holomorphically to $\mathbb{C} \setminus (M_{(z,\cdot)} \cap \overline{K}(c,\rho^-))$. Now, by Theorem 4.10.1, there exists a relatively closed pluripolar set $S = S_c \subset \mathbb{P}(a,\varepsilon) \times \overline{K}(c,\rho^-)$ such that:

- $S \cap ((A' \cap \mathbb{P}(a,\varepsilon)) \times \overline{K}(c,\rho^-)) \subset M$,
- any function $\widetilde{f}^-$ extends holomorphically to an $\approx{f}^- \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times \mathbb{C} \setminus S)$.

(8) Then we can take as $S_{k,a}$ the singular part of S with respect to the family $\{\widetilde{f} : f \in \mathcal{F}\}$ and $\widetilde{f}_{k,a} :=$ the extension of $\widetilde{f}$ to $V_{k,a} \setminus S_{k,a}$.

Since $\widetilde{f} = \widetilde{f}^+ + \widetilde{f}^-$, the function $\widetilde{f}$ extends holomorphically to a function $\widehat{f} = \widehat{f}_c \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times K(c,\rho^+) \setminus S)$. We may assume that the set S is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$. In particular, if $c', c'' \in K(R_0)$ and $C := K(c', \rho^+_{c'}) \cap K(c'', \rho^+_{c''}) \neq \varnothing$, then

$$S_{c'} \cap (\mathbb{P}(a,\eta) \times C) = S_{c''} \cap (\mathbb{P}(a,\eta) \times C), \quad \widehat{f}_{c'} = \widehat{f}_{c''} \text{ on } \mathbb{P}(a,\eta) \times C,$$

where $\eta := \min\{\varepsilon_{c'}, \varepsilon_{c''}\}$. Thus the functions $\widehat{f}_{c'}$, $\widehat{f}_{c''}$ and sets $S_{c'}$, $S_{c''}$ may be glued together.

Now, select $c_1, \dots, c_s \in K(R_0)$ so that $\overline{G}_k \subset \bigcup_{j=1}^s K(c_j, \rho^+_{c_j})$. Put

$$r = r_{k,a} := \min\{\varepsilon_{c_j} : j = 1, \dots, s\}, \quad V_{k,a} := \mathbb{P}(a, r) \times G_k.$$

Then $S := V_{k,a} \cap \bigcup_{j=1}^s S_{c_j}$ gives the required relatively closed pluripolar subset of $V_{k,a}$ such that $S \cap \boldsymbol{X} \subset M$ and for any $f \in \mathcal{F}$, the function $\widehat{f} := \bigcup_{j=1}^s \widehat{f}_{c_j}$ extends holomorphically f to $V_{k,a} \setminus S$. $\square$

5.3.2 Proof of Lemma 5.3.10 in the analytic case

Theorem 5.3.12. *Assume that:*

- $\mathbb{P}(r_0) \subset \mathbb{C}^p$, $K(R_0) \subset \mathbb{C}$, $R_0 > r_0$, $A \subset \mathbb{P}(r_0)$ *is locally pluriregular,*
- $\boldsymbol{Z} = \boldsymbol{K}(A, K(r_0); \mathbb{P}(r_0), K(R_0))$,
- $M := g^{-1}(0)$ *with* $g \in \mathcal{O}(\widehat{\boldsymbol{Z}})$, $g \not\equiv 0$.

Then for any $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M)$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}} \setminus M)$ *with* $\widehat{f} = f$ *on* $\boldsymbol{Z} \setminus M$.

Proof. First observe that $\mathcal{O}_s(\boldsymbol{Z} \setminus M) \subset \mathcal{O}(\mathbb{P}(r_0) \times K(r_0) \setminus M)$. Indeed, take an $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M) \subset \mathcal{O}(\mathbb{P}(r_0) \times K(r_0) \setminus M)$. Using the Hukuhara theorem (Theorem 2.2.2), one can easily prove that for any $a \in A'$, $b \in K(r_0) \setminus M_{(a,\cdot)}$, and $\mathbb{P}((a,b), r) \subset \mathbb{P}(r_0) \times K(r_0) \setminus M$, we have $f \in \mathcal{O}(\mathbb{P}((a,b),r))$. Now, let $(z_0, w_0) \in \mathbb{P}(r_0) \times K(r_0) \setminus M$ be arbitrary. Since $\mathbb{P}(r_0) \setminus M_{(\cdot, w_0)}$ is domain, we may find a subdomain $U \Subset \mathbb{P}(r_0) \setminus M_{(\cdot, w_0)}$ such that $z_0 \in U$ and $A' \cap U \neq \varnothing$. Take an $a \in A' \cap U$ and let $\varepsilon > 0$ be so small that $\mathbb{P}(a, \varepsilon) \subset U$ and $U \times K(w_0, \varepsilon) \subset \mathbb{P}(r_0) \times K(r_0) \setminus M$. In particular, $f \in \mathcal{O}(\mathbb{P}(a,\varepsilon) \times K(w_0, \varepsilon))$. Finally, using the Hartogs lemma (Lemma 2.1.8), we conclude that $f \in \mathcal{O}(U \times K(w_0, \varepsilon))$.

The main proof will be based on Theorem 5.3.4 (similarly as the proof of Theorem 5.3.11). As before, the "horizontal" case is simple. To prove the "vertical" case define A' as the set of all $a \in A \cap \mathbb{P}(r_0)$ which satisfy the following condition:

(†) For every $R' \in (r_0, R_0)$ there exist $R' < R'' < R_0$, $\delta > 0$, $m \in \mathbb{N}$, $c_1, \dots, c_m \in K(R'')$, $\varepsilon > 0$, and holomorphic functions $\varphi_\mu : \mathbb{P}(a, \delta) \longrightarrow K(c_\mu, \varepsilon)$, $\mu = 1, \dots, m$, such that:

- $\mathbb{P}(a, \delta) \subset \mathbb{P}(r_0)$,
- $K(c_\mu, \varepsilon) \Subset K(R'')$, $\mu = 1, \dots, m$,
- $\overline{K}(c_\mu, \varepsilon) \cap \overline{K}(c_\nu, \varepsilon) = \varnothing$ for $\mu \neq \nu$, $\mu, \nu = 1, \dots, m$,
- $\widetilde{H} := K(r_0) \cap H \neq \varnothing$, where $H := K(R'') \setminus \bigcup_{\mu=1}^m \overline{K}(c_\mu, \varepsilon)$,

- $(\mathbb{P}(a,\delta) \times K(R'')) \cap M = \bigcup_{\mu=1}^{m}\{(z,\varphi_\mu(z)) : z \in \mathbb{P}(a,\delta)\}$.

Notice that for every $a \in A'$ the fiber $M_{(a,\cdot)}$ is discrete.

Now we prove that $A \setminus A'$ is pluripolar. Write

$$M = \bigcup_{\nu=1}^{\infty}\{\zeta \in P_\nu : g_\nu(\zeta) = 0\},$$

where $P_\nu \Subset \mathbb{P}(r_0) \times K(R_0)$ is a polydisc and $g_\nu \in \mathcal{O}(P_\nu)$ is a defining function for $M \cap P_j$ (cf. [Chi 1989], § 2.9). Define

$$S_\nu := \Big\{(z,w) \in P_\nu : g_\nu(z,w) = \frac{\partial g_\nu}{\partial w}(z,w) = 0\Big\}$$

and observe that, by the implicit function theorem,

$$A \setminus \bigcup_{\nu=1}^{\infty} \mathrm{pr}_{\mathbb{C}^p}(S_\nu) \subset A'.$$

It is enough to show that each set $\mathrm{pr}_{\mathbb{C}^p}(S_\nu)$ is pluripolar. Fix a ν. Let S be an irreducible component of S_ν. We have to show that $\mathrm{pr}_{\mathbb{C}^p}(S)$ is pluripolar. If S has codimension ≥ 2, then $\mathrm{pr}_{\mathbb{C}^p}(S)$ is contained in a countable union of proper analytic sets (cf. [Chi 1989], § 3.8). Consequently, $\mathrm{pr}_{\mathbb{C}^p}(S)$ is pluripolar. Thus we may assume that S is of pure codimension one. The same argument as above shows that $\mathrm{pr}_{\mathbb{C}^p}(\mathrm{Sing}(S))$ is pluripolar. It remains to prove that $\mathrm{pr}_{\mathbb{C}^p}(\mathrm{Reg}(S))$ is pluripolar. Since g_ν is a defining function, for any $(z,w) \in \mathrm{Reg}(S)$ there exists a $k \in \{1,\dots,p\}$ such that $\frac{\partial g_\nu}{\partial z_k}(z,w) \neq 0$. Thus

$$\mathrm{Reg}(S) = \bigcup_{k=1}^{p} T_k,$$

where $T_k := \{(z,w) \in \mathrm{Reg}(S) : \frac{\partial g_\nu}{\partial z_k}(z,w) \neq 0\}$. We only need to prove that each set $\mathrm{pr}_{\mathbb{C}^p}(T_k)$ is pluripolar, $k = 1,\dots,p$. Fix a k. To simplify notation, assume that $k = 1$. Observe that, by the implicit function theorem, we can write

$$T_1 = \bigcup_{\ell=1}^{\infty}\{(z,w) \in Q_\ell : z_1 = \psi_\ell(z_2,\dots,z_p,w)\},$$

where $Q_\ell \subset P_\nu$ is a polydisc, $Q_\ell = Q'_\ell \times Q''_\ell \subset \mathbb{C} \times \mathbb{C}^p$, and $\psi_\ell : Q''_\ell \longrightarrow Q'_\ell$ is holomorphic, $\ell \in \mathbb{N}$. It suffices to prove that the projection of each set $T_{1,\ell} := \{(z,w) \in Q_\ell : z_1 = \psi_\ell(z_2,\dots,z_p,w)\}$ is pluripolar. Fix an ℓ. Since

$$g_\nu(\psi_\ell(z_2,\dots,z_p,w),z_2,\dots,z_p,w) = 0, \quad (z_2,\dots,z_p,w) \in Q''_\ell,$$

we conclude that $\frac{\partial \psi_\ell}{\partial w} \equiv 0$ and, consequently, ψ_ℓ is independent of w. Thus $\mathrm{pr}_{\mathbb{C}^p}(T_{1,\ell}) = \{z_1 = \psi_\ell(z_2,\dots,z_p)\}$ and therefore the projection is pluripolar. The proof that $A \setminus A'$ is pluripolar is completed.

Similarly as in the proof of Theorem 5.3.11, we only need to check that for any $a \in A'$ and $0 < R' < R_0$ there exists an $r > 0$ such that $\mathbb{P}(a,r) \subset \mathbb{P}(r_0)$ and for every $f \in \mathcal{F}$ there exists an $\widetilde{f} \in \mathcal{O}(\mathbb{P}(a,r) \times K(R') \setminus M)$ such that $\widetilde{f} = f$ on $(A' \cap \mathbb{P}(a,r)) \times K(R') \setminus M$.

Indeed, take an $a \in A'$ and apply (†). Put

$$\boldsymbol{Y} := \boldsymbol{K}(A \cap \mathbb{P}(a,\delta), \widetilde{H}; \mathbb{P}(a,\delta), H).$$

Notice that $\boldsymbol{Y}$ does not intersect M. In particular, $\widehat{f}|_{\boldsymbol{Y}} \in \mathcal{O}_s(\boldsymbol{Y})$. Hence, by Theorem 4.3.1, there exists an $\widehat{f}_1 \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ with $\widehat{f}_1 = \widehat{f}$ on $\boldsymbol{Y}$. Take $R''' \in (R', R'')$, and $\varepsilon'' > \varepsilon' > \varepsilon$, such that

- $K(c_\mu, \varepsilon'') \Subset K(R''')$, $\mu = 1, \dots, m$,
- $\overline{K}(c_\mu, \varepsilon'') \cap \overline{K}(c_\nu, \varepsilon'') = \varnothing$ for $\mu \neq \nu$, $\mu, \nu = 1, \dots, m$.

Then there exists $\delta' \in (0, \delta]$ such that

- $\mathbb{P}(a,\delta') \times H' \subset \widehat{\boldsymbol{Y}}$, where $H' := K(R''') \setminus \bigcup_{\mu=1}^{m} \overline{K}(c_\mu, \varepsilon')$.

In particular, $\widehat{f}_1 \in \mathcal{O}(\mathbb{P}(a,\delta') \times H')$.

Fix a $\mu \in \{1, \dots, m\}$. Then $\widehat{f}_1 \in \mathcal{O}(\mathbb{P}(a,\delta') \times \mathbb{A}(c_\mu, \varepsilon', \varepsilon''))$ (9) and

$$\widehat{f}_1(z, \cdot) \in \mathcal{O}(K(c_\mu, \varepsilon'') \setminus \{\varphi_\mu(z)\}), \quad z \in A \cap \mathbb{P}(a, \delta').$$

Using the biholomorphic mapping

$$\Phi_\mu : \mathbb{P}(a,\delta') \times \mathbb{C} \longrightarrow \mathbb{P}(a,\delta') \times \mathbb{C}, \quad \Phi_\mu(z,w) := (z, w - \varphi_\mu(z)),$$

we see that the function $g := \widehat{f}_1 \circ \Phi_\mu^{-1}$ is holomorphic in $\mathbb{P}(a,\delta'') \times \mathbb{A}(\eta', \eta''))$ for some $\delta'' \in (0, \delta']$ and $\varepsilon' < \eta' < \eta'' < \varepsilon''$. Moreover, $g(z, \cdot) \in \mathcal{O}(K_*(\eta''))$ (10) for any $z \in A \cap \mathbb{P}(a, \delta'')$. Using Theorem 4.3.1 for the cross

$$\boldsymbol{K}(A \cap \mathbb{P}(a,\delta''), \mathbb{A}(\eta', \eta''); \mathbb{P}(a,\delta''), K_*(\eta'')),$$

immediately shows that g extends holomorphically to $\mathbb{P}(a,\delta'') \times K_*(\eta'')$. Transforming the above information back via Φ_μ for all μ, we conclude that the function $\widehat{f}_1$ extends holomorphically to $\mathbb{P}(a,\delta''') \times K(R''') \setminus M$ for some $\delta''' \in (0, \delta'']$. Thus, $\widehat{f}_1$ extends holomorphically to $\mathbb{P}(a,\delta''') \times K(R') \setminus M$. □

5.4 Extension theorems with singularities in the case where $N \geq 3$

Throughout this section D_j denotes a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A'_j \times A''_j$ is pluripolar, $j = 1, \dots, N$, $N \geq 3$, $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$, $\boldsymbol{T} := \boldsymbol{GK}((A_j, D_j, \Sigma_j)_{j=1}^N)$.

(9) Recall that $\mathbb{A}(a, r^-, r^+) := \{z \in \mathbb{C} : r^- < |z - a| < r^+\}$, $\mathbb{A}(r^-, r^+) := \mathbb{A}(0, r^-, r^+)$.
(10) Recall that $K_*(r) := K(r) \setminus \{0\}$.

Definition 5.4.1. Let $M \subset \boldsymbol{T}$ be relatively closed. We say that a function $f : \boldsymbol{T} \setminus M \longrightarrow \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(\boldsymbol{T} \setminus M)$) if for any $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ the function $f(a'_j, \cdot, a''_j)$ is holomorphic in $D_j \setminus M_{(a'_j, \cdot, a''_j)}$.

To shorten the presentation, we concentrate our results in the form of the following general extension theorem with singularities, which will be later developed into four independent theorems.

Theorem 5.4.2 (Main extension theorem with singularities). *Let $\boldsymbol{W} \in \{\boldsymbol{X}, \boldsymbol{T}\}$ and let $M \subset \boldsymbol{W}$ be relatively closed and such that for every $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the fiber $M_{(a'_j, \cdot, a''_j)}$ is pluripolar.*

In the "analytic case" we additionally assume that:

- *M is a proper analytic set in an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{W}$,*
- *if $\boldsymbol{W} = \boldsymbol{X}$, then $\Sigma_j := \{(a'_j, a''_j) \in A'_j \times A''_j : M_{(a'_j, \cdot, a''_j)} \notin \mathcal{PLP}\}$, $j = 1, \ldots, N$.*

Let

$$\mathcal{F} \subset \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus M), & \text{if } \boldsymbol{W} = \boldsymbol{X} \\ \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M), & \text{if } \boldsymbol{W} = \boldsymbol{T} \end{cases},$$

where $\mathcal{C}^*(\boldsymbol{T} \setminus M)$ denotes the set of all functions $f : \boldsymbol{T} \setminus M \longrightarrow \mathbb{C}$ such that

(*) *for any $j \in \{1, \ldots, N\}$ and $b_j \in D_j$, the function*

$$A'_j \times A''_j \setminus (\Sigma_j \cup M_{(\cdot, b_j, \cdot)}) \ni (z'_j, z''_j) \longmapsto f(z'_j, b_j, z''_j)$$

is continuous (cf. condition (*) in Theorem 4.9.2). Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:

- *$\widehat{M} \cap \boldsymbol{T} \subset M$,*
- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *if for all $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the fiber $M_{(a'_j, \cdot, a''_j)}$ is thin in D_j, then $\widehat{M}$ is analytic,*
- *in the analytic case we additionally have $\widehat{M} \cap U_0 \subset U$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{W}$,*
- *in the analytic case with $U = \widehat{\boldsymbol{X}}$, the set $\widehat{M}$ coincides with the union of all irreducible components of M of codimension one.*

Remark 5.4.3. Notice that in the analytic case we have

$$\Sigma_j = \{(a'_j, a''_j) \in A'_j \times A''_j : M_{(a'_j, \cdot, a''_j)} \subsetneq D_j\}, \quad j = 1, \ldots, N.$$

In particular, $\boldsymbol{X} \setminus M = \boldsymbol{T} \setminus M$.

Notice that Theorem 5.4.2 contains in fact the following four result.

Theorem 5.4.4 (Extension theorem with analytic singularities). *Let M be a proper analytic set in an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$. Then there exists an analytic set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$,*
- *for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,*
- *if $U = \widehat{\boldsymbol{X}}$, then the set $\widehat{M}$ coincides with the union of all irreducible components of M of codimension one.*

Theorem 5.4.5 (Extension theorem for generalized crosses with analytic singularities). *Let M be a proper analytic set in an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{T}$. Assume that for every $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in A_j' \times A_j'' \setminus \Sigma_j$, the fiber $M_{(a_j', \cdot, a_j'')}$ is pluripolar. Then there exists an analytic set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{M} \cap U_0 \subset M$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{T}$,*
- *for any $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)\}$,*
- *if $U = \widehat{\boldsymbol{X}}$, then the set $\widehat{M}$ coincides with the union of all irreducible components of M of codimension one.*

Notice that Theorem 5.4.5 with $M = \varnothing$ reduces to Theorem 4.9.2.

Theorem 5.4.6 (Extension theorem with pluripolar singularities). *Let $M \subset \boldsymbol{X}$ be a relatively closed set such that for every $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in A_j' \times A_j'' \setminus \Sigma_j$, the fiber $M_{(a_j', \cdot, a_j'')}$ is pluripolar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{M} \cap \boldsymbol{T} \subset M$,*
- *for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$.*

Theorem 5.4.7 (Extension theorem for generalized crosses with pluripolar singularities). *Let $M \subset \boldsymbol{T}$ be a relatively closed set such that for every $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in A_j' \times A_j'' \setminus \Sigma_j$, the fiber $M_{(a_j', \cdot, a_j'')}$ is pluripolar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{M} \cap \boldsymbol{T} \subset M$,*
- *for any $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)\}$.*

Observe that Theorems 5.4.5, 5.4.7 are interesting also in the case where $M = \varnothing$.

Theorems 5.4.4, 5.4.6 say that the extension problem in Definition 5.1.2 has a solution with $T = \boldsymbol{X}$ and $T = \boldsymbol{T}$, respectively.

It is natural to ask how big is the class of all relatively closed sets $M \subset \boldsymbol{W}$ with pluripolar fibers, that are not pluripolar.

Proposition 5.4.8. *Let S be a d–dimensional $\mathcal{C}^1$–submanifold of an open set $\Omega \subset \mathbb{C}^n$ with $1 \leq d \leq 2n-2$. Then for every point $z_0 \in M$ there exist an open neighborhood U and a $\mathbb{C}$–linear isomorphism $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $L(U) = \mathbb{D}^n$ and if $M := L(S \cap U)$, then for any $a = (a_1, \dots, a_n) \in \mathbb{D}^n$ and $j \in \{1, \dots, n\}$, the fiber*

$$M_{(a'_j, \cdot, a''_j)} := \{\lambda \in \mathbb{C} : (a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) \in M\}$$

is finite.

Remark 5.4.9. (a) M satisfies the assumptions of Theorems 5.4.6, 5.4.7 for arbitrary cross with $N = n$.

(b) Notice that there are real analytic 2–dimensional submanifolds of $\mathbb{C}^2$ that are not locally pluripolar. For example (cf. [Sad 2005]):

$$S : \quad \begin{cases} y_1 = x_1 + x_2^2 \\ y_2 = x_1^2 + x_2 \end{cases} , \quad (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2.$$

Consequently, one may easily produce examples of sets M satisfying the assumptions of Theorems 5.4.6, 5.4.7 that are not pluripolar.

Proof of Proposition 5.4.8. [Will be completed. . . . . .] $\square$

As an elementary application of Theorem 5.4.5 and Proposition 5.4.8 we get the following extension theorem (cf. [Kar 1998]).

Theorem 5.4.10. *Let S be a connected d–dimensional $\mathcal{C}^1$–submanifold of a domain $D \subset \mathbb{C}^n$ with $1 \leq d \leq 2n-2$. Then every function $f \in \mathcal{O}(D \setminus S)$ extends holomorphically to D unless M is a complex submanifold of codimension one.*

Proof. Take a point $a \in S$. Using Proposition 5.4.8 we find a neighborhood $U_a \subset D$ of a and a $\mathbb{C}$–linear isomorphism L_a such that $L_a(U_a) = \mathbb{D}^n$ and all the one dimensional fibers of the manifold $M_a := L_a(S \cap U_a)$ are finite. Now we apply Theorem 5.4.6 with $N = n$, $D_j = A_j = \mathbb{D}$, $j = 1, \dots, n$, $M = M_a$. We get an analytic set $\widehat{S}_a \subset U_a$ with the following properties:

- $\widehat{S}_a \subset S \cap U_a$,
- every function $f \in \mathcal{O}(U_a \setminus S)$ extends to an $\widehat{f}_a \in \mathcal{O}(U_a \setminus \widehat{S}_a)$ with $\widehat{f}_a = f$ on $U_a \setminus S$,
- $\widehat{S}_a$ is either empty or of codimension one.

It is clear that if $d \leq 2n-3$, then each $\widehat{S}_a$ must be empty and consequently, S is removable. Thus assume that $d = 2n-2$.
[Will be completed. . . . .] $\square$

Observe that Remark 5.3.2(c, d) extends in a natural way to the case $N \geq 3$. The following proposition generalizes Remark 5.3.2(e, f).

Proposition 5.4.11. (a) *Let $S \subset \widehat{\boldsymbol{X}}$ be an analytic set of pure codimension one. Then $S \cap \boldsymbol{T} \neq \varnothing$.*

(b) *Let* $M \subsetneq \widehat{\boldsymbol{X}}$ *be an analytic set. Suppose that* $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *is an analytic set such that:*

- $\widehat{M} \cap U_0 \subset M$ *for an open neighborhood* $U_0 \subset \widehat{\boldsymbol{X}}$ *of* $\boldsymbol{T}$,
- *every function* $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)$ *extends to an* $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *with* $\widehat{f} = f$ *on* $\boldsymbol{T} \setminus M$,
- *the set* $\widehat{M}$ *is singular with respect to the family* $\{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)\}$.

Then $\widehat{M}$ *is the union of all irreducible components of* M *of codimension one.*

Proof. (a) Suppose that $S \cap \boldsymbol{T} = \varnothing$. Since S is of pure codimension one, the domain $\widehat{\boldsymbol{X}} \setminus S$ is a domain of holomorphy, and therefore, there exists a $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus S)$ such that $\widehat{\boldsymbol{X}} \setminus S$ is the domain of existence of g (cf. Proposition 3.1.20). Since $\boldsymbol{T} \subset \widehat{\boldsymbol{X}} \setminus S$, we conclude that $g|_{\boldsymbol{X}} \in \mathcal{O}_s(\boldsymbol{T}) \cap \mathcal{C}(\boldsymbol{T})$. By Theorem 5.4.5 with $M = \varnothing$, there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{g} = g$ on $\boldsymbol{T}$, and consequently, on $\widehat{\boldsymbol{X}} \setminus S$. Thus g extends holomorphically to $\widehat{\boldsymbol{X}}$; a contradiction.

(b) Let $\widetilde{M}$ be the union of all irreducible components of M of codimension one.

In the case where $\widetilde{M} \neq \varnothing$, similarly as in (a), there exists a non-continuable function $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widetilde{M})$. Then $g|_{\boldsymbol{T} \setminus M} \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}(\boldsymbol{T} \setminus M)$ and, therefore, there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{g} = g$ on $\boldsymbol{T} \setminus M$. Hence, $\widehat{g} = g$ on $\widehat{\boldsymbol{X}} \setminus (\widehat{M} \cup \widetilde{M})$. Since g is non-continuable, we conclude that $\widetilde{M} \subset \widehat{M}$. The set $\widehat{M}$, as a non-empty singular set, is also of pure codimension one. Since $\widehat{M} \cap U_0 \subset M$ and $S \cap U_0 \neq \varnothing$ for every irreducible component of $\widehat{M}$ (by (a)), we conclude (using the identity principle for analytic sets) that $\widehat{M} \subset M$ (cf. [Chi 1993], § 5.3). Consequently, $\widehat{M} \subset \widetilde{M}$

It remains to exclude the situation where $\widetilde{M} = \varnothing$ (i.e. the codimension of M is ≥ 2), but $\widehat{M} \neq \varnothing$ (i.e. the codimension of $\widehat{M}$ is 1). Then $\widehat{M} \subset M$ (as above), which obviously gives a contradiction. $\square$

The main "technical tool" in the proof of Theorem 5.4.2 is the following theorem.

Theorem 5.4.12 (Glueing theorem). *Let* $\boldsymbol{W} \in \{\boldsymbol{X}, \boldsymbol{T}\}$, $M \subset \boldsymbol{W}$, *and* $\mathcal{F}$ *be as in Theorem 5.4.2. If* $N \geq 4$, *then we additionally assume that Theorem 5.4.2 was already proved for all* $(N-2)$*–fold crosses.*

Let $(D_{j,k})_{k=1}^{\infty}$ *be an exhaustion sequence for* D_j *(in sense of Definition 2.2.5) such that each* $D_{j,k}$ *is a domain of holomorphy and* $A_{j,k} := A_j \cap D_{j,k} \neq \varnothing$, $k \in \mathbb{N}$, $j = 1, \dots, N$. *Put*

$$A'_{j,k} := A_{1,k} \times \cdots \times A_{j-1,k}, \quad A''_{j,k} := A_{j+1,k} \times \cdots \times A_{N,k},$$
$$\Sigma_{j,k} := \Sigma_j \cap (A'_{j,k} \times A''_{j,k}),$$
$$\boldsymbol{X}_k := \boldsymbol{K}((A_{j,k}, D_{j,k})_{j=1}^N) = \boldsymbol{X} \cap (D_{1,k} \times \cdots \times D_{N,k}),$$
$$\boldsymbol{T}_k := \boldsymbol{GK}((A_{j,k}, D_{j,k}, \Sigma_{j,k})_{j=1}^N) = \boldsymbol{T} \cap (D_{1,k} \times \cdots \times D_{N,k}),$$
$$\boldsymbol{W}_k := \boldsymbol{W} \cap (D_{1,k} \times \cdots \times D_{N,k}) \in \{\boldsymbol{X}_k, \boldsymbol{T}_k\}.$$

Let $\varXi_k := \boldsymbol{c}(\boldsymbol{T}_k) \setminus M$. *In the analytic case with* $\boldsymbol{W} = \boldsymbol{X}$ *we take* $\varXi_k := \boldsymbol{c}(\boldsymbol{X}_k) = A_{1,k} \times \cdots \times A_{N,k}$.

Assume that for any $k \in \mathbb{N}$, $j \in \{1, \dots, N\}$, *and* $a \in \varXi_k$, *there exist:*

- $r = r_{k,a} > 0$,
- *relatively closed pluripolar set* $S_{j,k,a} \subset \widehat{\mathbb{P}}(a'_j, r) \times D_{j,k} \times \widehat{\mathbb{P}}(a''_j, r) =: V_{j,k,a}$,

such that:

- $\widehat{\mathbb{P}}(a, r) \subset D_{1,k} \times \cdots \times D_{N,k}$ *and* $\widehat{\mathbb{P}}(a, r) \cap M = \varnothing$ *if* $a \notin M$,
- $S_{j,k,a} \cap \boldsymbol{T}_{j,k,a} \subset M$, *where*

$$\boldsymbol{T}_{j,k,a} := \{(z'_j, z_j, z''_j) \in (A'_{j,k} \cap \widehat{\mathbb{P}}(a'_j, r)) \times D_{j,k} \times (A''_{j,k} \cap \widehat{\mathbb{P}}(a''_j, r)) : \\ (z'_j, z''_j) \notin \Sigma_{j,k}\} \subset V_{j,k,a} \cap \boldsymbol{T}_k,$$

- *for any* $f \in \mathcal{F}$ *there exists an* $\widetilde{f}_{j,k,a} \in \mathcal{O}(V_{j,k,a} \setminus S_{j,k,a})$ *with* $\widetilde{f}_{j,k,a} = f$ *on* $\boldsymbol{T}_{j,k,a} \setminus M$,
- *in the analytic case we additionally assume that* $V_{j,k,a} \subset U$ *and* $S_{j,k,a} \subset M$.

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *such that:*

- $\widehat{M} \cap \boldsymbol{T} \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *with* $\widehat{f} = f$ *on* $\boldsymbol{T} \setminus M$,
- $\widehat{M}$ *is singular with respect to the family* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if each set* $S_{j,k,a}$ *is thin in* $V_{j,k,a}$, *then* $\widehat{M}$ *is analytic,*
- *in the analytic case we additionally have* $\widehat{M} \cap U_0 \subset U$ *for an open neighborhood* $U_0 \subset U$ *of* $\boldsymbol{W}$.

Thus, in order to prove Theorem 5.4.2, we only need to use induction on N and verify all the assumptions of Theorem 5.4.12.

Proof. Step 1: We may assume that each set $S_{j,k,a}$ is singular with respect to the family $\{\widetilde{f}_{j,k,a} : f \in \mathcal{F}\}$. In particular, $S_{j,k,a}$ is thin in $V_{j,ka}$ iff $S_{j,k,a}$ is analytic in $V_{j,k,a}$.

Step 2: If $N \geq 4$, then for any $1 \leq \mu < \nu \leq N$, define an auxiliary $(N-2)$–fold cross

$$\boldsymbol{Y}_{\mu,\nu} := \boldsymbol{K}((A_j, D_j)_{j \in \{1,\dots,\mu-1,\mu+1,\dots,\nu-1,\nu+1,\dots,N\}}).$$

We may assume that the number $r = r_{k,a}$ is so small that

$$\widehat{\mathbb{P}}((a_1, \dots, a_{\mu-1}, a_{\mu+1}, \dots, a_{\nu-1}, a_{\nu+1}, \dots, a_N), r) \subset \widehat{\boldsymbol{Y}}_{\mu,\nu}, \quad 1 \leq \mu < \nu \leq N.$$

Step 3: Fix a $k \in \mathbb{N}$. Put

$$V_k := \bigcup_{a \in \varXi_k} V_{j,k,a}, \quad S_k := \bigcup_{a \in \varXi_k} S_{j,k,a}, \quad \widetilde{f}_k := \bigcup_{a \in \varXi_k} \widetilde{f}_{j,k,a}.$$

In the case where $\varXi_k = \boldsymbol{c}(\boldsymbol{X}_k)$ we obviously have $\boldsymbol{X}_k \subset V_k$. Observe that in general we have $\boldsymbol{T}_k \subset V_k$.

Indeed, let $c \in \boldsymbol{T}_k$, e.g. $c = (a', c_N) \in (A'_{N-1,k} \setminus \Sigma_{N,k}) \times D_{N,k}$. Since $M_{(a',\cdot)}$ is pluripolar, there exists an $a_N \in A_{N,k} \setminus M_{(a',\cdot)}$. Then $a := (a', a_N) \in \boldsymbol{c}(\boldsymbol{T}_k) \setminus M = \Xi_k$ and $c \in \widehat{\mathbb{P}}(a', r_{k,a}) \times D_{N,k} = V_{N,k,a}$.

Note that in the analytic case we additionally have $V_k \subset U$ and $S_k \subset M$.

The main problem is to show that

(*) for arbitrary $a, b \in \Xi_k$, $i, j \in \{1, \dots, N\}$ with

$$W_{i,j,k,a,b} := V_{i,k,a} \cap V_{j,k,b} \neq \varnothing$$

we have $\widetilde{f}_{i,k,a} = \widetilde{f}_{j,k,b}$ on $W_{i,j,k,a,b} \setminus (S_{i,k,a} \cup S_{j,k,b})$ for all $f \in \mathcal{F}$.

Suppose for a moment that (*) is proved (the proof will be given in Step 5) and we finish the main proof.

Step 4: Since the sets $S_{i,k,a}$ and $S_{j,k,b}$ are singular, we conclude that

$$S_{i,k,a} \cap W_{i,j,k,a,b} = S_{j,k,b} \cap W_{i,j,k,a,b},$$

which implies that the function $\widetilde{f}_k$ is well defined on $V_k \setminus S_k$. Observe that:

- $S_{j,k,a} \cap \boldsymbol{T}_k \subset M$. Indeed, take a $c \in S_{j,k,a} \cap \boldsymbol{T}_k$. If $c \in \boldsymbol{T}_{j,k,a}$, then obviously $c \in M$. Suppose that $c \notin \boldsymbol{T}_{j,k,a}$. Then $c \in \boldsymbol{T}_{i,k,b}$ for some $i \in \{1, \dots, N\}$ and $b \in \boldsymbol{c}(\boldsymbol{T}_k) \setminus M$. In particular, $c \in W_{j,i,k,a,b}$. Thus $c \in S_{j,k,a} \cap W_{j,i,k,a,b} = S_{i,k,b} \cap W_{j,i,k,a,b}$. This means that $c \in S_{i,k,b} \cap \boldsymbol{T}_{i,k,b} \subset M$.
- $\widetilde{f}_{j,k,a} = f$ on $\boldsymbol{T}_k \cap V_{j,k,a} \setminus M$. Indeed, take a $c \in \boldsymbol{T}_k \cap V_{j,k,a} \setminus M$. If $c \in \boldsymbol{T}_{j,k,a}$, then obviously $\widetilde{f}_{j,k,a}(c) = f(c)$. Suppose $c \in \boldsymbol{T}_{i,k,b}$ for some $i \in \{1, \dots, N\}$ and $b \in \boldsymbol{c}(\boldsymbol{T}_k) \setminus M$. Then $c \in W_{j,i,k,a,b} \setminus (S_{i,k,a} \cup S_{j,k,b})$. Thus $f_{j,k,a}(c) = f_{i,k,b}(c) = f(c)$.

Moreover,

- S_k is a relatively closed pluripolar subset of V_k,
- $S_k \cap \boldsymbol{T}_k \subset M$,
- $\widetilde{f}_k \in \mathcal{O}(V_k \setminus S_k)$,
- $\widetilde{f}_k = f$ on $\boldsymbol{T}_k \setminus M$,
- S_k is singular with respect to the family $\{\widetilde{f}_k : f \in \mathcal{F}\}$,
- S_k is analytic provided that each set $S_{j,k,a}$ is analytic.

Let U_k denote the union of all connected component of $V_k \cap \widehat{\boldsymbol{X}}_k$ that intersect $\boldsymbol{T}_k$. Then $\widehat{\boldsymbol{X}}_k$ is the envelope of holomorphy of U_k.

Indeed, since $\widehat{\boldsymbol{X}}_k$ is a domain of holomorphy (Exercise 4.2.3(d)), we only need to show that any function from $\mathcal{O}(U_k)$ extends holomorphically to $\widehat{\boldsymbol{X}}_k$. Take a $g \in \mathcal{O}(U_k)$. Then $g|_{\boldsymbol{T}_k} \in \mathcal{O}_s(\boldsymbol{T}_k) \cap \mathcal{C}(\boldsymbol{T}_k)$. By Theorem 4.9.2, g extends to a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\widehat{g} = g$ on $\boldsymbol{T}_k$. Observe that $\boldsymbol{T}_k$ is locally pluriregular. In particular, $U \cap \boldsymbol{T}_k$ is not pluripolar for any connected component U of U_k. Hence, by the identity principle, $\widehat{g} = g$ on U_k [11].

By virtue of Theorem 4.11.1(c), there exists a relatively closed pluripolar set $\widehat{M}_k$ of $\widehat{\boldsymbol{X}}_k$, $\widehat{M}_k \cap U_k \subset S_k$, such that $\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k$ is the envelope of holomorphy

[11] Since $\widehat{\boldsymbol{X}}_k$ is a domain (Exercise 4.2.3(g)), U_k in fact must be connected.

of $U_k \setminus S_k$. Moreover, if S_k is analytic, then so is $\widehat{M}_k$. In particular, for each $f \in \mathcal{F}$ there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k)$ with $\widehat{f}_k = \widetilde{f}_k$ on $U_k \setminus S_k$. We may assume that $\widehat{M}_k$ is singular with respect to the family $\{\widehat{f}_k : f \in \mathcal{F}\}$. In particular, $\widehat{M}_{k+1} \cap \widehat{\boldsymbol{X}}_k = \widehat{M}_k$. Recall that $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$. Consequently:

- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{\boldsymbol{X}}$ with $\widehat{M} \cap \boldsymbol{T} \subset M$,
- for each $f \in \mathcal{F}$, the function $\widehat{f} := \bigcup_{k=1}^{\infty} \widehat{f}_k$ is holomorphic on $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,
- $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,
- if each set $S_{j,k,a}$ is analytic in $V_{j,k,a}$, then $\widehat{M}$ is analytic,
- in the analytic case, if $U_0 := \bigcup_{k=1}^{\infty} U_k$, then

$$\widehat{M} \cap U_0 = \bigcup_{k=1}^{\infty} \widehat{M}_k \cap U_k \subset \bigcup_{k=1}^{\infty} S_k \subset M.$$

This completes the proof of Theorem 5.4.12 modulo (*).

Step 5: We move to the proof of (*). Fix $a, b \in \Xi_k$ and $i, j \in \{1, \dots, N\}$ such that $W_{i,j,k,a,b} := V_{i,k,a} \cap V_{j,k,b} \neq \varnothing$, and $f \in \mathcal{F}$. We have the following two cases:
(a) $i \neq j$: We may assume that $i = N-1$, $j = N$. Write

$$w = (w', w'') \in (D_1 \times \dots \times D_{N-2}) \times (D_{N-1} \times D_N).$$

Observe that

$$W_{N-1,N,k,a,b} = \Big(\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})\Big) \times \widehat{\mathbb{P}}(b_{N-1}, r_{k,b}) \times \widehat{\mathbb{P}}(a_N, r_{k,a}).$$

Consider the following two subcases:

- $N = 3$: Then $W_{2,3,k,a,b} = (\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times \widehat{\mathbb{P}}(b_2, r_{k,b}) \times \widehat{\mathbb{P}}(a_3, r_{k,a})$. We are going to show that

$$\widetilde{f}_{N-1,k,a} = \widetilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times C) \setminus (S_{2,k,a} \cup S_{3,k,b}),$$

where $C \subset \widehat{\mathbb{P}}(b_2, r_{k,b}) \times \widehat{\mathbb{P}}(a_3, r_{k,a})$ is a non-pluripolar set; then, by the identity principle, we obtain $\widetilde{f}_{2,k,a} = \widetilde{f}_{3,k,b}$ on $W_{2,3,k,a,b} \setminus (S_{2,k,a} \cup S_{3,k,b})$.

Let

$$C := \{c \in ((A_{2,k} \cap \widehat{\mathbb{P}}(b_2, r_{k,b})) \times (A_{3,k} \cap \widehat{\mathbb{P}}(a_3, r_{k,a}))) \setminus \Sigma_{1,k} : \\ (S_{2,k,a})_{(\cdot,c)} \in \mathcal{PLP},\ (S_{3,k,b})_{(\cdot,c)} \in \mathcal{PLP}\}.$$

The set C is locally pluriregular (Exercise 5.3.3(c)). Fix a $c = (c_2, c_3) \in C$. Recall that $\widehat{\mathbb{P}}(a_1, r_{k,a}) \cup \widehat{\mathbb{P}}(b_1, r_{k,b}) \subset D_{1,k}$. Thus, the functions $\widetilde{f}_{3,k,b}(\cdot, c)$ and $f(\cdot, c)$ are holomorphic on

$$\widehat{\mathbb{P}}(b_1, r_{k,b}) \setminus ((S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

Moreover, they are equal on the non-pluripolar set $(A_{1,k} \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \setminus M_{(\cdot,c)}$. Hence, since the set $(S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}$ is polar, we get

$$\widetilde{f}_{3,k,b} = f(\cdot, c) \text{ on } \widehat{\mathbb{P}}(b_1, r_{k,b}) \setminus ((S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

An analogous argument shows that

$$\widetilde{f}_{2,k,a} = f(\cdot, c) \text{ on } \widehat{\mathbb{P}}(a_1, r_{k,a}) \setminus ((S_{2,k,a})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

Hence,

$$\widetilde{f}_{2,k,a}(\cdot, c) = \widetilde{f}_{3,k,b}(\cdot, c) \text{ on } (\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \setminus ((S_{2,k,a})_{(\cdot,c)} \cup (S_{3,k,b})_{(\cdot,c)}).$$

Consequently,

$$\widetilde{f}_{2,k,a} = \widetilde{f}_{3,k,b} \text{ on } ((\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times C) \setminus (S_{2,k,a} \cup S_{3,k,b}).$$

- $N \geq 4$: We are going to show that

$$\widetilde{f}_{N-1,k,a} = \widetilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times C) \setminus (S_{N-1,k,a} \cup S_{N,k,b}),$$

where $C \subset \widehat{\mathbb{P}}(b_{N-1}, r_{k,b}) \times \widehat{\mathbb{P}}(a_N, r_{k,a})$ is a non-pluripolar set; then, by the identity principle, we obtain

$$\widetilde{f}_{N-1,k,a} = \widetilde{f}_{N,k,b} \text{ on } W_{N-1,N,k,a,b} \setminus (S_{N-1,k,a} \cup S_{N,k,b}).$$

Let

$$B_{N-1} := \{c_{N-1} \in A_{N-1,k} \cap \widehat{\mathbb{P}}(b_{N-1}, r_{k,b}) : (\Sigma_N)_{(\cdot,c_{N-1})} \in \mathcal{PLP}\}.$$

By Proposition 3.3.27 the set B_{N-1} is locally pluriregular. Analogously, the set

$$B_N := \{c_N \in A_{N,k} \cap \widehat{\mathbb{P}}(a_N, r_{k,a}) : (\Sigma_{N-1})_{(\cdot,c_N)} \in \mathcal{PLP}\}$$

is locally pluriregular. Let

$$\begin{aligned} C := \{c \in B_{N-1} \times B_N : (S_{N-1,k,a})_{(\cdot,c)} \in \mathcal{PLP}, \\ (S_{N,k,b})_{(\cdot,c)} \in \mathcal{PLP},\ (\Sigma_\nu)_{(\cdot,c)} \in \mathcal{PLP},\ \nu = 1, \dots, N-2\}. \end{aligned}$$

The set C is also locally pluriregular. Fix a $c = (c_{N-1}, c_N) \in C$.

Observe that $\boldsymbol{T}_{(\cdot,c)} \supset \boldsymbol{T}_{N-1,N}(c)$, where

$$\boldsymbol{T}_{N-1,N}(c) := \boldsymbol{GK}((A_\nu, D_\nu, (\Sigma_\nu)_{(\cdot,c)})_{\nu=1}^{N-2}).$$

Put $A'''_\nu := A_{\nu+1} \times \dots \times A_{N-2}$, $\nu = 1, \dots, N-2$.

The $(N-2)$–fold cross $\boldsymbol{X}_{N-1,N}$, the sets $(\Sigma_\nu)_{(\cdot,c)}$, $\nu = 1, \dots, N-2$, and the set $M_{(\cdot,c)}$ satisfy the assumptions of Theorem 5.4.2.

Indeed,

- the sets $(\Sigma_\nu)_{(\cdot,c)}$, $\nu = 1, \dots, N-1$, are pluripolar,

- for any $\nu \in \{1, \dots, N-2\}$ and $(\zeta'_\nu, \zeta'''_\nu) \in (A'_\nu \times A'''_\nu) \setminus (\Sigma_\nu)_{(\cdot,c)}$,

$$(M_{(\cdot,c)})_{(\zeta'_\nu, \cdot, \zeta'''_\nu)} = M_{(\zeta'_\nu, \cdot, \zeta'''_\nu, c)}$$

is pluripolar.

Consequently, since Theorem 5.4.2 is true for $(N-2)$, there exists a relatively closed pluripolar set $\widehat{M}(c) \subset \widehat{\boldsymbol{Y}}_{N-1,N}$ such that:

- $\widehat{M}(c) \cap \boldsymbol{T}_{N-1,N}(c) \subset M_{(\cdot,c)}$,
- for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f}_c \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N-1,N} \setminus \widehat{M}(c))$ with $\widehat{f}_c = f(\cdot, c)$ on $\boldsymbol{T}_{N-1,N}(c) \setminus M_{(\cdot,c)}$.

Recall that $\widehat{\mathbb{P}}(a', r_{k,a}) \cup \widehat{\mathbb{P}}(b', r_{k,b}) \subset \widehat{\boldsymbol{Y}}_{N-1,N}$. Thus, the functions $\widetilde{f}_{N,k,b}(\cdot, c)$ and $\widehat{f}_c$ are holomorphic on

$$\widehat{\mathbb{P}}(b', r_{k,b}) \setminus ((S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)).$$

Moreover, they are equal to $f(\cdot, c)$ on the set $((\boldsymbol{T}_k)_{(\cdot,c)} \cap \boldsymbol{T}_{N-1,N}(c))) \setminus M_{(\cdot,c)} =: S$. Observe that S is not pluripolar.

Indeed, put $\widetilde{A}_\nu := A_{\nu,k} \cap \widehat{\mathbb{P}}(b_\nu, r_{k,b})$, $\nu = 1, \dots, N-2$. First observe that

$$(\widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-2}) \setminus (\Sigma_N)_{(\cdot, c_{N-1})} \subset (\boldsymbol{c}(\boldsymbol{T}_k))_{(\cdot,c)}.$$

On the other hand, $(\widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-2}) \setminus P \subset \boldsymbol{c}(\boldsymbol{T}_{N-1,N})$, where P is pluripolar. Hence, in view of the definition of the set B_{N-1}, we conclude that

$$(\widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-2}) \setminus Q \subset (\boldsymbol{c}(\boldsymbol{T}_k))_{(\cdot,c)} \cap \boldsymbol{c}(\boldsymbol{T}_{N-1,N}(c)),$$

where Q is pluripolar. In particular, the set

$$R := \{\xi \in \widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-3} : Q_{(\xi,\cdot)} \notin \mathcal{PLP}\}$$

is pluripolar. Moreover, for any $\xi \in (\widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-3}) \setminus (\Sigma_{N-2})_{(\cdot,c)}$, the fiber $(M_{(\cdot,c)})_{(\xi,\cdot)} = M_{(\xi,\cdot,c)}$ is pluripolar. Thus, for any

$$\xi \in (\widetilde{A}_1 \times \cdots \times \widetilde{A}_{N-3}) \setminus (R \cup (\Sigma_{N-2})_{(\cdot,c)})$$

the set $(Q \cup M_{(\cdot,c)})_{(\xi,\cdot)}$ is pluripolar. Now, we are in a position to apply Proposition 3.3.27 and to conclude that S is not pluripolar.

Hence, since the set $(S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)$ is pluripolar, we get

$$\widetilde{f}_{N,k,b} = \widehat{f}_c \text{ on } \widehat{\mathbb{P}}(b', r_{k,b}) \setminus ((S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)).$$

An analogous argument shows that

$$\widetilde{f}_{N-1,k,a} = \widehat{f}_c \text{ on } \widehat{\mathbb{P}}(a', r_{k,a}) \setminus ((S_{N-1,k,a})_{(\cdot,c)} \cup \widehat{M}(c)).$$

Hence,

$$\widetilde{f}_{N-1,k,a}(\cdot, c) = \widetilde{f}_{N,k,b}(\cdot, c)$$
$$\text{on } (\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \setminus ((S_{N-1,k,a})_{(\cdot,c)} \cup (S_{N,k,b})_{(\cdot,c)}).$$

Consequently,

$$\widetilde{f}_{N-1,k,a} = \widetilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times C) \setminus (S_{N-1,k,a} \cup S_{N,k,b}).$$

(b) $i = j$: We may assume that $i = j = N$. Write

$$w = (w', w_N) \in (D_1 \times \cdots \times D_{N-1}) \times D_N.$$

Observe that $\varnothing \neq W_{k,N,N,a,b} = (\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times D_{N,k}$. By (a) we know that

$$\widetilde{f}_{N,k,a} = \widetilde{f}_{N-1,k,a} \text{ on } (V_{N,k,a} \cap V_{N-1,k,a}) \setminus (S_{N,k,a} \cup S_{N-1,k,a}),$$
$$\widetilde{f}_{N-1,k,a} = \widetilde{f}_{N,k,b} \text{ on } (V_{N-1,k,a} \cap V_{N,k,b}) \setminus (S_{N-1,k,a} \cup S_{N,k,b}).$$

Hence $\widetilde{f}_{N,k,a} = \widetilde{f}_{N,k,b}$ on

$$\begin{aligned}(V_{N,k,a} \cap V_{N-1,k,a} \cap V_{N,k,b}) &\setminus (S_{N-1,k,a} \cup S_{N,k,a} \cup S_{N,k,b}) \\ &= ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times \widehat{\mathbb{P}}(a_N, r_{k,a})) \setminus (S_{N-1,k,a} \cup S_{N,k,a} \cup S_{N,k,b}),\end{aligned}$$

and finally, by the identity principle, $\widetilde{f}_{N,k,a} = \widetilde{f}_{N,k,b}$ on $W_{N,N,k,a,b} \setminus (S_{N,k,a} \cup S_{N,k,b})$.

The proof of (*) is completed. □

We move to the main proof of Theorem 5.4.2.

Proof that Theorem 5.3.1 $\Longrightarrow$ Theorem 5.4.2. Consider the general situation as in in Theorem 5.4.2. Our aim is to prove Theorem 5.4.2 via Theorem 5.4.12. We keep all the notations from Theorem 5.4.12.

Fix a $k \in \mathbb{N}$, $a \in \Xi_k$, and $j \in \{1, \dots, N\}$. We are going to construct, $r_{k,a}, S_{j,k,a}, \widetilde{f}_{j,k,a}$ with all of the properties listed in Theorem 5.4.12.

First assume that $j = N$. Let $r > 0$ be such that $\widehat{\mathbb{P}}(a, r) \Subset D_{1,k} \times \cdots \times D_{N,k}$ and $\widehat{\mathbb{P}}(a, r) \cap M = \varnothing$ in the case where $a \notin M$.

First consider the case where $\widehat{\mathbb{P}}(a, r) \cap M = \varnothing$. Put

$$\boldsymbol{W}_{k,a} := \boldsymbol{W} \cap \widehat{\mathbb{P}}(a, r) \in \{\boldsymbol{X}_{k,a},\ \boldsymbol{T}_{k,a}\}$$

with

$$\boldsymbol{X}_{k,a} := \boldsymbol{X} \cap \widehat{\mathbb{P}}(a, r) = \boldsymbol{K}((A_j \cap \widehat{\mathbb{P}}(a_j, r), \widehat{\mathbb{P}}(a_j, r))_{j=1}^N),$$
$$\boldsymbol{T}_{k,a} := \boldsymbol{T} \cap \widehat{\mathbb{P}}(a, r) = \boldsymbol{GK}((A_j \cap \widehat{\mathbb{P}}(a_j, r), \widehat{\mathbb{P}}(a_j, r), \Sigma_j \cap \widehat{\mathbb{P}}((a'_j, a''_j), r))_{j=1}^N).$$

Observe that for every function $f \in \mathcal{F} \in \{\mathcal{O}_s(\boldsymbol{X} \setminus M),\ \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)\}$, the function $f|_{\boldsymbol{W}_{k,a}}$ belongs to $\mathcal{O}_s(\boldsymbol{X}_{k,a})$ or to $\mathcal{O}_s(\boldsymbol{T}_{k,a}) \cap \mathcal{C}^*(\boldsymbol{T}_{k,a})$. Using Theorems 4.3.1 or 4.9.2, we know that f extends to an $\widetilde{f}_{k,a} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{k,a})$ with $\widetilde{f}_{k,a} = f$ on $\boldsymbol{T}_{k,a}$. Thus we may assume that the initial r is so small that every function $f \in \mathcal{F}$ extends to an $\widetilde{f}_{k,a} \in \mathcal{O}(\widehat{\mathbb{P}}(a, r))$ with $\widetilde{f}_{k,a} = f$ on $\boldsymbol{T} \cap \widehat{\mathbb{P}}(a, r)$.

Consider the special 2–fold crosses

$$\begin{aligned}\boldsymbol{Y}_{N,k,a} &:= \boldsymbol{K}((A'_N \setminus \Sigma_N) \cap \widehat{\mathbb{P}}(a'_N, r), \widehat{\mathbb{P}}(a_N, r); \widehat{\mathbb{P}}(a'_N, r), D_{N,k+1}) \\ &= \widehat{\mathbb{P}}(a, r) \cup (((A'_N \setminus \Sigma_N) \cap \widehat{\mathbb{P}}(a'_N, r)) \times D_{N,k+1}) \subset \widehat{\mathbb{P}}(a, r) \cup \boldsymbol{T},\end{aligned}$$

with

$$\begin{gathered} p := n_1 + \cdots + n_{N-1}, \quad q := n_N, \\ D := \widehat{\mathbb{P}}(a'_N, r), \quad G := D_{N,k+1}, \quad B := \widehat{\mathbb{P}}(a_N, r) \\ A = A' := A'_N \cap \widehat{\mathbb{P}}(a'_N, r) \setminus \Sigma_N, \end{gathered}$$

Observe that every function from $f \in \mathcal{F}$ may be identified with the function $f \cup \widetilde{f}_{k,a} \in \mathcal{O}_s(\boldsymbol{Y}_{N,k,a} \setminus M)$. Since the case $N = 2$ was proved, we get a relatively closed pluripolar set $S_{N,k,a} \subset \widehat{\boldsymbol{Y}}_{N,k,a}$ such that:

- $S_{N,k,a} \cap \boldsymbol{Y}_{N,k,a} \subset M$,
- for any function $f \in \mathcal{F}$ there exists an $\widetilde{f}_{N,k,a} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N,k,a} \setminus S_{N,k,a})$ such that $\widetilde{f}_{N,k,a} = f$ on $\boldsymbol{Y}_{N,k,a} \setminus M$,
- $S_{N,k,a}$ is singular with respect to the family $\{\widetilde{f}_{N,k,a} : f \in \mathcal{F}\}$; in particular, $S_{N,k,a} \cap \mathbb{P}(a, r) = \varnothing$,
- if all the fibers $M_{(z'_N, \cdot)}$, $z'_N \in A'$, are thin, then $S_{N,k,a}$ is analytic.

Note that $\{a'_N\} \times D_{N,k+1} \subset \widehat{\boldsymbol{Y}}_{N,k,a}$. Let $\rho = r_{N,k,a} \in (0, r)$ be so small that $V_{N,k,a} := \mathbb{P}(a'_N, \rho) \times D_{N,k} \subset \widehat{\boldsymbol{Y}}_{N,k,a}$. We substitute $S_{N,k,a}$ and $\widetilde{f}_{N,k,a}$ by $S_{N,k,a} \cap V_{N,k,a}$ and $\widetilde{f}_{N,k,a}|_{V_{N,k,a} \setminus S_{N,k,a}}$, respectively.

In the analytic case we argue in a little bit different way. We may assume that $\widehat{\mathbb{P}}(a'_N, r) \times D_{N,k+1} \subset U$. Consider the 2–fold crosses $\boldsymbol{W}_{N,k,a} \in \{\boldsymbol{X}_{N,k,a},\ \boldsymbol{T}_{N,k,a}\}$,

$$\begin{aligned}\boldsymbol{X}_{N,k,a} &:=:= \boldsymbol{K}(A'_N \cap \widehat{\mathbb{P}}(a'_N, r), A_{N,k+1}; \widehat{\mathbb{P}}(a'_N, r), D_{N,k+1}) \subset \boldsymbol{X}, \\ \boldsymbol{Z}_{N,k,a} &:= \boldsymbol{K}((A'_N \cap \widehat{\mathbb{P}}(a'_N, r)) \setminus \Sigma_N, A_{N,k+1}; \widehat{\mathbb{P}}(a'_N, r), D_{N,k+1}).\end{aligned}$$

Observe that every function $f \in \mathcal{F}$ belongs to $\mathcal{O}_s(\boldsymbol{X}_{N,k,a} \setminus M)$ or to $\mathcal{O}_s(\boldsymbol{Z}_{N,k,a} \setminus M)$. Our assumptions imply that every $f \in \mathcal{F}$ extends to an $\widehat{f}_{N,k,a} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N,k,a} \setminus M)$ with $\widehat{f}_{N,k,a} = f$ on $\boldsymbol{W}_{N,k,a} \setminus M$. Now we continue as in the pluripolar case and we end up with $S_{N,k,a} \subset M$.

It is clear that all the requirements from Theorem 5.4.12 are satisfied for $j = N$. We repeat the same procedure with respect to each $j \in \{1, \ldots, N-1\}$ and finally, we put $r_{k,a} := \min\{r_{j,k,a} : j = 1, \ldots, N\}$. □

Chapter 6

Separately meromorphic functions

6.1 Rothstein theorem

Theorem 6.1.1 (Cf. [Rot 1950]). *Let $f \in \mathcal{M}(\mathbb{D}^p \times \mathbb{D}^q)$. Assume that $A \subset \mathbb{D}^p$ be a locally pluriregular set such that for any $a \in \mathbb{D}^p$ we have $(S(f))_{(a,\cdot)} \neq \mathbb{D}^q$, where $S(f)$ denote the polar set of f, i.e. $S(f)$ is the union of the set of all poles of f and the set of all indeterminancy points of f; recall that $S(f)$ is analytic and $f \in \mathcal{O}(\mathbb{D}^p \times \mathbb{D}^q \setminus S(f))$ — cf. § 3.8. Let G be a Riemann domain over $\mathbb{C}^p$ such that $\mathbb{D}^q \subset G$ (1). Assume that for every $a \in A$ the function $f(a, \cdot)$ extends meromorphically to G. Then there exist an open neighborhood $\varOmega$ of $(\mathbb{D}^p \times \mathbb{D}^q) \cup (A \times G)$ and a function $\widehat{f} \in \mathcal{M}(\varOmega)$ such that $\widehat{f} = f$ on $\mathbb{D}^p \times \mathbb{D}^q$.*

Proof. (1) The case where $A = \mathbb{D}^p$ (2), $q = 1$, $G = K(R)$ $(R > 1)$, and $f \in \mathcal{O}(\mathbb{D}^p \times \mathbb{D})$:

The proof may be found for instance in [Siu 1974].

[Will be completed. . . . .]

(2) The case where $A = \mathbb{D}^p$, $q = 1$, and $G = K(R)$:

Recall that $(S(f))_{(a,\cdot)} \neq \mathbb{D}^q$ for any $a \in \mathbb{D}^p$, and therefore, for any $a \in \mathbb{D}^p$ there exists a $b \in \mathbb{D}^q$ such that f is holomorphic in a neighborhood of (a, b). By applying locally (1), we get the required result.

(3) The case where $A = \mathbb{D}^p$ and $G = \mathbb{P}_q(R)$:

Let R_0 denote the radius of the maximal polydisc $\mathbb{P}_q(R_0)$ such that f extends meromorphically to $\mathbb{D}^p \times \mathbb{P}_q(R_0)$. We only need to show that $R_0 \geq R$. Obviously $R_0 \geq 1$. Suppose that $R_0 < R$.

Let S_q be the set of all $(z, w') \in \mathbb{D}^p \times \mathbb{P}_{q-1}(R_0)$ such that $(S(f))_{(z,w',\cdot)} = \mathbb{D}$. It is well known that S_q is an analytic subset of $\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0)$. Moreover, our assumptions imply that $S_q \neq \mathbb{D}^p \times \mathbb{P}_{q-1}(R_0)$. Applying locally the Rothstein theorem to $(\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0) \setminus S_q) \times K(R) \subset \mathbb{C}^{p+q-1} \times \mathbb{C}$, we conclude that f extends meromorphically to $((\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0) \setminus S_q) \times K(R)) \cup (\mathbb{D}^p \times \mathbb{P}_q(R_0))$. Observe that, by the Levi extension theorem ([Jar-Pfl 2000], Prop. 3.4.5), the envelope of holomorphy of $((\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0) \times K(R)) \setminus (S_q \times K(R))) \cup (\mathbb{D}^p \times \mathbb{P}_q(R_0))$ equals $\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0) \times K(R)$. Consequently, the function f extends meromorphically to $\mathbb{D}^p \times \mathbb{P}_{q-1}(R_0) \times K(R)$. Repeating the same argument with respect to other variables in $\mathbb{C}^q$, we conclude that f extends meromorphically to the domain $\mathbb{D}^p \times H$, where

$$H = \bigcup_{j=1}^{q} \mathbb{P}_{j-1}(R_0) \times K(R) \times \mathbb{P}_{q-j}(R_0).$$

(1) That is, we identify $\mathbb{D}^q$ with certain "polydisc" $\widehat{\mathbb{P}}_G(b_0, r)$.

(2) Observe that if $A = \mathbb{D}^p$, then we have to prove that f extends meromorphically to $\mathbb{D}^p \times G$.

The envelope of holomorphy of $\mathbb{D}^p \times H$ has the form $\mathbb{D}^p \times \widehat{H}$, where $\widehat{H}$ contains a polydisc $\mathbb{P}_q(R_0')$ with $R_0' > R_0$. Thus f extends meromorphically to $\mathbb{D}^p \times \mathbb{P}_q(R_0')$; a contradiction — cf. the proof of Lemma 12 in [Jar-Pfl 2003b].

(4) The case where $A \subset \mathbb{D}^p$ is locally pluriregular and $G = \mathbb{P}_q(R)$:

For every $z \in \mathbb{D}^p$, let $\rho_f(z)$ denote the radius of the maximal polydisc $\mathbb{P}_q(r)$ such that $f(z,\cdot)$ extends meromorphically to $\mathbb{P}_q(r)$. Obviously, $\rho_f \geq 1$ on $\mathbb{D}^p$ and $\rho_f \geq R$ on A.

Using (3), one can easily conclude that f extends meromorphically to the Hartogs domain

$$D := \{(z,w) \in \mathbb{D}^p \times \mathbb{C}^q : |w| < (\rho_f)_*(z)\}.$$

Let $\widetilde{f} \in \mathcal{M}(D)$ be the meromorphic extension of f.

Moreover, $-\log(\rho_f)_* \in \mathcal{PSH}(\mathbb{D}^p)$.

Indeed, let $\widehat{D}$ denote the envelope of holomorphy of D. It is known that $\widehat{D} \subset \mathbb{D}^p \times \mathbb{C}^q$ is a Hartogs domain with complete q–circled fibers ([Jar-Pfl 2000], Remark 3.1.2(h)). Moreover, $\widetilde{f}$ extends meromorphically to $\widehat{D}$ ([Jar-Pfl 2000], Th. 3.6.6). In particular,

$$(\rho_f)_*(z) = \inf\{\delta_{\widehat{D},(0,\xi)}(z,0) : \xi \in \mathbb{C}^q,\ |\xi| = 1\}, \quad z \in \mathbb{D}^p,$$

where

$$\delta_{\widehat{D},(0,\xi)}(z,0) = \sup\{r > 0 : (z,0) + K(r)(0,\xi) \subset \widehat{D}\}.$$

Consequently, $-\log(\rho_f)_* \in \mathcal{PSH}(\mathbb{D}^p)$ ([Jar-Pfl 2000], Th. 2.2.9(iv)).

Thus $-\log(\rho_f)_* \in \mathcal{PSH}(\mathbb{D}^p)$. Recall that $\rho_f \geq R$ on A. Hence, using the local pluriregularity of A, we conclude that $(\rho_f)_* \geq R$ on A [3]. Thus $A \times \mathbb{P}_q(R) \subset D$, and therefore D is the required neighborhood.

(5) The general case where $A \subset \mathbb{D}^p$ is locally pluriregular and G is arbitrary:

Fix an $a \in A$. Let G_0 denote the set of all $b \in G$ such that there exist $r_b > 0$ and $f_b \in \mathcal{M}(\widehat{\mathbb{P}}((a,b),r_b))$, $\widehat{\mathbb{P}}((a,b),r_b) \subset \mathbb{D}^p \times G$, such that:

$\forall_{\alpha \in A \cap \widehat{\mathbb{P}}(a,r_b)} : f_b(\alpha,\cdot) = \widetilde{f(\alpha,\cdot)}$ on $\widehat{\mathbb{P}}(b,r_b)$ [4].

Obviously G_0 is open, $G_0 \neq \varnothing$ ($\mathbb{D}^q \subset G_0$). Using the Rothstein theorem with $G = \widehat{\mathbb{P}}_q(R)$, one can prove that G_0 is closed in G. Thus $G_0 = G$.

Moreover, one can also prove that if $\widehat{\mathbb{P}}(b',r_{b'}) \cap \widehat{\mathbb{P}}(b'',r_{b''}) \neq \varnothing$, then $f_{b'} = f_{b''}$ on $\widehat{\mathbb{P}}((a,b'),r_{b'}) \cap \widehat{\mathbb{P}}((a,b''),r_{b''})$. This gives a meromorphic extension of f to an open neighborhood of $\{a\} \times G$. Since a was arbitrary, we get the required neighborhood Ω. □

In the case where $A = \mathbb{D}^p$ the result may be strengthened as follows.

---

[3] Suppose that $h^*_{A,\mathbb{D}^p} = h_{A,\mathbb{D}^p}$ on $\mathbb{D}^p \setminus P$, where P is pluripolar. Put $u := \frac{-\log(\rho_f)_*}{\log R} + 1$. Then $u \leq 1$ and $u \leq 0$ on $A \setminus P$. Consequently, $u \leq h^*_{A\setminus P,\mathbb{D}^p} = h^*_{A,\mathbb{D}^p}$. In particular, $u \leq 0$ on A, i.e. $(\rho_f)_* \geq R$ on A.

[4] As before, $\widetilde{f(\alpha,\cdot)}$ denotes the meromorphic extension of $f(\alpha,\cdot)$.

Theorem 6.1.2 (Cf. [Rot 1950]). *Let $f \in \mathcal{M}(\mathbb{D}^p \times \mathbb{D}^q)$. Let G be a Riemann domain over $\mathbb{C}^q$ such that $\mathbb{D} \subset G$. Assume that for every $a \in \mathbb{D}^p$ the function $f(a,\cdot)$ extends meromorphically to G. Then there exists an $\widehat{f} \in \mathcal{M}(\mathbb{D}^p \times G)$ such that $\widehat{f} = f$ on $\mathbb{D}^p \times \mathbb{D}^q$.*

Proof. Let $S := \{z \in \mathbb{D}^p : \{z\} \times \mathbb{D}^q \subset S(f)\}$. It is known that S is a proper analytic set. Let $S_0 := S \times G$. Using locally Theorem 6.1.1 on $(\mathbb{D}^p \setminus S) \times G$, we easily conclude that f extends meromorphically to an $\widehat{f} \in \mathcal{M}((\mathbb{D}^p \times \mathbb{D}^q) \cup (\mathbb{D}^p \times G \setminus S_0))$. Using Proposition 3.1.25, we conclude that the envelope of holomorphy of the domain $(\mathbb{D}^p \times \mathbb{D}^q) \cup (\mathbb{D}^p \times G \setminus S_0)$ contains $\mathbb{D}^p \times G$. Finally, Theorem 3.8.2, we conclude that $\widehat{f}$ extends meromorphically on $\mathbb{D}^p \times G$. $\square$

6.2 Extension of separately meromorphic functions with singularities

It is known that the envelope of holomorphy (of any Riemann domain over $\mathbb{C}^n$) coincides with the envelope of meromorphy (cf. [Jar-Pfl 2000], Theorem 3.6.6). Thus it is natural to conjecture that in the above situation the domain $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is also the envelope of meromorphy of $\boldsymbol{X} \setminus M$ with respect to separate meromorphic functions.

Throughout this section D_j denotes a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A'_j \times A''_j$ is pluripolar, $j = 1, \dots, N$, $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$, $\boldsymbol{T} := \boldsymbol{GK}((A_j, D_j, \Sigma_j)_{j=1}^N)$. Moreover, $S \subset \boldsymbol{X}$, $M \subset \boldsymbol{T} \cap S$ are relatively closed sets such that for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the fiber $S_{(a'_j, \cdot. a''_j)}$ is pluripolar.

Definition 6.2.1. We say that a function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S)$ is *separately meromorphic on $\boldsymbol{T} \setminus M$* ($f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{T} \setminus M)$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the function

$$D_j \setminus S_{(a'_j, \cdot, a''_j)} \ni z_j \longmapsto f(a'_j, z_j, a''_j)$$

extends meromorphically to $D_j \setminus M_{(a'_j, \cdot, a''_j)}$.

Theorem 6.2.2 (Extension theorem for meromorphic functions). *Let $\widehat{S}$ and $\widehat{M}$ be constructed according to Theorems 5.4.6 and 5.4.7, respectively. Then for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{T} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{S}) \cap \mathcal{M}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{T} \setminus S$.*

The case $M = \varnothing$ was studied for instance in [Sak 1957], [Kaz 1976], [Kaz 1978b], [Kaz 1984], [Shi 1986], and [Shi 1989].

Proof. Obviously, by Theorem 5.4.6, every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{T} \setminus M)$ extends to an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{S})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus S$. We only need to show that $\widehat{f} \in \mathcal{M}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$. Observe that it suffices to show that there exists an open neighborhood $\Omega \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{T} \setminus M$ (indepenedent of f) such that every connected component of Ω intersects $\boldsymbol{T} \setminus M$ and $\widehat{f} \in \mathcal{M}(\Omega)$ for every f.

Indeed, suppose such an Ω is already constructed. By Theorem 5.4.7, the envelope of holomorphy of Ω coincides with $\widehat{\boldsymbol{X}} \setminus \widehat{M}$.

In fact, if $g \in \mathcal{O}(\Omega)$, then $g|_{\boldsymbol{T} \setminus M} \in \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}(\boldsymbol{T} \setminus M)$. Hence, there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus M)$ with $\widehat{g} = g$ on $\boldsymbol{T} \setminus M$. Consequently, $\widehat{g} = g$ on Ω because each connected component of Ω intersects $\boldsymbol{T} \setminus M$.

Thus, by Theorem 3.6.6 from [Jar-Pfl 2000], $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is the envelope of meromorphy of Ω, which means that $\widehat{f}$ extends meromorphically to an $\widetilde{\widehat{f}} \in \mathcal{M}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widetilde{\widehat{f}} = \widehat{f}$ on Ω. The indentity principle for meromorphic functions implies that $\widetilde{\widehat{f}} \equiv \widehat{f}$.

Indeed, take an $a \in \boldsymbol{T} \setminus M$. We may assume that

$$a = (a'_N, a_N) \in (A'_N \setminus \Sigma_N) \times D_N.$$

Fix a $b_N \in A_N \setminus S_{(a'_N, \cdot)}$ (recall that $S_{(a'_N, \cdot)} \in \mathcal{PLP}$). Put $b := (a'_N, b_N) \in \boldsymbol{c}(\boldsymbol{T}) \setminus S$. Let $r > 0$ be such that $\widehat{\mathbb{P}}(b, r) \subset \widehat{\boldsymbol{X}} \setminus \widehat{S}$. In particular, $\widehat{f} \in \mathcal{O}(\widehat{\mathbb{P}}(b, r))$ for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{T} \setminus M)$.

Since $M_{(a'_N, \cdot)}$ is pluripolar, there exists a domain $G_N \Subset D_N \setminus M_{(a'_N, \cdot)}$ with $a_N \in G_N$, $\widehat{\mathbb{P}}(b_N, r) \subset G_N$. Since M is relatively closed in $\boldsymbol{T}$, we may assume that r is so small that $((A'_N \setminus \Sigma_N) \cap \mathbb{P}(a'_N, r)) \times G_N \subset \boldsymbol{T} \setminus M$. By the Rothstein theorem (Theorem 6.1.1), there exists an open connected neighborhood $\Omega_a \subset \widehat{\boldsymbol{X}}$ of $\{a'_N\} \times G_N$ such that $\widehat{f} \in \mathcal{M}(\Omega_a)$ for every f.

We put $\Omega := \bigcup_{a \in \boldsymbol{T} \setminus S} \Omega_a$. Observe that any connected component of Ω intersects $\boldsymbol{c}(\boldsymbol{T}) \setminus M$. $\square$

6.3 The case $N = 2$

In the case where $N = 2$, $M = \varnothing$, Theorem 6.2.2 may be strengthened as follows.

Theorem 6.3.1 (Extension theorem for meromorphic functions). *Let D, G be Riemann domain of holomorphy over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, let $\varnothing \neq A \subset D$, $\varnothing \neq B \subset G$ be locally pluriregular sets, and let*

$$\boldsymbol{X} := \boldsymbol{K}(A, B; D, G) = (A \times G) \cup (D \times B).$$

Let $S \subset \boldsymbol{X}$ be a relatively closed set. Assume that:

(a) *for every $(a, b) \in A \times B$ we have* $\operatorname{int}_G S_{(a,\cdot)} = \varnothing$, $\operatorname{int}_D S_{(\cdot,b)} = \varnothing$,

(b) $A \times B \subset \overline{(A \times B) \setminus S}$ (5),

(5) In particular, for every $(a, b) \in A \times B$ and for every neighborhood $U \subset D \times G$ of (a, b) the set $(A \times B) \cap U \setminus S$ is not pluripolar.

there exist exhaustions $(D_k)_{k=1}^{\infty}$ *and* $(G_k)_{k=1}^{\infty}$ *of* D *and* G, *respectively, by domain of holomorphy such that:*

(c) $A_k := A \cap D_k \neq \varnothing$, $B_k := B \cap G_k \neq \varnothing$,

(d) *for every* $(a,b) \in A_k \times B_k$ *we have* $B_k \setminus S_{(a,\cdot)} \neq \varnothing$, $A_k \setminus S_{(\cdot,b)} \neq \varnothing$, $k \in \mathbb{N}$.

Then for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ *there exists a function* $\widehat{f} \in \mathcal{M}(\widehat{\boldsymbol{X}})$ *such that* $\widehat{f} = f$ *on* $\boldsymbol{X} \setminus S$.

Let Ω be a Riemann region over $\mathbb{C}^n$.

Definition 6.3.2. We say that a set $A \subset \Omega$ is *plurithin* at a point $a \in \Omega$ if either $a \notin \overline{A}$ or $a \in \overline{A}$ and $\limsup_{A\setminus\{a\}\ni z\to a} u(z) < u(a)$ for a function u plurisubharmonic in a neighborhood of a.

Remark 6.3.3. (a) ([Kli 1991], Corollary 4.8.4) If A, B are plurithin at a, then $A \cup B$ is plurithin at a.

(b) ([Arm-Gar 2001], Th. 7.2.2) Every polar set $P \subset \mathbb{C}$ is thin at any point $a \in \mathbb{C}$.

(c) If $A \subset \mathbb{C}$ is not thin at a point $a \in \overline{A}$, then for any polar set $P \subset \mathbb{C}$, the set $A \setminus P$ is not thin at a ((c) follows directly from (a) and (b)).

(d) If $A \subset \Omega$ is locally pluriregular at a point $a \in \overline{A}$, then A is not plurithin at a.

If $A \subset \mathbb{C}$ is not thin at a point $a \in \overline{A}$, then A is locally regular at a.

Indeed, suppose that $A \subset \Omega$ is locally pluriregular at a and

$$\limsup_{A\setminus\{a\}\ni z\to a} u(z) < c < u(a)$$

for some $u \in \mathcal{PSH}(V)$, where V is an open neighborhood of a. We may assume that $u \leq 0$ on V. Take an open neighborhood $U \subset V$ of a such that $u < c$ on $(A \setminus \{a\}) \cap U$. Put $v := \frac{u}{-c} + 1$. Then $v \leq 1$ on U and $v \leq 0$ on $(A \setminus \{a\}) \cap U$. Hence $v \leq h^*_{(A\setminus\{a\})\cap U,U} = h^*_{A\cap U,U}$ on U. In particular, $0 = v(a) = \frac{u(a)}{-c} + 1 < 0$; a contradiction.

Now, suppose that $A \subset \mathbb{C}$ is not thin at a and $h^*_{A\cap U,U}(a) > 0$ for some neighborhood U of a. Let $P \subset U$ be a polar set such that $h^*_{A\cap U,U} = h_{A\cap U,U}$ on $U \setminus P$ (cf. [Jar-Pfl 2000] Th. 2.1.41). In particular, $h^*_{A\cap U,U} = 0$ on $A \setminus P$. By (c), the set $A \setminus P$ is not thin at a. Hence $0 < h^*_{A\cap U,U}(a) = \limsup_{A\setminus P\ni z\to a} h^*_{A\cap U,U}(z) = 0$; a contradiction.

(e) ([Arm-Gar 2001], Th. 7.3.9) If $A \subset \mathbb{C}$ is thin at a point $a \in \overline{A}$, then there is a sequence $r_k \searrow 0$ such that $\{z \in A : |z - a| = r_k\} = \varnothing$, $k = 1, 2, \dots$.

Proof. It suffices to prove that for each k there exists an open neighborhood $\Omega_k \subset \widehat{\boldsymbol{X}}_k$ of the cross $\boldsymbol{X}_k := \boldsymbol{K}(A_k, B_k; D_k, G_k) = (A_k \times G_k) \cup (D_k \times B_k)$ such that there exists an $\widetilde{f}_k \in \mathcal{M}(\Omega_k)$ with $\widetilde{f}_k = f$ on $\boldsymbol{X}_k \setminus S$.

Indeed, the envelope of holomorphy of Ω_k coincides with $\widehat{\boldsymbol{X}}_k$ (cf. the proof of Theorem 5.4.12). Hence, by Theorem 3.6.6 from [Jar-Pfl 2000], the function $\widetilde{f}_k$ extends to a function $\widehat{f}_k \in \mathcal{M}(\widehat{\boldsymbol{X}}_k)$. Since $\boldsymbol{X}_k \setminus S$ is not pluripolar (by (a)), we conclude that $\widehat{f}_k = \widehat{f}_{k+1}$ on $\widehat{\boldsymbol{X}}_k$. Finally, we glue up the functions $(\widehat{f}_k)_{k=1}^{\infty}$ and we get the required extension.

Fix $(a, b) \in A_k \times B_k \setminus S$ and let $r > 0$ be such that $\widehat{\mathbb{P}}((a, b), r) \subset D_k \times G_k \setminus S$. Define $\boldsymbol{Y} := \boldsymbol{K}(A \cap \widehat{\mathbb{P}}(a, r), B \cap \widehat{\mathbb{P}}(b, r); \widehat{\mathbb{P}}(a, r), \widehat{\mathbb{P}}(b, r))$. Then $f \in \mathcal{O}_s(\boldsymbol{Y})$ and hence, by Theorem 4.3.1, $f|_{\boldsymbol{Y}}$ extends holomorphically on $\widehat{\boldsymbol{Y}}$. In particular, f extends holomorphically to an open neighborhood of (a, b).

By the Rothstein theorem (Theorem 6.1.1), we get an open set

$$\Omega_{k,a,b} = (\widehat{\mathbb{P}}(a, r_{a,b}) \times G_k) \cup (D_k \times \widehat{\mathbb{P}}(b, r_{a,b})) \subset \widehat{\boldsymbol{X}}_k \subset D_k \times G_k$$

for which there exists a function $\widehat{f}_{k,a,b} \in \mathcal{M}(\Omega_{k,a,b})$ such that $\widehat{f}_{k,a,b} = f$ on $\boldsymbol{X} \cap \Omega_{k,a,b} \setminus S$.

Now we show that if $\Omega_{k,a,b} \cap \Omega_{k,a',b'} \neq \varnothing$, then $\widehat{f}_{k,a,b} = \widehat{f}_{k,a',b'}$ on $\Omega_{k,a,b} \cap \Omega_{k,a',b'}$. Observe that

$$\begin{aligned}\Omega_{k,a,b} \cap \Omega_{k,a',b'} = &\Big((\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(a', r_{a',b'})) \times G_k\Big) \cup \Big(\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(b', r_{a',b'})\Big) \\ &\cup \Big(\widehat{\mathbb{P}}(a', r_{a',b'}) \times \widehat{\mathbb{P}}(b, r_{a,b})\Big) \cup \Big(D_k \times (\widehat{\mathbb{P}}(b, r_{a,b}) \cap \widehat{\mathbb{P}}(b', r_{a',b'}))\Big).\end{aligned}$$

First observe that $\widehat{f}_{k,a,b} = f = \widehat{f}_{k,a',b'}$ on $(A_k \times B_k) \cap (\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(b', r_{a',b'})) \setminus S$. Hence, by (a), $\widehat{f}_{k,a,b} = \widehat{f}_{k,a',b'}$ on $\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(b', r_{a',b'})$. The same argument works on $\widehat{\mathbb{P}}(a', r_{a',b'}) \times \widehat{\mathbb{P}}(b, r_{a,b})$.

If $\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(a', r_{a',b'}) \neq \varnothing$, then for any $\beta \in B_k$ we have $\widehat{f}_{k,a,b}(\cdot, \beta) = f(\cdot, \beta)$ on $A_k \cap \widehat{\mathbb{P}}(a, r_{a,b}) \setminus S_{(\cdot,\beta)}$. Hence $\widehat{f}_{k,a,b}(\cdot, \beta) = \widehat{f(\cdot, \beta)}$ on $\widehat{\mathbb{P}}(a, r_{a,b})$, and, consequently, $\widehat{f}_{k,a,b}(\cdot, \beta) = \widehat{f(\cdot, \beta)} = \widehat{f}_{k,a',b'}(\cdot, \beta)$ on $\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(a', r_{a',b'})$ for any $\beta \in B_k$. The identity principle implies that $\widehat{f}_{k,a,b} = \widehat{f}_{k,a',b'}$ on $(\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(a', r_{a',b'})) \times G_k$. The same argument works on $D_k \times (\widehat{\mathbb{P}}(b, r_{a,b}) \cap \widehat{\mathbb{P}}(b', r_{a',b'}))$.

It remains to observe that, by (d), $\Omega_k := \bigcup_{(a,b) \in A_k \times B_k \setminus S} \Omega_{k,a,b}$ is an open neighborhood of $\boldsymbol{X}_k$. $\square$

Corollary 6.3.4 (Cf. [Sak 1957]). *Let $S \subset \mathbb{D} \times \mathbb{D}$ be a relatively closed set such that:*

- $\operatorname{int} S = \varnothing$,
- *for every domain $U \subset \mathbb{D} \times \mathbb{D}$ the set $U \setminus S$ is connected* (6).

Let A (resp. B) denote the set of all $a \in \mathbb{D}$ (resp. $b \in \mathbb{D}$) such that $\operatorname{int}_{\mathbb{C}} S_{(a,\cdot)} = \varnothing$ *(resp.* $\operatorname{int}_{\mathbb{C}} S_{(\cdot,b)} = \varnothing$*). Put $\boldsymbol{X} := \boldsymbol{K}(A, B; \mathbb{D}, \mathbb{D}) = (A \times \mathbb{D}) \cup (\mathbb{D} \times B)$.*

Then for every function $f : \boldsymbol{X} \setminus S \longrightarrow \mathbb{C}$ which is separately meromorphic on $\boldsymbol{X}$, there exists an $\widehat{f} \in \mathcal{M}(\mathbb{D} \times \mathbb{D})$ such that $\widehat{f} = f$ on $\boldsymbol{X} \setminus S$.

Notice that the original proof of the above result is not correct — details may be found in [Jar-Pfl 2003c].

(6) We shortly say that S does not separate domains.

Proof. First we check that the sets A and B are not thin at any point of $\mathbb{D}$ (in particular, they are dense in $\mathbb{D}$).

Indeed, suppose that A is thin at a point $a \in \mathbb{D}$. By Remark 6.3.3(e), there exist a circle $C \subset \mathbb{D}$ such that $C \cap A = \varnothing$. Using a Baire category argument, we conclude that there exist a non-empty open arc $\Gamma \subset C$ and an open disc $\Delta \subset \mathbb{D}$ such that the 3–dimensional real surface $\Gamma \times \Delta$ is contained in S. Hence, since S is nowhere dense and does not separate domains, we get a contradiction.

Consequently, by Remark 6.3.3(d), the sets A and B are locally regular and $h^*_{A,\mathbb{D}} = h^*_{B,\mathbb{D}} = 0$. In particular, $\widehat{\boldsymbol{X}} = \mathbb{D} \times \mathbb{D}$.

Now, using the fact that A and B are dense in $\mathbb{D}$, one can easily check that all the assumptions of Theorem 6.3.1 ($D = G = \mathbb{D}$) are satisfied with arbitrary exhaustions $D_j := K(r_j)$, $G_j := K(r_j)$, $0 < r_j \nearrow 1$, which satisfy condition (c). □

Corollary 6.3.5 (Cf. [Shi 1989]). *Let $D, G, A, B, \boldsymbol{X}$ be as in Theorem 6.3.1. Assume that $S \subset \boldsymbol{X}$ is a relatively closed set such that*

- *the set $D \setminus A$ is of zero Lebesgue measure,*
- *for every $a \in A$ the fiber $S_{(a,\cdot)}$ is pluripolar,*
- *for every $b \in B$ the fiber $S_{(\cdot,b)}$ is of zero Lebesgue measure.*

Then for every function $f : \boldsymbol{X} \setminus S \longrightarrow \mathbb{C}$ which is separately meromorphic on $\boldsymbol{X}$, there exists an $\widehat{f} \in \mathcal{M}(D \times G)$ such that $\widehat{f} = f$ on $\boldsymbol{X} \setminus S$.

Proof. One can easily check that all the assumptions of Theorem 6.3.1 are satisfied. It remains to observe that $h^*_{A,D} \equiv 0$ (because $h^*_{A,D} = 0$ on A and the set $D \setminus A$ is of zero measure). Hence $\widehat{\boldsymbol{X}} = D \times G$. □

Chapter 7

General cross theorem with singularities

7.1 General extension theorem with singularities

Our aim is to generalize Theorems 5.3.1 5.4.6, and 5.4.7, by dropping the assumption that M is relatively closed.

Throughout this section $D_j \subset \mathbb{C}^{n_j}$ denotes a domain of holomorphy, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A'_j \times A''_j$ is pluripolar, $j = 1, \dots, N$, $\boldsymbol{X} := \boldsymbol{K}((A_j, D_j)_{j=1}^N)$, $\boldsymbol{T} := \boldsymbol{GK}((A_j, D_j, \Sigma_j)_{j=1}^N)$, $\boldsymbol{W} \in \{\boldsymbol{X}, \boldsymbol{T}\}$, $M \subset \boldsymbol{W}$ is such that for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ the fiber $M_{(a'_j, \cdot, a''_j)}$ is closed and pluripolar. If $\boldsymbol{W} = \boldsymbol{X}$, the we additionally assume that for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$ the fiber $M_{(a'_j, \cdot, a''_j)}$ is closed.

The definition of a separately holomorphic function $f : \boldsymbol{W} \longrightarrow \mathbb{C}$ extends easily to the above situation (cf. Definition 5.4.1).

The main result is the following theorem.

Theorem 7.1.1 (Extension theorem for generalized crosses with pluripolar singularities). *Let $\boldsymbol{W} \in \{\boldsymbol{X}, \boldsymbol{T}\}$, and let*

$$\mathcal{F} \subset \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus M), & \text{if } \boldsymbol{W} = \boldsymbol{X} \\ \mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M), & \text{if } \boldsymbol{W} = \boldsymbol{T} \end{cases}$$

be such that for every $a \in \boldsymbol{c}(\boldsymbol{W}) \setminus M$ there exist a polydisc $\mathbb{P}(a, r)$ such that for every $f \in \mathcal{F}$ there exists an $\widetilde{f}_a \in \mathcal{O}(\mathbb{P}(a, r(a)))$ with $\widetilde{f}_a = f$ on $\mathbb{P}(a, r(a)) \cap (\boldsymbol{T} \setminus M)$ ([1]).

Then there exist pluripolar sets $\Sigma'_j \subset A'_j \times A''_j$ with $\Sigma_j \subset \Sigma'_j$, $j = 1, \dots, N$, and relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:

- *$\widehat{M} \cap (\boldsymbol{T}' \cup \boldsymbol{c}(\boldsymbol{T})) \subset M$, where $\boldsymbol{T}' := \boldsymbol{GK}((A_j, D_j, \Sigma'_j)_{j=1}^N)$,*
- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T}' \cup \boldsymbol{c}(\boldsymbol{T}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *if for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j \setminus \Sigma_j$, the fiber $M_{(a'_j, \cdot, a''_j)}$ is*

thin in D_j, then $\widehat{M}$ is analytic.

The main "technical tool" in the proof of Theorem 7.1.1 is the following theorem.

([1]) Observe that if M is relatively closed in $\boldsymbol{W}$, then an argument as in the first part of the proof of Theorem 5.4.2 (cf. p. 112) shows that the above condition is satisfied with $\mathcal{F} := \mathcal{O}(\boldsymbol{X} \setminus M)$ or $\mathcal{F} := \mathcal{O}(\boldsymbol{T} \setminus M) \cap \mathcal{C}^*(\boldsymbol{T} \setminus M)$.

Theorem 7.1.2 (Glueing theorem). *Let $\boldsymbol{W} \in \{\boldsymbol{X}, \boldsymbol{T}\}$, $M \subset \boldsymbol{W}$, and $\mathcal{F}$ be as in Theorem 7.1.1. If $N \geq 4$, then we additionally assume that Theorem 7.1.1 was already proved for all $(N-2)$–fold crosses.*

Let $(D_{j,k})_{k=1}^{\infty}$ be an exhaustion sequence for D_j (in sense of Definition 2.2.5) such that each $D_{j,k}$ is a domain of holomorphy and $A_{j,k} := A_j \cap D_{j,k} \neq \varnothing$, $k \in \mathbb{N}$, $j = 1, \dots, N$. Put

$$A'_{j,k} := A_{1,k} \times \dots \times A_{j-1,k}, \quad A''_{j,k} := A_{j+1,k} \times \dots \times A_{N,k},$$
$$\Sigma_{j,k} := \Sigma_j \cap (A'_{j,k} \times A''_{j,k}),$$
$$\boldsymbol{X}_k := \boldsymbol{K}((A_{j,k}, D_{j,k})_{j=1}^N) = \boldsymbol{X} \cap (D_{1,k} \times \dots \times D_{N,k}),$$
$$\boldsymbol{T}_k := \boldsymbol{GK}((A_{j,k}, D_{j,k}, \Sigma_{j,k})_{j=1}^N) = \boldsymbol{T} \cap (D_{1,k} \times \dots \times D_{N,k}),$$
$$\boldsymbol{W}_k := \boldsymbol{W} \cap (D_{1,k} \times \dots \times D_{N,k}) \in \{\boldsymbol{X}_k, \boldsymbol{T}_k\}.$$

Let $\Xi_k := \boldsymbol{c}(\boldsymbol{W}_k) \setminus M$.

Assume that for any $k \in \mathbb{N}$, $j \in \{1, \dots, N\}$, and $a \in \Xi_k$, there exist:

- *$r = r_{k,a}$, $0 < r < r(a)$,*
- *relatively closed pluripolar set $S_{j,k,a} \subset \widehat{\mathbb{P}}(a'_j, r) \times D_{j,k} \times \widehat{\mathbb{P}}(a''_j, r) =: V_{j,k,a}$,*
- *a pluripolar set $P_{j,k,a} \subset A'_{j,k} \times A''_{j,k} \setminus \Sigma_j$,*

such that:

- *$\widehat{\mathbb{P}}(a, r) \subset D_{1,k} \times \dots \times D_{N,k}$,*
- *$S_{j,k,a} \cap \boldsymbol{T}'_{j,k,a} \subset M$, where*

$$\boldsymbol{T}'_{j,k,a} := \{(z'_j, z_j, z''_j) \in (A'_{j,k} \cap \widehat{\mathbb{P}}(a'_j, r)) \times D_{j,k} \times (A''_{j,k} \cap \widehat{\mathbb{P}}(a''_j, r)) : \\ (z'_j, z''_j) \notin \Sigma_{j,k} \cup P_{j,k,a}\} \subset V_{j,k,a} \cap \boldsymbol{T}_k,$$

- *for any $f \in \mathcal{F}$ there exists an $\widetilde{f}_{j,k,a} \in \mathcal{O}(V_{j,k,a} \setminus S_{j,k,a})$ with $\widetilde{f}_{j,k,a} = f$ on $\boldsymbol{T}'_{j,k,a} \setminus M$.*

Then there exist a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ and pluripolar sets $P_j \subset A'_j \times A''_j \setminus \Sigma_j$, $j = 1, \dots, N$, such that:

- *$\widehat{M} \cap \boldsymbol{T}' \subset M$, where $\boldsymbol{T}' := \boldsymbol{GK}((A_j, D_j, \Sigma_j \cup P_j)_{j=1}^{\infty})$,*
- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T}' \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *if each set $S_{j,k,a}$ is thin in $V_{j,k,a}$, then $\widehat{M}$ is analytic.*

Proof. (Cf. the proof of Theorem 5.4.12.)

Step 1: We may assume that each set $S_{j,k,a}$ is singular with respect to the family $\{\widetilde{f}_{j,k,a} : f \in \mathcal{F}\}$. In particular, $S_{j,k,a} \cap \boldsymbol{c}(\boldsymbol{T}_k) \subset M$ and $\widetilde{f}_{j,k,a} = f$ on $V_{j,k,a} \cap \boldsymbol{c}(\boldsymbol{T}_k) \setminus M$.

Step 2: If $N \geq 4$, then for any $1 \leq \mu < \nu \leq N$, define an auxiliary $(N-2)$–fold cross

$$\boldsymbol{Y}_{\mu,\nu} := \boldsymbol{K}((A_j, D_j)_{j \in \{1, \dots, \mu-1, \mu+1, \dots, \nu-1, \nu+1, \dots, N\}}).$$

We may assume that the number $r = r_{k,a}$ is so small that

$$\widehat{\mathbb{P}}((a_1, \dots, a_{\mu-1}, a_{\mu+1}, \dots, a_{\nu-1}, a_{\nu+1}, \dots, a_N), r) \subset \widehat{\boldsymbol{Y}}_{\mu,\nu}, \quad 1 \le \mu < \nu \le N.$$

Step 3: Fix a $k \in \mathbb{N}$. Put

$$V_k := \bigcup_{a \in \Xi_k} V_{j,k,a}, \quad S_k := \bigcup_{a \in \Xi_k} S_{j,k,a}, \quad \widetilde{f}_k := \bigcup_{a \in \Xi_k} \widetilde{f}_{j,k,a}.$$

Then $\boldsymbol{T}_k \subset V_k$ (the same proof as in Step 3 of Theorem 5.4.12).

The main problem is to show that

(*) for arbitrary $a, b \in \Xi_k$, $i, j \in \{1, \dots, N\}$ with

$$W_{i,j,k,a,b} := V_{i,k,a} \cap V_{j,k,b} \neq \varnothing$$

we have $\widetilde{f}_{i,k,a} = \widetilde{f}_{j,k,b}$ on $W_{i,j,k,a,b} \setminus (S_{i,k,a} \cup S_{j,k,b})$ for all $f \in \mathcal{F}$.

The proof of (*) is analogous as in the proof of Theorem 5.4.12 (and Theorem 5.3.4 for $N = 2$).

Step 4: Since the sets $S_{i,k,a}$ and $S_{j,k,b}$ are singular, we conclude that

$$S_{i,k,a} \cap W_{i,j,k,a,b} = S_{j,k,b} \cap W_{i,j,k,a,b},$$

which implies that the function $\widetilde{f}_k$ is well defined on $V_k \setminus S_k$. Moreover,

- S_k is a relatively closed pluripolar subset of V_k,
- $S_k \cap \boldsymbol{c}(\boldsymbol{T}_k) \subset M$,
- $\widetilde{f}_k \in \mathcal{O}(V_k \setminus S_k)$,
- $\widetilde{f}_k = f$ on $\boldsymbol{c}(\boldsymbol{T}_k) \setminus M$,
- S_k is singular with respect to the family $\{\widetilde{f}_k : f \in \mathcal{F}\}$,
- S_k is analytic provided that each set $S_{j,k,a}$ is analytic.

Let U_k denote the union of all connected component of $V_k \cap \widehat{\boldsymbol{X}}_k$ that intersect $\boldsymbol{T}_k$. Then $\widehat{\boldsymbol{X}}_k$ is the envelope of holomorphy of U_k ((the same proof as in Step 3 of Theorem 5.4.12). By virtue of Theorem 4.11.1(c), there exists a relatively closed pluripolar set $\widehat{M}_k$ of $\widehat{\boldsymbol{X}}_k$, $\widehat{M}_k \cap U_k \subset S_k$, such that $\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k$ is the envelope of holomorphy of $U_k \setminus S_k$. Moreover, if S_k is analytic, then so is $\widehat{M}_k$. In particular, for each $f \in \mathcal{F}$ there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k)$ with $\widehat{f}_k = \widetilde{f}_k$ on $U_k \setminus S_k$. We may assume that $\widehat{M}_k$ is singular with respect to the family $\{\widehat{f}_k : f \in \mathcal{F}\}$. In particular, $\widehat{M}_{k+1} \cap \widehat{\boldsymbol{X}}_k = \widehat{M}_k$. Recall that $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$. Consequently:

- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{\boldsymbol{X}}$ with $\widehat{M} \cap \boldsymbol{c}(\boldsymbol{T}) \subset M$,
- for each $f \in \mathcal{F}$, the function $\widehat{f} := \bigcup_{k=1}^{\infty} \widehat{f}_k$ is holomorphic on $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ with $\widehat{f} = f$ on $\boldsymbol{c}(\boldsymbol{T}) \setminus M$,
- $\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,
- if each set $S_{j,k,a}$ is analytic in $V_{j,k,a}$, then $\widehat{M}$ is analytic.

Observe that by Propositions 3.9.5 and 3.3.27, for any $j \in \{1, \dots, N\}$ there exists a pluripolar set $Q_j \subset A'_j \times A''_j \setminus \Sigma_j$ such that for all $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus (\Sigma_j \cup Q_j)$, the fiber $\widehat{M}_{(a'_j,\cdot,a''_j)}$ is singular with respect to the family $\{\widehat{f}(a'_j, \cdot, a''_j) : f \in \mathcal{F}\}$.

For any $k \in \mathbb{N}$ and $j \in \{1, \dots, N\}$ exists a countable set $I_k \subset \Xi_k$ such that $\Xi_k \subset \bigcup_{a \in I_k} \mathbb{P}(a, r_{k,a})$. Put $P_j := \bigcup_{k=1}^{\infty} \bigcup_{a \in I_k} P_{j,k,a} \in \mathcal{PLP}$.

Our construction shows that $\widehat{f}(a'_j, \cdot, a''_j) = f(a'_j, \cdot, a''_j)$ on $D_j \setminus M_{(a'_j,\cdot,a''_j)}$ for $(a'_j, a''_j) \in A'_j \times A''_j \setminus (\Sigma_j \cup P_j)$. Hence $\widehat{M}_{(a'_j,\cdot,a'_j)} \subset M_{(a'_j,\cdot,a''_j)}$ for $(a'_j, a''_j) \in A'_j \times A''_j \setminus (\Sigma_j \cup Q_j \cup P_j)$. Thus we only need to put $\Sigma'_j := \Sigma_j \cup Q_j \cup P_j$.

This completes the proof of Theorem 7.1.2. □

We move to the main proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. We are going to apply Theorem 7.1.2.

Assume for simplicity that $j = N$. For each $a \in \boldsymbol{c}(\boldsymbol{T}) \setminus M$ let $\mathbb{P}(a, r(a))$ and $\widetilde{f}_a \in \mathcal{O}(\mathbb{P}(a, r(a))$ be such that $\widetilde{f}_a = f$ on $\mathbb{P}(a, r) \cap (\boldsymbol{T} \setminus M)$, $f \in \mathcal{F}$.

For each $b'_N \in A'_N \setminus \Sigma_N$, let $\widetilde{M}_{N,b'_N}$ be the singular part of $M_{(b'_N,\cdot)}$ with respect to the family $\{f(b'_N, \cdot) : f \in \mathcal{F}\}$ (taken in the sense of § 3.1.8) and let $\widetilde{f}_{N,b'_N}$ stand for the holomorphic extension of $f(b'_N, \cdot)$ to $D_N \setminus \widetilde{M}_{N,b'_N}$. Observe that $\widetilde{f}_{N,b'_N} = \widetilde{f}_a(b'_N, \cdot)$ on $\mathbb{P}(a_N, r(a)) \setminus \widetilde{M}_{N,b'_N}$, because $\widetilde{f}_{N,b'_N} = f(b'_N, \cdot) = \widetilde{f}_a(b'_N, \cdot)$ on $\mathbb{P}(a_N, r(a)) \setminus M_{(b'_N,\cdot)}$. In particular, $\widetilde{M}_{N,b'_N} \cap \mathbb{P}(a_N, r(a)) = \varnothing$.

We are going to apply Lemma 3.9.6 with:

- $k := n_1 + \dots + n_{N-1}$, $\ell := n_N$,
- $D := \mathbb{P}(a'_N, r(a))$, $G_0 := \mathbb{P}(a_N, r(a))$, $G := D_N$,
- $A := (A'_N \setminus \Sigma_N) \cap \mathbb{P}(a'_N, r(a))$,
- $M(b'_N) := \widetilde{M}_{N,b'_N}$, $b'_N \in A$.

Notice that $\{\widetilde{f}_a : f \in \mathcal{F}\} \subset \mathcal{S}$, where $\mathcal{S}$ is defined in Lemma 3.9.6. Moreover, for every $b'_N \in A$ the set $M(b'_N)$ is singular with respect to the family $\{\widehat{g}(b'_N, \cdot) : g \in \mathcal{S}\}$, because it is singular with respect to the subfamily $\{\widetilde{f}_{N,b'_N} : f \in \mathcal{F}\}$.

Consequently, by Lemma 3.9.6, there exists a pluripolar set $P = P_{N,a}$ such that the set

$$M_{N,a} := \bigcup_{b'_N \in A \setminus P} \{b'_N\} \times M(b'_N)$$

is relatively closed in $(A \setminus P) \times G$.

Consider the special 2–fold crosses

$$\boldsymbol{Y}_{N,a} := \boldsymbol{K}(A \setminus P, G_0; D, G) = (D \times G_0) \cup ((A \setminus P) \times G) \subset (D \times G_0) \cup \boldsymbol{T}.$$

Since $M_{N,a}$ is relatively closed in $\boldsymbol{Y}_{N,a}$, we may apply Theorem 5.3.1 for the family $\mathcal{F}_{N,a} := \{g \in \mathcal{O}_s(\boldsymbol{Y}_{N,a} \setminus M_{N,a}) : \exists_{f \in \mathcal{F}} :\ g = \widetilde{f}_a \text{ on } D \times G_0\} \subset \mathcal{O}_s(\boldsymbol{Y}_{N,a} \setminus M_{N,a})$, and consequently, we get a relatively closed pluripolar set $S_{N,a} \subset \widehat{\boldsymbol{Y}}_{N,a}$ such that:

- $S_{N,a} \cap \boldsymbol{Z}_{N,a} \subset M_{N,a} \subset M$,
- for any function $f \in \mathcal{F}_{N,a}$ there exists an $\widehat{f}_{N,a} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N,k,a} \setminus S_{N,k,a})$ such that $\widehat{f}_{N,a} = f$ on $\boldsymbol{Z}_{N,k,a} \setminus M_{N,a}$,

- $S_{N,k,a}$ is singular with respect to the family $\{\widehat{f}_{N,a} : f \in \mathcal{F}_{N,a}\}$; in particular, $S_{N,a} \cap \mathbb{P}(a, r(a)) = \varnothing$,
- if all the fibers $M(b'_N)$, $b'_N \in A \setminus P$, are thin (in fact, analytic), then $S_{N,a}$ is analytic.

Note that $\{a'_N\} \times G \subset \widehat{\boldsymbol{Y}}_{N,a}$. Let $(D_{N,k})_{k=1}^{\infty}$ be an exhaustion of D_N by domains of holomorphy. Then for every $k \in \mathbb{N}$ there exists a $\rho = r_{N,k,a} \in (0, r(a))$ so small that $V_{N,k,a} := \mathbb{P}(a'_N, \rho) \times D_{N,k} \subset \widehat{\boldsymbol{Y}}_{N,a}$. Define $S_{N,k,a} := S_{N,a} \cap V_{N,k,a}$ and $\widetilde{f}_{N,k,a} := \widehat{f}_{N,a}|_{V_{N,k,a} \setminus S_{N,k,a}}$.

Obviously, an analogous construction may be dome for $j \in \{1, \ldots, N-1\}$. This is shows that all the assumptions of Theorem 7.1.2 are satisfied. $\square$

Chapter 8

Discs method

Chapter 9
Boundary cross theorems

Chapter 10
Generalized Hartogs theorems

Symbols

General symbols

$\mathbb{N} :=$ the set of natural numbers, $0 \notin \mathbb{N}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{N}_k := \{n \in \mathbb{N} : n \geq k\}$;
$\mathbb{Z} :=$ the ring of integer numbers;
$\mathbb{Q} :=$ the field of rational numbers;
$\mathbb{R} :=$ the field of real numbers;
$\mathbb{C} :=$ the field of complex numbers;
$\operatorname{Re} z :=$ the real part of $z \in \mathbb{C}$, $\operatorname{Im} z :=$ the imaginary part of $z \in \mathbb{C}$;
$\overline{z} := x - iy =$ the conjugate of $z = x + iy$;
$|z| := \sqrt{x^2 + y^2} =$ the modulus of a complex number $z = x + iy$;
$A^n :=$ the Cartesian product of n copies of the set A, e.g. $\mathbb{C}^n$;
set $A \subset \mathbb{C}$;
$x \leq y :\Longleftrightarrow x_j \leq y_j,\ j = 1, \dots, n,\ x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$;
$A_* := A \setminus \{0\}$, e.g. $\mathbb{C}_*, (\mathbb{C}^n)_*$; $A^n_* := (A_*)^n$, eg. $\mathbb{C}^n_*$;
$A_+ := \{a \in A : a \geq 0\}$, e.g. $\mathbb{Z}_+$, $\mathbb{R}_+$; $A^n_+ := (A_+)^n$, e.g. $\mathbb{Z}^n_+$, $\mathbb{R}^n_+$;
$A_- := \{a \in A : a \leq 0\}$;
$A_{>0} := \{a \in A : a > 0\}$, e.g. $\mathbb{R}_{>0}$; $A^n_{>0} := (A_{>0})^n$, e.g. $\mathbb{R}^n_{>0}$;
$A_{<0} := \{a \in A : a < 0\}$;
$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R}$, $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$;
$A + B := \{a + b : a \in A,\ b \in B\}$, $a + B := \{a\} + B$, $A, B \subset \boldsymbol{X}$, $a \in \boldsymbol{X}$, $\boldsymbol{X}$ is a vector space;
$A \cdot B := \{a \cdot b : a \in A,\ b \in B\}$, $A \subset \mathbb{C}, B \subset \mathbb{C}^n$;
$\delta_{j,k} := \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} =$ the Kronecker symbol;
$\boldsymbol{e} = (e_1, \dots, e_n) :=$ the canonical basis in $\mathbb{C}^n$, $e_j := (\delta_{j,1}, \dots, \delta_{j,n})$, $j = 1, \dots, n$;
$\mathbf{1} = \mathbf{1}_n := (1, \dots, 1) \in \mathbb{N}^n$; $\mathbf{2} := 2 \cdot \mathbf{1} = (2, \dots, 2) \in \mathbb{N}^n$;

$\langle z, w\rangle := \sum_{j=1}^n z_j \overline{w}_j =$ the Hermitian scalar product in $\mathbb{C}^n$;
$\overline{w} := (\overline{w}_1, \dots, \overline{w}_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$z \cdot w := (z_1 w_1, \dots, z_n w_n)$, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$e^z := (e^{z_1}, \dots, e^{z_n})$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$;
$\|z\| := \langle z, z\rangle^{1/2} = \left(\sum_{j=1}^n |z_j|^2\right)^{1/2} =$ the Euclidean norm in $\mathbb{C}^n$;
$\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\} =$ the maximum norm in $\mathbb{C}^n$;
$\|z\|_1 := |z_1| + \dots + |z_n| =$ the ℓ^1–norm in $\mathbb{C}^n$;

$\#A :=$ the number of elements of A;
$\operatorname{diam} A :=$ the diameter of the set $A \subset \mathbb{C}^n$ with respect to the Euclidean distance;
$\operatorname{conv} A :=$ the convex hull of the set A;
$A \Subset X$ $:\Longleftrightarrow A$ is relatively compact in X;

$\mathrm{pr}_X : X \times Y \longrightarrow X$, $\mathrm{pr}_X(x, y) := x$, or $\mathrm{pr}_X : X \oplus Y \longrightarrow X$, $\mathrm{pr}_X(x + y) := x$;

Euclidean balls:

$\mathbb{B}(a, r) = \mathbb{B}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ = the open Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{B}_n(a, 0) := \varnothing$; $\mathbb{B}(a, +\infty) := \mathbb{C}^n$;
$\overline{\mathbb{B}}(a, r) = \overline{\mathbb{B}}_n(a, r) := \overline{\mathbb{B}_n(a, r)} = \{z \in \mathbb{C}^n : \|z - a\| \leq r\}$ = the closed Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\overline{\mathbb{B}}_n(a, 0) := \{a\}$;
$\mathbb{B}(r) = \mathbb{B}_n(r) := \mathbb{B}_n(0, r)$; $\overline{\mathbb{B}}(r) = \overline{\mathbb{B}}_n(r) := \overline{\mathbb{B}}_n(0, r)$;
$\mathbb{B} = \mathbb{B}_n := \mathbb{B}_n(1)$ = the unit Euclidean ball in $\mathbb{C}^n$;
$K(a, r) := \mathbb{B}_1(a, r)$; $K(r) := K(0, r)$;
$\overline{K}(a, r) := \overline{\mathbb{B}}_1(a, r)$; $\overline{K}(r) := \overline{K}(0, r)$;
$K_*(a, r) := K(a, r) \setminus \{a\}$; $K_*(r) := K_*(0, r)$;
$\mathbb{D} := K(1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ = the unit disc;
$\mathbb{T} := \partial\mathbb{D}$;

Polydiscs:

$\mathbb{P}(a, r) = \mathbb{P}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\|_\infty < r\}$ = the polydisc with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{P}_n(a, +\infty) := \mathbb{C}^n$;
$\overline{\mathbb{P}}(a, r) = \overline{\mathbb{P}}_n(a, r) := \overline{\mathbb{P}_n(a, r)}$; $\overline{\mathbb{P}}_n(a, 0) := \{a\}$;
$\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0, r)$;
$\mathbb{P}_n := \mathbb{P}_n(1) = \mathbb{D}^n$ = the unit polydisc in $\mathbb{C}^n$;

$\mathbb{P}(a, \boldsymbol{r}) = \mathbb{P}_n(a, \boldsymbol{r}) := K(a_1, r_1) \times \cdots \times K(a_n, r_n)$ = the polydisc with center $a \in \mathbb{C}^n$ and multiradius (polyradius) $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$; notice that $\mathbb{P}(a, r) = \mathbb{P}(a, r \cdot \mathbf{1})$;
$\mathbb{P}(\boldsymbol{r}) = \mathbb{P}_n(\boldsymbol{r}) := \mathbb{P}_n(0, \boldsymbol{r})$;
$\partial_0\mathbb{P}(a, \boldsymbol{r}) := \partial K(a_1, r_1) \times \cdots \times \partial K(a_n, r_n)$ = the distinguished boundary of $\mathbb{P}(a, \boldsymbol{r})$;

Annuli:

$\mathbb{A}(a, r^-, r^+) := \{z \in \mathbb{C} : r^- < |z - a| < r^+\}$, $a \in \mathbb{C}$, $-\infty \leq r^- < r^+ \leq +\infty$, $r^+ > 0$; if $r^- < 0$, then $\mathbb{A}(a, r^-, r^+) = K(a, r^+)$; $\mathbb{A}(a, 0, r^+) = K(a, r^+) \setminus \{a\}$;
$\mathbb{A}(r^-, r^+) := \mathbb{A}(0, r^-, r^+)$;

Laurent series:

$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ $(0^0 := 1)$;
$\alpha! := \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$;
$|\alpha| := |\alpha_1| + \cdots + |\alpha_n|$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$;
$\binom{\alpha}{\beta} := \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}$, $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}_+$;
$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_+$;

Functions:

$\|f\|_A := \sup\{|f(a)| : a \in A\}$, $f : A \longrightarrow \mathbb{C}$;
$\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$ = the support of f;
$\mathcal{P}(\mathbb{C}^n)$:= the space of all polynomials $f : \mathbb{C}^n \longrightarrow \mathbb{C}$;
$\mathcal{P}_d(\mathbb{C}^n) := \{F \in \mathcal{P}(\mathbb{C}^n) : \deg F \leq d\}$;
$\mathcal{C}^\uparrow(\Omega)$:= the set of all upper semicontinuous functions $u : \Omega \longrightarrow \mathbb{R}_{-\infty}$;
$\frac{\partial f}{\partial z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) - i\frac{\partial f}{\partial y_j}(a)\Big)$, $\frac{\partial f}{\partial \overline{z}_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) + i\frac{\partial f}{\partial y_j}(a)\Big)$ = the formal partial derivatives of f at a;

$\operatorname{grad} u(a) := (\frac{\partial u}{\partial \overline{z}_1}(a), \dots, \frac{\partial u}{\partial \overline{z}_n}(a))$ = the gradient of u at a;
$D^{\alpha,\beta} := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} \circ (\frac{\partial}{\partial \overline{z}_1})^{\beta_1} \circ \dots \circ (\frac{\partial}{\partial \overline{z}_n})^{\beta_n}$;
$\mathcal{C}^k(X,Y)$:= the space of all $\mathcal{C}^k$–mappings $f : X \longrightarrow Y$, $k \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$ (ω stands for the real analytic case);
$\mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega, \mathbb{C})$;
$\mathcal{C}^k_0(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \operatorname{supp} f \Subset \Omega\}$;
$\mathcal{L}^N$:= Lebesgue measure in $\mathbb{R}^N$;
$L^p(\Omega)$:= the space of all p–integrable functions on Ω;
$\| \|_{L^p(\Omega)}$:= the norm in $L^p(\Omega)$;
$L^p(\Omega, \mathrm{loc})$:= the space of all locally p–integrable functions on Ω;
$\mathcal{O}(X,Y)$:= the space of all holomorphic mappings $f : X \longrightarrow Y$;
$\mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C})$ = the space of all holomorphic functions $f : \Omega \longrightarrow \mathbb{C}$;
$\frac{\partial f}{\partial z_j}(a) := \lim\limits_{\mathbb{C}_* \ni h \to 0} \frac{f(a+he_j)-f(a)}{h}$ = the j-th complex partial derivative of f at a;
$D^\alpha := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n}$ = α–th partial complex derivative;
$L^p_h(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega)$ = the space of all p–integrable holomorphic functions on Ω;
$\mathcal{H}^\infty(\Omega)$:= the space of all bounded holomorphic functions on Ω;
$\mathcal{H}(\Omega)$:= the space of all harmonic functions on Ω, $\Omega \subset \mathbb{C}$;
$\mathcal{SH}(\Omega)$:= the set of all subharmonic functions on Ω, $\Omega \subset \mathbb{C}$;
$\mathcal{PSH}(X)$:= the set of all plurisubharmonic functions on X;
$\mathcal{L}u(a;\xi) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a)\xi_j\overline{\xi}_k$ = the Levi form of u at a.

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