POLYTOPES WITH MASS LINEAR FUNCTIONS, PART I

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Abstract. We analyze mass linear functions on simple polytopes $\Delta$, where a mass linear function is an affine function on $\Delta$ whose value on the center of mass depends linearly on the positions of the supporting hyperplanes. On the one hand, we show that certain types of symmetries of $\Delta$ give rise to nonconstant mass linear functions on $\Delta$. We call mass linear functions which arise in this way inessential; the others are called essential. On the other hand, we show that most polytopes do not admit any nonconstant mass linear functions. Further, if every affine function is mass linear on $\Delta$, then $\Delta$ is affine equivalent to a product of simplices. Our main result is a classification of all 2-dimensional simple polytopes and 3-dimensional smooth polytopes which admit nonconstant mass linear functions. In particular there is only one family of smooth polytopes of dimension $\leq 3$ which admit essential mass linear functions. In part II, we will complete this classification in the 4-dimensional case.

These results have geometric implications. Fix a symplectic toric manifold $(M, \omega, T, \Phi)$ with moment polytope $\Delta = \Phi(M)$. Let $\text{Symp}_0(M, \omega)$ denote the identity component of the group of symplectomorphisms of $(M, \omega)$. Any linear function $H$ on $\Delta$ generates a Hamiltonian $\mathbb{R}$ action on $M$ whose closure is a subtorus $T_H$ of $T$. We show that if the map $\pi_1(T_H) \to \pi_1(\text{Symp}_0(M, \omega))$ has finite image, then $H$ is mass linear. By the claims described above, this implies that in most cases the induced map $\pi_1(T) \to \pi_1(\text{Symp}_0(M, \omega))$ is an injection. Moreover, the map does not have finite image unless $M$ is a product of projective spaces. Note also that there is a natural maximal compact connected subgroup $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$; there is a natural compatible complex structure $J$ on $M$, and $\text{Isom}_0(M)$ is the identity component of the group of symplectomorphisms that also preserve this structure. We prove that if the polytope $\Delta$ supports no nontrivial essential mass linear functions, then the induced map $\pi_1(\text{Isom}_0(M)) \to \pi_1(\text{Symp}_0(M, \omega))$ is injective. Therefore, this map is injective for all 4-dimensional symplectic toric manifolds. It is also injective in the 6-dimensional case unless $M$ is a $\mathbb{C}P^2$ bundle over $\mathbb{C}P^1$.

with Appendix written with V. Timorin

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1. Introduction

To begin, we shall give some basic definitions; we then state the main results. These are elaborated in §1.1; their geometric implications are described in §1.2.

Let \( t \) be a real vector space, let \( t^* \) denote the dual space, and let \( \langle \cdot, \cdot \rangle : t \times t^* \to \mathbb{R} \) denote the natural pairing. Let \( A \) be an affine space modeled on \( t^* \). A (convex) polytope \( \Delta \subset A \) is the bounded intersection of a finite set of affine half-spaces. Hence, \( \Delta \) can be written

\[
\Delta = \bigcap_{i=1}^{N} \{ x \in A \mid h_i(x) \leq \kappa_i \},
\]

where the \( h_i \) are affine functions on \( A \) and the support numbers \( \kappa_i \) lie in \( \mathbb{R} \) for all \( 1 \leq i \leq N \); the outward conormals are the unique vectors \( \eta_i \in t \) so that \( h_i(y) - h_i(x) = \langle \eta_i, y - x \rangle \) for all \( x \) and \( y \) in \( A \).

In this paper, we will always assume that \( A \) is the affine span of \( \Delta \) and that the span of each facet \( F_i := \Delta \cap \{ x \in A \mid h_i(x) = \kappa_i \} \) is a hyperplane. Moreover, the polytopes we consider are simple, that is, \( \dim t \) facets meet at every vertex.

We sometimes need stronger assumptions: Given an integer lattice \( \ell \subset t \), a polytope \( \Delta \subset A \) is rational if we can choose each outward conormal to lie in \( \ell \). A rational polytope is smooth (or Delzant) if, for each vertex of \( \Delta \), the primitive outward conormals to the facets which meet at that vertex form a basis for \( \ell \). Here, a vector \( \eta \in \ell \) is primitive if it is not a positive integer multiple of any other lattice element.
The chamber $C_\Delta$ of $\Delta$ is the set of $\kappa' \in \mathbb{R}^N$ so that the polytope

$$(1.2) \quad \Delta' = \Delta(\kappa') := \bigcap_{i=1}^N \{ x \in A \mid h_i(x) \leq \kappa'_i \}$$

is analogous to $\Delta$, that is, so that the intersection $\bigcap_{i \in I} F'_i$ is empty exactly if the intersection $\bigcap_{i \in I} F_i$ is empty for all $I \subset \{1, \ldots, N\}$. Since $\Delta$ is simple, $C_\Delta$ is a nonempty connected open set.

**Figure 1.1.** The polytope $Y_a$ constructed in Example 1.1.

**Example 1.1.** Let $\{e_i\}_{i=1}^n$ denote the standard basis for $\mathbb{R}^n$. Given $a = (a_1, a_2) \in \mathbb{R}^2$, define

$\eta_1 = -e_1$, $\eta_2 = -e_2$, $\eta_3 = e_1 + e_2$, $\eta_4 = -e_3$, $\eta_5 = e_3 + a_1 e_1 + a_2 e_2$, and

$${\mathcal C}_a = \left\{ \kappa \in \mathbb{R}^5 \mid \sum_{i=1}^3 \kappa_i > 0 \text{ and } \kappa_4 + \kappa_5 > -a_1 \kappa_1 - a_2 \kappa_2 + \max(0, a_1, a_2) \sum_{i=1}^3 \kappa_i \right\}.$$ 

Given $\kappa \in {\mathcal C}_a$, let

$$Y = Y_a(\kappa) = \bigcap_{i=1}^5 \{ x \in (\mathbb{R}^3)^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}.$$ 

It is illustrated in Figure 1.1, and has vertices

$$(-\kappa_1, -\kappa_2, -\kappa_4), (\kappa_2 + \kappa_3, -\kappa_2, -\kappa_4), (-\kappa_1, \kappa_1 + \kappa_3, -\kappa_4), (-\kappa_1, -\kappa_2, \kappa_5 + a_1 \kappa_1 + a_2 \kappa_2),$$

$$\left( \kappa_2 + \kappa_3, -\kappa_2, \kappa_5 - a_1 (\kappa_2 + \kappa_3) + a_2 \kappa_2 \right), \text{ and }$$

$$(-\kappa_1, \kappa_1 + \kappa_3, \kappa_5 + a_1 \kappa_1 - a_2 (\kappa_1 + \kappa_3)).$$

One can simplify the formulas here by translating $Y$ so that its first vertex is at the origin. Then $\kappa_1 = \kappa_2 = \kappa_4 = 0$ and we see that $Y_a(\kappa)$ really depends only on two parameters, $\kappa_3$ and $\kappa_5$. The polytope $Y_a(\kappa)$ is simple; it is smooth exactly if $a \in \mathbb{Z}^2$; and its chamber is $C_a$. 
Let \( c_\Delta : C_\Delta \to A \) denote the center of mass of \( \Delta \) (with respect to any affine identification of \( A \) with Euclidean space), considered as a function of the support numbers \( \kappa := (\kappa_1, \ldots, \kappa_N) \).

**Definition 1.2.** Let \( \Delta \subset A \) be a simple polytope. Let \( c_\Delta : C_\Delta \to A \) denote the center of mass of \( \Delta \). A **mass linear function** on \( \Delta \) is an affine function \( H : A \to \mathbb{R} \) so that \( H \circ c_\Delta : C_\Delta \to \mathbb{R} \) is linear, that is,

\[
H(c_\Delta(\kappa)) = \sum_{i=1}^{N} \gamma_i \kappa_i + C \quad \forall \ \kappa \in C_\Delta,
\]

where \( C \in \mathbb{R} \) and \( \gamma_i \in \mathbb{R} \) for all \( 1 \leq i \leq N \); we call \( \gamma_i \) the coefficient of the support number \( \kappa_i \) (in \( H \circ c_\Delta \)). In this situation, we also say that \( (\Delta, H) \) is a **mass linear pair**.

In practice, we usually choose an identification of \( A \) with the linear space \( t^* \) and consider \( H \) which are elements of the dual space \( t \); cf. Remarks 1.6 and 1.7.

Our primary goal in this paper is to analyze mass linear functions on simple polytopes. On the one hand, we show that certain types of symmetries of \( \Delta \) give rise to nonconstant mass linear functions on \( \Delta \). We call mass linear functions that arise in this way inessential (see Definition 1.13); the others are called essential. On the other hand, we show that most polytopes do not admit any nonconstant mass linear functions. Further, if every affine function is mass linear on \( \Delta \), then \( \Delta \) is affine equivalent to a product of simplices. In particular, the polytopes constructed in Example 1.1 are the only smooth polytopes of dimension \( \leq 3 \) which admit essential mass linear functions. In part II, we will complete this classification in the 4-dimensional case.

**Theorem 1.3.** A simple 2-dimensional polytope supports a nonconstant mass linear function exactly if it is a triangle, a trapezoid or a parallelogram. All such functions are inessential. The same statement holds in the smooth case.

**Theorem 1.4.** Let \( \Delta \) be a smooth 3-dimensional polytope. If there exists an essential mass linear function on \( \Delta \), then \( \Delta \) is affine equivalent to the polytope \( Y_a(\kappa) \) constructed in Example 1.1 for some \( a \in \mathbb{Z}^2 \) and \( \kappa \in C_Y \). Conversely, for any \( a \in \mathbb{Z}^2 \) there exists an essential mass linear function on the polytope \( Y_a \) exactly if \( a_1 a_2(a_1 - a_2) \neq 0 \).

We prove the second statement in Theorem 1.4 by a direct calculation; see Proposition 4.6. However the first claim relies on many of our other results.

These results have geometric implications. Fix a symplectic toric manifold \((M, \omega, T, \Phi)\) with moment polytope \( \Delta = \Phi(M) \). (See §1.2.) Let \( \text{Symp}_0(M, \omega) \) denote the identity component of the group of symplectomorphisms of \((M, \omega)\). Any linear function \( H \) on \( \Delta \) generates a Hamiltonian \( \mathbb{R} \) action on \( M \) whose closure is a subtorus \( T_H \) of \( T \). We show that if the map \( \pi_1(T_H) \to \pi_1(\text{Symp}_0(M, \omega)) \) has finite image, then \( H \) is mass linear. By the claims described above, this implies that in most cases the induced map \( \pi_1(T) \to \pi_1(\text{Symp}_0(M, \omega)) \)
is an injection. Moreover, the map does not have finite image unless $M$ is a product of projective spaces.

Note also that there is a natural maximal compact connected subgroup $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$; on $M$ and hence an associated metric $g_J := \omega(\cdot, J\cdot)$, and we define $\text{Isom}_0(M)$ to be the identity component of the group of isometries of the Riemannian manifold $(M, g_J)$. It is well known that $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$. By Theorem 1.24, if $\Delta$ admits no essential mass linear functions, then the natural map from $\pi_1(\text{Isom}_0(M))$ to $\pi_1(\text{Symp}_0(M, \omega))$ is an injection. Since the polytopes constructed in Example 1.1 are exactly the moment polytopes of $\mathbb{C}P^2$ bundles over $\mathbb{C}P^1$ (see Example 5.3), Theorems 1.3 and 1.4 have the following consequence.

**Theorem 1.5.** Let $(M, \omega, T, \Phi)$ be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. If $M$ is 2-dimensional, or if $M$ is 3-dimensional but is not a $\mathbb{C}P^2$ bundle over $\mathbb{C}P^1$, then the natural map from $\pi_1(\text{Isom}_0(M))$ to $\pi_1(\text{Symp}_0(M, \omega))$ is an injection.

We end this subsection with some important technical remarks.

**Remark 1.6.** The group of affine transformations of $A$ acts on simple polytopes $\Delta \subset A$ and affine functions $H: A \to \mathbb{R}$ by $a \cdot (\Delta, H) = (a(\Delta), H \circ a^{-1})$. In this case, we will say that $(\Delta, H)$ and $(a(\Delta), H \circ a^{-1})$ are affine equivalent. When we restrict to smooth polytopes we will only allow affine transformations $a: A \to A$ such that the associated linear transformation $\bar{a}: t^* \to t^*$ induces an automorphism of the integral lattice; here, $\bar{a}$ is the unique linear map so that $\bar{a}(x - y) = a(x) - a(y)$ for all $x$ and $y$ in $t^*$.

Except for the concepts of “volume” and “moment”, all the notions considered in this paper are affine invariant. For example, since $a(c_\Delta) = c_{a(\Delta)}$, $H$ is mass linear on $\Delta$ exactly if $H \circ a^{-1}$ is mass linear on $a(\Delta)$. Therefore, by identifying $A$ with $t^*$, we may restrict our attention to polytopes $\Delta \subset t^*$. Given two affine equivalent polytopes $\Delta$ and $\Delta'$, we will usually simply say that they are the same and write $\Delta = \Delta'$. For example, whether we work with simple or smooth polytopes, there is a unique equivalence class of $k$-dimensional polytopes with $k + 1$ facets, namely the $k$-simplex $\Delta_k$. (See Example 1.14.) Hence we say that every such polytope is $\Delta_k$. These conventions mean that, for example, the interpretation of the word “is” in Theorem 1.3 depends on whether we are considering smooth polytopes or simple ones.

There is another much weaker notion that is sometimes useful. Two polytopes $\Delta$ and $\Delta'$ are said to be combinatorially equivalent if there exists a bijection of facets $F_i \leftrightarrow F_i'$ so that $\bigcap_{i \in I} F_i \neq \emptyset$ exactly if $\bigcap_{i \in I} F_i' \neq \emptyset$ for all $I \subset \{1, \ldots, N\}$. We will never use this equivalence relation without explicitly saying so.

**Remark 1.7.** Each outward conormal of a simple polytope $\Delta \subset A$ is only well defined up to multiplication by a positive constant. (See Remark 1.12 for an important normalization convention.) The chamber $C_\Delta$ of $\Delta$ depends on these choices but the notion of mass linearity does not – it is well defined. On the other hand, if $H$ is mass linear the coefficients of the support numbers depend on the choice of the outward conormals.
Even once these outward conormals are determined, the affine functions $h_i$ of Equation (1.1) are only well defined up to addition by a constant. However, these choices have no significant consequence. In particular, if $\Delta \subset \mathfrak{t}^*$ we may assume that the $h_i$ are homogeneous. Similarly, we may restrict our attention to homogeneous linear functions $H \in \mathfrak{t}$.

1.1. Additional results on polytopes. Let us begin with a few definitions.

Definition 1.8. Fix an affine function $H : A \rightarrow \mathbb{R}$ and a simple polytope $\Delta \subset A$. We say that the facet $F$ is symmetric (or $H$-symmetric$^1$) if $\langle H, c_\Delta \rangle : C_\Delta \rightarrow \mathbb{R}$ does not depend on the support number of $F$. Otherwise we say that $F$ is asymmetric (or $H$-asymmetric). More generally, we say that a face $f$ is symmetric if it is the intersection of symmetric facets.

Definition 1.9. We say that a facet $F$ of a simple polytope $\Delta \subset A$ is pervasive if it meets every other facet of $\Delta$. We say that $F$ is flat if there is a hyperplane in $\mathfrak{t}$ that contains the conormal of every facet (other than $F$ itself) that meets $F$.

For example, in the polytopes constructed in Example 1.1, the triangular facets are flat but the quadrilaterals are not; the pervasive facets are the quadrilaterals.

In §2, we analyze the key features of symmetric and asymmetric facets. On the one hand, we prove that if $H$ is mass linear on $\Delta$, then $H$ is also mass linear on each symmetric face $f$ of $\Delta$. This allows us to analyze mass linear functions on polytopes which have symmetric faces by “induction” on the dimension of the polytope. On the other hand, we prove that if $H$ is a mass linear function on $\Delta$, then asymmetric facets are very special; they must be either pervasive or flat. We immediately conclude that most polytopes do not support any nonconstant mass linear functions.

Theorem 1.10. If $\Delta \subset A$ is a simple polytope that contains no pervasive facets and no flat facets then every mass linear function $H$ on $\Delta$ is constant.

On the other hand, there are some polytopes with nonconstant mass linear functions. The simplest type of such functions – inessential functions – arise from symmetries of the polytope.

Definition 1.11. Let $\Delta \subset A$ be a simple polytope. A symmetry of $\Delta$ is an affine map $a : A \rightarrow A$ so that $a(\Delta) = \Delta$. A robust symmetry of $\Delta$ is a linear map $\hat{a} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ so that for all $\kappa \in C_\Delta$ there exists an affine map $a_\kappa : A \rightarrow A$ such that

1. $a_\kappa$ is a symmetry of $\Delta(\kappa)$, and
2. $\hat{a}$ is the linear map associated to $a_\kappa$.

Finally, we say that two facets $F_i$ and $F_j$ are equivalent, denoted $F_i \sim F_j$, if there exists a robust symmetry $\hat{a}$ so that each $a_\kappa$ takes $F_i$ to $F_j$.

It is clear that this defines an equivalence relationship because the robust symmetries form a subgroup of the group of linear transformations. In §3.1 we give several easily verified criteria for facets to be equivalent. In Proposition 5.5, we show that in the smooth

$^1$When there is no danger of confusion, we will omit the $H$. 
case equivalent facets can also be characterized in terms of the symmetries of the associated toric manifold $M$. Moreover, in this case, $F_i \sim F_j$ exactly if the corresponding divisors in $M$ are homologous; cf. Remark 5.8.

**Remark 1.12.** Let $\Delta \subset A$ be a simple polytope and let $\hat{a}^* : t^* \to t^*$ denote the dual map. Each associated symmetry $a_\kappa$ takes the facet $F_i$ to the facet $F_j$ exactly if $\hat{a}^*(\eta_i)$ is a positive multiple of $\eta_j$. Moreover, since $\Delta$ is bounded, if each $a_\kappa$ takes the facet $F_i$ to itself then $\hat{a}^*(\eta_i) = \eta_i$. Therefore, two robust symmetries agree exactly if they induce the same permutation of the facets. Moreover, we can always choose the outward conormals so that $\hat{a}^*(\eta_i) = \eta_j$ for every robust symmetry $\hat{a}$ whose associated symmetries $a_\kappa$ take $F_i$ to $F_j$. For simplicity, we will always normalize the $\eta_i$ in this way.

If $\Delta \subset t^*$ is a smooth polytope, then by Lemma 3.3 every robust symmetry $\hat{a} : t^* \to t^*$ induces an isomorphism of the integral lattice. Therefore, $\hat{a}^*$ takes each primitive outward conormal to another primitive outward conormal. Hence, when we restrict to smooth polytopes we can (and will) always choose the unique primitive outward conormal to each facet; cf. Remark 3.4.

**Definition 1.13.** Let $\Delta = \bigcap_{i=1}^N \{ x \in t^* | \langle \eta_i, x \rangle \leq \kappa_i \}$ be a simple polytope; let $I$ denote the set of equivalence classes of facets. We say that $H \in t$ is **inessential** iff

$$H = \sum \beta_i \eta_i, \quad \text{where} \quad \beta_i \in \mathbb{R} \forall i \quad \text{and} \quad \sum_{i \in I} \beta_i = 0 \quad \forall I \in I.$$

Otherwise, we say that $H$ is **essential**.

If each equivalence class in $I$ has just one element then every inessential mass linear function on $\Delta$ vanishes. More generally, the set of inessential functions is an $(N - |I|)$-dimensional subspace of $t$; see Lemma 3.8.

**Example 1.14.** As an example, (here and subsequently) let $\Delta_k$ denote the standard $k$-simplex

$$\Delta_k = \{ x \in (\mathbb{R}^k)^* | x_i \geq 0 \forall i \quad \text{and} \quad \sum x_i \leq 1 \};$$

the outward conormals are $\{ \eta_i \}_{i=1}^{k+1}$, where $\eta_i = -e_i$ for all $i \in \{1, \ldots, k\}$ and $\eta_{k+1} = \sum_{i=1}^k e_i$. The linear map $(x_1, \ldots, x_k) \mapsto (-\sum x_i, x_1, \ldots, x_{k-1})$ is a robust symmetry of $\Delta$ whose transpose permutes the conormals $\eta_i$ cyclically. Hence, all $k+1$ facets of $\Delta_k$ are equivalent, and every $H \in \mathbb{R}^k$ is inessential. For example, $e_j = -\eta_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \eta_i$ for all $j \leq k$. Moreover, a direct computation shows that

$$\langle e_j, c_{\Delta_k}(\kappa) \rangle = -\kappa_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \kappa_i;$$

cf. Proposition 1.17.

**Example 1.15.** Consider the square

$$\Delta = \{ x \in (\mathbb{R}^2)^* | -1 \leq x_1 \leq 1 \quad \text{and} \quad -1 \leq x_2 \leq 1 \}.$$
The map \((x_1, x_2) \mapsto (x_2, -x_1)\) is a symmetry of \(\Delta\) which is not robust. In contrast, the map \((x_1, x_2) \mapsto (-x_1, -x_2)\) is a robust symmetry of \(\Delta\). Therefore, each edge is equivalent to the opposite edge, and every \(H \in \mathbb{R}^2\) is inessential.

More generally, consider a product of simplices \(\Delta = \Delta_k \times \Delta_{k'} \subset (\mathbb{R}^k)^* \times (\mathbb{R}^{k'}) = (\mathbb{R}^{k+k'})^*\). This polytope has \(k + k' + 2\) facets. The first \(k + 1\) facets are of the form \(F_i \times \Delta_{k'}\), and are all equivalent to each other. The remaining \(k' + 1\) facets are also equivalent, and are of the form \(\Delta_k \times F_i\). Again, every \(H \in \mathbb{R}^{k+k'}\) is inessential and mass linear;

\[
\langle e_j, c_\Delta(\kappa) \rangle = \begin{cases} 
-\kappa_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \kappa_i & \text{if } j \leq k \\
-\kappa_{j+1} + \frac{1}{k'+1} \sum_{i=1}^{k'+1} \kappa_{k+1+i} & \text{otherwise}
\end{cases}
\]

A nearly identical remark applies if \(\Delta\) is the product of three or more simplices.

Here, the product of simple polytopes \(\tilde{\Delta} \subset \tilde{A}\) and \(\Delta \subset A\) is \(\Delta = \tilde{\Delta} \times \Delta \subset \tilde{A} \times \tilde{A}\). This is a simple polytope; it is smooth exactly if \(\tilde{\Delta}\) and \(\tilde{A}\) are smooth.

**Example 1.16.** The triangular facets of the polytope \(Y = Y_a(\kappa)\) defined in Example 1.1 are equivalent because they are exchanged by the affine reflection \(a_\kappa: (\mathbb{R}^3)^* \rightarrow (\mathbb{R}^3)^*\) defined by

\[
a_\kappa(x_1, x_2, x_3) = (x_1, x_2, -a_1x_1 - a_2x_2 - x_3 + \kappa_5 - \kappa_4).
\]

Hence, \(H = \eta_4 - \eta_5 = -a_1e_1 - a_2e_2 - 2e_3\) is inessential.

Although the definition of an inessential function looks somewhat opaque at first glance, this is one of our most important notions. As we shall see in Corollary 1.27, in the smooth case inessential functions have a very natural interpretation in terms of the geometry of the associated toric manifold. Formula (1.3) is also closely tied to algebraic properties of robust symmetries; cf. Remark 3.5.

We now show that every inessential function is mass linear.

**Proposition 1.17.** Let \(\Delta \subset t^*\) be a simple polytope; let \(\mathcal{I}\) denote the set of equivalence classes of facets. If \(H \in t\) is inessential, write \(H = \sum \beta_i \eta_i\), where the \(\beta_i\) satisfy the conditions of Equation (1.3). Then

\[
\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i.
\]

**Proof.** Assume that \(F_i \sim F_j\). By definition, for all \(\kappa \in C_\Delta\) there exists an affine transformation \(a_\kappa: t^* \rightarrow t^*\) so that \(a_\kappa\) takes \(\Delta(\kappa)\) to itself and takes \(F_i\) to \(F_j\). Since \(a_\kappa(F_i) = F_j\),

\[
\langle \eta_i, x \rangle - \kappa_i = \langle \eta_j, a_\kappa(x) \rangle - \kappa_j \quad \forall x \in t^*;
\]

see Remark 1.12. On the other hand, since \(a_\kappa(\Delta(\kappa)) = \Delta(\kappa)\), \(a_\kappa\) fixes the center of mass \(c_\Delta(\kappa)\). Hence,

\[
\langle \eta_i - \eta_j, c_\Delta(\kappa) \rangle = \kappa_i - \kappa_j.
\]

Finally, every inessential \(H\) is a linear combination of terms of the form \(\eta_i - \eta_j\), where \(F_i \sim F_j\). \qed
Note that, in general, even if \( \sum \beta_i \eta_i \) is mass linear the function \( \langle H, c^\Delta(\kappa) \rangle \) need not equal \( \sum \beta_i \kappa_i \); indeed, the coefficients of the \( \eta_i \) are not uniquely determined by \( H \) while the coefficients of \( \kappa_i \) are. In contrast, as we shall see in Lemma 3.19, if \( H \) is mass linear and \( \langle H, c^\Delta(\kappa) \rangle = \sum \alpha_i \kappa_i \) then \( H = \sum \alpha_i \eta_i \).

In §3, we analyze polytopes with inessential functions. In particular, we show that there are exactly two types of polytopes which admit inessential functions: bundles over the \( k \)-simplex (see Definition 3.9) and \( k \)-fold expansions (see Definition 3.12). We then show that it is possible to use Proposition 1.17 to reduce the number of asymmetric facets that we need to consider; we can subtract an inessential function \( H' \) from a mass linear function \( H \) to find a new mass linear function \( \tilde{H} = H - H' \) with fewer asymmetric facets. For example, if \( H \) is a mass linear function on a simple polytope \( \Delta \) then, after possibly subtracting an inessential function, we may assume that every asymmetric facet is pervasive (Proposition 3.24).

Proposition 1.18. If \( \Delta \) is a simple polytope that contains no pervasive facets then every mass linear function on \( \Delta \) is inessential.

In Example 1.15, we saw that every affine function on a product of simplices is mass linear. Our next theorem shows that these are the only polytopes with this property. The equivalence of conditions (iii) and (iv) below is reminiscent of Nill’s Proposition 2.18 in [14]. However because Nill works only with rational polytopes the results are somewhat different.

Theorem 1.19. Let \( \Delta \) be a simple (or smooth) polytope. The following conditions are equivalent:

(i) every \( H \in t \) is mass linear on \( \Delta \);

(ii) every \( H \in t \) is inessential on \( \Delta \);

(iii) \( \Delta \) is a product of simplices;

(iv) \( \bigcap_{i \in I} F_i = \emptyset \) for every equivalence class of facets \( I \); in particular, \( |I| > 1 \).

In §4, we analyze mass linear functions on low dimensional polytopes. In particular, we prove Theorems 1.3 and 1.4.

In part II, we will give a complete list 4-dimensional polytopes which admit essential mass linear functions. As described above, the main technique is to use “induction” by looking at lower dimensional symmetric faces. In the Appendix, in a joint work with Timorin, we prove the key result that will enable us to begin this process.

Theorem 1.20. Let \( H \in t \) be a mass linear function on a smooth 4-dimensional polytope \( \Delta \subset t^* \). Then there exists an inessential function \( H' \in t \) so that the mass linear function \( \tilde{H} = H - H' \) has the following property: at least one facet of \( \Delta \) is \( \tilde{H} \)-symmetric.

To prove this, we show that every 4-dimensional simple polytope which admits a mass linear function with no symmetric facets is combinatorially equivalent to the product of simplices.

1.2. Geometric motivation. Our motivation for studying mass linear functions on polytopes is geometric; we wish to understand the fundamental group of the group of symplectomorphisms of a symplectic toric manifold. For other approaches to this question see

Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold, where \(M\) is a compact connected manifold of dimension \(2n\), \(\omega\) is a symplectic form on \(M\), \(T = t/\ell\) is a compact \(n\)-dimensional torus, and \(\Phi: M \to t^*\) is a moment map for an effective \(T\) action on \((M, \omega)\). Then the **moment polytope** \(\Delta = \Phi(M) \subset t^*\) is a smooth polytope. Conversely, every smooth polytope \(\Delta \subset t^*\) determines a canonical symplectic toric manifold \((M_\Delta, \omega_\Delta, \Phi_\Delta)\) with moment polytope \(\Delta = \Phi_\Delta(M_\Delta)\). (See §5 for more details.) Finally, any two symplectic toric manifolds with the same moment polytope are equivariantly symplectomorphic [6].

For example, the symplectic manifold associated to the standard \(n\)-simplex \(\Delta_n\) is complex projective space \(\mathbb{CP}^n\) with the standard (i.e. diagonal) \((S^1)^n\) action and suitably normalized Fubini-Study form. More generally, a product of simplices corresponds to a product of projective spaces. Similarly, the polytopes constructed in Example 1.1 correspond to \(\mathbb{CP}^2\) bundles over \(\mathbb{CP}^1\); cf. Example 5.3.

Let \(\text{Symp}_0(M, \omega)\) denote the identity component of the group of symplectomorphisms of \((M, \omega)\). Since the torus \(T\) is a subgroup of \(\text{Symp}_0(M, \omega)\), each \(H \in \ell\) induces a circle \(\Lambda_H \subset T \subset \text{Symp}_0(M, \omega)\). More generally, each \(H \in t\) generates a Hamiltonian \(\mathbb{R}\)-action on \(M\) whose closure is a subtorus \(T_H \subset T \subset \text{Symp}_0(M, \omega)\). The next proposition is proved in §5 by interpreting the quantity \(\langle H, c_\Delta(\kappa) \rangle\) as the value of Weinstein’s action homomorphism \(A_H: \pi_1(\text{Symp}_0(M_\Delta, \omega_\Delta)) \to \mathbb{R}/\mathbb{P}_\omega\).

**Proposition 1.21.** Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold with moment polytope \(\Delta \subset t^*\); fix \(H \in \ell\). If the image of \(\Lambda_H\) in \(\pi_1(\text{Symp}_0(M, \omega))\) is trivial, then

\[
\langle H, c_\Delta(\kappa) \rangle = \sum \alpha_i \kappa_i, \quad \text{where} \quad \alpha_i \in \mathbb{Z} \ \forall i
\]

for all \(\kappa\) in the chamber \(C_\Delta\).

**Definition 1.22.** Given a topological group \(G\), a subtorus \(K \subset G\) is **compressible** in \(G\) if the natural map \(\pi_1(K) \to \pi_1(G)\) has finite image.

**Corollary 1.23.** Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold with moment polytope \(\Delta \subset t^*\); fix \(H \in t\). If \(K \subset T\) is compressible in \(\text{Symp}_0(M, \omega)\), then every \(H \in t \subset t\) is mass linear.

**Theorem 1.24.** Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold with moment polytope \(\Delta\).

1. The map \(\pi_1(T) \to \pi_1(\text{Symp}_0(M, \omega))\) is an injection if there are no mass linear functions \(H \in \ell\) on \(\Delta\).

2. The torus \(T\) is compressible in \(\text{Symp}_0(M, \omega)\) exactly if \((M, \omega)\) is a product of projective spaces with a product symplectic form. In particular, the image of \(\pi_1(T)\) in \(\pi_1(\text{Symp}_0(M, \omega))\) is never zero.

**Proof.** Part (i) is an immediate consequence of Proposition 1.21. The first claim of part (ii) follows by combining Theorem 1.19 with Corollary 1.23. The second claim follows from Proposition 1.21 and Example 1.15. \qed
Remark 1.25. More generally, since “most” polytopes have no nonconstant mass linear functions, the above theorem implies that in most cases the map \( \pi_1(T) \to \pi_1(\text{Symp}_0(M, \omega)) \) is injective. For example, in the 4-dimensional case Theorem 1.3 implies that it is injective if \( \Delta \) has five or more facets, while in the 6-dimensional case Theorem 4.13 implies that it is injective. For example, in the 4-dimensional case Theorem 1.3 implies that it is injective.

Next, we claim that, given any smooth polytope \( \Delta = \Phi(M) \), then there are extra symmetries of \((M, \omega, T, \Phi)\) which preserve the canonical Kähler structure \((\omega, J, g_J)\), where \(g_J\) denotes the associated metric \(\omega(\cdot, J\cdot)\). We shall see in Proposition 5.5 that these robust symmetries in some sense generate \(\text{Isom}_0(M)\), the identity component of the group of isometries of the Riemannian manifold \((M, g_J)\). Moreover, \(\text{Isom}_0(M)\) is a maximal connected compact Lie subgroup of \(\text{Symp}_0(M, \omega)\). For example, if \(M = \mathbb{C}P^n\), \(\text{Isom}_0(M)\) is the projective unitary group \(PU(k + 1)\). Finally, note that \(T\) is also a subgroup of \(\text{Isom}_0(M)\).

Lemma 1.26. Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold, and let \(\text{Isom}_0(M)\) be the identity component of the associated Kähler isometry group. The map \(\pi_1(T) \to \pi_1(\text{Isom}_0(M))\) is surjective.

Moreover, let \(\Delta = \bigcap_{i=1}^N \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}\) be the moment polytope. Let \(\mathcal{I}\) denote the set of equivalence classes of facets of \(\Delta\), and fix \(H \in \ell\). The image of \(\Lambda_H\) in \(\pi_1(\text{Isom}_0(M))\) is trivial exactly if \(H\) can be written

\[
H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{Z} \forall i \text{ and } \sum_{i \in I} \beta_i = 0 \quad \forall I \in \mathcal{I}.
\]

The above lemma motivated the definition of inessential function. Its proof, given in §5, is independent of the intervening sections, except for the discussion of the equivalence relation in §3.1.

Corollary 1.27. Let \((M, \omega, T, \Phi)\) be a symplectic toric manifold, and let \(\text{Isom}_0(M)\) be the identity component of the associated Kähler isometry group. Given \(H \in t\), the torus \(T_H\) is compressible in \(\text{Isom}_0(M)\) exactly if \(H\) is inessential on \(\Delta\).

Proof. The Lie algebra \(t_H\) of \(T_H\) is the smallest rational subspace of \(t\) containing \(H\). Since the set of inessential \(H' \in t\) is clearly a rational subspace of \(t\), this implies that \(H\) is mass linear exactly if every \(H' \in \ell \cap t_H\) is mass linear. Therefore it suffices to prove the result.
for $H \in \ell$. But by Lemma 3.8, $H \in \ell$ is inessential exactly if $H$ can be written $H = \sum \beta_i \eta_i$, where $\beta_i \in \mathbb{Q}$ for all $i$ and $\sum_{i \in I} \beta_i = 0$ for all $I \in \mathcal{I}$. By Lemma 1.26, this implies that $H$ is inessential exactly if there exists a natural number $m$ such that $\Lambda_m H$ contracts in $\text{Isom}_0(M)$, that is, exactly if $\Lambda_H$ has finite order in $\text{Isom}_0(M)$.

Our final result concerns the map
$$\rho : \pi_1(\text{Isom}_0(M)) \to \pi_1(\text{Symp}_0(M, \omega))$$
induced by inclusion.

**Proposition 1.28.** Let $(M, \omega, T, \Phi)$ be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. If there are no essential mass linear functions $H \in \ell$ on the moment polytope $\Delta \subset \mathfrak{t}^*$, then the map $\rho : \pi_1(\text{Isom}_0(M)) \to \pi_1(\text{Symp}_0(M, \omega))$ is an injection.

**Proof.** Assume that $\rho$ is not injective. By Lemma 1.26, the map $\pi_1(\text{Isom}_0(M))$ is surjective. Therefore, there exists $H \in \ell$ such that the image of $\Lambda_H$ in $\pi_1(\text{Symp}_0(M, \omega))$ vanishes but the image of $\Lambda_H$ in $\pi_1(\text{Isom}_0(M))$ does not. By Proposition 1.21, $H$ is mass linear; moreover, $\langle H, c_\Delta \rangle = \sum \alpha_i \kappa_i$, where $\alpha_i \in \mathbb{Z}$ for all $i$.

Since we have assumed that $\Delta$ has no essential mass linear functions, $H$ must be inessential. Hence, we may write $H = \sum \beta_i \eta_i$ where $\sum_{i \in I} \beta_i = 0$ for each equivalence class of facets $I$. By Proposition 1.17 we must have $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. But then $\beta_i = \alpha_i$ because the variables $\kappa_i$ are independent. In particular $\beta_i \in \mathbb{Z}$. Lemma 1.26 then implies that the image of $\Lambda_H$ in $\pi_1(\text{Isom}_0(M))$ vanishes. This gives a contradiction. $\square$

**Example 1.29.** If $M = \mathbb{C}P^k$, then $H = e_1$ generates a circle $\Lambda_H$ with order $k + 1$ in $\pi_1(\text{Isom}_0(\mathbb{C}P^k)) = \pi_1(\text{PU}(k + 1)) = \mathbb{Z}/(k + 1)$. It follows from Proposition 1.21 (and Example 1.14) that $\Lambda_H$ also has order $k + 1$ in $\pi_1(\text{Symp}_0(\mathbb{C}P^k))$, a result first proved in Seidel [16]; see also [13].

**Remark 1.30.** The converse of Proposition 1.21 is open, that is, we do not know if the following statement holds:

Given $H \in \ell$ so that $\langle H, c_\Delta(\kappa) \rangle$ is a linear function with integer coefficients,

the image of $\Lambda_H$ in $\pi_1(\text{Symp}_0(M_\Delta, \omega_\Delta))$ is trivial.

This seems very likely to be the case, at least in dimension 3, and is the subject of ongoing research. By Remark 2.4, any mass linear function on a smooth polytope has rational coefficients. Therefore, if the statement above did hold, it would imply the converse to part (i) of Theorem 1.24 and hence we could conclude

The map $\rho : \pi_1(\text{Isom}_0(M)) \to \pi_1(\text{Symp}_0(M, \omega))$ is an injection exactly if there are no essential mass linear functions $H \in \ell$ on $\Delta$.

**Example 1.31.** Let $M$ be the one point blow up of $\mathbb{CP}^3$. Then $\text{Isom}_0(M) = U(3)$, and Theorem 1.5 implies that $\pi_1(U(3)) = \mathbb{Z}$ injects into $\pi_1(\text{Symp}_0(M, \omega))$, a result already noted by Viña [18]. In this case, we also understand the behavior of the higher homotopy groups, at least rationally; $\pi_3(U(3)) = \mathbb{Z}$ injects into $\pi_3(\text{Symp}_0(M, \omega))$ by Reznikov [15] (cf. also [12]), while $\pi_5(U(3)) \otimes \mathbb{Q}$ injects into $\pi_5(\text{Symp}_0(M, \omega)) \otimes \mathbb{Q}$ because it has nontrivial
image in $\pi_5(\mathbb{CP}^3) \otimes \mathbb{Q}$ under the composite of the point evaluation map $\text{Symp}_0(M, \omega) \to M$ with the blow down $M \to \mathbb{CP}^3$.

**Remark 1.32.** If $M$ is a $\mathbb{CP}^2$ bundle over $\mathbb{CP}^1$, the map $\pi_1(T) \to \pi_1(\text{Isom}_0(M))$ is not injective: the action of $SU(2)$ on the base $\mathbb{CP}^1$ lifts to an action on $M$. On the other hand, for generic toric manifolds $\text{Isom}_0(M) = T$. One might wonder whether the condition $\text{Isom}_0(M) = T$ is enough to imply that $\pi_1(T)$ injects into $\pi_1(\text{Symp}_0(M, \omega))$. However, this seems unlikely to be true since, as we shall see in Part II, there are mass linear pairs $(H, \Delta)$ where $H$ is not constant and $\Delta$ has no robust symmetries, i.e. is such that $\text{Isom}_0(M) = T$.

**Remark 1.33.** There seems to be very little work on simple polytopes that mentions the center of mass. A notable exception is Nill’s paper [14] on complete toric varieties. Interestingly enough, this paper is also concerned with questions about the symmetries of $M$, but considers the full group of biholomorphisms $\text{Aut}(M)$ rather than $\text{Isom}_0(M)$, which is one of its maximal connected compact subgroups. His work applies in the special case of smooth Fano varieties. The moment polytope $\Delta$ of such a variety contains a special point $p_\Delta$. Geometers usually normalize the polytope so that $p_\Delta$ lies at the origin but this point can be described purely in terms of the moment polytope; see Entov–Polterovich [8, Prop. 1.6]. Nill gives a direct combinatorial proof of the fact that if a Fano moment polytope $\Delta$ has the property that $p_\Delta$ is equal to the center of mass $c_\Delta$ of $\Delta$, then $\text{Aut}(M)$ is reductive. (For terminology, see Remark 5.8 below.) In fact, the difference between $p_\Delta$ and $c_\Delta$ is precisely the Futaki invariant, which is now known to vanish if and only if $(M, J)$ supports a Kähler–Einstein metric; cf. Wang–Zhu [19]. Moreover the automorphism group of any Kähler–Einstein manifold is reductive; this gives an alternate proof of Nill’s result.

However, the existence of mass linear functions, which is the question that concerns us here, is not related in any simple way to the vanishing of the Futaki invariant. For example, the Futaki invariant of the blow up of $\mathbb{CP}^2$ at three points vanishes. But the corresponding polytope supports no nontrivial mass linear functions by Theorem 1.3. Thus $\pi_1(T^3)$ injects into $\pi_1(\text{Symp}_0(M, \omega))$ in this case.

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## 2. Basic ideas

In this section, we introduce the basic ideas which underlie the results in this paper and its sequel. More specifically, we first introduce the volume and moment functions and then use them to analyze the key features of symmetric faces and asymmetric facets. Additionally, we prove Theorem 1.10 in §2.3.

Unless there is explicit mention to the contrary, we fix an identification of the affine space $A$ with $t^*$ and hence assume throughout that

$$\Delta = \bigcap_{i=1}^N \{ x \in t^* \ | \ \langle \eta_i, x \rangle \leq \kappa_i \}$$
is a simple polytope with facets $F_1, \ldots, F_N$ and outward conormals $\eta_1, \ldots, \eta_N$. Further, for each $I \subset \{1, \ldots, N\}$, we denote the intersection of the corresponding facets by

\begin{equation}
F_I := \bigcap_{i \in I} F_i.
\end{equation}

We will need to understand the faces of a polytope as polytopes in their own right. To this end, fix a simple polytope $\Delta$. Given a (nonempty) face $f = F_I$, let $P(f) \subset t^*$ denote the smallest affine subspace which contains $f$. Since $\Delta$ is simple, $P(f)$ has codimension $|I|$; indeed,

\begin{equation}
P(f) = \bigcap_{i \in I} \{ x \in t^* \mid \langle \eta_i, x \rangle = \kappa_i \}.
\end{equation}

Similarly, if $F$ is a facet of $\Delta$ which does not contain $f$, then $F \cap f$ is either empty or is a facet of $f$; conversely, every facet of $f$ has this form. Therefore,

\[ f = \bigcap_{j \in I_f} \{ x \in P(f) \mid \langle \eta_j, x \rangle \leq \kappa_j \}, \]

where $j \in \{1, \ldots, N\}$ lies in $I_f$ exactly if $j \not\in I$ and $F_j \cap f \neq \emptyset$. Let $\widehat{t}$ denote the quotient of $t$ by the span of the $\{\eta_i\}_{i \in I}$, and let $\pi: t \to \widehat{t}$ be the natural projection. Fix $x \in P(f)$ and define an isomorphism $j_x: \widehat{t}^* \to P(f)$ by $j_x(y) = \pi^*(y) + x$. Then

\[ f_x := j_x^{-1}(f) = \bigcap_{j \in I_f} \{ y \in \widehat{t}^* \mid \langle \pi(\eta_x), y \rangle \leq \kappa_j - \langle \eta_x, x \rangle \}; \]

in particular, $f_x \subset \widehat{t}^*$ is a convex polytope with outward conormals $\{\pi(\eta_j)\}_{j \in I_f}$. This leads to the following important facts:

- Every face of a simple polytope is itself a simple polytope.
- Every face of a smooth polytope is itself a smooth polytope.

2.1. **The volume and moment of $\Delta$.** The proofs in this section were greatly influenced by the point of view in Timorin [17].

Fix $H \in t$ and a simple polytope $\Delta \subset t^*$. Fix an identification of $t^*$ with Euclidean space, and consider the following polynomials of the support numbers:

\[ V = \int_{\Delta} dx \quad \text{and} \quad \mu = \int_{\Delta} H(x)dx. \]

That is, $V$ is the volume of $\Delta$ and $\mu$ is the first moment of $\Delta$ in the direction given by $H$; on $C_\Delta$ they are polynomials of degree at most $\dim \Delta$ and $\dim \Delta + 1$ respectively.

More generally, for any face $f$ of $\Delta$, fix an identification of $P(f)$ with Euclidean space. Let $V_f$ denote the volume of $f \subset P(f)$; analogously, let $\mu_f$ denote the integral of the function $H$ over the face $f$. On $C_{\Delta}$, $V_f$ and $\mu_f$ are polynomials of degree at most $\dim f$ and $\dim(f) + 1$ respectively. Let $\partial_i$ denote the operator of differentiation with respect to $\kappa_i$. 
Remark 2.1. If $\Delta$ is a smooth polytope, then it is natural to choose identifications of $t^*$ and $P(f)$ which respect the lattice. If we make this choice, the constant $K_f$ in the proposition below is 1 for every non-empty face.

However, if $\Delta$ is not rational then $P(f)$ has no lattice and there is no preferred identification with Euclidean space. While the center of mass of $f$ does not depend on this choice, the constant $K_f$ in the proposition below does.

Proposition 2.2. Let $f$ be the intersection of the facets $F_{i_1}, \ldots, F_{i_k}$. Then

$$\partial_{i_1} \cdots \partial_{i_k} V = K_f V_f \quad \text{and} \quad \partial_{i_1} \cdots \partial_{i_k} \mu = K_f \mu_f,$$

where $K_f$ is a positive real number that depends only on $f$. In particular, if the intersection of facets $F_{i_1}, \ldots, F_{i_k}$ is empty, then

$$\partial_{i_1} \cdots \partial_{i_k} V = \partial_{i_1} \cdots \partial_{i_k} \mu = 0.$$

Proof. Let $f = F_1 \cap \cdots \cap F_k$. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$, where $n = \text{dim} \ t$.

If $f$ is not empty, then $\eta_1, \ldots, \eta_k$ are linearly independent. Moreover, changing the identifications of $t^*$ and $P(f)$ with Euclidean space simply alters the measure by a positive constant; this only affects the value of $K_f$. Hence, we may identify $t^*$ with $\mathbb{R}^n$ so that $\eta_1, \ldots, \eta_k$ are identified with $e_1, \ldots, e_k$, and thus identify $P(f)$ with $\{0\} \times \mathbb{R}^{n-k}$. We can write $V$ as the integral

$$V = \int_{-\infty}^{\kappa_k} \cdots \int_{-\infty}^{\kappa_1} V(x_1, \ldots, x_k) \, dx_1 \cdots dx_k,$$

where $V(c_1, \ldots, c_k)$ is the volume of $\Delta \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n \mid x_i = c_i\}$. Therefore, $\partial_1 \cdots \partial_k V = V(\kappa_1, \ldots, \kappa_k) = V_f$ by the fundamental theorem of calculus. Similarly, we can write $\mu$ as the integral

$$\mu = \int_{-\infty}^{\kappa_k} \cdots \int_{-\infty}^{\kappa_1} \mu(x_1, \ldots, x_k) \, dx_1 \cdots dx_k,$$

where $\mu(c_1, \ldots, c_k)$ is the integral of $H$ over the polytope $\Delta \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n \mid x_i = c_i\}$. Hence, $\partial_1 \cdots \partial_k \mu = \mu(\kappa_1, \ldots, \kappa_k) = \mu_f$.

So suppose instead that $f$ is empty. Let $g = F_2 \cap \cdots \cap F_k$. We may assume by induction that $\partial_2 \cdots \partial_k V = K_g V_g$ for some positive constant $K_g$, and so $\partial_1 \cdots \partial_k V = K_g \partial_1 V_g$. However, since $F_1$ and $g$ do not intersect, $g$ does not depend on $\kappa_1$. Therefore, $\partial_1 V_g = 0$. A similar argument shows that $\partial_1 \cdots \partial_k \mu = 0$. \qed

Given $H \in t$ and a simple polytope $\Delta \subset t^*$, define a function $\widehat{H} : C_\Delta \to \mathbb{R}$ by

$$\widehat{H}(\kappa_1, \ldots, \kappa_N) = \langle H, e_\Delta(\kappa_1, \ldots, \kappa_N) \rangle,$$

where $e_\Delta$ denotes the center of mass of $\Delta$. We will repeatedly use the fact that

$$\mu = \widehat{H} V.$$

Remark 2.3. Fix $H \in t$ and a simple polytope $\Delta \subset t^*$. If the function $\widehat{H} : C_\Delta \to \mathbb{R}$ given by $\widehat{H}(\kappa) = \langle H, e_\Delta(\kappa) \rangle$ is linear on some open subset of $C_\Delta$, it is linear on all of $C_\Delta$. This
claim is a simple consequence of the fact that, since $V$ and $\mu$ are both polynomials, the function $\tilde{H} = \frac{\mu}{V}$ is a rational function.

**Remark 2.4.** If $\Delta \subset t^*$ is a smooth polytope, then every mass linear function $H \in \ell$ has rational coefficients, that is, if $\langle H, c_\Delta(\kappa) \rangle = \sum \gamma_i \kappa_i$ for all $\kappa \in C_\Delta$ then $\gamma_i \in \mathbb{Q}$.

To see this, note that by Proposition 2.2 and Remark 2.1, the volume $V : C_\Delta \to \mathbb{R}$ is a polynomial with rational coefficients. Similarly, since $\langle H, v \rangle : C_\Delta \to \mathbb{R}$ is a linear function with integer coefficients for every vertex $v \in \Delta$, the moment $\mu$ is also a polynomial with rational coefficients. So $\frac{\mu}{\eta}$ is a rational function with rational coefficients.

We include the following lemma for completeness, but make no use of it in this paper.

**Lemma 2.5.** Let $\Delta \subset t^*$ be a smooth polytope. Then the set of mass linear functions on $\Delta$ is a rational subspace of $t$.

**Proof.** For all $i \in \{1, \ldots, N\}$, let $\mu_i$ denote the moment in the $\eta_i$ direction. By the previous remark, the polynomial $P_i = \mu_i - V \kappa_i$ is a polynomial with rational coefficients for all $i \in \{1, \ldots, N\}$. Therefore, the set

$$X = \left\{ \beta \in \mathbb{R}^N \left| \sum_{i=1}^N \beta_i P_i = 0 \right. \right\}$$

is a rational subspace of $\mathbb{R}^N$. Hence, to prove the result, it is enough to show that $H \in t$ is mass linear exactly if $H = \sum \beta_i \eta_i$, where $\beta \in X$. To check this, note that $\mu_i/V = \langle \eta_i, c_\Delta(\kappa) \rangle$. Hence

$$\langle \sum \beta_i \eta_i, c_\Delta(\kappa) \rangle = \sum \beta_i \frac{\mu_i}{V}.$$ 

Therefore, if $\beta \in X$ then $\langle \sum \beta_i \eta_i, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. Conversely, consider $H \in t$ so that $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. By Lemma 3.19 below, this implies that $H = \sum \beta_i \eta_i$. But then $\sum \langle \beta_i \eta_i, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$, which implies that $\beta \in X$. \qed

### 2.2. Symmetric faces

We are now ready to consider symmetric faces; see Definition 1.8. In particular, we will analyze the restriction of a mass linear function to a symmetric face.

**Lemma 2.6.** Fix $H \in t$ and let $f$ be a symmetric face of a simple polytope $\Delta \subset t^*$. Then

$$\langle H, c_f(\kappa) \rangle = \langle H, c_\Delta(\kappa) \rangle \quad \forall \kappa \in C_\Delta,$$

where $c_f : C_\Delta \to t^*$ denotes the center of mass of $f$.

**Proof.** Let $f = F_f$ be a symmetric face; number the facets so that $J = \{1, \ldots, j\}$. Since $F_j$ is symmetric for each $1 \leq i \leq j$, $\partial_i \tilde{H} = 0$. Hence, by Proposition 2.2, if we apply the differential operator $\partial_1 \cdots \partial_j$ to the formula $\mu = \tilde{H}V$ we obtain

$$K_f \mu_f = K_f \tilde{H} V_f.$$ 

Since $K_f \neq 0$ and $\mu_f = \langle H, c_f \rangle V_f$, this proves that $\langle H, c_\Delta \rangle = \langle H, c_f \rangle$. \qed

**Lemma 2.7.** Fix $H \in t$ and let $\Delta \subset t^*$ be a simple polytope. If $f$ is a symmetric face and $F$ an asymmetric facet, then $f \cap F \neq \emptyset$; indeed, $F \cap f$ is a facet of $f$. 

Proof. Suppose on the contrary that $F \cap f = \emptyset$. Since $f(\kappa) \subseteq \Delta(\kappa)$ does not depend on support number of $F$; neither does $\langle H, cf(\kappa) \rangle : \mathcal{C}_\Delta \to \mathbb{R}$. By the lemma above, this implies that $\langle H, c_\Delta(\kappa) \rangle : \mathcal{C}_\Delta \to \mathbb{R}$ does not depend on support number of $F$, that is, $F$ is symmetric. This is a contradiction. Finally, we note that by Definition 1.8, $f \not\subseteq F$. Hence $F \cap f$ must be a facet of $f$. \hfill \Box

**Proposition 2.8.** Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subseteq \mathfrak{t}^*$. Let $f$ be a symmetric face of $\Delta$. Then the restriction of $H$ to $f$ is mass linear and the map $F \mapsto F \cap f$ induces a one-to-one correspondence between the asymmetric facets of $\Delta$ and the asymmetric facets of $f$. Moreover, the coefficient of the support number of $F$ in $\langle H, c_\Delta \rangle$ is the coefficient of the support number of $f \cap F$ in $\langle H, cf \rangle$.

Proof. Renumber the facets so that $F_j \cap f$ is a facet of $f$ exactly if $j \in \{1, \ldots, M\}$. In the statement of this proposition, we consider $f$ as a polytope in its own right, that is, $f = \bigcap_{j=1}^M \{ x \in P(f) \mid \langle n_j, x \rangle \leq \kappa_k \}$, where $P(f)$ denotes the affine span of $f$ as in Equation (2.3). Hence, we consider the center of mass of $f$ as a function on $\mathcal{C}_f \subseteq \mathbb{R}^M$. In contrast, in Lemma 2.6, we consider $f$ as a face of the polytope $\Delta$; hence there we consider $cf$ as a function on $\mathcal{C}_\Delta \subseteq \mathbb{R}^N$. However, by Lemma 2.7 every asymmetric facet $F$ intersects $f$; indeed, $F \cap f$ is a facet of $f$. This implies that $\langle H, c_\Delta \rangle : \mathcal{C}_\Delta \to \mathfrak{t}^*$ does not depend on the support number $\kappa_i$ for any $i > M$. Therefore, Lemma 2.6 implies that

\begin{equation}
\langle H, cf(\kappa_1, \ldots, \kappa_M) \rangle = \langle H, c_\Delta(\kappa_1, \ldots, \kappa_N) \rangle \quad \forall \kappa \in \mathcal{C}_\Delta.
\end{equation}

The claim now follows immediately from Remark 2.3. \hfill \Box

**Remark 2.9.** The following notion is sometimes useful. Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subseteq \mathfrak{t}^*$. We say that a face $f$ of $\Delta$ is **centered** if

\begin{equation}
\langle H, cf(\kappa) \rangle = \langle H, c_\Delta(\kappa) \rangle \quad \forall \kappa \in \mathcal{C}_\Delta.
\end{equation}

Lemma 2.6 shows that symmetric faces are centered. Conversely, if $F$ is a centered facet, then by definition $F$ is also symmetric. In contrast, the converse does not hold for faces of higher codimension. For example, let $\Delta_2 = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, \text{and } x_1 + x_2 \leq 1 \}$ denote the standard 2-simplex, and let $H(x) = x_1 - x_2$. Then the vertex $(0, 0)$ is centered, but it is not symmetric because the facets containing it are not symmetric. The proof of Lemma 2.7 can be adapted to show that every centered face must intersect every asymmetric facet.

### 2.3. Asymmetric facets

We now consider asymmetric facets. The main goal of this subsection is to prove Theorem 1.10 and the proposition below; see Definition 1.9.

**Proposition 2.10.** Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subseteq \mathfrak{t}^*$. Then every asymmetric facet is pervasive or flat (or both).

We will need the following lemmas.

**Lemma 2.11.** Let $\Delta \subseteq \mathfrak{t}^*$ be a simple polytope. The facet $F_i$ is flat if and only if the volume $V$ is a linear function of $\kappa_i$. 
Proof. If \( F_i \) is flat, a straightforward computation shows that the volume is a linear function of \( \kappa_i \). So assume that \( F_i \) is not flat. At each vertex \( v \) of \( F_i \) there is a unique edge \( e_v \) of \( \Delta \) that does not lie in \( F_i \). Since \( F_i \) is not flat, these edges cannot all be parallel. In particular, there is an edge \( e \) joining two vertices \( v \) and \( v' \) of \( \Delta \) that does not lie in \( F_i \). Since the edges \( e_v \) and \( e_{v'} \) are not parallel, \( V_{F_j} \) is not a linear function of \( \kappa_1 \); it has degree 2. But by Proposition 2.2, \( \partial_j \cdots \partial_{j_{n-2}} V = K_{F_j} V_{F_j} \). Hence, \( V \) is not a linear function of \( \kappa_1 \).

Lemma 2.12. Fix a nonzero \( H \in \mathfrak{t} \) and a simple polytope \( \Delta \subset \mathfrak{t}^* \). Then \( \Delta \) has at least two asymmetric facets.

Proof. Let \( F_i \) be any facet. Since \( \Delta \) has a nonempty interior, \( c_\Delta \) does not lie on \( F_i \). Therefore, by translating and then rescaling, we can find a dilation of \( \mathfrak{t}^* \) that fixes the plane \( P(F_i) \) but moves \( c_\Delta \) to an arbitrary nearby point. Hence \( \langle H, c_\Delta \rangle \) depends on at least one \( \kappa_j \) for \( j \neq i \).

Proof of Proposition 2.10. Let \( F_1 \) be an asymmetric facet which is not flat and let \( F_2 \) be a facet which does not meet \( F_1 \). Since \( H \) is mass linear, the function \( \tilde{H} \) of Equation (2.4) is linear, and so its second derivatives all vanish. Hence, since \( F_1 \cap F_2 = \emptyset \), by Proposition 2.2 if we apply the differential operator \( \partial_1 \partial_2 \) to the formula \( \mu = \tilde{H} V \) we obtain

\[
0 = (\partial_1 \tilde{H}) V_{F_2} + (\partial_2 \tilde{H}) V_{F_1}.
\]

Applying the operator \( \partial_1 \) again to the equation above, we obtain

\[
(\partial_2 \tilde{H}) (\partial_1 V_{F_1}) = 0.
\]

On the one hand, Lemma 2.7 implies that \( F_2 \) is asymmetric, that is, \( \partial_2 \tilde{H} \neq 0 \). On the other hand, since \( F_1 \) is not flat Lemma 2.11 implies that \( \partial_1 V_{F_1} \neq 0 \); this gives a contradiction.

Proof of Theorem 1.10. Let be \( H \in \mathfrak{t} \) be a nonzero mass linear function on a simple polytope \( \Delta \subset \mathfrak{t}^* \). By Lemma 2.12, \( \Delta \) contains an asymmetric facet \( F \). By Proposition 2.10, \( F \) must be pervasive or flat.

3. Polytopes with inessential functions

The main results of this section are Proposition 3.15, which characterizes polytopes with nonconstant inessential functions, and Theorem 1.19, which characterizes polytopes with the property that all linear functions are mass linear.

3.1. Equivalent facets. In this subsection, we prove several useful criteria for determining if facets are equivalent.

Lemma 3.1. Let \( F_i \) and \( F_j \) be distinct facets of a simple polytope \( \Delta \subset \mathfrak{t}^* \). The following are equivalent:

1. \( F_i \) is equivalent to \( F_j \).
2. There exists a vector \( \xi \in \mathfrak{t}^* \) satisfying

\[
\langle \eta_i, \xi \rangle > 0 > \langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k.
\]
(iii) There exists a vector \( \xi \in t^* \) satisfying
\[
\langle \eta_i, \xi \rangle = 1 = -\langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k.
\]

(iv) There exists a reflection \( \hat{a}: t^* \to t^* \) which is a robust symmetry of \( \Delta \); for all \( \kappa \in \mathcal{C}_\Delta \) the associated symmetry \( a_\kappa \) exchanges \( F_i \) and \( F_j \) and otherwise takes each facet to itself.

**Proof.** It is obvious that (iii) implies (ii) and that (iv) implies (i).

Our first step is to show that (ii) implies (iii) and (iv). Assume that there exists \( \xi \in t^* \) such that \( \langle \eta_1, \xi \rangle > 0 > \langle \eta_2, \xi \rangle \) and \( \langle \eta_k, \xi \rangle = 0 \) for all \( k > 2 \). Let
\[
\eta_1' = \frac{\eta_1}{\langle \eta_1, \xi \rangle} \quad \text{and} \quad \eta_2' = \frac{\eta_2}{-\langle \eta_2, \xi \rangle};
\]
clearly \( \langle \eta_1', \xi \rangle = 1 = -\langle \eta_2', \xi \rangle \). Define a reflection \( \hat{a}: t^* \to t^* \) by
\[
\hat{a}(x) = x - \langle \eta_1' - \eta_2', x \rangle \xi.
\]
Given \( \kappa \in \mathcal{C}_\kappa \), define an affine reflection \( a_\kappa: t^* \to t^* \) by
\[
a_\kappa(x) = \hat{a}(x) + (\kappa_1' - \kappa_2') \xi, \quad \text{where} \quad \kappa_1' = \frac{\kappa_1}{\langle \eta_1, \xi \rangle} \quad \text{and} \quad \kappa_2' = \frac{\kappa_2}{-\langle \eta_2, \xi \rangle}.
\]
It is easy to check that \( a_\kappa \) carries \( \Delta(\kappa) \) to itself, exchanging \( F_1 \) and \( F_2 \) and otherwise taking each facet to itself. In particular, \( \hat{a} \) is a robust symmetry of \( \Delta \). By our normalization convention (see Remark 1.12), this implies that \( \hat{a}^* \eta_1 = \eta_2 \). Since also \( \hat{a}^* (\eta_1') = \eta_2' \), this implies that \( \langle \eta_1, \xi \rangle = -\langle \eta_2, \xi \rangle \). Hence, \( \langle \eta_1, \xi' \rangle = 1 = -\langle \eta_2, \xi' \rangle \), where \( \xi' = \frac{\xi}{\langle \eta_1, \xi \rangle} \).

Finally, we will prove that (i) implies (ii). Let \( \hat{a}: t^* \to t^* \) be a robust symmetry and let \( \hat{a}^*: t \to t \) denote the dual map. For each \( \kappa \in \mathcal{C}_\Delta \), let \( a_\kappa: t^* \to t^* \) be the associated symmetry of \( \Delta(\kappa) \). We can naturally partition the set of facets of \( \Delta \) into \( a_\kappa \)-orbits; let \( O \) denote the set of such orbits. Assume that each \( a_\kappa \) takes \( F_1 \) to \( F_2 \).

Since \( a_\kappa(\Delta) = \Delta \), \( a_\kappa^k = \text{id} \) for some \( k \). Define \( p: t \to t \) by
\[
p = 1 + \hat{a}^* + \cdots + (\hat{a}^*)^{k-1}.
\]
Since \( a^k = \text{id} \), \( p \circ p = k \cdot p \). Therefore,
\[
t = \ker p \oplus p(t).
\]

Our first claim is that \( \ker p \) has dimension \( N - |O| \). To see this, first note that, by our normalization convention, \( p(\eta_j) \) is a positive multiple of \( \sum_{i \in O} \eta_i \) for all \( j \), where \( O \) is the orbit containing \( F_j \). Moreover, since \( \Delta \) is bounded, the \( \eta_i \) span \( t \). Hence,
\[
(3.1) \quad \ker p = \left\{ \sum \beta_i \eta_i \in t \left| \sum_{i \in O} \beta_i = 0 \quad \forall O \in O \right. \right\}.
\]
Now fix \( \beta \in \mathbb{R}^N \) such that \( \sum_{i \in O} \beta_i = 0 \) for all \( O \in O \), and let \( H = \sum \beta_i \eta_i \). Then Proposition 1.17 implies that
\[
(3.2) \quad \langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i.
\]
Since this equality holds for all $\kappa$ in the open set $C_\Delta$, this implies that $H \neq 0$. The claim follows immediately.

Since $t = \ker p \oplus p(t)$, this implies that $p(t)$ has codimension $N - |O|$. On the other hand, because the $\eta_j$ span $t$,

$$p(t) = \left\{ \sum_{O \in O} \beta_O \left( \sum_{i \in O} \eta_i \right) \middle| \beta_O \in \mathbb{R} \forall O \in O \right\}.$$ 

By a dimension count, this implies that, since $F_1$ and $F_2$ are in the same $a_\kappa$-orbit, the subspace spanned by $\eta_1 + \eta_2$ and $\{\eta_k\}_{k \geq 2}$ cannot contain the vector $\eta_1 - \eta_2 \in \ker p$ and so has codimension 1. Therefore, there exists $\xi \in t^*$ so that

$$\langle \eta_1, \xi \rangle > 0 > \langle \eta_2, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \forall k.$$ 

This completes the proof. \qed

The above lemma implies that the equivalence relation $\sim$ depends only on the set of conormals of $\Delta$ and not on the intersection pattern of its facets. In other words, it depends only on the 1-skeleton of the corresponding fan, not on its higher dimensional simplices.

This lemma (or more precisely, its proof) has the following corollary, which will be useful in §5.

**Corollary 3.2.** Let $\Delta \subset t^*$ be a simple polytope. Let $\hat{a}: t^* \to t^*$ be a linear map so that for all $\kappa'$ in some neighborhood $U$ of $\kappa$ in $C_\Delta$ there exists an affine map $a_\kappa$ that satisfies conditions (1) and (2) of Definition 1.11. Then $\hat{a}$ is a robust symmetry of $\Delta$.

**Proof.** Suppose that the $a_\kappa$ take $F_i$ to $F_j$. The above proof that (i) implies (ii) goes through in this situation since all we use is that equation (3.2) holds for all $\kappa$ in some open set $U$. Since (ii) implies (iv), there is a robust symmetry that exchanges $F_i$ and $F_j$ but takes every other facet to itself. By composing such maps, we can construct a robust symmetry $\hat{b}$ so that $\hat{a}$ and $\hat{b}$ induce the same permutation on the facets. As in Remark 1.12, this implies that $\hat{a} = \hat{b}$. \qed

We next consider the properties of robust symmetries of smooth polytopes. 

**Lemma 3.3.** Let $\hat{a}: t^* \to t^*$ be a robust symmetry of a smooth polytope $\Delta \subset t^*$. Then $\hat{a}$ induces an isomorphism of the integral lattice.

**Proof.** Let $\eta_1, \ldots, \eta_N$ be the outward conormals to $\Delta$. Recall that, by our convention, these outward conormals are normalized so that dual map to every robust symmetry takes each $\eta_i$ to some $\eta_j$. In contrast, let $\eta'_1, \ldots, \eta'_N$ be the primitive outward conormals. Since $\Delta$ is smooth, these primitive outward conormals generate the integer lattice. Therefore, it is enough to fix an arbitrary facet $F_i$ and prove that the dual map $\hat{a}^*$ takes $\eta'_i$ to another primitive outward conormal $\eta'_j$. As we have already seen, if each associated symmetry $a_k$ takes the facet $F_i$ to itself, then $\hat{a}^*(\eta'_i) = \eta'_i$. So assume that each $a_k$ takes the facet $F_i$ to a different facet $F_j$. Then $F_i$ and $F_j$ are equivalent. By Lemma 3.1, this implies that there exists a vector $\xi \in t^*$ satisfying

$$\langle \eta_i, \xi \rangle > 0 > \langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \forall k.$$
Since each $\eta_i$ is a positive multiple of $\eta_k'$, this implies that also

$$\langle \eta_i, \xi \rangle > 0 > -\langle \eta_j', \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k', \xi \rangle = 0 \ \forall k.$$ 

On the other hand, the definition of smooth, these primitive outward conormals at any vertex of $\Delta$ form a basis for $\ell$. By looking at vertices in $F_i \setminus F_j$ and $F_j \setminus F_i$ one sees that there is a basis of $\ell$ that contains $\eta_i'$ but not $\eta_j'$ and also a basis that contains $\eta_j'$ but not $\eta_i'$. Hence there is a vector $\xi' \in \ell^*$ satisfying

$$\langle \eta_i', \xi' \rangle = 1 = -\langle \eta_j', \xi' \rangle \quad \text{and otherwise} \quad \langle \eta_k', \xi' \rangle = 0 \ \forall k.$$ 

As in the proof above, the reflection $\hat{b}: t^* \to t^*$ defined by

$$\hat{b}(x) = x - \langle \eta_i' - \eta_j', x \rangle \xi'$$

is a robust symmetry of $\Delta$ and the dual map $\hat{b}^*$ takes $\eta_i'$ to $\eta_j'$. Since $\hat{a}$ is also a robust symmetry, this implies that $\hat{a}^*$ must also take $\eta_i'$ to $\eta_j'$. \qed

**Remark 3.4.** Note that the lemma above does not hold if $\Delta$ is rational polytope which is not smooth. For example, let $\Delta \subset (\mathbb{R}^2)^*$ be the triangle with vertices $(0,0)$, $(3,0)$, and $(0,2)$. The linear map $(x_1, x_2) \mapsto (-x_1 - \frac{3}{2} x_2, \frac{2}{3} x_1)$ is a robust symmetry, but it does not preserve the integral lattice. Therefore, the outward conormals we use will not be primitive, they will be $\lambda(-3,0), \lambda(0,-2)$, and $\lambda(3,2)$ for some $\lambda > 0$; cf. Remark 5.8. However, this is not too troubling; when we consider polytopes which are not smooth we will always ignore the integer lattice.

**Remark 3.5.** Let $\Delta$ be a simple polytope. We claim that $H \in t$ is inessential on $\Delta$ exactly if there exists a robust symmetry $\hat{a}$ of $\Delta$ so that $H$ is in the kernel of the associated projection

$$p = 1 + \hat{a}^* + \cdots + (\hat{a}^*)^{k-1},$$

that is, so that $H$ has zero average with respect to the action of $\hat{a}^*$. Here, $\hat{a}^*: t^* \to t^*$ is the dual map and $k > 0$ is chosen so that $a^k = \text{id}$.

To see this, note that the lemma above immediately implies that there exists a robust symmetry $\hat{a}$ of $\Delta$ whose set $\mathcal{O}$ of orbits of facets is precisely the set $\mathcal{I}$ of equivalence classes of facets. Moreover, given any robust symmetry $\hat{a}$ of $\Delta$, all the facets in any orbit $O \in \mathcal{O}$ are equivalent. The claim now follows from the formulas (1.3) and (3.1).

**Example 3.6.** Given simple polytopes $\tilde{\Delta} \subset \tilde{t}^*$ and $\hat{\Delta} \subset \hat{t}^*$, consider the product $\Delta = \tilde{\Delta} \times \hat{\Delta}$. It follows immediately from the lemma above that two facets $\tilde{F}_i \times \hat{\Delta}$ and $\tilde{F}_j \times \hat{\Delta}$ of $\Delta$ are equivalent exactly if $\tilde{F}_i$ and $\tilde{F}_j$ are equivalent as facets of $\tilde{\Delta}$. Similarly, two facets $\hat{\Delta} \times \hat{F}_k$ and $\hat{\Delta} \times \hat{F}_l$ of $\Delta$ are equivalent exactly if $\hat{F}_l$ and $\hat{F}_k$ are equivalent as facets of $\hat{\Delta}$. Therefore, $(\tilde{H}, \hat{H}) \in \tilde{t} \times \hat{t}$ is inessential on $\Delta$ exactly if $\tilde{H}$ is inessential on $\tilde{\Delta}$ and $\hat{H}$ is inessential on $\hat{\Delta}$.

The next lemma gives a very useful criterion for a set of facets to be equivalent.

**Lemma 3.7.** Let $\Delta \subset t^*$ be a simple polytope. Given a subset $I \subset \{1, \ldots, N\}$, $F_i \sim F_j$ for all $i$ and $j$ in $I$ exactly if the subspace $V \subset t$ spanned by the outward conormals $\eta_k$ for $k \notin I$
has codimension $|I| - 1$. Moreover, in this case the linear combination $\sum_{i \in I} c_i \eta_i$ lies in $V$ if and only if $c_i = c_j$ for all $i$ and $j$.

Proof. Assume that $F_i \sim F_j$ for all $i$ and $j$ in $I$. By Lemma 3.1, for each such pair there exists a vector $\xi_{ij} \in \ell^*$ satisfying $\langle \eta_i, \xi_{ij} \rangle = 1 = -\langle \eta_j, \xi_{ij} \rangle$ and otherwise $\langle \eta_k, \xi_{ij} \rangle = 0$ for all $k$. Fix $i \in I$. Then the vectors $\xi_{ij}$, $j \in I \setminus \{i\}$, span an $(|I| - 1)$-dimensional subspace of $\ell^*$; let $V \subset \ell$ be its annihilator. By construction, $\eta_k$ lies in $V$ for all $k \not\in I$. On the other hand, since $\Delta$ is bounded the positive span of the $\eta_k$ contains all of $t$. (Recall that the positive span of $\eta_1, \ldots, \eta_N$ is the set $\{ \sum a_i \eta_i \mid a_i > 0 \forall i \}$. ) In particular, any $\eta_k$ can be written as a linear combination of the other outward conormals. Hence, the subspace spanned by the $\eta_k$ for $k \not\in I$ has codimension at most $|I| - 1$; so it must be $V$. Finally, since $\langle \sum_{k \in I} c_k \eta_k, \xi_{ij} \rangle = c_i - c_j$, $\sum_{k \in I} c_k \eta_k$ lies in $V$ exactly if $c_i = c_j$ for all $i$ and $j$.

Conversely, assume that the plane $V \subset t$ spanned by the outward conormals $\eta_k$ for $k \not\in I$ has codimension $|I| - 1$. By a dimension count, for any pair $i \neq j$ in $I$ there exists $\xi \in t$ so that $\langle \eta_k, \xi \rangle = 0$ for all $k$ except $i$ and $j$. Since the positive span of the $\eta_k$ contains all of $t$, the positive span of the numbers $\langle \eta_k, \xi \rangle$, namely $\{ \sum a_k \langle \eta_k, \xi \rangle \mid a_k > 0 \}$, contains all of $\mathbb{R}$. In particular, after possibly replacing $\xi$ by $-\xi$, we have $\langle \eta_i, \xi \rangle > 0 > \langle \eta_j, \xi \rangle$. By Lemma 3.1, this implies that $F_i$ and $F_j$ are equivalent. \hfill $\square$

Lemma 3.8. Let $\Delta \subset \ell^*$ be a simple polytope; let $\mathcal{I}$ denote the set of equivalence classes of facets. The set of inessential functions is an $(N - |\mathcal{I}|)$-dimensional subspace of $t$. Moreover, if $\Delta$ is smooth then this is a rational subspace, and every inessential $H \in \ell$ can be written

$$H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{Q} \forall i \text{ and } \sum_{i \in \mathcal{I}} \beta_i = 0 \forall I \in \mathcal{I}.$$ 

Proof. Let $H \in \ell$ be inessential. By Definition 1.13 we may write $H = \sum \beta_i \eta_i$, where the $\beta_i$ satisfy all the above conditions except that they may be in $\mathbb{R}$ not $\mathbb{Q}$. Fix $i \in I \in \mathcal{I}$. By Lemma 3.1, for all $j \in I \setminus \{i\}$, there exists $\xi_{ij} \in \ell^*$ so that

$$\langle \eta_i, \xi_{ij} \rangle = 1 = -\langle \eta_j, \xi_{ij} \rangle \quad \text{and otherwise } \quad \langle \eta_k, \xi_{ij} \rangle = 0 \forall k.$$ 

Then

$$|I| \beta_i = |I| \beta_i - \sum_{j \in \mathcal{I}} \beta_j = \sum_{j \in \mathcal{I}} (\beta_i - \beta_j) = \sum_{j \in \mathcal{I} \setminus \{i\}} \langle H, \xi_{ij} \rangle.$$ 

If $H = 0$, this implies immediately that $\beta_i = 0$; this proves the first claim.\footnote{This also follows from Proposition 1.17, as we saw while proving Lemma 3.1.}

On the other hand, if $\Delta$ is smooth then, because the $\{\eta_k\}$ form a basis for $\ell$, $\xi_{ij}$ lies in the lattice $\ell^*$ for all $j \in I \setminus \{i\}$. Hence, if $H \in \ell$, then $|I| \beta_i$ is an integer for every $i \in I \in \mathcal{I}$; in particular, $\beta_i \in \mathbb{Q}$.

$\square$

3.2. Bundles and expansions. We will now define two important class of polytopes: bundles and $k$-fold expansions. Bundles over the simplex $\Delta_k$ and $k$-fold expansions are very similar; in particular, both admit nonconstant inessential functions. Our main result is that these two classes of polytopes are the only ones which admit nonconstant inessential
functions. The following definition uses the notion of combinatorial equivalence defined in Remark 1.6.

**Definition 3.9.** Let \( \tilde{\Delta} = \bigcap_{j=1}^{\tilde{N}} \{ x \in \mathbb{T}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j \} \) and \( \hat{\Delta} = \bigcap_{i=1}^{\hat{N}} \{ y \in \hat{T}^* \mid \langle \hat{\eta}_i, y \rangle \leq \hat{\kappa}_i \} \) be simple polytopes. We say that a simple polytope \( \Delta \subset t^* \) is a **bundle with fiber** \( \Delta \) over the base \( \hat{\Delta} \) if there exists a short exact sequence

\[
0 \to \tilde{t} \to t \xrightarrow{\pi} \hat{t} \to 0
\]

so that

- \( \Delta \) is combinatorially equivalent to the product \( \hat{\Delta} \times \tilde{\Delta} \).
- If \( \tilde{\eta}_j' \) denotes the outward conormal to the facet \( \tilde{F}_j' \) of \( \Delta \) which corresponds to \( \tilde{F}_j \times \hat{\Delta} \subset \tilde{\Delta} \times \hat{\Delta} \), then \( \tilde{\eta}_j' = i(\tilde{\eta}_j) \) for all \( 1 \leq j \leq \tilde{N} \).
- If \( \hat{\eta}_i' \) denotes the outward conormal to the facet \( \hat{F}_i' \) of \( \Delta \) which corresponds to \( \tilde{\Delta} \times \hat{F}_i \subset \tilde{\Delta} \times \hat{\Delta} \), then \( \pi(\hat{\eta}_i') = \hat{\eta}_i \) for all \( 1 \leq i \leq \hat{N} \).

The facets \( \tilde{F}_1', \ldots, \tilde{F}_{\tilde{N}}' \) will be called **fiber facets**, and the facets \( \hat{F}_1', \ldots, \hat{F}_{\hat{N}}' \) will be called **base facets**.

If \( \Delta \) is smooth then \( \tilde{\Delta} \) and \( \hat{\Delta} \) are both smooth. However, the converse does not hold; see Example 1.1.

**Remark 3.10.** This terminology is justified by the following fact. As we explain in greater detail in §5, there is a manifold \( M_\Delta \) associated to every smooth polytope \( \Delta \). By Remark 5.2, if \( \Delta \) is a bundle with fiber \( \tilde{\Delta} \) over the base \( \hat{\Delta} \) in the sense defined above, then \( M_\Delta \) is a bundle with fiber \( M_{\hat{\Delta}} \) over the base \( M_{\tilde{\Delta}} \) in the usual sense.

The example below illustrates an important, but slightly confusing, point. The fiber facets are almost never equivalent in any sense to the fiber polytope \( \tilde{\Delta} \). Indeed, if the fiber \( \tilde{\Delta} \) is a one-simplex, the fiber facets \( \tilde{F}_j' \) are analogous to the base polytope \( \tilde{\Delta} \). In other words we may identify \( P(\tilde{F}_j') \) with \( \mathbb{T}^* \) in such a way as to set up a combinatorial equivalence between \( \tilde{F}_j' \) and \( \tilde{\Delta} \) in which corresponding facets are parallel; cf. the discussion just before Example 1.1. More generally, if the fiber \( \tilde{\Delta} \) has dimension \( k \) and the base \( \hat{\Delta} \) has dimension \( n \), then the non-empty intersection of any \( k \) fiber facets is analogous to the base polytope and may be considered as a section of the bundle. Similarly, the non-empty intersection of any \( n \) base facets is affine equivalent to the fiber polytope.

**Example 3.11.** The polytope \( Y_a \) defined in Example 1.1 is a \( \Delta_2 \) bundle over \( \Delta_1 \) because \( \Delta \) is combinatorially equivalent to \( \Delta_1 \times \Delta_2 \) and the conormals \((-1,0,0),(0,-1,0), \) and \((1,1,0)\) are linearly dependent. However, it is not a \( \Delta_1 \) bundle over \( \Delta_2 \) unless \( a_1 = a_2 = 0 \).

In contrast, the polytope

\[
\{ x \in \mathbb{R}^3 \mid x_i \geq 0 \forall i, \ x_1 + x_2 \leq \lambda + bx_3, \ \text{and} \ x_3 \leq h \}
\]

is a \( \Delta_1 \) bundle over \( \Delta_2 \) for all \( h > 0 \) and \( \lambda > \max(0,-bh) \), but is not a \( \Delta_2 \) bundle over \( \Delta_1 \) unless \( b = 0 \). See Figure 3.1.
Figure 3.1. (a) is a $\Delta_2$ bundle over $\Delta_1$ with a base facet shaded; (b) is a $\Delta_1$ bundle over $\Delta_2$ with a fiber facet shaded. Notice that the fiber facets all contain at least one common vector, vertical in (a) and horizontal in (b).

**Definition 3.12.** Let $\Delta = \bigcap_{j=1}^{\tilde{N}} \{ x \in t^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j \}$ be a simple polytope. Given a natural number $k$, a polytope $\Delta \subset t^*$ is the $k$-fold expansion of $\Delta$ along the facet $\tilde{F}_1$ if there is an identification $t = \tilde{t} \oplus \mathbb{R}^k$ so that

$$\Delta = \bigcap_{j=2}^{\tilde{N}} \{ x \in t^* \mid \langle (\tilde{\eta}_j,0), x \rangle \leq \tilde{\kappa}_j \} \cap \bigcap_{i=1}^{k+1} \{ x \in t^* \mid \langle \hat{\eta}_i, x \rangle \leq \hat{\kappa}_i \},$$

where $\hat{\eta}_i = (0, -e_i)$ and $\hat{\kappa}_i = 0$ for $1 \leq i \leq k$, $\hat{\eta}_{k+1} = (\tilde{\eta}_1, \sum_i e_i)$ and $\hat{\kappa}_{k+1} = \tilde{\kappa}_1$. We shall call the facet $\tilde{F}_j^t$ of $\Delta$ with outward conormal $(\tilde{\eta}_j,0)$ the fiber-type facet (associated to $\tilde{F}_j$) and the facet $\hat{F}_i$ with outward conormal $\hat{\eta}_i$ a base-type facet.

Figure 3.2. (a) is the 1-fold expansion of the shaded polygon along $f$; (b) is the 2-fold expansion of the heavy line at the vertex $v$.
It is easy to check that $\Delta$ is simple, and is smooth exactly if $\tilde{\Delta}$ is. The base-type facets intersect. Indeed, the face $\bigcap_{i=1}^{k+1} \tilde{F}_i$ can be identified with $\tilde{F}_1$. Similarly, the face $\bigcap_{i \neq n} \tilde{F}_i$ can be identified with $\tilde{\Delta}$ for all $1 \leq n \leq k + 1$. See Remark 5.4 for a geometric interpretation of 1-fold expansions.

**Example 3.13.** The $k$-fold expansion of an $n$-simplex along one of its facets is an $(n + k)$-simplex.

**Remark 3.14.** For $\epsilon > 0$ sufficiently small, the $\epsilon$-blow up $\Delta'$ of a polytope $\Delta$ along a face $f = F_I = \bigcap_{i \in I} F_i$ of codimension at least 2 is defined to be the intersection

$$\Delta' = \Delta \cap \{ x \in t' \mid \langle \eta'_0, x \rangle \leq \kappa'_0 \},$$

where $\eta'_0 := \sum_{i \in I} \eta_i$ and $\kappa'_0 = \sum_{i \in I} \kappa_i - \epsilon$. Thus $\Delta'$ is obtained from $\Delta$ by cutting out a small neighborhood of $f$ by a suitably angled cut.\(^3\) Suppose now that $\Delta$ is the $k$-fold expansion of $\tilde{\Delta}$ along $\tilde{F}_1$, and consider its blow up along the face $f = \bigcap_{i=1}^{k+1} \tilde{F}_i$. It is straightforward to check directly that $\Delta'$ is a $\tilde{\Delta}$ bundle over $\Delta_k$ and that the base facets are $\tilde{F}_1 \cap \Delta', \ldots, \tilde{F}_{k+1} \cap \Delta'$; this justifies our terminology.

By Lemma 3.7, the base facets of a bundle over a simplex are equivalent, and the base-type facets of a $k$-fold expansion are equivalent. Conversely, the next lemma shows that if a polytope has equivalent facets, it is an expansion or a bundle over a simplex.

**Proposition 3.15.** Let $\Delta \subset t'$ be a simple (or smooth) polytope. Let $I \in \mathcal{I}$ be an equivalence class of facets and define $I' := I \setminus \{ n \}$ for some $n \in I$.

(i) If $F_I = \emptyset$, then $\Delta$ is a $F_{I'}$ bundle over $\Delta_{|I|-1}$ with base facets $\{ F_i \}_{i \in I}$.

(ii) If $F_I \neq \emptyset$, then $\Delta$ is the $(|I|-1)$-fold expansion of $F_I$ along $F_I = F_i \cap F_{I'}$ with base-type facets $\{ F_i \}_{i \in I}$.

**Proof.** Renumber the facets so that $I = \{ 1, \ldots, |I| \}$ and $n = |I|$. By Lemma 3.7, the plane $\tilde{t} \subset t$ spanned by the $\eta_j$ for all $j > |I|$ has codimension $|I|-1$.

Fix any $J \subset \{ |I| + 1, \ldots, N \}$ so that $F_J$ is not empty and is minimal, in the sense that it has no lower dimensional face $F_{J'} \subsetneq F_J$, where $J' \subset \{ |I| + 1, \ldots, N \}$. We claim that $F_J$ is affine equivalent to the (standard) $(|I|-1)$-simplex with facets $F_i \cap F_{I'}$ for $1 \leq i \leq |I|$; moreover, the $\eta_j$ for $j \in J$ span $\tilde{t}$. To see this, note that the only possible facets of $F_J$ have the form $F_i \cap F_{J'}$, where $1 \leq i \leq |I|$. In particular, $F_J$ has at most $|I|$ facets. On the other hand, the plane $P(F_J)$ contains a translate of the annihilator of $\tilde{t}$, which implies that $\dim(F_J) \geq |I|-1$. The claim follows.

If $F_I = \emptyset$, the proposition follows immediately. So assume instead that $F_I \neq \emptyset$. By the paragraph above, $\sum_{1 \leq i \leq |I|} \eta_i$ lies in $\tilde{t}$ and $\{ \eta_i \}_{1 \leq i < |I|}$ descends to a basis for the quotient of $t$ by $\tilde{t}$. Hence, we can identify $t$ with $\tilde{t} \oplus \mathbb{R}^{|I|-1}$ so that $\eta_i = (0, -e_i)$ for all $1 \leq i < |I|$ and $\eta_i = (\alpha, \sum_{i=1}^{|I|-1} e_i)$ for some $\alpha \in \tilde{t}$. Then the outward conormal to the facet $F_i \cap F_{I'}$ of the polytope $F_{I'}$ is the image of $\eta_i$ in the quotient of $t$ by the span of $\{ \eta_i \}_{i \in I'}$. This is $\alpha \in \tilde{t}$, as required.\(\square\)

\(^3\)We give more details of this construction in Part II.
Corollary 3.16. Let $\Delta$ be a simple polytope such that all its facets are equivalent. Then $\Delta = \Delta_k$.

Proof. In this case there is just one equivalence class of facets $I$. Since $\Delta$ is bounded, $F_I = \emptyset$. Now apply part (i) of the above proposition. 

3.3. Subtracting inessential functions. In this subsection, we show that it is often possible to use Proposition 1.17 to simplify a mass linear function by subtracting an inessential function; the resulting function will have fewer asymmetric facets.

Lemma 3.17. Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. If $F_1, \ldots, F_m$ are equivalent facets, there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following properties:

- For all $i < m$, the facet $F_i$ is $\tilde{H}$-symmetric.
- For all $i > m$, the facet $F_i$ is $\tilde{H}$-symmetric iff it is $H$-symmetric.

Proof. Since $H$ is mass linear, $\langle H, c_\Delta \rangle = \sum_{i=1}^N \beta_i \kappa_i$ for some $\beta_i \in \mathbb{R}$. Define $\alpha_i = \beta_i$ for all $1 \leq i < m$ and define $\alpha_m = -\sum_{i=1}^{m-1} \beta_i$. Then $\sum_{i=1}^m \alpha_i = 0$, and so by definition $H' = \sum_{i=1}^m \alpha_i \eta_i$ is inessential. By Proposition 1.17, $\langle H', c_\Delta \rangle = \sum_{i=1}^m \alpha_i \kappa_i$. Therefore, if $H = H - H'$ then

$$\langle \tilde{H}, c_\Delta \rangle = \left( \sum_{i=1}^m \beta_i \right) \kappa_m + \sum_{i=m+1}^N \beta_i \kappa_i,$$

as required.

Corollary 3.18. Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. If the asymmetric facets are all equivalent, then $H$ is inessential.

Proof. By Lemma 3.17, there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has at most one asymmetric facet. By Lemma 2.12, this implies that $\tilde{H} = 0$. 

This method works particularly well when the equivalent facets do not intersect, that is, when our polytope is a bundle over a simplex. To prove this, we will need the following lemmas. The first is a partial converse to Proposition 1.17.

Lemma 3.19. Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. If $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$, then

$$H = \sum \beta_i \eta_i.$$ 

Proof. Given any $\xi \in \mathfrak{t}^*$, let $\Delta' = \Delta + \xi$, the translate of $\Delta$ by $\xi$. Then

$$\Delta' = \Delta(\kappa') := \bigcap_{i=1}^N \{ x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i' \}, \text{ where } \kappa_i' = \kappa_i + \langle \eta_i, \xi \rangle \quad \forall i.$$
Hence, by assumption
\[
\langle H, c_{\Delta}(\kappa') \rangle = \sum \beta_i \kappa'_i \\
= \sum \beta_i \kappa_i + \sum \beta_i \langle \eta_i, \xi \rangle \\
= \langle H, c_{\Delta}(\kappa) \rangle + \sum \langle \beta_i \eta_i, \xi \rangle.
\]
Since clearly \(\langle H, c_{\Delta}(\kappa') \rangle = \langle H, c_{\Delta}(\kappa) \rangle + \langle H, \xi \rangle\), this implies that
\[
\langle H, \xi \rangle = \sum \langle \beta_i \eta_i, \xi \rangle \quad \forall \xi \in t^*.
\]

**Lemma 3.20.** Let \(H \in t\) be a mass linear function on a simple polytope \(\Delta \subset t^*\). Assume that \(\Delta\) is a \(\Delta\) bundle over the simplex \(\Delta_k\), and let \(\bar{t} \subset t\) be the subspace spanned by the conormals to the fiber facets. Then \(H\) lies in \(\bar{t}\) exactly if the base facets are symmetric.

**Proof.** Let \(\tilde{\eta}_1, \ldots, \tilde{\eta}_N\) be the outward conormals to the fiber facets, and let \(\tilde{\eta}_1, \ldots, \tilde{\eta}_{k+1}\) be the outward conormals to the base facets. Let \(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_N\) and \(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{k+1}\) be the associated support numbers.

If the base facets are symmetric then \(H\) lies in \(\bar{t}\) by Lemma 3.19. To prove the converse, assume that \(H\) lies in \(\bar{t}\). Because \(\Delta\) is bounded, the positive span of the \(\tilde{\eta}_i\) contains all of \(t\). In particular, \(H = \sum \beta_i \tilde{\eta}_i\), where \(\beta_i > 0\) for all \(i\). Because \(\langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j\) for all \(x \in \Delta\), we have \(\langle \tilde{\eta}_j, c_{\Delta} \rangle \leq \tilde{\kappa}_j\) for all \(j\). Hence \(\langle H, c_{\Delta} \rangle \leq \sum \beta_j \tilde{\kappa}_j\). The number \(\sum \beta_j \tilde{\kappa}_j\) does not depend on the \(\tilde{\kappa}_i\). On the other hand, because the base is a simplex, we can make any \(\tilde{\kappa}_i\) arbitrarily large without changing the combinatorics of the polytope \(\Delta\). Since \(\langle H, c_{\Delta} \rangle\) is a linear function on \(C_\Delta\), this implies that it cannot depend positively on any \(\tilde{\kappa}_i\). A similar argument shows that \(\langle H, c_{\Delta} \rangle\) cannot depend negatively on any \(\tilde{\kappa}_i\).

**Proposition 3.21.** Let \(H \in t\) be a mass linear function on a simple polytope \(\Delta \subset t^*\) which is a bundle over the simplex \(\Delta_k\). There exists an inessential function \(H' \in t\) so that the mass linear function \(\bar{H} = H - H'\) has the following properties:

- The base facets are \(\bar{H}\)-symmetric
- A fiber facet is \(\bar{H}\)-symmetric iff it is \(H\)-symmetric.

**Proof.** Let \(\tilde{\eta}_1, \ldots, \tilde{\eta}_{k+1}\) denote the outward conormals to the base facets. By Lemma 3.7 and the definition of a bundle, the base facets are equivalent. Moreover, we may decompose the vector \(H \in t\) uniquely as
\[
H = \bar{H} + \sum_{i=1}^{k+1} \alpha_i \tilde{\eta}_i, \quad \text{where} \quad \sum \alpha_i = 0 \quad \text{and} \quad \bar{H} \in \bar{t},
\]
where \(\bar{t} \subset t\) is the subspace spanned by the conormals of the fiber facets. The function \(H' = \sum_i \alpha_i \tilde{\eta}_i\) is inessential by definition. Hence, by Proposition 1.17, \(H'\) is mass linear and \(\langle H', c_{\Delta} \rangle\) depends only on the support numbers of the base facets. Thus \(\bar{H}\) is mass linear.
and the fiber facets are $\tilde{H}$-symmetric exactly if they are $H$-symmetric. The base facets are $\tilde{H}$-symmetric by Lemma 3.20.

3.4. Flat facets. In this subsection, we consider flat facets. The main result (Proposition 3.24) allows us to reduce to the case that every asymmetric facet is pervasive.

If $\Delta$ is a bundle over $\Delta_1$ then the base facets are flat and do not intersect. The lemma below proves the converse.

**Lemma 3.22.** Let $\Delta \subset t^*$ be a simple polytope. Let $F_1$ and $F_2$ be flat facets of $\Delta$ which do not intersect. Then $\Delta$ is a bundle over the simplex $\Delta_1$ with base facets $F_1$ and $F_2$.

**Proof.** By definition, since $F_1$ is flat there exists a nonzero $\xi_1 \in t^*$ so that $\langle \eta_j, \xi_1 \rangle = 0$ for every facet $F_j$ (other than $F_1$ itself) which meets $F_1$. Since $F_1$ is bounded, these $\eta_j$ span $t/\eta_1$; hence $\langle \eta_1, \xi_1 \rangle \neq 0$ and so we can renormalize $\xi_1$ so that $\langle \eta_1, \xi_1 \rangle = 1$. Similarly, there exists $\xi_2 \in t^*$ so that $\langle \eta_2, \xi_2 \rangle = 1$ but $\langle \eta_j, \xi_2 \rangle = 0$ for every (other) facet $F_j$ which meets $F_2$.

We claim that

$$\langle \eta_k, \xi_1 \rangle \leq 0 \quad \forall k \neq 1 \quad \text{and} \quad \langle \eta_k, \xi_2 \rangle \leq 0 \quad \forall k \neq 2.$$  

To see this, pick a point $x$ which lies in some $F_k$ but not in $F_1$. By definition $\langle \eta_k, x \rangle = \kappa_k$, $\langle \eta_1, x \rangle < \kappa_1$, and $\langle \eta_k, x \rangle \leq \kappa_1$ for all $i$. Define $y := x + (\kappa_1 - \langle \eta_1, x \rangle)\xi_1$. An easy calculation shows that $\langle \eta_1, y \rangle = \kappa_1$ and $\langle \eta_k, y \rangle = \kappa_k + (\kappa_1 - \langle \eta_1, x \rangle)\langle \eta_k, \xi_1 \rangle$. If $F_i \neq F_1$ is any facet which meets $F_1$ then $\langle \eta_i, \xi \rangle = 0$, and hence $\langle \eta_i, y \rangle = \langle \eta_i, x \rangle \leq \kappa_i$. Therefore, $y \in F_1 \subset \Delta$. In particular, $\langle \eta_k, y \rangle \leq \kappa_k$ for all $k$. Since $\langle \eta_1, x \rangle < \kappa_1$, this implies that $\langle \eta_k, \xi_1 \rangle \leq 0$, as required.

Next, we claim that $\xi_2$ is a multiple of $\xi_1$. To see this, let $J$ denote the set of $j \in \{2, 3, \ldots, N\}$ so that $F_j$ meets $F_1$. On the one hand, since $\langle \eta_j, \xi_1 \rangle = 0$ for all $j \in J$ and since $F_1$ is bounded, the positive span of the $\eta_j$ for $j \in J$ contains $\xi_1$, the annihilator of $\xi_1$. On the other hand, since $F_1 \cap F_2 = \emptyset$, $2$ is not in $J$. Hence (3.4) implies that $\langle \eta_j, \xi_2 \rangle \leq 0$ for all $j \in J$. Therefore, $\langle \eta_i, \xi_2 \rangle \leq 0$ for all $\alpha \in \xi_1^\perp$; the claim follows immediately.

Since $\langle \eta_2, \xi_2 \rangle = 1$ and $\langle \eta_1, \xi_2 \rangle \leq 0$ by (3.4), $\xi_2$ cannot be a positive multiple of $\xi_1$; so it must be a negative multiple. Hence, for all $k$ except 1 and 2, we have $\langle \eta_k, \xi_1 \rangle \leq 0$ and $\langle \eta_k, \xi_1 \rangle \geq 0$, that is, $\langle \eta_k, \xi_1 \rangle = 0$.

To see that $F_1$ and $F_2$ are the base facets of a bundle over $\Delta_1$ it now suffices to check that $\Delta$ is combinatorially equivalent to the product $F_1 \times \Delta_1$. Equivalently, we must check that every non-empty face $F_J$, where $J \subset \{3, \ldots, N\}$, meets both $F_1$ and $F_2$. To see this, pick $x \in F_J$ and define $y := x + (\kappa_1 - \langle \eta_1, x \rangle)\xi_1$. Since $\langle \eta_1, \xi_1 \rangle = 1$, $\langle \eta_2, \xi_1 \rangle < 0$, and otherwise $\langle \eta_j, \xi_1 \rangle = 0$, $y \in F_1 \cap F_J$. A similar argument shows that $F_2 \cap F_J \neq \emptyset$. □

**Corollary 3.23.** Let $H \in t$ be a mass linear function on a simple polytope $\Delta \subset t^*$. If $F$ is an asymmetric facet which is not pervasive, then $\Delta$ is an $F$ bundle over $\Delta_1$.

**Proof.** Let $G$ be any facet which is disjoint from $F$. By Lemma 2.7, $G$ is also asymmetric. Hence, Proposition 2.10 implies that $F$ and $G$ are flat. By Lemma 3.22, this implies that $\Delta$ is an $F$ bundle over $\Delta_1$. □

Here is the main result of this section.
Proposition 3.24. Let \( H \in \mathfrak{t} \) be a mass linear function on a simple polytope \( \Delta \subset \mathfrak{t}^* \). There exists an inessential function \( H' \in \mathfrak{t} \) so that the mass linear function \( \tilde{H} = H - H' \) has the following property: every \( \tilde{H} \)-asymmetric facet is pervasive.

Proof. We will argue this by induction on the number of asymmetric facets which are not pervasive. Let \( F \) be an \( H \)-asymmetric facet which is not pervasive. By Corollary 3.23, \( \Delta \) is an \( F \) bundle over \( \Delta_1 \). By Proposition 3.21 there exists an inessential function \( H' \in \mathfrak{t} \) so that the mass linear function \( \tilde{H} = H - H' \) has the following properties: the base facets (including \( F \)) are \( \tilde{H} \)-symmetric, and the fiber facets are \( \tilde{H} \)-symmetric exactly if they are \( H \)-symmetric. Hence, there are fewer \( \tilde{H} \)-asymmetric facets which are not pervasive. Since the sum of inessential functions is inessential, this completes the proof. □

3.5. Polytopes such that all linear functions are mass linear. This subsection is devoted to a proof of Theorem 1.19. First, note that (ii) \( \Rightarrow \) (i) by Proposition 1.17. Next, note that by Example 1.15, if \( \Delta \subset \mathfrak{t}^* \) is a product of simplices, then every \( H \in \mathfrak{t} \) is an inessential mass linear function on \( \Delta \), and \( \bigcap_{i \in I} F_i = \emptyset \) for every equivalence class of facets \( I \). Thus (iii) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (iv).

To complete the proof, we will repeatedly need the following elementary lemmas.

Lemma 3.25. Let \( \Delta \subset \mathfrak{t}^* \) be a simple polytope. Each facet \( F_i \) is \( \eta_i \)-asymmetric.

Proof. Assume that \( F_i \) is \( \eta_i \)-symmetric. By Lemma 2.6, this implies that \( \langle \eta_i, c_{F_i} \rangle = \langle \eta_i, c_{\Delta} \rangle \). Since \( \langle \eta_i, c_{F_i} \rangle = \kappa_i \), this contradicts the assumption that \( F_i \) is \( \eta_i \)-symmetric. □

Lemma 3.26. Let \( F_i \) be a pervasive facet of a simple polytope \( \Delta \subset \mathfrak{t}^* \), and let \( j, k \neq i \). Then \( F_{ji} = F_j \cap F_i \) and \( F_{ki} = F_k \cap F_i \) are equivalent as facets of \( F_i \) exactly if \( F_j \) and \( F_k \) are equivalent as facets of \( \Delta \).

Proof. Since \( F_i \) is pervasive, \( F_{ki} \) is a facet of \( F_i \) for every \( k \neq i \). By Lemma 3.7 \( F_j \sim F_k \) exactly if there is a vector \( \xi \in \mathfrak{t}^* \) that lies in the annihilator \( \eta_i^\circ \) of the conormal \( \eta_i \) for all \( \ell \neq j, k \). But \( P(F_i) \) is naturally affine isomorphic to \( \eta_i^\circ \). The claim is now easily verified. □

Proof that (i) \( \Rightarrow \) (iii) in Theorem 1.19.

Let \( \Delta \subset \mathfrak{t}^* \) be a simple polytope; let \( I \) denote the set of equivalence classes of facets. Assume that every \( H \in \mathfrak{t} \) is mass linear on \( \Delta \). We will prove that \( \Delta \) is a product of simplices. The proof is divided into a sequence of steps.

Claim I: If \( \dim \Delta \leq 2 \), then \( \Delta \) is a product of simplices.

The claim is trivial in dimension 1. So assume that \( \dim \Delta = 2 \) and that \( \Delta \neq \Delta_2 \). Since \( \Delta \) has more than three edges, no edge is pervasive. Moreover, each outward conormal \( \eta_i \) is mass linear by assumption, and each edge \( F_i \) is \( \eta_i \)-asymmetric by the lemma above. Therefore, Proposition 2.10 implies that each edge \( F_i \) of \( \Delta \) is flat. But this is possible only if \( \Delta \) is the product \( \Delta_1 \times \Delta_1 \).

We will now assume that \( \dim \Delta \geq 3 \) and that (i) \( \Rightarrow \) (iii) for all lower dimensional polytopes.

Claim II: Each proper face of \( \Delta \) is a product of simplices.
Since we have assumed that (i) \( \implies \) (iii) holds for lower dimensional polytopes, it is enough to prove every facet \( F_i \) of \( \Delta \) satisfies condition (i); that is, we must show that every \( H' \) in \( t/\langle \eta_i \rangle \), which is naturally the dual space to \( P(F_i) \), is mass linear on \( F_i \). To see this, let \( \mathcal{H}_i \) denote the set of \( H \in t \) so that the facet \( F_i \) is \( H \)-symmetric. We saw above that \( \eta_i \notin \mathcal{H}_i \). On the other hand, every \( H \in t \) is mass linear on \( \Delta \). It follows that \( \mathcal{H}_i \subset t \) is a linear subspace of codimension one, so that the natural projection from \( \mathcal{H}_i \) to \( t/\langle \eta_i \rangle \), is surjective. Finally, by Proposition 2.8, each \( H \in \mathcal{H}_i \) restricts to a mass linear function on \( F_i \).

**Claim III:** If \( F_1 \) is not pervasive, \( \Delta \) is a bundle over the 1-simplex \( \Delta_1 \).

Since \( \eta_1 \in t \) is a mass linear function on \( \Delta \) and \( F_1 \) is \( \eta_1 \)-asymmetric, \( \Delta \) is a bundle over \( \Delta_1 \) by Corollary 3.23.

**Claim IV:** If every facet of \( \Delta \) is pervasive, \( \bigcap_{i \in I} F_i = \emptyset \) for all \( I \in \mathcal{I} \).

Let \( \{F_1, \ldots, F_{k+1}\} \) be an equivalence class of facets. If \( \Delta \) has no other facets, then the conclusion holds because \( \Delta \) is bounded. (Alternatively, \( \Delta = \Delta_k \) by Corollary 3.16.) Otherwise, fix \( j > k + 1 \). Since \( F_j \) is pervasive, \( \{F_i\}_{i=1}^{k+1} \) is a nonempty equivalence class of facets of \( F_j \) by Lemma 3.26. Moreover, by Claim II, \( F_j \) is the product of simplices. Hence, by Example 1.15, \( \bigcap_{i=1}^{k+1} F_{ij} = \emptyset \). In particular, \( k > 0 \). Therefore, a nearly identical argument shows that \( \bigcap_{i=2}^{k+1} F_{i1} = \emptyset \). But \( \bigcap_{i=2}^{k+1} F_{i1} = \bigcap_{i=1}^{k+1} F_i \).

**Claim V:** \( \Delta \) is a bundle over the \( k \)-simplex \( \Delta_k \), where \( k \neq \dim \Delta - 1 \).

If some facet of \( \Delta \) is not pervasive, then \( \Delta \) is a bundle over \( \Delta_1 \) by Claim III. Since \( \dim \Delta \geq 3 \), \( \dim \Delta - 1 \neq 1 \). So assume instead that every facet is pervasive. By Claim IV, \( \bigcap_{i \in I} F_i = \emptyset \) for all \( I \in \mathcal{I} \). Therefore, applying Proposition 3.15, we see that \( \Delta \) is a bundle over the \( k \)-simplex \( \Delta_k \). Finally, if \( k = \dim \Delta - 1 \) then the fiber is 1-dimensional; hence it is \( \Delta_1 \). This contradicts the claim that every facet is pervasive.

**Claim VI:** \( \Delta \) is a product of simplices.

By Claim V, \( \Delta \) is a bundle over the \( k \)-simplex \( \Delta_k \), where \( k \neq \dim \Delta - 1 \). Label the base facets \( \{F_1, \ldots, F_{k+1}\} \). If \( k = \dim \Delta \) we are done, so assume that \( k \leq \dim \Delta - 2 \).

Since \( k + 1 < \dim \Delta \) and \( \Delta \) is bounded, there exists a facet \( F_i \) so that \( \eta_i \) does not lie in the span of \( \eta_1, \ldots, \eta_{k+1} \). On the one hand, \( F_i \) is also a bundle over \( \Delta_k \) with base facets \( F_1 \cap F_i, \ldots, F_{k+1} \cap F_i \). On the other hand, \( F_i \) is a product of simplices by Claim II. Together, these facts imply that the projections of \( \eta_1, \ldots, \eta_{k+1} \) to \( t/\langle \eta_i \rangle \) are linearly dependent. Since \( \eta_i \) does not lie in the span of \( \eta_1, \ldots, \eta_{k+1} \), this implies that \( \eta_1, \ldots, \eta_{k+1} \) are themselves linearly dependent, that is, \( \Delta \) is a trivial bundle over \( \Delta_k \). Finally, the fiber of this bundle is naturally isomorphic to the face \( \bigcap_{i=2}^{k+1} F_i \). By Claim II, this face is a product of simplices.

\( \square \)

**Proof that (iv) \( \implies \) (iii) in Theorem 1.19.**

Let \( \Delta \subset t^* \) be a simple polytope; let \( \mathcal{I} \) denote the set of equivalence classes of facets. Assume that \( \bigcap_{i \in I} F_i = \emptyset \) for all \( I \in \mathcal{I} \). We will prove that \( \Delta \) is a product of simplices. As before, the proof is divided into a sequence of steps.
Claim I: If $\dim \Delta \leq 2$, then $\Delta$ is a product of simplices.

If $\dim \Delta = 2$ then Proposition 3.15 implies that either $\Delta = \Delta_2$, or each edge $F_i$ of $\Delta$ is flat. As in the previous case, the claim follows.

Claim II: Every proper face of $\Delta$ is a product of simplices.

Since we have assumed that $(iv) \implies (iii)$ holds for lower dimensional polytopes, it is enough to prove that every facet $F_i$ satisfies condition $(iv)$; in other words, we must show that $\bigcap_{i \in I} F_i = \emptyset$ for every equivalence class $\{F_i\}_{i \in I}$ of facets of $F_1$. Even if $F_1$ is not pervasive, it is clear from the definitions that if $F_i$ and $F_j$ are equivalent facets of $\Delta$ then $F_i$ and $F_j$ are equivalent facets of $\Delta$. Therefore, for each equivalence class $I$ of facets of $F_1$, the set $\{F_1\} \cup \{F_i\}_{i \in I}$ contains an equivalence class of facets of $\Delta$. By assumption, this implies that $F_1 \cap \bigcap_{i \in I} F_i = \emptyset$. But $F_1 \cap \bigcap_{i \in I} F_i = \bigcap_{i \in I} F_i$.

Claim III: If $F_1$ is not pervasive, $\Delta$ is a bundle over the 1-simplex $\Delta_1$.

Let $I \in \mathcal{I}$ be the equivalence class of facets which contains $F_1$. By assumption, $\bigcap_{i \in I} F_i = \emptyset$. By Proposition 3.15, this is only possible if $\Delta$ is a bundle over $\Delta_1$.

The rest of the proof now follows exactly as in the previous case. (Although in fact Claim IV is true by assumption.)

4. Polytopes in 2 and 3 dimensions

In this section, we give a complete classification of polygons and smooth 3-dimensional polytopes which admit mass linear functions. Here, a (convex) polygon is a simple 2-dimensional polytope.

4.1. Polygons. In this subsection, we describe mass linear functions on polygons, showing that they are all inessential.

We shall need the following lemma; since every edge has only two facets, it is an immediate consequence of Lemmas 2.7 and 2.12.

Lemma 4.1. Fix a nonzero $H \in t$ and a simple polytope $\Delta \subset t^*$. If $\Delta$ has a symmetric edge then it has exactly two asymmetric facets.

Here is our main result; it is an elaboration of Theorem 1.3.

Proposition 4.2. Let $H \in t$ be a nonzero mass linear function on a polygon $\Delta \subset t^*$. Then one of the following statements holds:

- $\Delta$ is a triangle; at most one edge is symmetric.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_1$; the base facets are the asymmetric edges.
- $\Delta$ is the product $\Delta_1 \times \Delta_1$; each edge is asymmetric.

In any case, $H$ is inessential.

Proof. Assume that $\Delta$ has a symmetric edge. By Lemma 4.1, $\Delta$ has two asymmetric edges. Moreover, by Lemma 2.7, every symmetric edge must meet both asymmetric edges; in particular, there are at most two symmetric edges. If there is one symmetric edge, $\Delta$ is a triangle. If there are two symmetric edges, then the asymmetric edges are not pervasive.
and so by Proposition 2.10 they are flat. This implies that $\Delta$ is a $\Delta_1$ bundle over $\Delta_1$, and that the fiber facets are symmetric. In either case, the asymmetric facets are equivalent. Hence, $H$ is inessential by Corollary 3.18.

So assume that every edge is asymmetric. If $\Delta$ has three edges, it is a triangle. Otherwise, since none of the edges are pervasive Proposition 2.10 implies that they are all flat. By definition, this is impossible unless $\Delta$ is the product $\Delta_1 \times \Delta_1$. Moreover, all the facets of a triangle are equivalent, and the two pairs of opposite facets of $\Delta_1 \times \Delta_1$ are equivalent. It is easy to check that in either case every $H' \in t$ is inessential. □

The next corollary will be useful in Part II.

**Corollary 4.3.** Let $H \in t$ be a mass linear function on a polygon $\Delta \subset t^\ast$. If two edges $F_i$ and $F_j$ do not intersect then $\gamma_i + \gamma_j = 0$, where $\gamma_k$ is the coefficient of the support number of $F_k$ in the linear function $\langle H, c_\Delta \rangle$.

**Proof.** If $H \neq 0$, then the lemma above implies that $H$ is inessential and $F_i$ and $F_j$ are opposite edges of a quadrilateral. Hence while they may (or may not) be equivalent to each other, neither is equivalent to any other edge. Therefore Proposition 1.17 implies that either $\gamma_i = \gamma_j = 0$ (if $F_i \not\sim F_j$) or $\gamma_i + \gamma_j = 0$. □

### 4.2. Polytopes with two asymmetric facets.

We now show that any mass linear function on a simple polytope with at most two asymmetric facets is inessential. In this paper, we need this result for the 3-dimensional case, but it is valid in all dimensions.

**Proposition 4.4.** Let $H \in t$ be a mass linear function on a simple polytope $\Delta \subset t^\ast$ with exactly two asymmetric facets. Then the asymmetric facets are equivalent.

**Proof.** Let $F_1$ and $F_2$ be the asymmetric facets.

First consider a 2-dimensional symmetric face $Y$ with two symmetric edges $e$ and $e'$. By Proposition 2.8, $Y$ has exactly two asymmetric edges $F_1 \cap Y$ and $F_2 \cap Y$, and the restriction of $H$ to $Y$ is mass linear. Therefore by Proposition 4.2, $Y$ is a $\Delta_1$ bundle over $\Delta_1$ and $e, e'$ are fiber facets. Hence they are parallel.

More generally, define a graph $\Gamma$ as follows: let $V := V(\Gamma)$ be the set of vertices in $F_1 \setminus (F_1 \cap F_2)$ and let $E := E(\Gamma)$ be the set of edges which join such vertices. By Lemma 2.7 every symmetric face intersects $F_1$. Hence, since $\Delta$ is simple, intersection with $F_1$ induces a one-to-one correspondence between the set of symmetric edges and $V$, and also a one-to-one correspondence between the set of 2-dimensional symmetric faces with two symmetric edges and $E$. Hence, if $e$ and $e'$ are symmetric edges so that $e \cap F_1$ and $e' \cap F_1$ lie in the same component of $\Gamma$, then $e$ and $e'$ are parallel. Since $\Gamma$ is clearly connected, this implies that all symmetric edges are parallel.

Every symmetric facet $G$ contains a symmetric face $g$ which is minimal in the sense that it has no symmetric faces. By Proposition 2.8, $g$ is a polytope with exactly two facets $F_1 \cap g$ and $F_2 \cap g$. This is only possible if $g$ is a symmetric edge. Therefore, the conormals to the symmetric facets lie in a codimension 1 subspace. By Lemma 3.7, this implies that the asymmetric facets are equivalent. □

This proposition has the following important consequence.
Corollary 4.5. Let \( H \in t \) be a mass linear function on a simple polytope \( \Delta \subset t^* \) with exactly two asymmetric facets \( F_1 \) and \( F_2 \). Then \( H \) is inessential, and exactly one of the following occurs.

- \( \Delta \) is a \( F_1 \) bundle over \( \Delta_1 \); the base facets are the asymmetric facets, or
- \( \Delta \) is the expansion of \( F_1 \) along \( F_{12} \); the base-type facets are the asymmetric facets.

Proof. By Proposition 4.4, \( F_1 \) and \( F_2 \) are equivalent. Thus the claim follows immediately from Corollary 3.18 and Proposition 3.15. \( \square \)

4.3. Mass linear functions on the polytopes defined in Example 1.1. In this subsection, we calculate all mass linear functions on the polytopes defined in Example 1.1. In particular, we show that these functions are not all inessential. Let’s recall this example. Given \( \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \), define

\[ \eta_1 = -e_1, \ \eta_2 = -e_2, \ \eta_3 = e_1 + e_2, \ \eta_4 = -e_3, \ \eta_5 = e_3 + a_1 e_1 + a_2 e_2, \ \text{and} \]

\[ \mathcal{C}_a = \left\{ \kappa \in \mathbb{R}^5 \bigg| \sum_{i=1}^{3} \kappa_i > 0 \text{ and } \kappa_4 + \kappa_5 > -a_1 \kappa_1 - a_2 \kappa_2 + \max(0, a_1, a_2) \sum_{i=1}^{3} \kappa_i \right\}. \]

Given \( \kappa \in \mathcal{C}_a \), let

\[ Y = Y(\kappa) = \bigcap_{i=1}^{5} \{ x \in (\mathbb{R}^3)^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}. \]

Here is the main result of this section.

Proposition 4.6. Let \( Y \) be the polytope defined in Example 1.1. Then \( H \in t \) is mass linear function on \( Y \) exactly if

\[ H = \sum_{i=1}^{5} \gamma_i \eta_i, \ \text{where } \gamma_1 + \gamma_2 + \gamma_3 = 0, \ \gamma_4 + \gamma_5 = 0, \ \text{and} \ a_1 \gamma_1 + a_2 \gamma_2 = 0; \]

in this case, \( \langle H, c_Y \rangle = \sum_{i=1}^{5} \gamma_i \kappa_i \). Moreover, \( H \) is inessential exactly if at least one of the following conditions holds: \( a_1 a_2 (a_1 - a_2) = 0 \), or \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \).

Remark 4.7. In particular, all vectors \( H \in t \) are mass linear on \( Y \) (and are inessential) exactly if the condition \( a_1 \gamma_1 + a_2 \gamma_2 = 0 \) is always satisfied, i.e. \( a_1 = a_2 = 0 \) and \( Y \) is the product \( \Delta_1 \times \Delta_2 \). This is consistent with Theorem 1.19. Otherwise, \( \Delta \) admits a 2-dimensional family of mass linear functions. If \( a_1 a_2 (a_1 - a_2) = 0 \), these functions are all inessential. Otherwise, there is a 1-dimensional family of inessential functions; the rest are essential.

The proof of this proposition rests mainly on the following direct calculation.

Lemma 4.8. Let \( Y \) be the polytope defined in Example 1.1, and let \( H = \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3 \), where \( \gamma_1 + \gamma_2 + \gamma_3 = 0 \). Then \( H \) is mass linear on \( Y \) if and only if

\[ \gamma_1 a_1 + \gamma_2 a_2 = 0; \ \text{in this case} \ \langle H, c_Y \rangle = \sum_{i=1}^{3} \gamma_i \kappa_i. \]
Proof. As a first step, fix $\kappa_1 = \kappa_2 = \kappa_4 = 0$, and let $\kappa_3 = \lambda$ and $\kappa_5 = h$. Let $\Delta^2_3 \subset \mathbb{R}^2$ denote the 2-simplex given by the inequalities

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \text{and} \quad x_1 + x_2 \leq \lambda.$$  

An elementary calculation shows that for any non-negative integers $i_1$ and $i_2$

$$\int_{\Delta^2_3} x_1^{i_1} x_2^{i_2} = \frac{i_1! i_2! \lambda^{i_1 + 2}}{(I + 2)!},$$

where $I = i_1 + i_2$ and where by convention $0! = 1$. Further, both here and below we integrate with respect to the standard measure $dx_1 dx_2$ on $\mathbb{R}^2$.

Since $Y$ is a $\Delta^2_3$ bundle over $\Delta_1$, $Y$ has volume

$$V = \int_{\Delta^2_3} (h - a_1 x_1 - a_2 x_2) = \frac{3h\lambda^2 - (a_1 + a_2)\lambda^3}{3!}.$$  

For $j \neq 3$, the moment $\mu_j$ of $Y$ along the $x_j$ axis is

$$\mu_j = \int_{\Delta^2_3} ((h - a_1 x_1 - a_2 x_2)x_j) = \frac{4h\lambda^3 - (a_j + a_1 + a_2)\lambda^4}{4!}.$$  

Let $c_j := \mu_j / V$ denote the $j$’th component of the center of mass. For $j \neq 3$,

$$c_j = \lambda \frac{4h - \lambda(a_j + a_1 + a_2)}{3h - \lambda(a_1 + a_2)}.$$  

Since $\gamma_1 + \gamma_2 + \gamma_3 = 0$, a straightforward calculation shows that

$$\langle H, c_Y \rangle = \sum_{i=1}^{2} (\gamma_3 - \gamma_i) c_i = \lambda \left( \gamma_3 + \frac{\lambda(\gamma_1 a_1 + \gamma_2 a_2)}{12h - 4\lambda(a_1 + a_2)} \right).$$  

This is a linear function of $h$ and $\lambda$ exactly if $\gamma_1 a_1 + \gamma_2 a_2 = 0$. Hence if $H$ is mass linear, this condition must be satisfied.

To prove the converse, assume that $\gamma_1 a_1 + \gamma_2 a_2 = 0$. Given $\kappa \in C_a$, note that by Equation (3.3) in Lemma 3.19,

$$Y(\kappa) = Y(0, 0, \lambda, 0, h) - (\kappa_1, \kappa_2, \kappa_4),$$  

where $\lambda = \kappa_1 + \kappa_2 + \kappa_3$ and $h = \kappa_4 + \kappa_5 + a_1 \kappa_1 + a_2 \kappa_2$.

Hence,

$$\langle H, c_Y(\kappa) \rangle = \langle H, c_Y(0, 0, \lambda, 0, h) \rangle - \langle H, (\kappa_1, \kappa_2, \kappa_4) \rangle = \sum_{i=1}^{3} \kappa_i \gamma_i,$$

as required. \qed

Remark 4.9. Lemma 4.8 implies the following fact. If $H = \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$ is mass linear on $Y$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$, then $\langle H, c_Y \rangle = \sum_{i=1}^{3} \gamma_i \kappa_i$. In fact, there is an easier proof of this fact. As we saw in Example 3.11, $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$. By Lemma 3.20, the base facets $F_4$ and $F_5$ are symmetric. By Proposition 2.8 the restriction of $H$ to $F_4$ is mass linear and the coefficient of the support number of $F_i$ in $\langle H, c_Y \rangle$ is the coefficient of
the support number of $F_i \cap F_j$ in $\langle H, c_{F_i} \rangle$ for all $1 \leq i \leq 3$. But since $F_4$ is the 2-simplex, every $H$ is inessential. The claim now follows from Proposition 1.17.

We are now ready to prove our main proposition.

**Proof of Proposition 4.6.** First, note that every $H \in t$ can be written uniquely as $H = \sum_{i=1}^{5} \gamma_i \eta_i$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\gamma_4 + \gamma_5 = 0$. By Example 1.16, $F_4$ and $F_5$ are equivalent, and so

$$\langle \gamma_4 \eta_4 + \gamma_5 \eta_5, cy \rangle = \gamma_4 \kappa_4 + \gamma_5 \kappa_5$$

by Proposition 1.17. Hence $H$ is mass linear exactly if $\tilde{H} := \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$ is mass linear. On the other hand, by Lemma 4.8, $\tilde{H}$ is mass linear exactly if $a_1 \gamma_1 + a_2 \gamma_2 = 0$, in which case

$$\langle \tilde{H}, cy \rangle = \gamma_1 \kappa_1 + \gamma_2 \kappa_2 + \gamma_3 \kappa_3.$$

The first claim follows immediately.

To prove the last statement, it is convenient to introduce a dummy variable $a_3$ that we set equal to 0. By Lemma 3.7, $F_4$ is equivalent to $F_5$, but neither is equivalent to $F_1$, $F_2$, or $F_3$. Moreover, for any $\{i,j\} \subset \{1,2,3\}$, $F_i$ is equivalent to $F_j$ exactly if $a_i = a_j$. Hence, if $\gamma_4 + \gamma_5 = 0$, then $H = \gamma_4 \eta_4 + \gamma_5 \eta_5$ is inessential. Similarly, if $a_1 = a_2$, for example, and $\gamma_1 + \gamma_2 = 0$, then $H = \gamma_1 \eta_1 + \gamma_2 \eta_2$ is inessential. The cases where $a_1 = a_3$ or $a_2 = a_3$ follow similarly. In contrast, if $a_1 a_2 (a_1 - a_2) \neq 0$ then every mass linear $H$ is essential unless $\gamma_1 = \gamma_2 = \gamma_3$. □

**4.4. Smooth 3-dimensional polytopes.** In this subsection we describe mass linear functions on 3-dimensional smooth polytopes. In particular, we show that the only 3-dimensional smooth polytopes which admit essential mass linear functions are $\Delta_2$ bundles over $\Delta_1$.

We will need the following lemma, which holds by a straightforward calculation. Note that the claim is false without the smoothness assumption; this lemma is the one extra piece of information that we use in the smooth case.

**Lemma 4.10.** Let $\Delta$ be a smooth polytope which is combinatorially equivalent to $\Delta_k \times \Delta_n$. Then $\Delta$ is either a $\Delta_k$ bundle over $\Delta_n$, or a $\Delta_n$ bundle over $\Delta_k$.

**Lemma 4.11.** Let $H \in t$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset t^\ast$ with more than two asymmetric facets. If every asymmetric facet is pervasive, then one of the following statements holds:

- $\Delta$ is the simplex $\Delta_3$.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_2$; the base facets are the asymmetric facets.
- $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$; the fiber facets are the asymmetric facets.

**Proof.** Let $A$ be the set of asymmetric facets and let $S$ be the set of symmetric facets. By Lemma 4.1, $\Delta$ has no symmetric edges. Since the intersection of two symmetric facets is by definition a symmetric face, this implies that the symmetric facets are disjoint. Since every asymmetric facet is pervasive, there are $\frac{1}{2} |A| (|A| - 1) + |A||S|$ edges. Since the polytope is
simple and 3-dimensional, there are two-thirds as many vertices. Since the Euler number of $\Delta$ is 2,
\[
|A| + |S| - \frac{1}{6}|A|(|A| - 1) - \frac{1}{3}|A||S| = 2,
\]
that is,
\[
(|A| - 3) ((|A| + 2|S| - 4) = 0.
\]
Assume first that $|A| + 2|S| = 4$. Since every 3-dimensional polytope has at least four facets, this implies that $|A| = 4$ and $|S| = 0$; hence $\Delta$ is the simplex $\Delta_3$.

So assume that $|A| = 3$; label the asymmetric facets $F_1, F_2$, and $F_3$. Since no 3-dimensional polytope has only three facets, $|S| \neq 0$. Since every symmetric facet $G$ intersects each asymmetric facet, $G$ is a triangle, and hence also intersects $F_1 \cap F_2$, $F_1 \cap F_3$, and $F_2 \cap F_3$. Since the edge $F_1 \cap F_2$ has two vertices, $|S|$ is 1 or 2. In the first case, $\Delta$ is again a tetrahedron. In the second case, $\Delta$ is combinatorially equivalent to the product $\Delta_2 \times \Delta_1$. The claim now follows from Lemma 4.10.

**Remark 4.12.** More generally, let $\Delta$ be a 3-dimensional simple polytope; assume that every facet is pervasive. As in the argument above, the fact that there are two-thirds as many vertices as edges and that the Euler number of $\Delta$ is 2 implies that $\Delta$ is combinatorially equivalent to a 3-simplex.

In contrast, there are many 4-dimensional simple polytopes with the property that every facet is pervasive. For example, $C_4(N)$, the 4-dimensional cyclic polytope with $N$ vertices is simplicial (each facet contains exactly 4 vertices) and 2-neighborly (each pair of vertices is joined by an edge); cf. [10]. Therefore, its dual polytope is a 4-dimensional simple polytope with $N$ facets, each of which is pervasive.

We are now ready to classify 3-dimensional smooth polytopes with essential mass linear functions.

**Proof of Theorem 1.4.** By Lemma 2.12, if $H \neq 0$, then $\Delta$ has at least two asymmetric facets. If it has exactly two, then the claim follows by Corollary 4.5. So assume that $\Delta$ has more than two asymmetric facets. By Proposition 3.24, after possibly subtracting an inessential function, we may assume that every asymmetric facet is pervasive. Hence, by Lemma 4.11, $\Delta$ is either the simplex $\Delta_3$, a $\Delta_1$ bundle over $\Delta_2$ with symmetric fiber facets, or a $\Delta_2$ bundle over $\Delta_1$ with symmetric base facets. In either of the first two cases, the asymmetric facets are equivalent. Hence $H$ is inessential by Corollary 3.18. Therefore, $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$, that is, $\Delta$ is the polytope described in Example 1.1.

With a little more effort, we can give a complete list of all smooth 3-dimensional polytopes which admit nonconstant mass linear functions.

**Theorem 4.13.** Let $H \in \mathfrak{t}$ be a nonzero mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$. One of the following statements holds:

- $\Delta$ is a bundle over a simplex.
- $\Delta$ is a 1-fold expansion.

**Proof.** The 3-simplex $\Delta_3$ is the expansion of the 2-simplex $\Delta_2$ along an edge. Moreover, a $\Delta_2$ bundle over $\Delta_1$, a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$, and the product $\Delta_1 \times \Delta_1 \times \Delta_1$ are all bundles
over the 1-simplex $\Delta_1$. Therefore, this result follows immediately from Proposition 4.14 below.

**Proposition 4.14.** Let $H \in t$ be a nonzero mass linear function on a smooth 3-dimensional polytope $\Delta \subset t^*$. One (or more) of the following statements holds:

- $\Delta$ is a bundle over $\Delta_1$; the base facets are the asymmetric facets.
- $\Delta$ is a 1-fold expansion; the base-type facets are the asymmetric facets.
- $\Delta$ is the simplex $\Delta_3$.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_2$; the base facets are the asymmetric facets.
- $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$; if either base facet is asymmetric then both are.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$; the base facets are the asymmetric facets.
- $\Delta$ is the product $\Delta_1 \times \Delta_1 \times \Delta_1$; every facet is asymmetric.

**Proof.** By Lemma 2.12, $\Delta$ has at least two asymmetric facets. If it has exactly two, then the claim follows by Corollary 4.5. So assume that $\Delta$ has more than two asymmetric facets.

If all the asymmetric facets are pervasive, then the claim holds by Lemma 4.11. So assume that there is an asymmetric facet $F$ which is not pervasive.

Corollary 3.23 implies that $\Delta$ is bundle over $\Delta_1$ with base facets $F$ and $G$. By Lemma 2.7, $G$ is also asymmetric. By Proposition 3.21, there exists a mass linear function $\tilde{H} \in t$ so that $\tilde{H} - H$ is inessential, the base facets are $\tilde{H}$-symmetric, and at least one fiber facet is still $\tilde{H}$-asymmetric. By Proposition 2.8, the restriction of $\tilde{H}$ to the 2-dimensional polygon $F$ is mass linear and at least one edge of $F$ is $\tilde{H}$-asymmetric. By Proposition 4.2, this implies that $F$ has at most four edges.

If $F$ is a triangle, then $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$ and both base facets are $H$-asymmetric (although they are $H$-symmetric).

If $F$ is a quadrilateral, $\Delta$ is combinatorially equivalent to the product $\Delta_1 \times \Delta_1 \times \Delta_1$. Suppose that $\Delta$ has an $H$-symmetric facet. Since every symmetric facet intersects every asymmetric facet and there are at least three asymmetric facets, this implies that $\Delta$ has exactly two symmetric facets and they do not intersect. Since none of the facets are pervasive, Proposition 2.10 implies that every asymmetric facet is flat, that is, there exists a hyperplane which contains the conormal of every facet that meets it. If $F$ and $F'$ are asymmetric facets which intersect, then the associated hyperplanes must be distinct, and so they intersect in a line $L$. Since both symmetric facets intersect $F$ and $F'$, the two symmetric faces have conormals in $L$ and hence are parallel. Moreover each is affine equivalent to the product $\Delta_1 \times \Delta_1$. Hence, $\Delta$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$ and the fiber facets are symmetric. On the other hand, if none of the facets are $H$-symmetric then a very similar argument proves that every pair of opposite faces is parallel. Hence, $\Delta$ is the product $\Delta_1 \times \Delta_1 \times \Delta_1$.

We will need the following result in the next paper.
Lemma 4.15. Let $H \in \mathfrak{t}$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$. Then
\[ \sum_{i=1}^{N} \gamma_i = 0, \]
where $\gamma_i$ is the coefficient of the support number of the facet $F_i$ in the linear function $\langle H, c_\Delta \rangle$.

Proof. If $H$ is inessential, then $\sum \gamma_i = 0$ by Proposition 1.17. Otherwise, by Theorem 1.4, $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$. The result now follows immediately from Proposition 4.6. (See also Remark 4.9.)

5. Relationship to Geometry

We begin by constructing the symplectic toric manifold associated to each smooth polytope. Let $\Delta = \bigcap_{i=1}^{N} \{ x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ be a smooth polytope. Let $e_1, \ldots, e_N$ denote the standard basis for $\mathbb{R}^N$, and $e_1^*, \ldots, e_N^*$ denote the dual basis for $(\mathbb{R}^N)^*$. Let $\pi: \mathbb{R}^N \to \mathfrak{t}$ denote the linear function given by $\pi(e_i) = \eta_i$ for all $i$; let $\ell$ denote its kernel. Since $\Delta$ is bounded, $\pi$ is surjective. Thus we have dual short exact sequences
\[
(5.1) \quad 0 \to \ell \xrightarrow{\mu} \mathbb{R}^N \xrightarrow{\pi} \mathfrak{t} \to 0, \quad \text{and}
\]
\[
(5.2) \quad 0 \to \mathfrak{t}^* \xrightarrow{\pi^*} (\mathbb{R}^N)^* \xrightarrow{\ell^*} \mathfrak{t}^* \to 0.
\]

Since $\eta_i \in \ell$ for all $i$, $\pi$ induces a surjective map from $(S^1)^N$ to $T := \mathfrak{t}/\ell$; since $\eta_1, \ldots, \eta_N$ generate $\ell$, the kernel $K$ of this map is connected. The moment map $\mu: \mathbb{C}^N \to \mathfrak{t}^*$ for the natural $K$ action on $\mathbb{C}^N$ is given by $\mu(z) = i^* \left( \frac{1}{2} |z_1|^2, \ldots, \frac{1}{2} |z_N|^2 \right)$. Let $(M_\Delta, \omega_\Delta)$ be the symplectic quotient $\mathbb{C}^N/K := \mu^{-1}(t^*(\kappa))/K$. The $(S^1)^N$ action on $\mathbb{C}^N$ descends to an effective $T$ action on $M_\Delta$: the moment map $\Phi_\Delta: M_\Delta \to \mathfrak{t}^*$ for this action is given by $\Phi_\Delta([z]) = (\pi^*)^{-1} \left( \frac{1}{2} |z_1|^2 - \kappa_1, \ldots, \frac{1}{2} |z_N|^2 - \kappa_N \right)$ for all $[z] \in M_\Delta$. Hence, $\Phi_\Delta(M_\Delta) = \Delta$, as required.

Note also that $M_\Delta$ has a canonical $\omega_\Delta$-compatible complex structure $J_\Delta$, and hence has a natural Kähler structure. To define $J_\Delta$, denote by $(\omega_0, J_0)$ the standard symplectic form and complex structure on $\mathbb{C}^N$ and define $V := \mu^{-1}(t^*(\kappa))$. Then the bundle $TV \cap J_0(TV)$ is $\omega_0$-orthogonal to the $K$-orbits and its complex structure descends to $J_\Delta$ on $T(M_\Delta)$. For further details see [2], for example.

Remark 5.1. There is an alternate way to construct $M_\Delta$. Given $z \in \mathbb{C}^N$, let $I_z$ be the set of $i \in \{1, \ldots, N\}$ so that $z_i = 0$. Define a set
\[ \mathcal{U} := \left\{ z \in \mathbb{C}^N \mid \bigcap_{i \in I_z} F_i \neq \emptyset \right\}. \]
As a complex manifold, $M_\Delta = \mathcal{U}/K_\mathbb{C}$, where $K_\mathbb{C}$ denotes the complexification of $K$. Moreover, under this quotient map, the $i$'th coordinate hyperplane in $\mathbb{C}^N$ corresponds to $\Phi^{-1}(F_i)$, the preimage of the $i$'th facet under the moment map. For more details, see Audin [1, Ch VII] or Cox–Katz [5]. From this description, it is clear that $J_\Delta$ does not depend on $\kappa$. 

□
The cohomology $H^2(M; \mathbb{R})$ is naturally isomorphic to $\mathfrak{k}^*$. Under this isomorphism, the cohomology class of the symplectic form $\omega$ corresponds to $\iota^*(\kappa) \in \mathfrak{k}^*$. The integral $\frac{1}{n!} \int_M \omega^n$ is (up to a normalizing constant) the volume of $\Delta$ with respect to standard Lebesgue measure.

Many of the terms that we introduce for polytopes have geometric interpretations. For example, a facet $F$ of a smooth polytope $\Delta$ is flat exactly if the corresponding divisor $\Phi^{-1}(F)$ in $M$ has self-intersection zero.

**Remark 5.2.** Polytopes which are bundles in the sense of Definition 3.9 correspond to bundles of symplectic toric manifolds. To see this, let $\Delta \subset \mathfrak{k}^*$ be a bundle with fiber $\Delta \subset \mathfrak{t}^*$ over the base $\Delta \subset \mathfrak{t}^*$; we will use the notation of Definition 3.9. Construct the associated toric manifolds $M_\Delta = U/K_C$, $\hat{M}_\Delta = \hat{U}/\hat{K}_C$, and $M_\Delta = \hat{U}/\hat{K}_C$ as in Remark 5.1; identify $\mathbb{C}^{\hat{N}+\hat{N}}$ with $\mathbb{C}^{\hat{N}} \times \mathbb{C}^{\hat{N}}$. Since $\Delta$ is combinatorially equivalent to $\Delta \times \hat{\Delta}$, $U = \hat{U} \times \hat{U}$. Moreover, since $\tilde{\eta}_j' = \iota(\tilde{\eta}_j)$ for all $j$, $\pi(\tilde{\eta}_j') = \tilde{\eta}_j$ for all $i$, $\tilde{K}$ is the intersection of $K$ with $(S^1)^{\hat{N}} \subset (S^1)^{\hat{N}} \times (S^1)^{\hat{N}}$, and $\tilde{K}$ is the image of $K$ under the natural projection $(S^1)^{\hat{N}} \times (S^1)^{\hat{N}} \rightarrow (S^1)^{\hat{N}}$; in particular, $\tilde{K} = K/\tilde{K}$. Therefore, the natural projection from $K \subset (S^1)^{\hat{N}} \times (S^1)^{\hat{N}}$ to $(S^1)^{\hat{N}}$ induces a homomorphism $\rho$ from $\tilde{K} \simeq K/\tilde{K}$ to $\tilde{T} = (S^1)^{\hat{N}}/\tilde{K}$. The toric manifold $M_\Delta$ is the associated bundle

$$M_\Delta = M_\Delta \times K_C \hat{U},$$

where $\rho$ describes the action of $K_C$ on $M_\Delta$.

**Example 5.3.** Given $a = (a_1, a_2) \in \mathbb{R}^2$, let $Y_a$ be the polytope defined in Example 1.1. As we saw in Example 3.11, $Y_a$ is a $\Delta_2$ bundle over $\Delta_1$. Recall that the outward conormals $\eta_1, \ldots, \eta_5$ are given by

$$(-1, 0, 0), \ (0, -1, 0), \ (1, 1, 0), \ (0, 0, -1), \ \text{and} \ (a_1, a_2, 1);$$

these satisfy

$$\eta_1 + \eta_2 + \eta_3 = 0 \ \text{and} \ a_1 \eta_1 + a_2 \eta_2 + \eta_4 + \eta_5 = 0.$$ 

Thus, by definition of the map $\pi$ in Equation (5.1), $K$ is the subtorus of $(S^1)^5$ generated by the elements $(1, 1, 1, 0, 0)$ and $(a_1, a_2, 0, 1, 1)$ in $\mathbb{Z}^5$. Moreover, by Definition 5.1,

$$U = (\mathbb{C}^3 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}).$$

Therefore $M_{Y_a}$ is the $\mathbb{C}P^2$ bundle

$$\mathbb{C}P^2 \times_{C^*} \left( \mathbb{C}^2 \setminus \{0\} \right)$$

over $\mathbb{C}P^1$, where $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ acts on $\mathbb{C}P^2$ by

$$\lambda \cdot [z_1 : z_2 : z_3] = [\lambda^{a_1} z_1 : \lambda^{a_2} z_2 : z_3],$$

that is, $M_{Y_a} = \mathbb{P}(\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(0))$. 

Remark 5.4. Polytopes which are 1-fold expansions in the sense of Definition 3.12 correspond to symplectic toric manifolds which are smooth symplectic pencils; that is, they are swept out by a family \( \tilde{M}_\lambda, \lambda \in \mathbb{C}P^1 \), of (real) codimension 2 symplectic submanifolds (often called fibers) that intersect transversally along a codimension 4 symplectic submanifold called the axis or base locus. To see this, let \( \Delta \subset t^* \) be the 1-fold expansion of a polytope \( \tilde{\Delta} \subset \hat{t}^* \). Construct the associated toric manifolds \( M_\Delta = \mathcal{U}/K_\mathcal{C} \) and \( M_\tilde{\Delta} = \hat{\mathcal{U}}/\hat{K}_\mathcal{C} \), as in Remark 5.1. We will use the notation of Definition 3.12 and identify \( \mathbb{C}^N/\mathbb{C}^{N-2} \) with \( \mathbb{C}^2 \times \mathbb{C}^N \). There is a well-defined map from \( \{ z \in \mathcal{U} \mid z_1z_2 \neq 0 \} \) to \( \mathbb{C}^2 \setminus \{0\} \) given by \( z \mapsto (z_1, z_2) \). Since the \( \tilde{\eta}_j \) all lie in the hyperplane \( t \), this descends to an equivariant holomorphic map from \( M_\Delta \setminus \Phi^{-1}(\hat{F}_1 \cap \hat{F}_2) \) to \( \mathbb{C}P^1 \). Therefore, \( M_\Delta \) is a pencil with axis \( \Phi^{-1}(\hat{F}_1 \cap \hat{F}_2) \). Notice that the toric manifold obtained by blowing up \( M_\Delta \) along this axis as in Remark 3.14 is a fibration over \( \mathbb{C}P^1 \) with fiber \( \tilde{M}_\Delta \).

Our proof of Proposition 1.21 is based on Weinstein’s action homomorphism
\[
\mathcal{A}_\omega: \pi_1(\text{Ham}(M, \omega)) \to \mathbb{R}/\mathcal{P}_\omega.
\]
Here, \( \text{Ham}(M, \omega) \) denotes the group of Hamiltonian symplectomorphisms of \((M, \omega)\) and \( \mathcal{P}_\omega \subset \mathbb{R} \), the period group of \( \omega \), is the image of \( \pi_2(M) \) under the homomorphism \( \alpha \mapsto \int_\alpha \omega \). The homomorphism \( \mathcal{A}_\omega \) is defined as follows: If the loop \( \{ \phi_t \}_{t \in S^1} \) in \( \text{Ham}(M, \omega) \) is generated by the mean normalized Hamiltonian\(^4 \) \( K_t \) for \( t \in [0, 1] \), then
\[
\mathcal{A}_\omega(\{ \phi_t \}) := \int_0^1 K_t(\phi_t(p)) dt - \int_D u^*(\omega)
\]
where \( p \) is any point in \( M \) and \( u: D^2 \to M \) is any smooth map such that \( u(e^{2\pi it}) = \phi_t(p) \). If the loop \( \{ \phi_t \}_{t \in S^1} \) is a circle subgroup of \( \text{Ham}(M, \omega) \) generated by a mean normalized Hamiltonian \( K: M \to \mathbb{R} \) then we may take \( p \) to be a fixed point of the action and \( D \) to be the constant disc. Hence \( \mathcal{A}_\omega(\Lambda) \) is the image in \( \mathbb{R}/\mathcal{P}_\omega \) of any critical value of \( K \). This is well defined because for any fixed points \( p \) and \( p' \) the difference \( K(p) - K(p') \) is the integral of \( \omega \) over the 2-sphere formed by rotating an arc from \( p \) to \( p' \) by the \( S^1 \)-action.

Proof of Proposition 1.21. First, note that since toric manifolds are simply connected, \( \text{Symp}_0(M, \omega) \) is \( \text{Ham}(M, \omega) \). Given \( H \in \ell \), the mean normalized Hamiltonian for the action of \( \Lambda_H \) on \( M \) is
\[
K = \langle H, \Phi \rangle - \langle H, c_\Delta \rangle,
\]
where \( c_\Delta \) denotes the center of mass of \( \Delta \). Since vertices of \( \Delta \) correspond to fixed points in \( M \), this implies that
\[
\mathcal{A}_\omega(\Lambda_H) = \langle H, v \rangle - \langle H, c_\Delta \rangle \in \mathbb{R}/\mathcal{P}_\omega
\]
for any vertex \( v \) of \( \Delta \).

Clearly if \( \Lambda_H \) vanishes in \( \pi_1(\text{Symp}_0(M, \omega)) \), then \( \mathcal{A}_\omega(\Lambda_H) = 0 \). In fact, it is immediate from Moser’s homotopy argument that if \( \Lambda_H \) contracts in \( \text{Symp}_0(M, \omega) \), then it also contracts for all sufficiently small perturbations of \( \omega \). Since changing \( \kappa \) corresponds to

\(^4\)We define the sign of \( K_t \) by requiring that \( \dot{\phi}_t = X_K \), where \( \omega(X_K, \cdot) = -dK \). We say that \( K \) is mean normalized iff \( \int_M K \omega^n = 0 \).
changing the symplectic form on $M$, for any vertex $v$ of $\Delta$ the image of $\Lambda_H$ under the action homomorphism

$$A_{\omega(\kappa)}(\Lambda_H) = \langle H, v \rangle - \langle H, c_\Delta(\kappa) \rangle$$

must lie in $P_{\omega(\kappa)}$ for all $\kappa$ in some open set of $\mathbb{R}^N$. Since $P_{\omega(\kappa)}$ is generated by the lengths of the edges of the polytope $\Delta$ (considered as a multiple of a primitive vector), it is a finitely generated subgroup of $\mathbb{R}$ whose generators are linear functions of the $\kappa_i$ with integer coefficients. Similarly, $\langle H, v \rangle$ is a linear function of the $\kappa_i$ with integer coefficients. Since the function $\kappa \mapsto A_{\omega(\kappa)}(\Lambda_H)$ is continuous, it follows that the function $\kappa \mapsto \langle H, c_\Delta(\kappa) \rangle$ is also a linear function of the $\kappa_i$ with integer coefficients as $\kappa$ varies in some open set. Since this function is rational (see Remark 2.3), it is a linear function of $\kappa_i$ with integer coefficients for all $\kappa \in \mathcal{C}_K$. \hfill $\Box$

The next proposition gives a geometric interpretation of the equivalence relation on the facets of $\Delta$ in terms of $\text{Isom}_0(M)$, the identity component of the group of isometries of $(M,g_I)$. Recall that these act by biholomorphisms of $(M,J)$ and hence also preserve $\omega$. To see this, note that the Kähler condition implies that $J$ is invariant under parallel translation with respect to the Levi–Civita connection of $g_I$, while, for each path $\phi_t$, $t \in [0,1]$, in $\text{Isom}_0(M)$ with $\phi_0 = id$ and each $x \in M$, the derivative $d\phi_t(x) : T_xM \rightarrow T_{\phi_t(x)}M$ is the linear map given by parallel translation along the path $\phi_s(x)$, $s \in [0, t]$.

**Proposition 5.5.** Let $(M,\omega,T,\Phi)$ be an $n$-dimensional symplectic toric manifold with moment polytope $\Delta \subset t^*$, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. Let $\mathcal{I}$ denote the set of equivalence classes of facets of $\Delta$. Then

$$\text{Isom}_0(M) \cong \left( \prod_{I \in \mathcal{I}} U(|I|) \right) / K,$$

where, for each equivalence class $I \in \mathcal{I}$, the unitary group $U(|I|) \subset U(N)$ acts on the subspace of $\mathbb{C}^N$ spanned by $\{e_i\}_{i \in I}$. Under this identification, $T = (\mathbb{S}^1)^N / K$ and $\prod_{I \in \mathcal{I}} U(|I|)$ is the centralizer of $K$ in $U(N)$. Moreover, $\text{Isom}_0(M)$ is a maximal connected compact subgroup of $\text{Symp}_0(M,\omega)$.

**Proof.** Let $(M_\Delta,\omega_\Delta,J_\Delta,\Phi_\Delta) = \mathbb{C}^N / K$ be the Kähler toric manifold associated to the moment polytope $\Delta \subset t^*$, as above. Since $M$ and $M_\Delta$ are equivariantly symplectomorphic by [6], we may identify $M$ and $M_\Delta$.

Let $Z(K)$ denote the centralizer of $K \subset (\mathbb{S}^1)^N$ in $U(N)$. Our first claim is that $Z(K)$ is the product $\prod_{I \in \mathcal{I}} U(|I|)$. To see this, note first that since $\eta_k = \pi(e_k)$ for all $k$, $\xi \in t^*$ satisfies

$$\langle \eta_k, \xi \rangle = 1 = -\langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \ \forall k$$

exactly if

$$\langle e_i, \pi^*(\xi) \rangle = 1 = -\langle e_j, \pi^*(\xi) \rangle \quad \text{and otherwise} \quad \langle e_k, \pi^*(\xi) \rangle = 0 \ \forall k,$$

that is, exactly if $\pi(\xi) = e_i^* - e_j^*$. Therefore, Lemma 3.7 implies that two facets $F_i$ and $F_j$ of $\Delta$ are equivalent exactly if $e_i^* - e_j^* \in \pi^*(t^*)$. But this holds exactly if $\langle e_i - e_j, \xi \rangle = 0$, that is,
iff $\iota^*(e^*_i) = \iota^*(e^*_j)$. Since the weights for the $K$ action on $\mathbb{C}^N$ are exactly $\iota^*(e^*_1), \ldots, \iota^*(e^*_N)$, the claim follows immediately.

Now observe that $Z(K) \subset U(N)$ acts on $\mathbb{C}^N$, preserving the standard symplectic and complex structures on $\mathbb{C}^N$. This implies that the quotient group $Z(K)/K$ acts on $M_\Delta$, preserving $\omega_\Delta$ and $J_\Delta$ and hence also the Kähler metric $g_J$. Thus, because $Z(K)$ is connected, $Z(K)/K \subset \text{Isom}_0(M_\Delta)$. But $\text{Isom}_0(M_\Delta)$ is a compact connected Lie group. Therefore, to complete the proof it is enough to prove that $Z(K)/K$ is a maximal compact connected subgroup of $\text{Symp}_0(M_\Delta, \omega_\Delta)$.

To this end, let $G$ be any connected compact subgroup of $\text{Symp}_0(M_\Delta, \omega_\Delta)$ that contains $Z(K)/K$. Then $G$ is a Lie group. By Lemma 5.6 below, $T$ is the maximal torus of $G$ and the Weyl group of $G$ acts on $t^*$ as a subgroup of the group of robust symmetries of $\Delta$. By the same argument, $T$ is the maximal torus of $Z(K)/K$ and the Weyl group of $Z(K)/K$ acts as a subgroup of the group of robust symmetries of $\Delta$. Moreover, given any two equivalent facts $F_i$ and $F_j$, it is easy to check that there is an element of the Weyl group of $Z(K)/K$ which exchanges $F_i$ and $F_j$ and otherwise takes each facet to itself. Therefore, the Weyl group of $Z(K)/K$ acts on $t^*$ as the group of robust symmetries of $\Delta$. Since, $Z(K)/K \subset G$, this implies that the two groups have the same Weyl group. But, because $Z(K)/K = \prod_{I \in \mathcal{I}} U(|I|)/K$, each of its simple factors has a root system of type $A_{|I|-1}$. Therefore, there is no way to extend its root system without either increasing the rank or enlarging the Weyl group. It follows that $G$ has the same roots as $Z(K)/K$, so that the two groups must coincide. \hfill\Box

**Lemma 5.6.** Let $(M, \omega, T, \Phi)$ be a symplectic toric manifold. Let $G \subset \text{Symp}(M, \omega)$ be a compact connected Lie subgroup that contains $T$. Then $T$ is a maximal torus of $G$, and the Weyl group $W$ of $G$ acts on $t^*$ as a subgroup of the group of robust symmetries of the moment polytope $\Delta \subset t^*$.

**Proof.** Since $M$ is simply connected, there is a moment map $\Phi_G : M \to g^*$ for the $G$ action. Since $\Phi_G$ is equivariant, the moment image $\Phi_G(M) \subset g^*$ is invariant under the coadjoint action of $G$ on $g^*$.

Let $\iota : t \to g$ denote the inclusion map. The dual map $\iota^* : g^* \to t^*$ is equivariant with respect to the coadjoint action of $N(T)$, where $N(T)$ denotes the normalizer of $T$ in $G$. Moreover, adding a constant if necessary, we may assume that $\Phi_T = \iota^* \circ \Phi_G$. Therefore, the moment polytope $\Phi_T(M) \subset t^*$ is also invariant under the coadjoint action of $N(T)$, that is, $\Phi_T(M)$ is invariant under the action of the Weyl group $W = N(T)/T$.

Finally, let $\Delta = \bigcap_{i=1}^N \{ \eta \in t^* \mid \langle \eta, x \rangle \leq \kappa_i \}$. By the construction explained in the beginning of this section, for all $\kappa' \in \mathcal{C}_\Delta$, there exists a $T$-invariant symplectic form $\omega_{\kappa'}$ with corresponding moment polytope $\Delta(\kappa')$. Let $\alpha \in \Omega^2(M; \mathbb{R})$ be any $G$-invariant 2-form in the cohomology class $[\omega'] - \omega$. For $\epsilon > 0$ sufficiently small, the form $\omega + \epsilon \alpha$ is a $G$-invariant symplectic form with moment polytope $\Delta(\kappa + \epsilon \kappa')$. By the previous paragraph, this implies that, up to translation, this polytope is also invariant under the action of the Weyl group. The conclusion now follows from Corollary 3.2. \hfill\Box
Proof of Lemma 1.26. Let $Z(K) = \prod_{I \in \mathcal{I}} U(|I|)$. Then the natural map from $\pi_1((S^1)^N)$ to $\pi_1(Z(K))$ is surjective. Moreover, $H' = \sum_i \beta_i e_i \in Z^N$ generates a trivial element of $\pi_1(Z(K)) = \pi_1\left(\prod_{I \in \mathcal{I}} U(|I|)\right)$ exactly if it lies in the Lie algebra of $\prod_{I \in \mathcal{I}} SU(|I|)$, that is, exactly if $\sum_{i \in I} \beta_i = 0$ for all $i \in I$.

Since $K$ is connected, the natural maps from $\pi_1((S^1)^N)$ to $\pi_1(T)$ and from $\pi_1(Z(K))$ to $\pi_1(Z(K)/K)$ are surjective. Therefore, the natural map from $\pi_1(T)$ to $\pi_1(Z(K)/K)$ is surjective. Moreover, $H \in \ell$ generates a trivial element of $\pi_1(Z(K)/K)$ exactly if there exists $H' = \sum_i \beta_i e_i \in \pi^{-1}(H) \subset Z^N$ that generates a trivial element of $\pi_1(Z(K))$, that is, exactly if $H = \sum_i \beta_i \eta_i \in \ell$, where $\beta_i \in Z$ for all $i$ and $\sum_{i \in I} \beta_i = 0$ for all $I \in \mathcal{I}$. The result now follows from Proposition 5.5. \qed

Example 5.7. Given $a = (a_1, a_2) \in \mathbb{R}^2$, let $Y_a$ be the polytope defined in Example 1.1 Construct $M_{Y_a} = U/K$ as in Example 5.3; let $Z(K)$ be the centralizer of $K$ in $U(N)$. Recall that $K$ is the subtorus of $(S^1)^5$ generated by the elements $(1, 1, 1, 0, 0)$ and $(a_1, a_2, 0, 1, 1)$.

Therefore,

- If $a_1 a_2 (a_1 - a_2) \neq 0$, then $Z(K) = S^1 \times S^1 \times S^1 \times U(2)$, and so $\pi_1(Z(K)/K)$ has rank 2.
- If $a_1 = a_2 \neq 0$ (or if $a_1 = 0 \neq a_2$ or $a_1 \neq 0 = a_2$) then $Z(K) = U(2) \times S^1 \times U(2)$ and $\pi_1(Z(K)/K)$ has rank 1.
- If $a_1 = a_2 = 0$ then $Z(K) = U(3) \times U(2)$ and $\pi_1(Z(K)/K)$ has rank 0.

Thus the structure of $Z(K)/K$ depends on the coefficients $(a_1, a_2)$. This is consistent with the calculation of the equivalence relation on the facets via Lemma 3.7 given in the proof of Proposition 4.6. Also, by Proposition 5.5, $\text{Isom}_0(M) = Z(K)/K$. It is hard to check this independently except in the case $a_1 = a_2 = 0$ when $\Delta$ is a product. However, one can verify the consistency of Lemma 1.26, which implies that the set of inessential functions should have a corresponding dependence on $(a_1, a_2)$. This is the case, as the reader may check by looking at Remark 4.7.

Remark 5.8. (i) A rational polytope $\Delta$ defines an algebraic variety $X := X_\Delta$. In this case, it is customary to consider a refinement $\sim'$ of the equivalence relation $\sim$ that takes multiplicities into account. One way of doing this is to consider the Chow group $A_{n-1}(X)$ of divisors in $X$, and to set $F_i \sim' F_j$ exactly if $F_i$ and $F_j$ have the same image in $A_{n-1}(X)$; cf. Cox [4, §4]. If $\Delta$ is smooth then $X = M$, $A_{n-1}(X)$ is naturally identified with $H_{2n-2}(M; \mathbb{Z})$ and $F_i \sim' F_j$ exactly if the submanifolds $\Phi^{-1}(F_i)$ and $\Phi^{-1}(F_j)$ of $M$ are homologous. It then follows from standard facts about toric manifolds that the two equivalence relations are the same. But they differ in general, for example for the polygon with conormals $(-1, 0), (0, -\frac{2}{3}), (1, \frac{2}{3})$ corresponding to the weighted projective space $\mathbb{C}P^2(1, 2, 3)$.

(ii) Consider the group $\text{Aut}(M)$ of biholomorphisms of the complex manifold $(M, J_\Delta)$. Its structure was worked out by Demazure [7]. As explained in Cox [4], for any rational polytope $\Delta$, its identity component $\text{Aut}_0(X)$ is isomorphic to a semidirect product $R_u \rtimes G_s$ where the unipotent radical $R_u$ is isomorphic to $\mathbb{C}^k$ for some $k \geq 0$ and the reductive part $G_s$ is isomorphic to the product $(\prod_{I \in \mathcal{I}} GL(|I|, \mathbb{C}))/K_C$, where $K_C$ is the complexification of
$K$ and $\mathcal{I}'$ is the set of equivalence classes with respect to the relation $\sim'$ discussed in (i). Moreover, the elements of $\pi_0(\text{Aut}(X))$ act nontrivially on homology.

Let us call a facet $F_i$ of $\Delta$ **convex** if all the conormals except for $\eta_i$ lie in a closed half-space of $t$. It follows easily from the description of Demazure roots in [4, §4] that in the smooth case the unipotent radical $R_u$ is trivial (i.e. $\text{Aut}(M)$ is reductive) exactly if each convex facet of $\Delta$ is equivalent to at least one other facet. This fact about the structure of $\text{Aut}(M)$ does not seem relevant in our situation, although it is in other geometric contexts; cf. Remark 1.33.

**Remark 5.9.** The first claim in Proposition 5.5 is that $\text{Isom}_0(M) \cong Z(K)/K$. One can also prove this by thinking of $\text{Isom}_0(M)$ as the identity component of $\text{Symp}_0(M) \cap \text{Aut}_0(M)$, where $\text{Aut}_0(M)$ is as in the previous remark. Since $R_u$ contains no nontrivial compact subgroup, $\text{Isom}_0(M)$ consists of the unitary elements in $(\prod_{I \in \mathcal{I}} \text{GL}(|I|, \mathbb{C}))/K\mathbb{C}$ and so is $Z(K)/K$.

**Appendix A. Powerful facets**

*Appendix written with Vladlen Timorin*

The purpose of this appendix is to analyze mass linear function on 4-dimensional simple polytopes where every facet is asymmetric. In part II, we will need these results in order to classify mass linear functions on 4-dimensional smooth polytopes. We need one last important definition.

**Definition A.1.** We say that a facet $F$ of a simple polytope $\Delta$ is **powerful** if it is connected to each vertex of $\Delta$ by an edge.

We will adopt the notations of §2.1. In particular, given $H \in t$ and a simple polytope $\Delta \subset t^*$, fix an identification of $t^*$ with Euclidean space and define

$$V = \int_{\Delta} dx, \quad I = \int_{\Delta} H(x)dx, \quad \text{and} \quad \widehat{H}(\kappa) = \langle H, e_\Delta(\kappa) \rangle.$$ 

Let $\partial_i$ denote the operator of differentiation with respect to $\kappa_i$.

**Proposition A.2.** Let $H \in t$ be a mass linear function on a simple polytope $\Delta \subset t^*$. Every asymmetric facet is powerful.

**Proof.** Let $F_1$ be an asymmetric facet and let $v = F_J$ be a vertex which is not connected to $F_1$ by any edge; number the facets so that $J = \{2, \ldots, n+1\}$. Since $\widehat{H}$ is linear, its second derivatives all vanish. By assumption, the intersection $F_1 \cap e_j$ is empty for each $j \in J$, where $e_j$ is the edge $\bigcap_{i \in J \setminus \{j\}} F_i$. Hence, by Theorem 2.2, if we apply the differential operator $\partial_1 \cdots \partial_{n+1}$ to the formula $I = \widehat{H}V$ we obtain

$$0 = K_v \partial_1 \widehat{H},$$

where $K_v$ is a positive real number. Since $F_1$ is an asymmetric facet $\partial_1 \widehat{H} \neq 0$; this gives a contradiction. 

With this motivation, we analyze polytopes with powerful facets.
**Lemma A.3.** Let $\Delta$ be a simple polytope. If two powerful facets $F$ and $G$ do not intersect, then $\Delta$ is combinatorially equivalent to the product $\Delta_1 \times F$. If every facet is powerful, then $\Delta$ is combinatorially equivalent to a product $\Delta_1 \times \cdots \times \Delta_1 \times \Delta'$, where every facet of $\Delta'$ is both powerful and pervasive.

*Proof.* Assume that two powerful facets $F$ and $G$ do not intersect. Then every vertex in $F$ has a unique edge which does not lie in $F$, and that edge must meet $G$. Conversely every edge which meets (but does not lie in) $G$ must meet $F$. Since $\Delta$ is connected, this implies that $\Delta$ is combinatorially equivalent to $\Delta_1 \times F$.

The second claim follows by induction. $\Box$

**Remark A.4.** We can now give an alternate proof of Corollary 3.23. Let $H \in t$ be a mass linear function on a simple polytope $\Delta \subset t^*$. Let $F$ be an asymmetric facet, and let $G$ be any facet which is disjoint from $F$. By Lemma 2.7, $G$ is also asymmetric. Hence Proposition A.2 implies that $F$ and $G$ are powerful. Next, Lemma A.3 implies that $\Delta$ is combinatorially equivalent to $\Delta_1 \times F$.

In the proof of our main proposition we need the following well known technical lemma. We include a proof for the convenience of the reader.

**Lemma A.5.** Fix vectors $\eta_1, \ldots, \eta_N$ in an $n$-dimensional vector space $t$. Assume that their positive span contains $t$ and that no subset of $n$ vectors is linearly dependent. After possibly renumbering, the positive span of $\eta_1, \ldots, \eta_{n+1}$ contains $t$.

*Proof.* After possibly renumbering, there exists $k$ so that all of $t$ lies in the positive span of $\eta_1, \ldots, \eta_k$, but not in the positive span of any proper subset. Assume that $k > n + 1$. Write $0 = \sum_{i=1}^{k} a_i \eta_i$, where $a_i > 0$ for all $i$. There exists a linear relation $\sum_{i=1}^{n+1} d_i \eta_i = 0$; let $d_i = 0$ for all $n + 1 < i \leq k$. Since $k > n + 1$, by adding an appropriate multiple of $(d_1, \ldots, d_k)$ to $(a_1, \ldots, a_k)$, we can find non-negative numbers $b_1, \ldots, b_k$ – some of which are zero and some of which are not – so that $\sum_{i=1}^{k} b_i \eta_i = 0$. After renumbering again, this implies that $\sum_{i=1}^{\ell} b_i \eta_i = 0$, where $\ell < k$ and $b_i > 0$ for all $i$. Since no set of $n$ vectors is linearly dependent, this implies that every $\alpha \in t$ can be written as a (not necessarily positive) linear combination $\alpha = \sum_{i=1}^{\ell} c_i \eta_i$. Hence, for large enough $t$, $\alpha$ can be written as a positive sum $\alpha = \sum_{i=1}^{\ell} (c_i + tb_i)$. This contradicts the claim that $k$ is minimal. $\Box$

**Proposition A.6.** Let $\Delta$ be a $n$-dimensional simple polytope; assume that every facet is powerful. If $n \leq 4$, then $\Delta$ is combinatorially equivalent to the product of simplices.

*Proof.* The polytope $\Delta$ has at most $2n$ facets. To see this, pick a vertex $v$ of $\Delta$. Since $\Delta$ is simple, $v$ lies on $n$ facets and $n$ edges. Since every facet of $\Delta$ is powerful, there is an edge from $v$ to every facet that does not contain $v$.

By Lemma A.3, we may assume that every facet of $\Delta$ is pervasive. By Remark 4.12, if $d = 3$ this implies that $\Delta$ is combinatorially equivalent to a simplex; the same claim is obvious if $d < 3$ or if $d = 4$ and $\Delta$ has five facets. So assume that $\Delta$ is 4-dimensional and has 6, 7, or 8 facets.
Write $\Delta = \bigcap_{i=1}^{N} \{ x \in \mathbb{R}^d \mid \langle \eta_i, x \rangle \leq \kappa_i \}$. Since our statement is purely combinatorial we may slightly perturb the facets so that the polytope is generic. In particular, no four outward conormals are linearly dependent; hence, after possibly renumbering, the lemma above implies that the positive span of $\eta_1, \ldots, \eta_5$ contains all of $t$. Now define

$$ P := \bigcap_{i=1}^{6} \{ x \in \mathbb{R}^d \mid \langle \eta_i, x \rangle \leq \kappa_i \} \quad \text{and} \quad H^+_\ell := \{ x \in \mathbb{R}^d \mid \langle \eta_\ell, x \rangle \leq \kappa_\ell \} \quad \text{for} \quad 7 \leq \ell \leq N. $$

Clearly,

$$ \Delta = P \cap \bigcap_{\ell=7}^{N} H^+_\ell. $$

By construction, $P$ is bounded; by the genericity assumption, $P$ is simple, as is $P \cap H^+_\ell$ for any $\ell$. It is well known that any $d$-dimensional polytope with $d + 2$ facets, such as $P$, is a product of simplices; for a proof in the current setting see [17, Prop 1.1.1]. This solves the case $N = 6$. Since every facet of $\Delta$ is pervasive, and every facet of $P$ contains a facet of $\Delta$, the facets of $P$ are also pervasive. Hence, $P$ is combinatorially equivalent to $\Delta_2 \times \Delta_2$.

Label the facets of $P$ as $F_k$ and $F'_k$ for $0 \leq k \leq 2$, so that $F_0 \cap F_1 \cap F_2 = F'_0 \cap F'_1 \cap F'_2 = \emptyset$. Let $V$ be the set of all vertices of $P$; label them

$$ v_{ij} = \bigcap_{m \neq i} F_m \cap \bigcap_{n \neq j} F'_n. $$

We will think of the vertices as lying on a $3 \times 3$ grid, where the first subscript determines the row and the second determines the column.

Now, fix $\ell$ such that $7 \leq \ell \leq N$. Let $V^+ = V \cap H^+_\ell$, and let $V^- = V$ be the complement of $V^+$. Since the facets of $\Delta$ are pervasive, the facets $F_0 \cap \Delta$ and $F_1 \cap \Delta$ must intersect. A fortiori, $F_0 \cap F_1 \cap H^+_\ell \neq \emptyset$. Since $H^+_\ell$ is a half space, this implies that (at least) one of the three vertices $v_{20}, v_{21}$ or $v_{22}$ of $F_0 \cap F_1$ lies in $V^+$. A similar argument implies that $V^- \cap F^+_\ell$ contains at least one vertex in each row and at least one in each column.

On the other hand, suppose that $v_{00}$ and $v_{11}$ are in $V^+$. Consider the 2-dimensional face $F_2 \cap F'_2$; it is a quadrilateral with vertices $v_{00}, v_{01}, v_{11}$, and $v_{10}$. The intersection of this face with the half space $H^+_\ell$ cannot contain just the two opposite vertices $v_{00}$ and $v_{11}$; it must also contain (at least) one of the vertices $v_{10}$ or $v_{01}$. More generally, if $V^+ \cap F_\ell$ contains the vertices $v_{ij}$ and $v_{mn}$ where $i \neq m$ and $j \neq n$, then it also contains $v_{in}$ or $v_{mj}$. A brief analysis shows that this, together with the paragraph above, implies that $V^+ \cap F^+_\ell$ contains at least five vertices.

On the other hand, the facets $F_i \cap \Delta$ and $F'_i \cap \Delta$ must also intersect the new facet $F'_\ell$ for each $i$. This implies that at least one vertex of $F_i$ and at least one vertex of $F'_i$ lies in $V^- \cap F_\ell$ for each $i$. It is straightforward to check that this is only possible if $V^- \cap F_\ell$ contains two vertices $v_{ij}$ and $v_{mn}$, where $i \neq m$ and $j \neq n$. As in the previous paragraph, this implies that $V^- \cap F_\ell$ contains two vertices which lie in the same column, and two vertices which lie in the same row.
Now suppose that \( v_{00} \) is a vertex of \( \Delta \). Every edge of \( \Delta \) which meets \( v_{00} \) is of the form \( e \cap \Delta \), where \( e \) is an edge of \( P \) which meets \( v_{00} \), that is, where \( e = \overline{v_{00}v_{10}}, \overline{v_{00}v_{20}}, \overline{v_{00}v_{01}}, \) or \( \overline{v_{00}v_{02}} \). Moreover, consider – for example – the edge \( e = \overline{v_{00}v_{10}} \). If \( e \cap \Delta \) connects \( v_{00} \) to \( F_0 \cap \Delta \), then \( v_{10} \) must lie in \( \Delta \); otherwise \( e \cap \Delta \) will not intersect \( F_0 \cap \Delta \). On the other hand, if \( e \cap \Delta \) connects \( v_{00} \) to the new facet \( F_\ell \) of \( \Delta \), then \( v_{10} \) must lie in \( V_\ell^- \). Since \( F_0 \cap \Delta \) is powerful, there must be an edge from \( v_{00} \) to \( F_0 \cap \Delta \). This edge must be of the form \( e \cap \Delta \), where \( e \) is an edge in \( P \) from \( v_{00} \) to \( F_0 \), that is, \( \overline{v_{00}v_{10}} \) or \( \overline{v_{00}v_{20}} \). Hence, (at least) one of the vertices \( v_{10} \) and \( v_{20} \) lies in \( \Delta \). On the other hand, the new facet \( F_\ell \) is also powerful. Hence, \( V_\ell^- \) contains (at least) one of the vertices \( v_{10}, v_{20}, v_{02}, \) or \( v_{01} \). A similar argument shows that \( \Delta \) cannot contain exactly one vertex in any row or column, and that if \( \Delta \) contains \( v_{ij} \) then \( V_\ell^- \) contains some vertex which either lies in the same row or in the same column.

We now specialize, and assume that \( N = 7 \). In this case, \( v_{ij} \) lies in \( \Delta \) exactly if \( v_{ij} \in V_7^+ \). Therefore, since \( V_7^+ \) contains at least one vertex in each column, the paragraph above implies that it contains at least two vertices in each column. But this contradicts the fact that \( V_\ell^- \) contains two vertices in the same column.

So we may assume that \( N = 8 \). In this case, \( v_{ij} \) lies in \( \Delta \) exactly if \( v_{ij} \in V_7^+ \cap V_8^+ \). Hence, \( V_7^+ \cap V_8^+ \) cannot contain exactly one vertex in any row or column. On the other hand, \( V_7^- \) contains two vertices in the same column and also contains two vertices in the same row, so \( V_7^+ \cap V_8^+ \) cannot contain two or more vertices in every column or in every row. Together, these imply that \( V_7^+ \cap V_8^+ \) must not contain any vertices in at least one row and at least one column. Assume that these are the last row and column. Since \( V_7^+ \) and \( V_8^+ \) each contain at least five vertices, \( V_7^+ \cap V_8^+ \) is not empty. This implies that, \( V_7^+ \cap V_8^+ = \{v_{00}, v_{01}, v_{10}, v_{11}\} \). But then \( V_7^- \) must contain at least one vertex in the same row or column as each of \( v_{00}, v_{01}, v_{10} \) and \( v_{11} \). After possibly swapping the columns with the rows, this implies that \( V_7^- \) contains \( v_{02} \) and \( v_{12} \). Since \( V_7^+ \) contains at least one vertex in every column, this implies that \( V_7^+ \) contains \( v_{22} \). A similar argument implies that \( V_8^+ \) contains \( v_{22} \). This contradicts the fact that \( v_{22} \) does not lie in \( V_7^+ \cap V_8^+ \).

**Remark A.7.** It is natural to wonder if there is an analog of this result in higher dimensions. Chen [3] recently found a 7-dimensional polytope with all facets pervasive and powerful that is not combinatorially equivalent to a product. However, since it is not clear whether his example can be taken to be smooth, the following question is still open:

Let \( \Delta \) be a smooth polytope; assume that every facet is powerful. Is \( \Delta \) combinatorially equivalent to the product of simplices? If not, is there a structure theorem for such polytopes?

Note that, by Lemma A.3, it is enough to consider the case that every facet of \( \Delta \) is pervasive. Moreover, the proof above shows that if \( \Delta \) is \( n \)-dimensional, then it has at most \( 2n \) facets.

Combined, Propositions A.2 and A.6 have the following important corollary.

**Corollary A.8.** Let \( H \in \mathcal{I} \) be a mass linear function on a simple 4-dimensional polytope \( \Delta \subset \mathcal{I} \) with no symmetric facets. Then \( \Delta \) is combinatorially equivalent to a product of simplices.
This allows us to complete the first step in classification of mass linear function on smooth 4-dimensional polytopes.

**Theorem A.9.** Let $H \in t$ be a mass linear function on a smooth 4-dimensional polytope $\Delta \subset t^*$. Then there exists an inessential function $H' \in t$ so that the mass linear function $\tilde{H} = H - H'$ has the following property: at least one facet of $\Delta$ is $\tilde{H}$-symmetric.

**Proof.** Assume that $\Delta$ has no symmetric facets. By Corollary A.8, $\Delta$ is combinatorially equivalent to a product of simplices. Moreover, by Proposition 3.24 we may assume that every facet is pervasive; hence, $\Delta$ is combinatorially equivalent to $\Delta_2 \times \Delta_2$. By Lemma 4.10, this implies that $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$. The result now follows immediately from Proposition 3.21. \qed

**REFERENCES**


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