Abstract. Consider a Hamiltonian action of a compact Lie group $G$ on a compact symplectic manifold $(M, \omega)$ and let $\mathcal{G}$ be a subgroup of the diffeomorphism group $\text{Diff} M$. We develop techniques to decide when the maps on rational homotopy and rational homology induced by the classifying map $BG \to B\mathcal{G}$ are injective. For example, we extend Reznikov’s result for complex projective space $\mathbb{CP}^n$ to show that both in this case and the case of generalized flag manifolds the inclusion $H_\ast(\text{BSU}(n+1)) \to H_\ast(B\mathcal{G})$ is injective, where $\mathcal{G}$ denotes the group of all diffeomorphisms that act trivially on cohomology. We also show that if $\lambda$ is a Hamiltonian circle action that contracts in $\mathcal{G} := \text{Ham}(M, \omega)$ then there is an associated nonzero element in $\pi_3(\mathcal{G})$ that deloops to a nonzero element of $H_4(B\mathcal{G})$. This result (as well as many others) extends to c-symplectic manifolds $(M, a)$, i.e. $2n$-manifolds with a class $a \in H^2(M)$ such that $a^n \neq 0$. The proofs are based on calculations of certain characteristic classes and elementary homotopy theory.

1. Introduction

This paper investigates the homotopy properties of smooth compact group actions. Although its main motivation is the attempt to understand symplectic group actions, many of the results do not use the fact that the action preserves a symplectic structure. Rather it is enough for the manifold to be cohomologically (or c-)symplectic, in the sense that there is a cohomology class $a \in H^2(M; \mathbb{R})$ such that $a^n \neq 0$ (where $2n = \dim M$.) We shall denote by $\mathcal{G}$ an (infinite dimensional) subgroup of the diffeomorphism group $\text{Diff} M$. For example, $\mathcal{G}$ might be the identity component $\text{Diff}_0 M$ of $\text{Diff} M$, or, if $(M, \omega)$ is a symplectic manifold, the group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of $(M, \omega)$, or the symplectomorphism group $\text{Symp}(M, \omega)$. In the c-symplectic case, it is also useful to allow $\mathcal{G}$ to denote certain covers of $\text{Diff}_0(M)$. In particular, consider the $a$-Flux
homomorphism

$$\text{Flux}^a : \pi_1(\text{Diff}_0(M)) \to H^1(M;\mathbb{R}),$$

which is defined by the requirement that for each loop $\lambda$ in $\text{Diff}_0(M)$ the value of the cohomology class $\text{Flux}^a(\lambda)$ on the 1-cycle $\gamma$ is given by evaluating $a$ on the 2-cycle defined by the map $T^2 \to M$, $(s,t) \mapsto \lambda(s)(\gamma(t))$. We denote the corresponding covering space of $\text{Diff}_0(M)$ by $\text{Ham}(M,a)$.

If $\omega$ is a symplectic form in class $a$ then the inclusion $\text{Symp}_0(M,\omega) \to \text{Diff}_0(M)$ lifts to a map $\text{Ham}(M,\omega) \to \text{Ham}(M,a)$. Further $\text{Ham}(M,a) = \text{Diff}_0(M)$ if $H^1(M) = 0$. (For more detail see §3.1.)

We also suppose that a connected compact Lie group $G$ acts smoothly on $M$. To avoid trivial cases, we assume that the map $r : G \to G$ has finite kernel. This paper develops techniques for understanding the induced map on rational homotopy

$$r_* : \pi_*(G) \otimes \mathbb{Q} \to \pi_*(G) \otimes \mathbb{Q}.$$

Our guiding idea is that if $r(G)$ is a maximal compact subgroup of $G$ then $r_*$ should usually be injective, and that if it is not there are associated nonzero elements in the homotopy of $G$. This accords with the situation for $M = S^2 \times S^2$ studied in Abreu–McDuff [1] in which all the (rational) homotopy in $\text{Ham}(M,\omega)$ is generated by its Lie subgroups. We also study the composite

$$h \circ R_* : \pi_*(BG) \otimes \mathbb{Q} \to H_*(BG)$$

where $h$ denotes the Hurewicz map $\pi_*(BG) \otimes \mathbb{Q} \to H_*(BG)$. For a Lie group the Hurewicz map $h$ is always injective, but as shown in [1] this need not hold for $G$ because Samelson products need not vanish in $G$. Therefore the injectivity of $h \circ R_*$ implies that of $r_*$ but not vice versa. In Section §4.1 we give natural examples of inclusions $SU(k) \to G$ for which $r_*$ is not injective. However, so far we know of no examples in which $r_*$ is injective but $h \circ R_*$ is not.

Another general question (addressed in §5) is the extent to which the existence of the action of a Lie group $G$ on $M$ forces the point evaluation map $ev_* : \pi_*(G) \to \pi_*(M)$ to be nontrivial. Again we are particularly interested in nonzero elements created by Whitehead (or Samelson) products.

The main applications concern either the case $G = S^1$ or the case when $G := SU(\ell)$ acts transitively on $M$. Throughout $H_*$ denotes homology with rational coefficients and we work over $\mathbb{Q}$ unless explicit mention is made otherwise. Elements in the kernel of $r_*$ are called (rationally) inessential in $G$. Note that even if $G$ is a maximal compact subgroup of $G$ the induced map on rational homotopy need not be injective. For example, the natural automorphism group $G_T$ of a symplectic toric manifold $(M,T,\omega)$ is maximal in $\text{Symp}_0(M,\omega)$, and McDuff–Tolman [15] give examples in which the induced map on $\pi_1 \otimes \mathbb{Q}$ is not injective. Both [15] and [14] concentrate
on finding criteria to ensure that a given Hamiltonian $S^1$ action is essential, using the Seidel representation in quantum homology. In contrast, this paper discusses properties of inessential $S^1$-actions, and develops methods to detect nontrivial higher homotopy classes. The methods of proof are elementary.

### 1.1. Circle actions

Our first result concerns the consequences of the existence of an inessential circle action. In the symplectic case, any such circle action must lie in the kernel of the flux homomorphism $\pi_1(\text{Symp}(M,\omega)) \to H^1(M)$ and hence be Hamiltonian. In the c-symplectic case, the action must be c-Hamiltonian, i.e. in the kernel of the $a$-Flux homomorphism. This is equivalent to saying that $M$ is a Hamiltonian $S^1$-manifold in the sense of Ginzburg–Guillemin–Karshon [6]: see §3.

**Theorem 1.1.** Let $(M,a)$ be a c-symplectic manifold of dimension $2n$ and set $\mathcal{G} := \text{Ham}(M,a)$. Suppose given a nonconstant homomorphism $\lambda : S^1 \to \mathcal{G}$ that represents the zero element in $\pi_1(\mathcal{G})$ and so extends to a map $\tilde{\lambda} : D^2 \to \mathcal{G}$. Define $\rho \in \pi_3(\mathcal{G}) \otimes \mathbb{Q}$ by

$$S^3 := (D^2 \times S^1) / ((D^2 \times \{1\}) \cup (\partial D^2 \times S^1)) \to \mathcal{G},$$

where the bracket $\langle \phi, \psi \rangle$ represents the commutator $\phi \psi \phi^{-1} \psi^{-1}$. Then $\rho \neq 0$ and is independent of the choice of extension $\tilde{\lambda}$. Moreover, $\rho$ transgresses to an element $\overline{\rho} \in \pi_4(B\mathcal{G}) \otimes \mathbb{Q}$ with nonzero image in $H_4(B\mathcal{G})$.

The map $D^2 \times S^1 \to \mathcal{G}$ descends to $S^3$ precisely because it contracts the boundary $\partial D^2 \times S^1$ to a point. Thus it is crucial here that $G = S^1$ is abelian and that $\lambda$ is a homomorphism. (Indeed the statement makes no sense in the context of arbitrary inessential smooth maps $\lambda : S^1 \to \mathcal{G}$ because the only special element in the homotopy class of such maps is the constant map.) Any two extensions $\tilde{\lambda}$ differ by an element $\beta \in \pi_2(\mathcal{G})$, and it is easy to see that this change $\rho$ by the Samelson product (or commutator) $\langle \beta, \lambda \rangle$ defined in (2.1) below. But this vanishes since $\lambda = 0$ in $\pi_1(\mathcal{G})$.

**Remark 1.2.** (i) If $\lambda$ is a smooth inessential circle action on $M$ then one can use formula (1.1) to define an element $\rho \in \pi_3(\text{Diff}(M))$. However, in this generality we have no way of proving that $\rho \neq 0$.

(ii) We detect the nontriviality of $\overline{\rho}$ by using the characteristic classes of Reznikov [16] and Januszkiewicz–Kędra [10]. These classes extend beyond the group $\text{Ham}(M,a)$ to appropriate $H$-spaces $\mathcal{H}$ that act on $M$. For example, we show in §3 below that if $H^1(M) = 0$ then $\overline{\rho}$ has nonzero image in $H_4(B\mathcal{H})$ where $\mathcal{H}$ denotes the $H$-space $\mathcal{H}_a := (M^M)_a$ formed by all smooth homotopy equivalences $M \to M$ that fix the class $a$.

If $H^1(M) \neq 0$ it is not clear whether $\overline{\rho}$ must have nonzero image in $H_4(B\mathcal{H}_0)$, where $\mathcal{H}_0$ denotes the identity component of $\mathcal{H}_a$. However it
does have nonzero image in the cover \( \tilde{\mathcal{H}}_0 \) of \( \mathcal{H}_0 \) that is determined by the \( a \)-Flux homomorphism. If \( \lambda \) is a circle action on \((M,a)\) that is inessential in \( \mathcal{H}_0 \) but not in \( \text{Diff}_0 \), then Theorem 1.1 as stated no longer makes sense, since formula (1.1) has no meaning when inverses do not exist. However, the construction in \( \S 2.1 \) does not use inverses, and it follows from Proposition 2.1 and Corollary 3.7 that there is still an associated nonzero element in \( H_4(B\tilde{\mathcal{H}}_0^a) \). Another way of dealing with this point is to note that the \( H \)-space \( \mathcal{H}_0 \) is grouplike because it is path connected, and hence it has inverses up to homotopy.

(iii) Our result extends work by Reznikov in the following way. Reznikov proved in [16] that any Hamiltonian action of \( SU(2) \) induces a nonzero map on \( \pi_3 \otimes \mathbb{Q} \). Moreover, Lemma 2.5 below implies that the element \( \rho \) created from any circle subgroup of \( SU(2) \) lies in the image of \( \pi_3(SU(2)) \). Thus the nontriviality statement in Theorem 1.1 follows from Reznikov’s work in the case when \( G = \text{Ham}(M,\omega) \) and the circle \( \lambda \) contracts in \( G \) by virtue of the fact that it is contained in a simply connected Lie subgroup \( G \) of \( \mathcal{G} \).

(iv) Let \( \text{ev}_*: \pi_*(G) \otimes \mathbb{Q} \to \pi_*(M) \otimes \mathbb{Q} \) denote the map obtained by evaluating at the base point \( p \). By looking at the \( SU(2) \)-action on \( M := \mathbb{CP}^n \) for \( n = 1,2 \) one sees from (iii) that the element \( \text{ev}_*(\rho) \in \pi_3(M) \) is sometimes zero and sometimes nonzero. We show in Proposition 5.3 that if \( H^1(M) = 0 \) then \( \text{ev}_*(\rho) \neq 0 \) only if there is a nonzero quadratic relation \( \sum_{ij} c_i c_j = 0 \) among the classes \( c_i \in H^2(M) \). In the symplectic case this is no surprise since it follows from the work of Lalonde-McDuff [11] that the map

\[
h \circ \text{ev}_*: \pi_*(\text{Ham}(M,\omega)) \to H_*(M)
\]

is zero. Hence, by minimal model theory, any element in the image of \( \text{ev}_* \) in this low degree must give rise to a relation in \( H^*(M) \). However, the arguments in [11] do not apply in the c-symplectic case.

A Hamiltonian \( S^1 \)-action always has fixed points \( p \) and one can consider its image in the subgroup \( \text{Diff}_p \) of diffeomorphisms that fix \( p \). When \( H^1(M) \neq 0 \) we shall need to consider the corresponding subgroup \( G_p \) of \( \mathcal{G} := \text{Ham}(M,\omega) \) which is the fiber of the evaluation map \( \mathcal{G} \to M \) at \( p \). The following result is proved in \( \S 3.2 \).

**Lemma 1.3.** Suppose that \( \lambda \) is a Hamiltonian circle action with moment map \( H : M \to \mathbb{R} \), and set \( \mathcal{G} := \text{Ham}(M,\omega) \). Then \( \lambda \) is essential in \( G_p \) for every fixed point \( p \) such that \( \int_M (H - H(p)) \omega^n \neq 0 \). In particular, \( \lambda \) is essential in \( G_p \) if \( H \) assumes its maximum or minimum at \( p \).

A similar statement holds in the c-Hamiltonian case.

### 1.2 Higher homotopy groups

The proof of Theorem 1.1 is based on a general construction of a secondary product on the homotopy of topological groups. Given a point \( p \in M \) we shall write \( G_p, \mathcal{G}_p \) for the subgroups of \( G, \mathcal{G} \) that fix \( p \).
Proposition 1.4. Consider the induced map
\[ r_\ast : \pi_{k-1}(G_p) \otimes Q \to \pi_{k-1}(G) \otimes Q. \]
For each pair \( f \in \ker r_\ast, f' \in \pi_{m-1}(G_p) \) there is an element
\[ \{f, f'\} \in \pi_{k+m-1}(G)/(\text{im } r_{k+m-1}) \]
that is well defined modulo Samelson products of the form \( \langle \beta, r \circ f' \rangle, \beta \in \pi_k(G) \).

In §3 we use characteristic classes to calculate \( \{f, f'\} \) in various cases, in particular when \( f \) is given by a circle action. We also use these classes to investigate \( \ker r_\ast \).

The next proposition is an easy consequence of Theorem 1.1. It extends to \( BH_0^\sigma \) a result that Reznikov proved for \( B\text{Ham}(M, \omega) \) and is proved in in §3.2. (Recall from Remark 1.2(ii) that if \((M,a)\) is c-symplectic then \( \tilde{H}_0^\sigma \) denotes the cover of \( H_0 \) corresponding to the \( a \)-Flux homomorphism and hence contains \( \text{Ham}(M,a) \).)

Proposition 1.5. Suppose that \( G := SU(2) \) acts smoothly and with finite kernel on a c-symplectic manifold \((M,a)\). Then the induced map
\[ \pi_4(BG) \to H_4(B\tilde{H}_0) \]
is injective. Further, if \( H^1(M) = 0 \) then the image of \( \pi_4(BG) \) in \( H_4(BH_a) \) is nonzero, where \( H_a \) is the space of all homotopy equivalences of \( M \) fixing \( a \).

If \((M,a)\) supports an action of \( G := SU(\ell) \) with \( \ell > 2 \) then one can also try to understand the maps \( \pi_{2i}(BG) \otimes Q \to H_{2i}(BG) \) for \( 2 < i \leq \ell \) and appropriate \( G \). In view of Proposition 1.5 and Lemma 3.6 it is tempting to conjecture that for any Hamiltonian \( G \)-action the maps \( \pi_{2i}(BG) \to \pi_{4i}(B\text{Ham}(M, \omega)), i \geq 1 \), are injective. However, we show in §4.1 that this is not the case. It is still possible that the injectivity of \( \pi_{2i}(BG) \otimes Q \to \pi_{2i}(BG) \) implies that of \( \pi_{2i}(BG) \otimes Q \to H_{2i}(BG) \).

Here is one result in this direction. Again we denote \( H_0 := (M^M)_0 \), the space of (smooth) self maps of \( M \) that are homotopic to the identity. Since the point evaluation map \( ev_\ast : \pi_\ast(G) \to \pi_\ast(M) \) factors through \( \pi_\ast(H_0) \) the kernel of \( r_\ast : \pi_\ast(G) \to \pi_\ast(H_0) \) is contained in the kernel of \( ev_\ast \). We now describe conditions under which the kernel of the corresponding map on homology is contained in \( \ker ev_\ast \).

We say that \( M \) satisfies the c-splitting condition for an \( H \)-space \( H \) that acts on \( M \) if the associated (Hurewicz) fibration \( M \to M_H \to BH \) is c-split, that is, if \( \pi_1(BH) \) acts trivially on \( H^*(M) \) and the Leray–Serre spectral sequence for cohomology degenerates at the \( E_2 \)-term. For example, Blanchard showed in [4] that a simply connected Kähler manifold satisfies the c-splitting condition for \( H_0 \); see Lemma 4.4. Further, it was conjectured in [11] that any symplectic manifold satisfies the c-splitting condition for \( \text{Ham}(M, \omega) \). \( M \) is called nilpotent if the action of \( \pi_1(M) \) on \( \pi_\ast(M) \otimes Q \) is
nilpotent. The following result is proved in §4.2. For $\alpha \in \pi_{2k-1}G$ we denote by $\overline{\alpha}$ the corresponding element in $\pi_{2k}(BG)$.

**Proposition 1.6.** Let $G$ be a compact, connected, and simply connected Lie group that acts (with finite kernel) on a nilpotent manifold $(M, a)$. Suppose further that $M$ satisfies the c-splitting condition with respect to an $H$-space $\mathcal{H}$ containing $G$. Then for all $\alpha \in \pi_*(G)$

$$\text{ev}_*(\alpha) \neq 0 \text{ in } \pi_{2k-1}(M) \Rightarrow h \circ R_*(\overline{\alpha}) \neq 0 \text{ in } H_{2k}(B\mathcal{H}).$$

See Proposition 4.6 for an alternative version that is valid in the symplectic case with weaker conditions on $M$.

In the next results $\mathcal{H}_H$ denotes the space of self-maps of $M$ that act trivially on cohomology.

**Corollary 1.7.** Consider the action of $G := SU(\ell)$ on the manifold $M := G/T$ of complete flags, where $T$ is the maximal torus in $G$. Then the map $\pi_*(BG) \to H_*(B\mathcal{H}_H)$ is injective.

**Proof.** It follows from Lemma 4.4 that $M$ satisfies the c-splitting condition for $\mathcal{H}_H$. Further $\text{ev}_*: \pi_*(G) \to \pi_*(M)$ is injective. \hfill \square

A similar result holds for the action of $G = SU(n+1)$ on $M := \mathbb{CP}^n$.

**Proposition 1.8.** In the case of the action of $G = SU(n+1)$ on $M := \mathbb{CP}^n$, the induced map $h \circ R_* : \pi_*(BG) \to H_*(B\mathcal{H}_H)$ is injective.

The proof of this result relies on the fact that the so called coupling class is well defined in $H^2(M_{\mathcal{H}_H})$ and hence gives rise to characteristic classes on $B\mathcal{H}_H$ by means of integration over the fiber: see §3.1. This result extends work by Reznikov [16]. He proved the injectivity of the map $H_*(BSU(n+1)) \to H_*(BSymp(M, \omega))$ by explicitly constructing and calculating some characteristic classes in the Lie algebra cohomology of the Fréchet Lie group $Symp(M)$: see §3.3 below.

A general flag manifold can be written as

$$M(m_1, \ldots, m_k) := U(\ell)/U(m_1) \times \cdots \times U(m_k), \quad m_1 \geq \cdots \geq m_k,$$

where $\ell = \sum m_i$. In Proposition 4.8 below we extend the result of Corollary 1.7 to arbitrary flag manifolds.

The last section §5 investigates the image $\text{ev}_*(\rho)$ of the element $\rho$ of Theorem 1.1 under the evaluation map, and shows its relation to certain Whitehead products in $\pi_*(M)$. For example, in Proposition 5.3 we give necessary and sufficient conditions for $\text{ev}_*(\rho)$ to be nonzero, while Proposition 5.5 gives conditions under which the map $\text{ev}_* : \pi_5(G) \to \pi_5(M)$ is nonzero.

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2. A secondary product on homotopy groups of classifying spaces

If $G$ is a topological group the Samelson product $\langle f, f' \rangle \in \pi_{i+j}(G)$ of the elements $f \in \pi_i(G), f' \in \pi_j(G)$ is given by the map

$$S^{i+j} = S^i \times S^j / S^i \vee S^j \to G : (x, y) \mapsto \langle f(x), f'(y) \rangle.$$  

Here $\langle a, b \rangle$ denotes the commutator $aba^{-1}b^{-1}$, and we take the base point of $G$ to be the identity element so that the commutator vanishes on the wedge $S^i \vee S^j$. The main fact we shall use about this product is that (up to sign) it is the desuspension (or transgression) of the Whitehead product. Thus the obvious isomorphism $\pi_*(G) \cong \pi_{*+1}(BG)$ takes $\pm \langle f, f' \rangle$ to the Whitehead product $[F, F']$ of the images $F, F'$ of $f, f'$. Another important fact is that the Whitehead product vanishes rationally on $BG$ when $G$ is a compact connected Lie group. This follows from minimal model theory: nonzero Whitehead products lie in the kernel of the Hurewicz homomorphism and give rise to relations in rational cohomology, but $H^*(BG; \mathbb{Q})$ is a free algebra. Hence the rational Samelson product also vanishes on Lie groups, but need not vanish when $G$ is a symplectomorphism group.

All results in this section are concerned with rational homotopy. One could therefore work in the rational homotopy category, in which case the notation $S^k$ does not denote a sphere but rather its image $S^k_\mathbb{Q}$ in this category. Alternatively, one can take sufficiently high multiples of all maps so that they are null homotopic rather than zero in $\pi_* \otimes \mathbb{Q}$, and then work in usual category. We adopt the latter approach.

2.1. A general construction. In this section we prove Proposition 2.1, a more precise version of Proposition 1.4. In Corollary 2.3 we establish the uniqueness (but not the nontriviality) of the element $\rho$ of Theorem 1.1.

Consider the sequence of fibrations associated with this action:

$$G_p \xrightarrow{i} G \xrightarrow{ev} M \xrightarrow{j} B\mathcal{G}_p = M\mathcal{G} \xrightarrow{\pi} BG,$$

where $M\mathcal{G}$ denotes the total space of the universal $M$-bundle over $BG$.

Let $f : S^{k-1} \to \mathcal{G}_p$ and $f' : S^{m-1} \to \mathcal{G}_p$ be maps satisfying the following assumptions:

(A1) The Samelson product is trivial, $\langle f, f' \rangle = 0$;

(A2) The map $i \circ f$ is null homotopic in $\mathcal{G}$.

Equivalently, for the corresponding maps to the classifying space, $F : S^k \to B\mathcal{G}_p$ and $F' : S^m \to B\mathcal{G}_p$ we have

(B1) The Whitehead product is trivial, $[F, F'] = 0$;

(B2) The map $\pi \circ F$ is null homotopic in $BG$; in other words $F = j \circ \alpha$, where $\alpha : S^k \to M$. 

These assumptions permit the construction of the following commutative diagram:

\[
\begin{array}{ccc}
S^k \times S^m & \xrightarrow{F \times F'} & B\mathcal{G}_p \\
\downarrow & & \downarrow \pi \\
(S^k \times S^m / S^k) \simeq S^{k+m} \cup S^m & \xrightarrow{(F,F') \vee F'} & B\mathcal{G}
\end{array}
\]

Here \( F \times F' \) is some extension of \( F \vee F' \), and exists because \([F,F'] = 0\). By (B2) this map descends to \( S^k \times S^m / S^k \to B\mathcal{G} \). But \( S^k \times S^m / S^k \simeq S^{m+k} \cup S^m \) because the attaching map of the top cell in the quotient \( S^k \times S^m / S^k \) is null homotopic. (It is the Whitehead product of the trivial map on \( S^k \) with the identity map on \( S^m \)). We denote the homotopy class of the induced map on \( S^{m+k} \) by \( \{F,F'\} \). The element of \( \pi_{k+m-1}(\mathcal{G}) \) corresponding to \( \{F,F'\} \) under transgression is denoted by \( \{f,f'\} \).

Note that the homotopy class \( \{F,F'\} \) may depend on the choice of extension of \( F \vee F' \) to \( S^k \times S^m \) as well as on the chosen null homotopy \( H_t, t \in [0,1] \), of \( \pi \circ F \). Further it is not symmetric: indeed the conditions under which \( \{F',F\} \) is defined are different from those for \( \{F,F'\} \).

This construction is a particular case of the secondary Whitehead product. For any pair of spaces \((X,A)\), let \( \hat{F} \in \pi_{k+1}(X,A), F' \in \pi_m(A) \) be such that \([\partial \hat{F}, F'] = 0 \in \pi_{m+k-1}(A)\). Consider the following diagram

\[
\begin{array}{cccc}
\pi_{k+1}(A) & \xrightarrow{i} & \pi_{k+1}(X) & \xrightarrow{\partial} & \pi_k(A) \\
[.,F] & & [.,F'] & & [.,F'] \\
\pi_{m+k}(A) & \xrightarrow{i} & \pi_{m+k}(X) & \xrightarrow{\partial} & \pi_{m+k-1}(A)
\end{array}
\]

Then \([\hat{F}, F'] \in \pi_{m+k}(X,A)\) has trivial image in \( \pi_{m+k-1}(A)\) and so lifts to \( \pi_{m+k}(X)\). The secondary Whitehead product

\[
\{\hat{F}, F'\} \in \pi_{m+k}(X) / i(\pi_{m+k}(A))
\]

is defined to be such a lift. In our situation \( X = B\mathcal{G} \) and \( A = B\mathcal{G}_p \). We are given two maps \( F,F' \in \pi_{*}(A) \) where \( F \) has trivial image in \( \pi_{*}(X) \). Therefore we must choose a lift \( \hat{F} \) of \( F \) to \( \pi_{k+1}(X,A) \) which adds an extra indeterminacy \([\pi_{k+1}(X), F']\) to the element \( \{F,F'\} := \{\hat{F}, F'\} \).

We next specialize further to the case when \( A = B\mathcal{G}_p \), where \( \mathcal{G}_p \) is a Lie group. Then all Whitehead products in \( \pi_{*}(A) \otimes \mathbb{Q} \) vanish, and we obtain the following result.

**Proposition 2.1.** Suppose that the maps \( f, f' \) take values in a Lie group \( G_p \) that maps to \( \mathcal{G}_p \) via \( r \). Then for each \( f \in \pi_{k-1}(G_p) \) such that \( r \circ f \) is nullhomotopic in \( \mathcal{G} \) the above construction gives a homomorphism

\[
\pi_{m-1}(G_p) \to \pi_{k+m}(BG) \otimes \mathbb{Q} / \mathcal{K}, \quad f' \mapsto \{F,F'\} + \mathcal{K}.
\]
Here $K$ is the subgroup of $\pi_{k+m}(BG) \otimes \mathbb{Q}$ generated by $R_*(\pi_{k+m}(BG_p))$ and the Whitehead products $[\beta, R_*(\pi_{k+m}(BG_p))]$, where $\beta \in \pi_{k+1}(BG)$ and $R : BG_p \to BG$ is induced by $r$. In particular, if $r \circ f'$ is also nullhomotopic in $G$ then $\{F, F'\}$ is well defined modulo the image of $\pi_{k+m}(BG_p)$. Further in this case $\{F', F\}$ is defined and equal to $\pm\{F, F'\}$ modulo the image of $\pi_{k+m}(BG_p)$.

**Proof.** The first two statements are an immediate consequence of the previous remarks. To see that $\{F', F\} = \pm\{F, F'\}$ when $r \circ f$ and $r \circ f'$ are both null homotopic, observe that the two relative Whitehead products

$$\begin{align*}
(D^{i+j}, S^{i+j-1}) & \xrightarrow{W} (D^{i+1} \cup S^i, S^i \cup S^j) \\
(D^{i+j}, S^{i+j-1}) & \quad \to (S^i \cup D^{i+1}, S^i \cup S^j)
\end{align*}$$

become homotopic in $(D^{i+1} \cup D^{i+1}, S^i \cup S^j)$ since taking the boundary gives an isomorphism $\pi_{i+j}(D^{i+1} \cup D^{i+1}, S^i \cup S^j) \to \pi_{i+j-1}(S^i \cup S^j)$. Hence one gets the same answer if one defines the relative product $\{F, F'\}$ by using the null homotopy of either variable. The sign might change when one reverses the order. \hfill \Box

We next give an explicit formula for $\{f, f'\}$ under the assumptions (A1), (A2). Choose a map $c : D^{k+m-1} \to G_p$ such that $\partial c = \langle f, f' \rangle$ and an extension $\tilde{f} : D^k \to G$. Define an element $\beta := \langle \tilde{f}, f' \rangle \in \pi_{k+m-1}(G)$ as follows.

$$\beta : (D^k \times S^{m-1}) \cup D^{k+m-1} \to G,$$

$$\beta(x, t) = \langle \tilde{f}(x), f'(t) \rangle \quad \text{for } (x, t) \in D^k \times S^{m-1}$$

$$\beta(z) = c(z) \quad \text{for } z \in D^{k+m-1},$$

where $\langle \cdot, \cdot \rangle$ is the commutator as before. In what follows we shall assume that the maps $f, f'$ take values in $G_p$ and we choose $c$ also to take values in $G_p$. Then $\beta$ is well defined modulo the image of $\pi_{k+m-1}(G_p)$ and Samelson products of the form $\langle \gamma, f' \rangle$, where $\gamma \in \pi_k(G)$.

**Proposition 2.2.** Let $f, f'$ be as in Proposition 2.1. Denote by $K'$ the subgroup of $\pi_{k+m-1}(G)$ generated by the image of $\pi_{k+m-1}(G_p)$ and Samelson products of the form $\langle \gamma, f' \rangle$, where $\gamma \in \pi_k(G)$. Then, if we define $\beta = \langle \tilde{f}, f' \rangle$ by (2.2) taking $c$ also to have values in $G_p$, $\beta$ represents the transgressive image $\{f, f'\} \in \pi_{k+m-1}(G)/K'$ of the map $\{F, F'\}$ of Proposition 2.1.

Before giving the proof we derive some easy corollaries.

**Corollary 2.3.** Let $\lambda : S^1 \to G_p$ be a circle action that is inessential in $G$. Then $\{\lambda, \lambda\}$ is defined and equals the element $\rho \in \pi_3(G)$ given by (1.1). Further, it is independent of choices.

**Proof.** Take $G = S^1$. Then it is immediate that $\{\lambda, \lambda\}$ is defined. It is independent of choices by Proposition 2.1 and the fact that $\pi_3(G) = 0$. It equals $\rho$ by Proposition 2.2. \hfill \Box
Corollary 2.4. Suppose given a homomorphism \( h : S^1 \times K \to \mathcal{G} \), where \( K \) is a Lie group. Let \( \lambda := h|_{S^1 \times \{1\}} \) and \( f' \in \pi_m(K) \). Suppose that \( \lambda \) is inessential in \( \mathcal{G} \). Then we may construct the element \( \{\lambda, f'\} \in \pi_{2+m}(\mathcal{G}) \) so that it is well defined modulo the Samelson products \( \langle \beta, f' \rangle \), \( \beta \in \pi_2(\mathcal{G}) \).

Proof. Define \( \{\lambda, f'\} \) using formula (2.2) noting that we may take \( c \) to be constant since, by assumption, the maps \( \lambda \) and \( f' \) are homotopic to the constant map. This homotopy exists because the Whitehead product \( \langle \lambda, f' \rangle \) is the interior of the top cell in \( BS^1 \) and the left hand vertical map induced by collapsing the spheres of dimensions \( i - 1 \) and \( j - 1 \). Further \( \langle \Omega(t_1), \Omega(t_2) \rangle \) is the Samelson product of the maps obtained by looping the inclusions on each factor.

For readers who are unfamiliar with such homotopy theoretic arguments we now give a more explicit proof that describes the relevant homotopies in more detail. We begin by describing the map \( \{F, F'\} : S^{k+m} \to BG \).

Consider \( S^{k+m} \) as the union of three pieces \( D_1 \cup D_0 \cup D_{-1} \) where:

(a) \( D_1 \) is the interior of the top cell in \( S^k \times S^m \), and \( \{F, F'\}|_{D_1} \) equals \( F \times F' \);

(b) \( D_0 \) is a cylinder \( S^{k+m-1} \times [0, 1] \) which is mapped by the composite

\[
S^{k+m-1} \times [0, 1] \xrightarrow{\text{id} \times e} (S^k \vee S^m) \times [0, 1] \xrightarrow{H_t \vee F'} BG,
\]

where \( e : S^{k+m-1} \to S^k \vee S^m \) is the attaching map of the top cell in \( S^k \times S^m \), and \( H_t \) is a homotopy from \( H_1 = \pi \circ F \) to the constant map \( H_0 \); and

(c) \( D_{-1} \) is a capping disc, mapped by a fixed homotopy from \( (H_0 \vee F') \circ e \) to the constant map. This homotopy exists because the Whitehead product \( [H_0, F'] \) vanishes. Since \( F'(S^m) \subset BG_p \) we may assume that the restriction of \( \{F, F'\} \) to \( D_{-1} \) factors through a map \( D_{-1} \to BG_p \). Thus here we think of the domain as

\[
S^{k+m} = S^{k+m-1} \times [-1, 2]/\sim,
\]
where the equivalence relation \( \sim \) collapses each sphere
\[ S^{k+m-1} \times \{ -1 \}, \quad S^{k+m-1} \times \{ 2 \} \]
to a single point and we identify \( D_i \) with \( S^{k+m-1} \times [i, i+1] \).

Now choose any model for \( BG \) such that the induced map \( BG_p \to BG \) is injective, and denote by \( EG \) the space of paths in \( BG \) with fixed initial point. The map \( \Psi : S^{k+m} \to G \) lifts to
\[ \tilde{\Psi} : D^{k+m} := D^{k+m-1} \times [-1, 2]/\sim \to EG. \]

We must show how to construct \( \Psi \) so that the restriction of \( \tilde{\Psi} \) to the boundary
\[ S^{k+m-1} := \partial D^{k+m-1} \times [-1, 2]/\sim \]
agrees with formula (2.2). To do this we construct \( \Psi \) using data coming from \( G \) rather than \( BG \). Again, we think of \( S^{k+m-1} \) as divided into three pieces, a central cylinder \( C_0 := \partial D^{k+m-1} \times [0, 1] \) together with two capping discs \( C_{-1}, C_1 \).

Choose the null homotopy \( H_t : S^k \to BG, t \in [0, 1], \) to be the image under transgression of the null homotopy \( h_t : S^{k-1} \to G \), where \( h_t \) is the restriction of \( \tilde{f} \) to the sphere of radius \( t \) in \( D^k \). Then the Whitehead product \( [H_t, F^t] \) desuspends to the Samelson product \( \langle h_t, f' \rangle \) for each \( t \in [0, 1] \). Hence we may define \( \Psi \) on the cylinder \( D_0 \) to have a lift \( \tilde{\Psi} \) that restricts on \( C_0 := \partial D^{k+m-1} \times [0, 1] \) to the family of maps \( \langle h_t, f' \rangle \) defined as in (2.1).

When \( t = 0, 1 \) the Samelson products \( \langle h_t, f' \rangle \) lie in \( G_p \) and we may choose \( \Psi \) and its lift \( \tilde{\Psi} \) so that \( \tilde{\Psi} \) maps the discs \( D^{k+m-1} \times \{ 0 \} \) into \( EG_p \). Moreover, \( \langle h_0, f' \rangle \) is constant, since \( h_0 \) is the constant map at the identity element. Thus the restriction of \( \tilde{\Psi} \) to \( D^{k+m-1} \times \{ 0 \} \) descends to a map \( D^{k+m-1} \to EG_p \). Then \( \tilde{\Psi} \) extends to the ball \( D^{k+m} \) because \( EG_p \) is contractible, and we define \( \Psi \) on \( D_{-1} \equiv D^{k+m} \) to be the projection of \( \tilde{\Psi} \) to \( BG \), followed by the map \( BG_p \to BG \).

Finally consider the disc \( D_1 \). The chosen null homotopy \( c \) of \( \langle f, f' \rangle \) defines a map of the disc \( D' = \partial D^{k+m-1} \times [1, 2]/\sim \to G_p \) that agrees with the restriction of \( \tilde{\Psi} \) to \( \partial D' = \partial D^{k+m-1} \times \{ 1 \} \). This gives a map of the \( (k+m) \)-sphere \( D' \cup (D^{k+m-1} \times \{ 1 \}) \) into the contractible space \( EG_p \). Again, choose \( \tilde{\Psi} \) on the disc \( D^{k+m-1} \times [1, 2]/\sim \) to be any extension and define \( \Psi|_{D_1} \) to be the corresponding map to \( BG \).

2.2. Calculations in \( SU(n+1) \). We end this section by further investigating the construction in Corollary 2.4 in the case when the homomorphism \( h : S^1 \times K \) takes values in a Lie group \( H_p \). For simplicity we restrict to the case when \( H \) is a subgroup of \( G := SU(n+1) \) and \( H_p \) is contained in \( G_p := U(n) \). (Thus \( G/G_p = \mathbb{C}^p \).) Unless explicit mention is made to the contrary, we embed \( SU(k) \) in \( SU(k+1) \) as the subgroup that acts on the first \( k \) coordinates, and similarly for inclusions \( U(k) \subset U(k+1) \).
For $k = 1, \ldots, n$ consider the commuting circle actions
\begin{equation}
\lambda_k : S^1 \to SU(k+1) \subset SU(n)
\end{equation}
where $\lambda_1$ has weights $(1, -1, 0, \ldots, 0)$, $\lambda_2$ has weights $(1, 1, -2, 0, \ldots, 0)$ and $\lambda_k$ has weights $(1, \ldots, 1, -k, 0, \ldots, 0)$. Since $\lambda_1$ contracts in $SU(2)$ and $\pi_2(U(k)) = 0$, the construction of Corollary 2.4 gives a well defined element
\[
\alpha_3 := \{\lambda_1, \lambda_1\} \in \pi_3(U(2)).
\]
Because $\lambda_2$ commutes with $SU(2)$ and contracts in $SU(3)$ we may repeat to get a well defined element
\[
\alpha_5 := \{\lambda_2, \alpha_3\} = \langle \tilde{\lambda}_2, \alpha_3 \rangle \in \pi_5(U(3)),
\]
where $\tilde{\lambda}_2$ is a contraction of $\lambda_2$ and we use formula (2.2). This is a version of the construction in Corollary 2.4. To see this in general, define the homomorphism $h_k : S^1 \times U(k) \to U(k+1)$ to be the identity on the second factor and to take $U(1)$ to the center of $U(k+1)$ in the obvious way, and define $\iota_k : U(k) \to SU(k+1)$ by
\[
\iota_k(A) = \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}.
\]
Then the composite $\iota_k \circ h_k : S^1 \to SU(k+1)$ is precisely $\lambda_k$, and assuming that $\alpha_{2k-1} \in \pi_{2k-1}(SU(k))$ is already defined we may set:
\[
\alpha_{2k+1} := \{\lambda_k, \alpha_{2k-1}\} = \langle \tilde{\lambda}_k, \alpha_{2k-1} \rangle \in \pi_{2k+1}(SU(k+1)),
\]
where $\tilde{\lambda}_k : D^2 \to SU(k+1)$ is a contraction of $\lambda_k$.

**Lemma 2.5.** $\alpha_{2k+1} \in \pi_{2k+1}(SU(k+1))$ is nonzero for $k = 1, \ldots, n$.

**Proof.** Let $T^n$ be the diagonal torus in $SU(n+1)$ and denote by $\Lambda_k : S^2 \to BT^k$ the desuspension of $\lambda_k$. Because Whitehead products vanish in $BT^k$, the map $\Lambda_1 \vee \Lambda_1 \vee \Lambda_2 \vee \cdots \vee \Lambda_k : \vee_{k+1} S^2 \to BT^k$ has an extension to
\[
\prod_{i=1}^{k+1} S^2 \xrightarrow{g_k = \Lambda_1 \times \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_k} BT^k
\]
that is unique up to homotopy. Taking the composite with the inclusion $BT^k \to BU(k+1)$ gives a rank $(k+1)$ bundle over $\prod S^2$. We show below that it suffices to show that this bundle has some nontrivial Chern classes in dimension $2k+2$. We then calculate the pullback of $c_{k+1}$.

We first claim that for each $k \in \{1, \ldots, n\}$ there is a homotopy commutative diagram
\[
\begin{array}{ccc}
\prod_{i=1}^{k+1} S^2 & \xrightarrow{g_k} & BT^k \\
\downarrow{\gamma_k} & & \downarrow \\
S^{2k+2} \vee J_k & \xrightarrow{F_k} & BSU(k+1),
\end{array}
\]
in which the CW complex $J_k$ has dimension $\leq 2k$. We construct this diagram by induction on $k$. The only difficulty is to construct the left vertical map $\gamma_k$. When $k = 1$ the diagram reduces to

$$
\begin{array}{ccc}
S^2 \times S^2 & \xrightarrow{\Lambda_1 \times \Lambda_1} & BT^1 \\
\downarrow \gamma_1 & & \downarrow \\
S^4 \vee S^2 & \xrightarrow{F_1} & BSU(2) \subset BSU(n+1),
\end{array}
$$

and is a special case of the situation discussed before Proposition 2.1. For $k > 1$ we may assume by induction that the map $\prod_{i=1}^{k+1} S^2 \rightarrow BSU(k+1)$ given by going horizontally to $BT^k$ and then vertically to $BSU(k+1)$ factors through

$$
S^2 \times (S^{2k} \vee J_k) \xrightarrow{\Lambda_k \times F_{k-1}} BZ_k \times BSU(k) \rightarrow BSU(k+1) \subset BSU(n+1),
$$

where $Z_k$ denotes the commutator of $SU(k)$ in $SU(k+1)$. Since $\Lambda_k$ contracts in $BSU(k+1)$ the product $\Lambda_k \times F_{k-1}$ is homotopic via maps to $BSU(k+1)$ to a map that takes the sphere $S^2 \times \{pt\}$ to a single point. But when one contracts the first sphere in the product $S^2 \times (S^{2k} \vee J_{k-1})$ one obtains a space of the form $S^{2k+2} \vee J_k$. Thus the above diagram exists for all $k$.

Next observe that $g_k$ factors through the inclusion $BU(k) \rightarrow BSU(k+1)$ given by

$$
A \mapsto \left( \begin{array}{cc} A & 0 \\ 0 & \det A^{-1} \end{array} \right).
$$

Therefore we may apply Proposition 2.2 to conclude that the restriction of $F_k$ to the top sphere $S^{2k+2}$ transgresses to $\alpha_{2k+1} \in \pi_{2k+1} SU(k+1)$ modulo the subgroup $\mathcal{K}'$ generated by the image of $\pi_{2k+1}(U(k))$ and certain Samelson products. Since $K' = 0$, we find that $\alpha_{2k+1} \neq 0$ if and only if $F_k|_{S^{2k+2}}$ is nonzero. This will be the case precisely when the bundle represented by the composite

$$
\prod S^2 \xrightarrow{g_k} BT^k \longrightarrow BSU(k+1)
$$

has nontrivial top dimensional Chern classes.

We now check this by calculating the pullback of $c_{k+1}$. Denote by $T := (S^1)^{k+1}$ the diagonal subgroup in $U(k+1)$ and by $t_1, \ldots, t_{k+1} \in H^2(BT)$ the obvious generators of $H^*(BT)$. Then $c_{k+1}$ pulls back to $t_1 \ldots t_{k+1} \in H^{2k+2}(BT)$. The map $BT^k \rightarrow BU(k+1)$ factors through $BT$ and it suffices to show that the pull back of $t_1 \ldots t_{k+1}$ by the composite

$$
\Phi : \prod S^2 \xrightarrow{g_k} BT^k \longrightarrow BT
$$

does not vanish. For $j = 0, 1, \ldots, k$ denote by $y_j$ the pullback to $\prod_{j=0}^k S^2$ of the generator of $H^2(S^2)$ by the projection onto the sphere that is mapped

\[\text{\footnotesize{Because the map } BSU(k+1) \rightarrow BU(\infty) \text{ is } (2k+2)-\text{connected, one can equally well phrase this calculation in terms of the structure of the induced stable bundle on } \prod S^2.}\]
by $\Lambda_j$ (where we set $\Lambda_0 := \Lambda_1$.) Then
\[
\Phi^*(t_1 \ldots t_{k+1}) = \left( y_0 + y_1 + y_2 + \cdots + y_k \right) \left( -y_0 - y_1 + y_2 + \cdots + y_k \right) \times \\
\times \prod_{j=2}^{k} \left( -jy_j + \sum_{i>j} y_i \right) \\
= 2(-1)^k k! y_0 y_1 y_2 \ldots y_k.
\]

since $y_j^2 = 0$ for all $j$. Since this is nonzero, the proof is complete. \qed

**Remark 2.6.** One can construct a nontrivial element in $\pi_{2k+2}(BSU(k+2))$ for $k \geq 1$ by the following inductive procedure. Let $E_2 \to S^2$ be the complex line bundle classified by the map $\Lambda_2 : S^2 \to BU(1)$. Note that its first Chern class is equal to the generator $y \in H^2(S^2)$. Suppose also that we have already constructed a map $\Lambda_{2k} : S^2 \to BU(k)$ that classifies a bundle $E_{2k}$ with nontrivial Chern class $c_{2k}(E_{2k})$. Then construct a homotopy commutative diagram

\[
\begin{array}{c}
S^2 \times S^{2k} \\
\downarrow \Lambda_2 \times \Lambda_{2k} \\
S^{2k+2} \lor S^{2k} \\
\downarrow F \\
BU(1) \times BU(k) \\
\downarrow \\
BU(k+1) \\
\downarrow \\
BSU(k+2)
\end{array}
\]

in which the top row classifies the bundle $E_2 \times E_{2k} \to S^2 \times S^{2k}$ whose Chern class $c_{k+1} = p_1^* c_1(E_2) \cup p_2^* c_{2k}(E_{2k}) \neq 0$. Since $\Lambda_2$ contracts in $BSU(k+1)$ the bottom row of the diagram is constructed in the usual way, and the induced map $S^{2k+2} \to BSU(k+2)$ is homotopically nontrivial. But note that it does not transgress to the element of $\pi_{2k+1}(SU(k+2))$ constructed via commutators as in Corollary 2.4. For one can choose the contraction of $\Lambda_2$ to lie in a copy of $SU(2) \subset SU(k+2)$ that commutes with the image of $U(k)$. This is not a contradiction since Proposition 2.2 states only that the induced map should be given by (2.2) modulo elements of $K'$. But now $K'$ is the whole of $\pi_{2k+1}(SU(k+2))$ since it includes the image of $\pi_{2k+1}(U(k+1))$.

3. **Characteristic classes**

Let $G = \text{Ham}(\mathcal{M}, \omega)$ and consider the universal bundle $M \xrightarrow{j} M_G \xrightarrow{\pi} BG$. There is a unique class $[\Omega] \in H^2(M_G; \mathbb{R})$ called the **coupling class** that extends the fiberwise symplectic class $[\omega]$ and has the property that the fiberwise integral $\int_M [\Omega]^{n+1} \in H^2(BG; \mathbb{R})$ vanishes. Following Januszkiewicz–Kędra [10], we define the classes

\[
(3.1) \quad \mu_k := \int_M [\Omega]^{n+k} \in H^{2k}(BG).
\]

In this section we first generalize these classes to other groups and $H$-spaces, and then discuss computations and applications. When $H^1(M) \neq 0$ we shall work only with connected groups and $H$-spaces. The issues that arise in the
general case are discussed from different perspectives in Gal–Kedra [5] and McDuff [12].

3.1. The classes $\mu_k$. Our first aim is to define the classes $\mu_k$ in as general a context as possible. Thus if $\mathcal{H}$ is an $H$-space that acts on a c-symplectic manifold $(M, a)$ we need to determine conditions on $\mathcal{H}$ that guarantee that the class $a \in H^2(M)$ has a well defined extension to a class $\tilde{a} \in H^2(M_\mathcal{H})$. As before we denote by $\mathcal{H}_0$ the identity component of the space of smooth maps $M \to M$ and by

$$H_a$$

the space of smooth homotopy equivalences that fix $a$. Further $M_\mathcal{H}$ denotes the total space of the universal $M$-bundle over $B\mathcal{H}$. Thus the projection $\pi : M_\mathcal{H} \to B\mathcal{H}$ is a Hurewicz fibration.

Recall that the a-Flux homomorphism $\text{Flux}^a : \pi_1(\mathcal{H}_0) \to H^1(M; \mathbb{R})$ is given by

$$\text{Flux}^a(\lambda)(\gamma) = \langle a, \text{tr}_\lambda(\gamma) \rangle$$

for $\gamma \in H_1(M)$, where $\text{tr}_\lambda(\gamma) \in H_2(M)$ is represented by $T^2 \to M, (s, t) \mapsto \lambda(s)(\gamma(t))$. In the symplectic case $\text{Flux}^\omega$ extends to a homomorphism defined on the identity component $\text{Symp}_0(M, \omega)$ of the symplectomorphism group, and the Hamiltonian group is its kernel. In the c-symplectic case there is no extension of $\text{Flux}^a$ to $\text{Diff}_0(M)$ (which is a simple group), and the analog $\text{Ham}(M, a)$ of the Hamiltonian group is the covering group of $\text{Diff}_0(M)$ corresponding to the kernel of $\text{Flux}^a : \pi_1(\text{Diff}_0) \to H^1(M; \mathbb{R})$. (In the symplectic case, $\text{Ham}(M, \omega)$ is homotopy equivalent to the corresponding covering of $\text{Symp}_0(M, \omega)$. Further there is an inclusion $\text{Ham}(M, \omega) \to \text{Ham}(M, a)$ given by taking the element $f \in \text{Ham}(M, \omega)$ to the pair $[f, \{f_t\}]$, where $f_t, t \in [0, 1]$, is any path in the connected group $\text{Ham}(M, \omega)$ from the identity to $f$. Since two such paths differ by a loop with zero a-flux, this recipe defines a unique element $[f, \{f_t\}] \in \text{Ham}(M, a)$.) Sometimes we will also work with the covering space

$$\tilde{\mathcal{H}}_0^a$$

of $\mathcal{H}_0$ corresponding to the kernel of $\text{Flux}^a$. This is an $H$-space.

A smooth circle action on a c-symplectic manifold $(M, a)$ is said to be c-Hamiltonian if it is in the kernel of $\text{Flux}^a$. As shown by Allday [2] such actions do have certain geometric properties; for example their fixed point set has at least two connected components, though these components need not be c-symplectic as would happen in the symplectic case.

**Proposition 3.1.** (i) Given $a \in H^2(M)$ such that $\int_M a^n \neq 0$, consider the fibration $M \to M_\mathcal{H} \to B\mathcal{H}$ where $\mathcal{H} := \tilde{\mathcal{H}}_0^a$. Then there is a unique element $\tilde{a} \in H^2(M_\mathcal{H})$ that restricts to $a \in H^2(M)$ and is such that $\int_M \tilde{a}^{n+1} = 0$.

(ii) If $H^1(M) = 0$, such a class $\tilde{a}$ exists on $M_\mathcal{H}$.

The above class $\tilde{a}$ is called the coupling class.
Proof. (i) Consider the Leray–Serre cohomology spectral sequence for the (Hurewicz) fibration $M \to M_H \to BH$. Since $\mathcal{H}$ is connected the $E_2$-term is a product and the class $a$ lies in $E_2^{2,0} = H^2(M) \otimes H^0(B\mathcal{H})$. As explained for example in [11, Lemma 2.2] the differential

$$d_2 : E_2^{2,0} \to E_2^{1,2} = H^1(M) \otimes H^2(B\mathcal{H})$$

is determined by the flux homomorphism. More precisely, if we consider the elements of $H^1(M) \otimes H^2(B\mathcal{H})$ as homomorphisms $H_2(B\mathcal{H}) \to H^1(M)$, then

$$d_2(a)(\overline{\lambda}) = \text{Flux}^a(\lambda),$$

where $\overline{\lambda} \in \pi_2(B\mathcal{H}) \cong H_2(B\mathcal{H})$ corresponds to the loop $\lambda \in \pi_1(\mathcal{H})$. Hence, because $\lambda$ is c-Hamiltonian, $d_2(a) = 0$. But the image of $a$ under $d_3 : E_3^{2,0} \to E_3^{3,3}$ must vanish: since $a^{n+1} = 0$ in $H^*(M)$,

$$0 = d_3(a^{n+1}) = (n+1)a^n \otimes d_3(a) \in H^{2n}(M) \otimes H^3(B\mathcal{H}),$$

which is possible only if $d_3(a) = 0$. Therefore $a$ survives into the $E_\infty$-term of the spectral sequence, and so has some extension $u \in H^2(M_\mathcal{H})$.

To prove uniqueness note that because $B\mathcal{H}$ is simply connected the kernel of the restriction map $H^2(M_\mathcal{H}) \to H^2(M)$ is isomorphic to the pullback of $H^2(B\mathcal{H})$. A short calculation shows that we may take

$$\tilde{a} := u - \frac{1}{n+1}\pi^*\pi_1(u^{n+1}),$$

where $\pi_1$ denotes integration over the fiber. This proves (i).

Now consider (ii). Because the elements in $\mathcal{H} := \mathcal{H}_a$ preserve $a$ there is an element $a$ in the $E_2^{0,2}$-term in the Leray–Serre spectral sequence for the Hurewicz fibration $M_{\mathcal{H}_a} \to B\mathcal{H}_a$. Since $H^1(M) = 0$ $d_2$ must vanish and the argument showing that $d_3(a) = 0$ still holds. Further the uniqueness proof goes through as before: although $B\mathcal{H}$ need no longer be simply connected, the fact that $H^1(M) = 0$ implies that $E_\infty^{1,1} = 0$. Hence the kernel of the restriction map $H^2(M_{\mathcal{H}}) \to H^2(M)$ is still isomorphic to the pullback of $H^2(B\mathcal{H})$. This proves (ii). \hfill \Box

Remark 3.2. (i) If the first Chern class of $(M, \omega)$ is a nonzero multiple of the symplectic class $a$, then there is an easier way to find an extension $u \in H^2(M_{\mathcal{H}})$ of the symplectic class: simply take it to be an appropriate multiple of the first Chern class of the vertical tangent bundle. This construction applies whenever $\mathcal{H}$ acts in such a way that the vertical bundle has a complex structure. The most natural choice for $\mathcal{H}$ is the group of symplectomorphisms. Note that we cannot take it to be the diffeomorphism group.

(ii) Let $(M, a)$ be any c-symplectic manifold and consider an $M$-bundle $P \to B$. We saw above that if $a$ survives into the $E_2$ term of the Leray–Serre spectral sequence then $d_2(a) = 0$ and $a$ survives to $E_\infty$. This shows that the obstruction to the existence of an extension $\tilde{a}$ of $a$ depends only on the restriction of $P \to B$ over the 2-skeleton of $B$. 
Definition 3.3. Let \((M,a)\) be a c-symplectic manifold and let \(\mathcal{H}\) denote either the H-space \(\mathcal{H}_a^0\) or, if \(H^1(M) = 0\), the H-space \(\mathcal{H}_a\). We define \(\mu_k \in H^{2k}(BH)\) by
\[
\mu_k := \pi_1(\tilde{a}^{n+k}),
\]
where \(\tilde{a}\) is the coupling class constructed in Proposition 3.1.

These classes \(\mu_k\) extend those defined in equation (3.1). Note that \(\mu_1 \equiv 0\) by definition. We will see in the next section that the pullback of \(\mu_{2k}\) to \(H^*(BS^1)\) is nonzero for all \(k > 0\) and for every nontrivial c-Hamiltonian circle action on \(M\).

Proof of Proposition 1.8. We must show that if \(G = SU(n+1)\) acts on \(M := \mathbb{C}P^n\) then the induced map \(R_* : H_*(BSU(n+1)) \to H_*(BH)\) is injective where \(\mathcal{H} := \mathcal{H}_a\). For \(2 \leq k \leq n+1\) choose \(s_{2k} : S^{2k} \to BSU(n+1)\) so that \(\alpha_k := s_{2k}^*(c_k)\) generates \(H^{2k}(S^{2k})\), where \(c_k\) is the \(k\)th Chern class.

Consider the bundle \(\mathbb{C}P^n \to P \overset{p}{\rightarrow} S^{2k}\) associated with this element. (This is simply the projectivization of the corresponding rank \((n+1)\) vector bundle \(E \to S^{2k}\).) The Leray–Hirsch theorem states that the cohomology of \(P\) is a free \(H^*(S^{2k})\)-module generated by the powers \(1 = c^0, \ldots, c^n\) of the first Chern class \(c\) of the tautological line bundle. Since \(2 \leq k \leq n+1\) there is \(\beta_k \in H^{2k}(S^{2k})\) so that
\[
c^{n+1} = p^*(\beta_k) \cup c^{n+1-k}.
\]
The theory of characteristic classes implies that \(\beta_k = c_k(E) = s_{2k}^*(c_k) =: \alpha_k\). Multiplying this equality by \(c^{k-1}\) we obtain
\[
c^{n+k} = p^*(\alpha_k) \cup c^n,
\]
which implies that \(p_!(c^{n+k}) = \text{const.}\alpha_k \neq 0\). Now observe that because \(k \geq 2\), \(H^2(P)\) has dimension 1, so that \(c\) equals the coupling class \(\tilde{a} \in H^2(P)\). Hence
\[
\left\langle \mu_k, R_*([S^{2k}]) \right\rangle = \text{const.} \left\langle \alpha_k, [S^{2k}] \right\rangle \neq 0.
\]
Therefore \(\pi_*(BG) \to H_*(BH)\) is injective. Since \(H^*(BG)\) is freely generated by the duals of the spherical classes, the induced map \(R^* : H^*(BH) \to H^*(BG)\) is surjective, and so the corresponding map on homology is injective.

Remark 3.4. Let \(\mathcal{G} := \text{Ham}(M,a)\) and denote by \(\mathcal{G}_p\) the homotopy fiber of the evaluation map \(\mathcal{G} \to M\). Thus if the elements of \(\mathcal{G}\) are pairs \((g, \{g_t\})\) where \(\{g_t\}\) is an equivalence class of paths from \(id\) to \(g\), then \(\mathcal{G}_p\) is the subgroup of \(\mathcal{G}\) consisting of all pairs such that \(g(p) = p\). The restriction \(M_{\mathcal{G}_p} \to B\mathcal{G}_p\) of the universal bundle \(M \to BG\) to \(B\mathcal{G}_p \subset BG\) has a canonical section \(\sigma : B\mathcal{G}_p \to M_{\mathcal{G}_p}\) whose image \(\sigma(b)\) at \(b \in B\mathcal{G}_p\) is the point in the fiber
$M_b$ corresponding to $p$. Thus there is a homotopy commutative diagram

$$
\begin{array}{ccc}
M_{G_p} & \xrightarrow{\iota} & M_G \\
\sigma \downarrow & & \downarrow \\
BG_p & \xrightarrow{\sigma} & BG
\end{array}
$$

such that the composite $\iota \circ \sigma : BG_p \to M_G$ is a homotopy equivalence. By Proposition 3.1, the fiberwise symplectic class $a$ extends to $M_{G_p} \subset M_G$. We shall denote by $\tilde{a}_p$ the extension that is normalized by the requirement that $\sigma^*(\tilde{a}_p) = 0$. Correspondingly there are characteristic classes

$$
\nu_k := \pi_! \left( (\tilde{a}_p)^{k+n} \right) \in H^{2k}(B\text{Ham}(M,a)_p), \quad k \geq 1.
$$

We show at the end of §3.2 that $\nu_1$ need not vanish.

### 3.2. Calculations for circle actions.

We suppose that $S^1$ acts smoothly on a connected almost symplectic manifold $(M,a)$. By averaging we may construct an $S^1$-invariant closed representative $\omega$ of the class $a$. If $\xi$ is the generating vector field on $M$ for the action then the identity

$$
0 = L_\xi(\omega) = d(t_\xi \omega) + t_\xi(d\omega)
$$

implies that the 1-form $t_\xi \omega$ is closed. To say the action is c-Hamiltonian is equivalent to saying that this 1-form is exact. The requirements

$$
dH = t_\xi \omega, \quad \int_M H\omega^n = 0
$$

define a unique function $H : M \to \mathbb{R}$ that is called the normalized (Hamiltonian) $\omega$-moment map. The following useful result is presumably well known.

**Lemma 3.5.** If the action is nontrivial (i.e. nonconstant) $H$ cannot be identically zero.

**Proof.** There is a subset $M_0$ of full measure in $M$ where the $S^1$ action gives rise to a fibration $S^1 \to M_0 \to N$. If $\omega$ is both invariant and such that $t_\xi \omega \equiv 0$ then $\omega|_{M_0}$ pulls back from $N$ and so cannot satisfy the condition $\int_{M_0} \omega^n \neq 0$.

We first describe a cocycle representing the coupling class in the Cartan model of the equivariant cohomology (see [9] for details). Recall that the Cartan model is the following DGA

$$
\Omega^*_G(M) := (S(\text{Lie}(S^1)^*) \otimes \Omega^*(M))^{S^1},
$$

with differential $d := 1 \otimes d_M - x \otimes t_\xi$, where $\xi \in \text{Lie}(S^1)$ is a basis vector, $x \in S(\text{Lie}(S^1)^*)$ its dual and $\xi$ is the induced vector field on $M$. The coupling class is represented by $\Omega := 1 \otimes \omega - x \otimes H$, where $H : M \to \mathbb{R}$ is the normalized (i.e. $\int_M H\omega^n = 0$) $\omega$-moment map. Note that, by definition of the moment map,

$$
d\Omega = -x \otimes t_\xi \omega - x \otimes d_M H = 0.
$$
Thus \( \Omega \) is closed.

The fiber integration \( \pi : \Omega^*_{S^1}(M) \to \Omega^*_{S^1}(pt) = S(Lie(S^1)^*) \) corresponds to the (equivariant) constant map \( M \to pt \) in the following way. An equivariant differential form \( \alpha \in \Omega^*_{S^1}(M) \) can be regarded as a polynomial map \( \alpha : \text{Lie}(S^1) \to \Omega^*(M) \). Then \( \pi(\alpha) \) is a polynomial map \( \text{Lie}(S^1) \to \mathbb{R} \) given by

\[
(\pi(\alpha))(\xi) := \int_M \alpha(\xi),
\]

since the (nonequivariant) fiber integration corresponding to the constant map is just the usual integration over the manifold \( \int_M : \Omega^*(M) \to \Omega^*(pt) = \mathbb{R} \) (cf. Theorem 10.1.1 in [9]).

**Lemma 3.6.** Suppose that \((M,a)\) is a c-symplectic manifold of dimension \(2n\) and consider a c-Hamiltonian circle action \(\lambda\) on \(M\) with normalized \(\omega\)-moment map \(H\). Denote by \(\mu_k(\lambda) \in H^*_S(pt)\) the pullback of the characteristic class \(\mu_k\) to \(BS^1\) by the classifying map of the associated bundle \(M \to P \to BS^1\). Then

\[
(\mu_k(\lambda))(\xi) := (-1)^n \left( \frac{n+k}{n} \right) \int_M H^k \omega^{n+k} \cdot x^k \in S(\text{Lie}(S^1)^*) = H^*_S(pt).
\]

In particular, \(\mu_k(\lambda) \neq 0\) for even \(k\) whenever the circle action is nontrivial.

**Proof.** Because \(\omega^m = 0\) for \(m > n\) for dimensional reasons,

\[
\Omega^{n+k} = (1 \otimes \omega - x \otimes H)^{n+k} = \sum_{m=0}^{n} (-1)^m \left( \frac{n+k}{m} \right) (x^{n+k-m} \otimes \omega^m H^{n+k-m}).
\]

Now apply the above discussion to see that the only term that contributes to the integral is the one containing the \(n\)-th power of the symplectic form. This establishes formula (3.2). Now observe that the function \(H^k\) is nonnegative for even \(k\) and so by Lemma 3.5 its integral vanishes only if the action is trivial. \(\square\)

We now complete the proof of Theorem 1.1.

**Corollary 3.7.** Let \(\mathcal{H}\) be as in Definition 3.3 and suppose that \(\lambda \subset \mathcal{H}\) is an inessential (nontrivial) circle action on \(M\). Denote by \(\bar{\Lambda} \in \pi_4(B\mathcal{H})\) the element \(\{\Lambda, \Lambda\}\) that is formed from the suspension \(\Lambda : S^2 \to B\mathcal{H}\) of \(\lambda\) as in Proposition 2.1. Then

\[
\mu_2(\bar{\Lambda}) \neq 0.
\]

**Proof.** Denote by \(r : S^1 \to \mathcal{H}\) the homomorphism with image \(\lambda\). As remarked after Corollary 2.4 it suffices to show that the corresponding map \(R : BS^1 \to B\mathcal{H}\) is nontrivial on \(H_4\). But it follows from Lemma 3.6 that the pullback \(R^*(\mu_2) = [\pi(\Omega^{n+2})]\) is nonzero, where \(\mu_2\) is as in (3.1). \(\square\)

The next result is an immediate consequence of Lemma 3.6 and the proof of Lemma 2.5.
Lemma 3.8. Suppose that $G := SU(\ell)$ acts smoothly on the $2n$-manifold $(M,a)$, and let $\mathcal{H}$ be as in Definition 3.3. Denote by $\lambda_k, k = 1, \ldots, \ell - 1$, the circles in $SU(\ell)$ defined by (2.3) and by $H_k, k = 1, \ldots, \ell - 1$, the $\omega$-moment maps for $\lambda_k$ where $\omega$ is a $G$-invariant representative for $a$. Then $\pi_{2k}(BG)$ has nonzero image in $H_{2k}(B\mathcal{H})$ if and only if

$$
\int_M (H_1)^2 H_2 \cdots H_{k-1} \omega^n \neq 0.
$$

Proof of Proposition 1.5. This is the special case with $\ell = 2$. Then the integral is $\int H_2 \omega^n$ which is always nonzero by Lemma 3.5.

One can also deduce Proposition 1.5 directly from Theorem 1.1 since by Lemma 2.5 and the uniqueness statement in Theorem 1.1 the element $\rho$ formed from any circle subgroup in $SU(2)$ is a generator of $\pi_3(SU(2))$.

Proof of Lemma 1.3. Consider a $c$-Hamiltonian $S^1$-action with moment map $H$, and choose a fixed point $p$. Because $S^1$ maps to a loop in $Diff_0$ with trivial flux the inclusion $S^1 \to Diff_0$ lifts to a homomorphism $S^1 \to G_p$, where $G_p$ is as defined in Remark 3.4. Moreover, the section $\sigma$ pulls back over the corresponding classifying map $BS^1 \to BG_p$ to the section of $M_{S^1} \to BS^1$ with image $BS^1 \times \{p\}$. Therefore the pullback of $\tilde{a}_p$ to $M_{S^1}$ may be represented in the Cartan model by the element $\Omega := 1 \otimes x \otimes (H - H(p))$. Hence the proof of Lemma 3.6 shows that for some nonzero constant $c$

$$
\nu_1(\lambda) = c \int_M (H - H(p)) \omega^n.
$$

If this integral is nonzero, the 2-skeleton $S^2$ of $BS^1$ maps nontrivially to $BG_p$ under the classifying map $BS^1 \to BG_p$. This is equivalent to saying that $\lambda$ is essential in $G_p$.

3.3. Relation to Reznikov’s classes and the $\kappa$-classes. Above we defined classes $\mu_k$ in $H^*(B\mathcal{H})$ for $H$-spaces $\mathcal{H}$ whose action is $c$-Hamiltonian, i.e. the $c$-symplectic class $a$ extends over $M_{\mathcal{H}}$. If we restrict to the case $\mathcal{G} = Ham(M,\omega)$ then these classes have a geometric interpretation. We now show that they can be constructed by the usual Chern–Weil process. As a consequence they correspond to classes in $H^{2k-1}(Ham(M,\omega))$ that are the restrictions of Reznikov’s classes in the smooth Lie algebra cohomology $H^{2k-1}_{sm}(Lie Ham(M,\omega))$.

To understand this, recall that $Lie Ham(M,\omega)$ can be identified with the space of functions $C_0(M)$ of zero mean on $M$ (with respect to the volume form $\omega^n$). Moreover, it has a nondegenerate bilinear form (a Killing form) defined by

$$
\langle F, G \rangle := \int_M FG \omega^n
$$

that is invariant under the adjoint action of $\mathcal{G} := Ham(M,\omega)$. Now the coupling class $[\Omega]$ may be represented by a closed differential form $\Omega$ on $M_\mathcal{G}$ that extends the fiberwise symplectic form. (Here we are working with den
Rham theory on $M_G$ which is not, strictly speaking, a manifold. However, one can make everything precise by considering an arbitrary smooth map $B \to BG$ and looking at the associated smooth pullback bundle over $B$.) The form $\Omega$ defines a connection on the bundle $\pi : M_G \to BG$ whose horizontal distribution $Hor_x, x \in M_G$, is given by the $\Omega$-orthogonals to the vertical tangent bundle. As shown by Guillemin–Lerman–Sternberg [7], the corresponding parallel transport maps preserve the fiberwise symplectic form and the holonomy is Hamiltonian. Moreover, given vector fields $v, w \in T_bBG$ with horizontal lifts $v^\flat, w^\flat$, the function $\Omega(v^\flat, w^\flat)(x)$ restricts on each fiber $M_b := \pi^{-1}(b)$ to an element of $\text{Lie}(\mathcal{G})$ that represents the curvature of this connection at $(v, w)$. In other words, the closed 2-form $\Omega$ on $M$ defines a 2-form $\tilde{\Omega}$ on the base $BG$ of the fibration that takes values in the Lie algebra $\text{Lie}(\mathcal{G})$ of the structural group $\mathcal{G}$ and is a curvature form in the usual sense. (For more detail, see McDuff–Salamon [13, Ch 6].)

Any $\text{Ad}$-invariant polynomial $T^k : \text{Lie}(\mathcal{G})^\otimes k \to \mathbb{R}$ therefore gives rise to a characteristic class $c^\mathcal{G}_k$ in $H^*(BG)$, namely the class represented by the closed real-valued 2k-form $T^k \circ \tilde{\Omega}^k$. In the case at hand we may take

$$T^k(F_1 \otimes \cdots \otimes F_k) := \int_M F_1 \cdots F_k \omega^n.$$  

**Lemma 3.9.** This class $c^\mathcal{G}_k$ equals $\text{const.}\mu_k$.

**Proof.** Let $v_1, \ldots, v_{2k}$ be vector fields on $BG$ with horizontal lifts $v_1^\flat, \ldots, v_{2k}^\flat$. Then, if the $w_j$ are tangent to the fiber at $x \in M_G$ we find

$$\begin{align*}
\Omega^{n+k}(w_1, \ldots, w_{2n}, v_1^\flat, \ldots, v_{2k}^\flat)(x) &= \sum_{\sigma} \varepsilon(\sigma) \binom{n+k}{n} \times \\
F_{1,\sigma}(x) \cdots F_{k,\sigma}(x) \omega^n(w_1, \ldots, w_{2n}),
\end{align*}$$

where, for each permutation $\sigma$ of $\{1, \ldots, 2k\}$, $\varepsilon(\sigma)$ denotes its signature and

$$F_{j,\sigma}(x) := \Omega(v_{\sigma(2j-1)}^\flat, v_{\sigma(2j)}^\flat)(x) = \tilde{\Omega}(v_{\sigma(2j-1)}, v_{\sigma(2j)})(x).$$

Therefore $(\pi^*\Omega^{n+k})(v_1, \ldots, v_{2k}) = \text{const.}T^k \circ \tilde{\Omega}^k(v_1, \ldots, v_{2k})$ as claimed. □

Now suppose given a homomorphism from a Lie group $G$ to $\text{Ham}(M, \omega)$. The classes $\mu_k \in H^*(B\text{Ham}(M, \omega))$ pull back under the map $R : BG \to B\text{Ham}(M, \omega)$ to some elements in $H^*(BG)$, which by definition are $G$-characteristic classes. One can figure out which classes they are by investigating the invariant polynomial

$$\text{Lie}(G)^\otimes k \to C_0(M)^\otimes k \xrightarrow{T^k} \mathbb{R}.$$  

Reznikov [16] did this calculation (in a slightly different context) for the case of the action of $SU(n+1)$ on $\mathbb{C}P^n$, and concluded that the $\mu_k$ pull back to algebraically independent elements in $H^*(BSU(n+1))$. The advantage of our approach is that the classes extend to $B\text{Ham}_0^k$ (and when $H^1(M) = 0$ to $B\text{Ham}_0$) since one does not use the action of the Lie algebra in their definition.
There are other characteristic classes in $H^*(BG)$ arising from the characteristic classes on $M$ preserved by the elements of $G$, i.e. the Chern classes when $G = \text{Symp}(M,\omega)$ or $\text{Ham}(M,\omega)$, the Pontriagin classes when $G = \text{Diff}(M)$, and the Euler class in the case of $\tilde{H}^*_0$. For example, in the Hamiltonian (but not the c-Hamiltonian) case we can use the Chern classes of the tangent bundle $TM$. Each Chern class $c_i(M)$ has a natural extension $\tilde{c}_i$ to $M_G$, namely the $i$th Chern class of the vertical tangent bundle. Hence for each multi-index $I := (m_1, \ldots, m_n)$ there is a class

$$
\mu_{k,I} := \pi_! \left( \Omega^k (\tilde{c}_1)^{m_1} \cdots (\tilde{c}_n)^{m_n} \right) \in H^*(BG).
$$

As indicated above, initially these classes live on the connected group $G := \text{Ham}(M,\omega)$. If $H^1(M) = 0$ they extend to the full symplectomorphism group $\text{Symp}(M)$, but in general only the classes with $k = 0$ extend over this group. If $M$ is a Riemann surface $\Sigma$ and $I = m$ then the classes $\kappa_m := \mu_{0,m+1}$ are known as the Miller–Morita–Mumford $\kappa$-classes. If $\Sigma$ has genus $g > 1$ with orientation class $a$ normalized by $\int_{\Sigma} a = 1$, then $\text{Symp}(M)$ and $\mathcal{H}_a$ are homotopy equivalent to the (orientation preserving) mapping class group $\pi_0(\text{Symp}(\Sigma))$. Further

$$
\tilde{a} = \frac{1}{2 - 2g} (\tilde{c}_1 + \frac{1}{4g - 4} \pi^*(\kappa_1)).
$$

Hence the $\mu_{k,k > 1}$, together with the single class $\kappa_1$ contain the same information as the $\kappa$-classes.

4. HIGHER HOMOTOPY GROUPS

We begin with examples of nontrivial $G = SU(\ell)$ actions for which the map $\pi_*(G) \to \pi_*(\tilde{G})$ is not injective, and then discuss an easy way to detect elements in $\pi_*(\tilde{G})$. §4.2 proves Proposition 1.6 and §4.3 concerns flag manifolds.

4.1. Examples.

**Example 4.1** (Action of $G = SU(\ell)$ with $r_* = 0$ in dimensions 1 mod 4). Let $2\ell \leq n + 1$ and consider the action of $SU(\ell)$ on $\mathbb{C}P^n$ given by restricting the standard action of $SU(n+1)$ to the image of the homomorphism

$$
\rho : G = SU(\ell) \to SU(n+1) : A \mapsto (A, \overline{A}) \in SU(\ell) \times SU(\ell) \subset SU(n+1),
$$

where $\overline{A}$ denotes the conjugate of $A$. Since conjugation is an automorphism of $SU(\ell)$ that acts by $-1$ on the homotopy groups in degrees $\equiv 1$ (mod 4), the induced map $\rho_* : \pi_j(G) \to \pi_j(SU(n+1))$ is zero when $j \equiv 1$ (mod 4). Hence the image of $\pi_j(G)$ in $\pi_j(\text{Ham}(\mathbb{C}P^n))$ vanishes for these $j$. Note that the pullback of the universal bundle $E_{n+1} \to BU(n+1)$ by $B\rho : BU(\ell) \to BU(n+1)$ is the sum $E_{\ell} + E_{2\ell}^* \oplus \mathbb{C}^{n+1-2\ell}$. Thus this example reformulates the fact that the odd Chern classes of a bundle of the form $E_{\mathbb{R}} \otimes \mathbb{C}$ vanish (rationally), where $E_{\mathbb{R}}$ denotes the real bundle underlying $E$. 
One can also find representations \( \rho : U(\ell) \to U(n+1) \) that kill homotopy in dimensions \( 4i + 3 \), and hence actions of \( U(\ell) \) on projective space with the same property.

**Example 4.2** (Action of \( SU(4) \) with \( r_* = 0 \) in dimension 7). If \( E \to BU(4) \) is the universal bundle and \( \gamma : S^8 \to BU(4) \) generates \( \pi_8(BU(4)) \), then \( c_4(E)(\gamma) = 6 \) by definition. We claim that
\[
c_4(\Lambda^2(E))(\gamma) = -24.
\]
To check this, pull back \( c_4(\Lambda^2(E)) \) back to \( H^*\left( B\mathbb{T} \right) \), where \( \mathbb{T} \) is the maximal 4-torus in \( U(4) \). If \( t_1, \ldots, t_4 \) denote the obvious generators of \( H^2\left( B\mathbb{T} \right) \) then the pullback of \( c_4(\Lambda^2(E)) \) to \( H^8\left( B\mathbb{T} \right) \) is
\[
2s_{3,1} + 5s_{2,2} + 13s_{2,1,1,1} + 30s_{1,1,1,1}.
\]
where, for each partition \( I = (i_1, \ldots, i_r) \) of the number 4, \( s_I \) denotes the symmetric function \( \sum t^{i_1}_1 \cdots t^{i_r}_r \). When we express this in terms of the elementary symmetric functions \( \sigma_i \) and pull back by \( \gamma \), the only contribution comes from the terms in \( \sigma_4 \). Thus
\[
c_4(\Lambda^2(E))(\gamma) = 6(2 \cdot 4 + 5 \cdot 2 + 13 \cdot (-4) + 30) = -24.
\]
Therefore the homomorphism \( U(4) \to U(22) \) that classifies the sum of four copies of the rank 4 bundle \( E \) with the rank 6 bundle \( \Lambda^2(E) \) kills \( \pi_7(U(4)) \). A similar statement holds for the corresponding action of \( SU(4) \) on \( \mathbb{CP}^{21} \).

The next result gives a way of detecting the image in \( \pi_*(G) \) of some of the elements that come from \( \pi_*(G_p) \). Consider the commutative diagram
\[
\begin{array}{c}
M \\
\downarrow j
\end{array}
\quad
\begin{array}{c}
BG_p \\
\downarrow q
\end{array}
\xrightarrow{R} \quad
\begin{array}{c}
BG_p \xrightarrow{d} \mathcal{B}L \\
\downarrow \pi
\end{array}
\quad
\begin{array}{c}
BG \\
\downarrow R
\end{array}
\]
where \( G \) denotes either \( \text{Ham}(M, \omega) \) or \( \text{Diff}_0(M) \) as appropriate, and \( \mathcal{L} \) is the group formed by the linearized action of \( G_p \) on \( T_pM \). Thus when \( G \) consists of symplectomorphisms \( \mathcal{L} = Sp(2n) \simeq U(n) \), while in the smooth case \( \mathcal{L} = GL(2n, \mathbb{R}) \simeq SO(2n) \). The homomorphism \( d : G_p \to \mathcal{L} \) is given by taking the derivative, so that the composite \( d \circ j : M \to BG_p \to \mathcal{B}L \) classifies the tangent bundle of \( M \). In degrees in which this vanishes in homotopy we can use the map \( d \circ R \) to detect the image of \( R_* \). For clarity, we state the next lemma in the symplectic case. Thus \( G := \text{Ham}(M, \omega) \) and \( \mathcal{L} := U(n) \) where \( 2n = \dim M \).
Lemma 4.3. Suppose that the Lie group $G := SU(\ell)$ acts on $M$, and consider the associated map $r : G \to \mathcal{G}$. Assume that $(d \circ j)_* : \pi_k(M) \to \pi_k(B\mathcal{L})$ vanishes, and that there is $\overline{\alpha} \in \pi_k(BG_\rho)$ such that both $q_* (\overline{\alpha}) \in \pi_k(BG)$ and $(d \circ R)_* (\overline{\alpha}) \in \pi_k(B\mathcal{L})$ are nonzero. Then $R_* : \pi_k(BG) \otimes \mathbb{Q} \to \pi_k(B\mathcal{G}) \otimes \mathbb{Q}$ is injective.

Proof. The proof is an easy diagram chase. It suffices to show that

$$(R \circ q)_* (\overline{\alpha}) = (\pi \circ R)_* (\overline{\alpha})$$

is nonzero. But otherwise $R_* (\overline{\alpha})$ would be in the kernel of $\pi_*$ and hence in the image of $j_*$. This would imply $d_* (R_* (\overline{\alpha})) = d_* (j_* (\beta)) = 0$, which contradicts the assumption that $d_* (R_* (\overline{\alpha})) \neq 0$. \hfill \Box

We apply this to flag manifolds in Lemma 4.7 below.

4.2. c-split manifolds and the evaluation map. We now prove Proposition 1.6 that detects the image of elements in $\pi_*(G)$ that map nontrivially under the evaluation map. Recall from Lalonde–McDuff [11] that a fibration $M \to P \to B$ is said to be c-split if $\pi_1(B)$ acts trivially on $H^*(M)$ and the Leray–Serre spectral sequence degenerates at the $E_2$ term. This is equivalent to the condition that the inclusion induces an injective map $H_*(M) \to H_*(P)$. Further a c-symplectic manifold $(M, a)$ is said to satisfy the hard Lefschetz condition if

$$a^k \cup : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for all $1 \leq k \leq n$.

Lemma 4.4. Suppose that $(M, a)$ satisfies the hard Lefschetz condition and denote by $\mathcal{H}$ either $\mathcal{H}_0$ or (if $H^1(M) = 0$) the $H$-space $\mathcal{H}_H$ of all homotopy equivalences that act trivially on rational homology. Then $M \to M_\mathcal{H} \to B\mathcal{H}$ is c-split.

Proof. It is a classical result due to Blanchard [4] that every fibration over a base $B$ such that $\pi_1(B)$ acts trivially on the homology of the fiber is c-split provided that the class $a$ extends to $P$ and the fiber $(M, a)$ satisfies the hard Lefschetz condition. Hence we need only check that $a$ extends to $M_\mathcal{H}$, which is true by Proposition 3.1. \hfill \Box

Proof of Proposition 1.6. Consider the fibration $M \to M_\mathcal{H} \to B\mathcal{H}$. The boundary map $\partial : \pi_*(B\mathcal{H}) \to \pi_{*-1}(M)$ of its long exact homotopy sequence is essentially the same as the evaluation map $ev_* : \pi_*(\mathcal{H}) \otimes \mathbb{Q} \to \pi_*(M) \otimes \mathbb{Q}$: more precisely, $ev = \partial \circ \tau$ where $\tau : \alpha \to \overline{\alpha}$ is the transgression. Therefore, we must show that if a connected group $G$ acts on a nilpotent manifold $(M, a)$ that satisfies the c-splitting condition for $\mathcal{H}$, then for every element $\alpha \in \pi_{2k-1}(G), k > 1$, with $ev_* (\alpha) \neq 0$ the element $h(\overline{\alpha}) \in H_*(B\mathcal{H})$ is nonzero. To this end, let $P \to S^{2k}$ be the pullback of $M_\mathcal{H} \to B\mathcal{H}$ by $\overline{\alpha}$, and
consider the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{=} & M \\
\downarrow & & \downarrow \\
P & \longrightarrow & M_H \\
\pi & \downarrow & \downarrow \\
S^{2k} & \longrightarrow & B\mathcal{H}.
\end{array}
\]

By hypothesis the fibration \( M \to M_H \to B\mathcal{H} \) is c-split. Hence the elements of any additive basis \( b_0 := 1, b_1, \ldots, b_q \) for \( H^*(M) \) extend to elements \( \tilde{b}_0 := 1, \ldots, \tilde{b}_q \) of \( H^*(M_H) \). Moreover, if \( \deg b_i < 2k \) the restriction of \( \tilde{b}_i \) to \( P \), which we denote by \( \tilde{b}_i^P \), is uniquely determined.

Let \( e \in H^{2k}(S^{2k}) \) be a generator. Then \( \pi^*(e) \neq 0 \) since the fibration \( P \to S^{2k} \) c-splits. We claim that there is a polynomial \( f \) such that the following relation holds in \( H^*(P) \) for suitable \( i_1, \ldots, i_s \):

\[
(4.1) \quad \pi^*(e) = f(\tilde{b}_{i_1}^P, \ldots, \tilde{b}_{i_s}^P), \quad \deg b_{i_j}^P < 2k, \ \forall j.
\]

To see this, build the KS model \((A_P, D)\) for the fibration \( M \to P \to S^{2k} \). According to Tralle–Oprea [17], this has the form \((\Lambda(e, t) \otimes A_M, D)\) where \((A_M, d_M)\) is a minimal model for \( H^*(M) \), \((\Lambda(e, t), d)\) is a minimal model for \( S^{2k} \) (so that \( \deg e = 2k, \deg t = 4k - 1 \)), and \( D \) extends \( d_M \). Since \( M \) and hence \( P \) is nilpotent, this KS model calculates \( H^*(P) \). Since it has generators in dimensions \( \geq 2 \) that are dual to the rational homotopy, it agrees with \( A_M \) in dimensions \( < 2k - 1 \). In dimension \( 2k - 1 \) we may choose generating cochains for \( A_M^{2k-1} \) so that precisely one of them, say \( x \), does not vanish on the image of \( \partial : \pi_{2k}(S^{2k}) \to \pi_{2k-1}(M) \). Then \( D(x) = e + r \) where \( r = d_M(x) \) is a product of elements in \( A_M \) of degrees \( < 2k \). Since \( 0 = D^2(x) = D(e) + D(r) = D(r), r \) is a cocycle in \( A_P \). Moreover, because \( \pi^*(e) \neq 0 \), \( r \) must represent a nontrivial class in \( H^*(P) \). Therefore if \( f \) is the polynomial such that \( f(b_{i_1}, \ldots, b_{i_s}) = -r \), relation (4.1) holds.

Now consider the corresponding element \( f(\tilde{b}_{i_1}, \ldots, \tilde{b}_{i_s}) \in H^*(M_H) \). Since \( \pi : M_H \to B\mathcal{H} \) is c-split, we may apply the Leray–Hirsch theorem. Therefore this can be written uniquely in the form

\[
(4.2) \quad f(\tilde{b}_{i_1}, \ldots, \tilde{b}_{i_s}) = \sum_{j=0}^q \pi^*(z_j)\tilde{b}_j,
\]

where \( z_j \in H^*(B\mathcal{H}) \). If we pull this relation back to \( P \) we obtain the unique expression for \( f(\tilde{b}_{i_1}^P, \ldots, \tilde{b}_{i_s}^P) \) in terms of the Leray–Hirsch basis \( \tilde{b}_j^P \) for \( H^*(P) \). Comparing with (4.1), we find that \( z_0 \in H^{2k}(B\mathcal{H}) \) must extend \( e \). Thus \( \pi \) has nonzero image in \( H_*(B\mathcal{H}) \). \( \square \)
Remark 4.5. (i) Equation (4.1) implies that \( \pi_!(f(\tilde{b}_{P_1}^0, \ldots, \tilde{b}_{P_s}^0)) = e. \) In fact the fiber integral simply picks out the coefficient of \( b_0^0 := 1 \) in the Leray–Hirsch decomposition for an element in \( H^*(M_H) \).

(ii) The c-splitting hypothesis in Proposition 1.6 is satisfied when \( M \) is a c-symplectic manifold with a transitive action of a connected and simply connected compact Lie group \( G \). For if \( \omega \) is a \( G \)-invariant representative for the c-symplectic class \( a \in H^2(M) \), then \( \omega^n \) has constant rank and hence does not vanish. Thus \( \omega \) is symplectic. Our assumptions imply that \( G \) is semisimple and that \( \pi_1(M) = 0. \) Therefore \( M \) has a Kähler structure with Kähler form \( \omega \) by a theorem of Borel that is stated for example in Tralle–Oprea [17, Ch 5, Thm 2.1]. Therefore the claim follows from Lemma 4.4.

We now give a version of Proposition 1.6 in which the c-splitting condition is relaxed. All we need is that the cohomology classes in \( M \) occurring in the relation (4.1) extend to \( H^*(M_H) \). In particular, if in the statement below \( H^*(X) \) is generated by the Chern classes of \( TM \) then we can forget about the class \([\omega]\) and take \( G := \text{Symp}(M) \) even if \( H^1(M) \neq 0. \)

Proposition 4.6. Suppose that a simply connected group \( G \) acts on \((M, \omega)\) in such a way that the subring of \( H^*(M) \) generated by \([\omega]\) and the Chern classes of \( TM \) surjects onto \( H^*(X) \), where \( X := G/G_p \) is some orbit of \( G \). Let \( \mathcal{G} \) be a subgroup of \( \text{Symp}(M) \) that contains \( G \) and is such that \([\omega]\) extends to \( H^*(M_G) \). Then the map

\[
h \circ R_* : \pi_*(BG) \rightarrow H_*(B\mathcal{G})
\]

is nonzero on all elements \( \overline{\alpha} \in \pi_{2k}(BG) \) such that \( ev_*(\alpha) \in \pi_{2k-1}(M) \) is nonzero.

Proof. Suppose given \( \alpha \in \pi_{2k-1}(G) \) such that \( ev_*(\alpha) \neq 0 \) in \( \pi_{2k-1}(M) \) and hence also in \( \pi_{2k-1}(X) \). Consider the corresponding diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & M \\
\downarrow & & \downarrow \\
Q & \longrightarrow & P \longrightarrow M_G \\
\pi & \downarrow & \pi \\
S^{2k} & \longrightarrow & S^{2k} \longrightarrow B\mathcal{G},
\end{array}
\]

Our hypotheses ensure that there is a basis \( b_0 := 1, b_1, \ldots, b_N \) for \( H^*(X) \) consisting of elements that extend to \( H^*(M_G) \). Hence the fibration \( \pi : Q \rightarrow S^{2k} \) is c-split and the result follows by arguing as in the proof of Proposition 1.6.

4.3. Flag manifolds. Consider the general flag manifold

\[
M(m_1, \ldots, m_k) := U(\ell)/U(m_1) \times \cdots \times U(m_k), \quad m_1 \geq \cdots \geq m_k,
\]
where $\ell = \sum m_i$. We shall denote
\[
G' := U(\ell), \quad G'_p := U(m_1) \times \cdots \times U(m_k), \quad G := SU(\ell), \quad G_p := G'_p \cap G.
\]
If $E_i \to U(m_i)$ denotes the universal bundle, then the vertical tangent bundle of the fibration
\[
M \xrightarrow{j} B G_p \xrightarrow{q} B G
\]
is the pullback from $B G'_p$ to $B G_p$ of the bundle
\[
\bigoplus_{1 \leq i < j \leq k} \text{Hom}(E_i, E_j) = \bigoplus_{1 \leq i < j \leq k} E_i^* \otimes E_j
\]
Further, the cohomology ring of $M$ is generated by the Chern classes of the $E_i$ with defining relations coming from the fact that the restriction of $\bigoplus_i E_i$ to $M$ is trivial. In particular, in the case of a Grassmannian (i.e. $k = 2$) the cohomology of $M$ is generated by the Chern classes of its tangent bundle so that $\text{Symp}(M, \omega)$ acts trivially on $H^*(M)$.

The first part of the next lemma illustrates the use of Lemma 4.3, though in this case stronger conclusions may be obtained by other methods (see Proposition 4.8). Recall that $\mathcal{H}_H$ denotes the space of homotopy equivalences that act trivially on $H^*(M)$.

**Lemma 4.7.** Let $M$ be the flag manifold $M(m_1, \ldots, m_k)$ and set $G := SU(\ell)$ where $\ell = \sum m_i$ and $m_1 \geq \cdots \geq m_k$. Then:

(i) $r_* : \pi_{2i-1}(G) \to \pi_{2i-1}(\text{Symp}(M, \omega))$ is injective for $m_2 < i \leq m_1$.

(ii) $h \circ R_* : \pi_{2i}(BG) \to H_{2i}(B \mathcal{H}_H)$ is injective for $i = 2$ and $m_1 < i \leq \ell$.

**Proof.** The existence of the fibration
\[
M \to B G_p \to B G
\]
implies that the generators of $\pi_*(M)$ divide into two groups. Those in dimensions $2i$, $1 \leq i \leq m_2$, map to the elements in the kernel of $\pi_*(B G_p) \to \pi_*(B G)$, while the generators in the image of $\text{ev}_*$ come from $\pi_{i+1}(B G)$ and lie in odd dimensions $2i - 1$ for $m_1 < i \leq \ell$. Therefore because $\pi_{2i}(M) = 0$ in the range $m_2 < i \leq m_1$, the first statement will follow from Lemma 4.3 if we show that $(d \circ r)_* : \pi_{2i}(B G_p) \to \pi_{2i}(B \mathcal{L})$ is surjective for $m_2 < i \leq m_1$. To prove this it suffices to check that the Chern classes of the vertical tangent bundle of the fibration $M \to B G_p \to B G$ are nonzero on $\pi_{2i}(B G_p)$ for $i$ in this range. This holds because the restriction of this bundle to $B SU(m_1)$ is simply a sum of copies of $E_1^*$. This proves (i).

Since $\text{ev}_* : \pi_*(G) \to \pi_*(M)$ is injective in dimensions $* > 2m_1 - 1$, (ii) follows immediately from Propositions 1.5 and 1.6 and Lemma 4.4. \qed

We next show that the map $h \circ R_* : \pi_*(BG) \to H_*(B \mathcal{H}_H)$ is always injective for flag manifolds. Our argument imitates the proof for $\mathbb{C} P^n$, considering the coefficients of the higher degree elements $\tilde{b}_j$, $j > 0$, in (4.2).
Proposition 4.8. Let $M$ be the flag manifold $M(m_1, \ldots, m_k)$ and set $G := SU(\ell)$ where $\ell = \sum m_i$ and $m_1 \geq \cdots \geq m_k$. Then the inclusion $BSU(\ell) \to BH_H$ induces an injection on $H^*$.

Proof. Let us first consider the case of a Grassmannian $M(m, k)$ where $\ell = m+k, m \geq k$, and set $G := SU(\ell)$. It is convenient to work with cohomology. Therefore we aim to show that the Chern classes $c_i, i = 2, \ldots, \ell$ in $H^*(BG)$ extend to classes in $H^*(BH_H)$.

As remarked above, the ring $H^*(B(U(m) \times U(k)))$ is freely generated by elements $x_1, \ldots, x_m$ (the pullbacks of the Chern classes of the universal bundle $E := E_1 \to BU(m)$) and $y_1, \ldots, y_k$ (the pullbacks of the Chern classes of the universal bundle $F := E_2 \to BU(k)$). Thus $H^*(BG_p) = H^*(BS(U(m) \times U(k)))$ is the quotient of this free ring by the relation $x_1 + y_1 = 0$.

Consider the fibration $\pi : BG_p \to BG$. Since $E \oplus F$ is the pullback of the universal bundle over $BG := BSU(\ell)$ its Chern classes are the pullbacks $Q_i := \pi^*(c_i)$ of the Chern classes in $H^*(BG)$. Thus $Q_0 := 1$ and $Q_1 = 0$. Taking the total Chern class of $E \oplus F$ we find:

$$\tag{4.3} (1 + x_1 + \cdots + x_m)(1 + y_1 + \cdots + y_k) = 1 + Q_1 + \cdots + Q_{m+k}. $$

Since the restriction of $E \oplus F$ to $M$ is trivial, the above identity gives $m+k$ relations among the restrictions of the $x_i, y_j$ to $H^*(M)$. The first $m$ of these should be interpreted as defining the $x_i$'s in terms of the generators $y_i$ of $H^*(M)$, while the equations $Q_{m+i} = 0$ give the relations in $H^*(M)$. In particular, there are no relations among the monomials in the $y_i$ in degrees $\leq 2m$. Hence we may choose an additive basis $b_0 := 1, b_1, \ldots, b_N$ for $H^*(M)$ whose elements in degrees $\leq 2m$ consist of all the monomials in the $y_i$. In these degrees we may therefore write $b_v = y_{L_v}$, and will extend $b_v \in H^*(M)$ to $H^*(M_G)$ by identifying it with $y_{L_v} \in H^*(M_G)$. As in Proposition 1.6, the $b_v$ also extend to elements $\tilde{b}_v \in H^*(M_H)$. We denote by $\tilde{b}_v^G$ the restriction of $\tilde{b}_v$ to $M_G = BG_p$. These two extensions $\tilde{b}_v^G$ and $b_v$ of $b_v \in H^*(M)$ to $H^*(M_G)$ need not agree, but they do agree modulo the ideal $\langle Q_2, \ldots, Q_m \rangle$ generated by the elements of $\pi^*(BG)$. In other words, if we identify $H^*(BG_p)$ with the free algebra generated by the $y_j$ and $Q_i, 2 \leq i \leq m$, we have

$$b_v \in \tilde{b}_v^G + \langle Q_2, \ldots, Q_m \rangle.$$

Formally inverting $1 + y_1 + \cdots + y_k$ in equation (4.3) we obtain

$$1 + x_1 + \cdots + x_m = (1 + Q_1 + \cdots + Q_{m+k})(1 + f_1(y_1, \ldots, y_k) + \cdots),$$

where the $f_i$ are the homogeneous terms of degree $2i$ in $(1 + y_1 + \cdots + y_k)^{-1}$. The terms of degree $2m + 2$ give the relation

$$-Q_{m+1} = \sum_{i=1}^{m} Q_{m-i+1} f_i(y_1, \ldots, y_k).$$
Notice also that the coefficient of $y^i_1$ in $f_i$ equals $(-1)^i$. Since the polynomials $f_i$ have degree $2i \leq 2m$ they are sums $\sum \alpha_i b_c$ of the basis monomials $b_c = y Q^m$. Therefore we have

$$\pi^*(c_{m+1}) = Q_{m+1} = -\sum_{i=1}^m Q_{m-i+1} f_i(y_1, \ldots, y_k)$$

$$= -\sum_{i=1}^m Q_{m-i+1} \left( \sum_{\nu} \alpha_i b_{c_{\nu}} \right)$$

$$= \sum_{\nu} \pi^*(c_{\nu}) \tilde{b}_{c_{\nu}}$$

where each $c_{\nu} \in H^s(BG)$ is a polynomial in the Chern classes with a nonzero linear term. Further, if $b_{c_{\nu}}$ denotes the basis element $y^m c_{\nu}$ of $H^s(M)$ for $1 < i \leq m$, then the coefficient of $\tilde{b}_{c_{\nu}}$ in the above expression has the form $\mu c_{\nu}$ decomposables, for some nonzero number $\mu$.

It follows from Proposition 1.6 that there is a class $u \in H^{2m+2}(BH)$ whose restriction to $BG$ does not vanish on $\pi_{2m+2}(BG)$. Hence we may choose $u$ so that it restricts to $c_{m+1} + p(c) \in H^{2m+2}(BG)$ where $p(c)$ is some polynomial in the $c_i$, $2 \leq i \leq m$. Thus $\pi^*(u)$ restricts to $Q_{m+1} + \pi^*(p(c))$ where $\pi^*(p(c))$ is a polynomial in the $Q_i$, $i \leq m$. Since $\pi^*(u)$ vanishes in the fiber $M$ and there is a unique relation in $H^{2m+2}(M)$ namely $Q_{m+1} = 0$, $\pi^*(p(c))$ must be a multiple of $Q_{m+1}$ and the argument may be completed as before. The Leray–Hirsch theorem implies that $\pi^*(u)$ may be written uniquely as

$$\pi^*(u) = \sum_{\nu} \pi^*(u_{c_{\nu}}) \tilde{b}_{c_{\nu}} \in H^s(M_H),$$

where $u_{c_{\nu}} \in H^s(BH)$. Comparing with the expression previously found for $Q_{m+1}$ we see that a multiple of $u_{c_{\nu}}$ extends a class which equals $c_i$ modulo products of $c_j$, $j < i$. An easy inductive argument now shows that each $c_i$ must extend to $BH$. This completes the proof for Grassmannians.

The proof for the flag manifold $M(m_1, \ldots, m_k), k > 2$, is very similar. We denote by $x_i$ the Chern classes of the universal bundle $E_1 \to BU(m_1)$ and by $y_{a\alpha}$ the Chern classes of $E_\alpha \to BU(m_\alpha)$ for $2 \leq \alpha \leq m$. Further let $y_i$ be the Chern classes of the sum $F := E_2 \oplus \cdots \oplus E_m$. Since $E_1 \oplus F$ is pulled back from $BU(\ell)$ it is trivial on $M$. Further the equation (4.3) holds as before. It should be interpreted as first defining the $x_i$ in terms of the generators $y_{a\alpha}$ for $H^s(M)$ and then giving the relations in $H^s(M)$. The rest of the argument goes through without essential change.

5. THE EVALUATION MAP AND WHITEHEAD PRODUCTS

In this section we consider a nontrivial $S^1$ action $\lambda$ that is inessential in $G$. For simplicity, we suppose throughout this section that $M$ is simply
connected, though versions of the first lemmas extend to the general case. We shall suppose either that we are in the symplectic category so that \( G = \text{Ham}(M, \omega) \) or that \((M, \alpha)\) is c-symplectic, the action is smooth, and \( G := \Diff_0(M) \). Let \( \rho \in \pi_3(G) \) be the nonzero element constructed in Theorem 1.1. Our first aim is to understand what it means for \( \text{ev}_* (\rho) \in \pi_3(M) \) to vanish.

We denote by \( \partial \) the boundary map in the long exact sequence of the fibration

\[ G_p \longrightarrow G \xrightarrow{\text{ev}} M. \]

**Lemma 5.1.** Let \( M, G \) be as above and suppose that \( \lambda \) is inessential in \( G \) but essential in \( G_p \). Then there is a unique \( \alpha \in \pi_2(M) \otimes \mathbb{Q} \) such that \( \partial \alpha = \lambda \). Moreover, \( h(\alpha) \in H_2(M) \) is nonzero.

**Proof.** The first statement holds because the evaluation map

\[ \text{ev}_* : \pi_2(G) \otimes \mathbb{Q} \to \pi_2(M) \otimes \mathbb{Q} \]

is zero so that \( \partial \) is injective. The second is an immediate consequence of the Hurewicz theorem: because \( \pi_1(M) = 0 \) the map \( \pi_2(M) \otimes \mathbb{Q} \to H_2(M) \) is an isomorphism. \( \square \)

Note that the element \( \alpha \) may depend on the choice of \( p \). For example the loop in \( \text{Ham}(\mathbb{C}P^2, \omega) \) given by \([z_0 : z_1 : z_2] \mapsto [e^{2\pi it} z_0 : e^{-2\pi it} z_1 : z_2] \) is nullhomotopic in \( G_p \) when \( p = [0 : 0 : 1] \) but is essential when \( p = [1 : 0 : 0] \). Therefore if we work with the corresponding diagonal circle action on \( M = \mathbb{C}P^2 \times \mathbb{C}P^2 \) we can find points \( p_1, p_2 \) such that \( \lambda \) is essential in \( G_{p_i} \) for \( i = 1, 2 \) but the elements \( \alpha_1, \alpha_2 \) are different.

**Lemma 5.2.** Let \( \lambda, \alpha \) and \( p := [\lambda, \lambda] \) be as in Lemma 5.1. Then \( \text{ev}_* (\rho) \) is the Whitehead product \([\alpha, \alpha]\).

**Proof.** Consider the fibration \( \Omega M \to \tilde{G}_p \to G \), where \( \Omega M \) is the based loop space of \( M \) and \( \tilde{G}_p \) is the space of all pairs \((h, \gamma)\) where \( h \in G \) and \( \gamma \) is a path in \( M \) from the base point \( p \) to \( h(p) \). The choice of a contraction \( \tilde{\lambda} \) of \( \lambda \) in \( G \) determines a homotopy from \( \lambda \) to a loop \( \ell : s \mapsto \ell(s) \) in \( \Omega M \): if \( \tilde{\lambda} \) is given by a map

\[ [0, 1] \times [0, 1] \to G, \quad (s, \nu) \mapsto \tilde{\lambda}(s, \nu), \]

\[ \tilde{\lambda}(s, 0) = \tilde{\lambda}(0, \nu) = \tilde{\lambda}(1, \nu) = id, \quad \tilde{\lambda}(s, 1) = \lambda(s), \]

then for each \( s \in [0, 1] \), \( \ell(s) \) is the loop \( \nu \mapsto \tilde{\lambda}(s, \nu)(p) \).

It suffices to show that the image of \( \rho \) under the boundary map \( \partial : \pi_3(G) \to \pi_2(\Omega M) \) is the Samelson product \( \langle \ell, \ell \rangle \) since this is the desuspension of \([\alpha, \alpha]\). But it follows from the definition of \( \rho \) as a map \( D^2 \times S^1 / \sim \longrightarrow G \) that \( \partial \rho \) is represented by the map

\[ S^1 \times S^1 / \sim \longrightarrow \Omega M, \quad (s, t) \mapsto \langle \tilde{\lambda}(s, \cdot), \lambda(t) \rangle(p), \]
where \((s, t) \in S^1 \equiv \mathbb{R}/\mathbb{Z}\). But this is homotopic to the map \((t, \ell) : (s, t) \mapsto \gamma(s, t)\) where
\[
\gamma(s, t)(\nu) := \left\langle \tilde{\lambda}(s, \nu), \tilde{\lambda}(t, \nu) \right\rangle (p)
\]
via the homotopy \(\gamma_r(s, t)(\nu) := \left\langle \tilde{\lambda}(s, \nu), \tilde{\lambda}(t, r + \nu(1 - r)) \right\rangle (p)\). \(\square\)

A similar argument shows that if the product \(\{f, f'\}\) is as defined in Proposition 2.1 then \(ev_*\{f, f'\} = [\alpha, \alpha']\), provided that \(f, f'\) both vanish in \(G\) so that they correspond to elements \(\alpha, \alpha' \in \pi_1(M)\). If only \(f\) vanishes in \(G\), \(ev_*\{f, f'\}\) need not be a Whitehead product. For example, in the case of the action of \(SU(n + 1)\) on \(\mathbb{C}P^n\) we have (in the notation of Lemma 2.5) \(ev_*\{\lambda_1, \lambda_3\} = 0\), but \(ev_*\{\lambda_1, \lambda_{2n-1}\} \neq 0\). In this particular example, \(ev_*\{\lambda_1, \lambda_{2n-1}\}\) is a higher order Whitehead product. We now investigate the extent to which this generalizes. To get clean statements we need to assume that the homotopy of \(M\) has some of the characteristics of \(\mathbb{C}P^n\). For the first result we only need \(M\) to be simply connected, but later on need more assumptions.

**Proposition 5.3.** Let \(\lambda\) be a nontrivial \(S^1\)-action on a simply connected \(c\)-symplectic manifold \((M, \alpha)\). Suppose that there is \(\alpha \in \pi_2(M)\) such that \(\partial \alpha = \lambda \in \pi_1(G_p)\). Then \(ev_*\alpha \neq 0\) if and only if any element \(c \in H^2(M)\) such that \(c(\alpha) \neq 0\) has the property that \(c^2 = 0\) modulo the ideal \(I\) in \(H^*(M)\) generated by the kernel of \(\alpha^* : H^2(M) \rightarrow H^2(S^2)\).

**Proof.** If \(ev_*\alpha = 0\) then Lemma 5.2 implies that the map \(\alpha \vee \alpha : S^2 \vee S^2 \rightarrow M\) extends to \(S^2 \times S^2\). Hence if \(c \in H^2(M)\) is such that \(c(\alpha) \neq 0\) its square \(c^2\) has nonzero pullback to \(S^2 \times S^2\) and hence is nonzero modulo \(I\).

The converse follows from from Lemma 5.2 and minimal model theory. When building a minimal model \(A := (A^k, d)_{k \geq 0}\) for \(H^*(M)\) we may choose a basis \(c_0, \ldots, c_k\) for the 2-cochains \(A^2\) such that \(c_i(\alpha) = 0, i > 0\). The differential \(d : A^1 \rightarrow (A^2)^2 \subset A^4\) is dual to the Whitehead product \(\pi_2(M) \times \pi_2(M) \rightarrow \pi_3(M)\). We may choose a basis for \(\pi_3(M)\) whose first element is \(ev_*\alpha\). Then if \(r \in A^3\) is the first element in the dual basis, \(dr = \sum \mu_{ij} c_i c_j\) where \(\mu_{00} = 1\). Hence \(dr = (c_0)^2\) modulo \(I\). \(\square\)

We now explore what happens when \(ev_*\alpha = 0\). This means that the Whitehead product \([\alpha, \alpha]\) is zero. Thus we can consider the higher order Whitehead product \([\alpha, \alpha, \alpha] \in \pi_3(M)\) which is defined as follows (see [3] for details). Let \(W \subset S^2 \times S^2 \times S^2\) denote the fat wedge, that is it consists of triples with at least one coordinate at the base point. Then \([\alpha, \alpha, \alpha] := \{f_\alpha(u) \in \pi_3(M)\} \) where \(f : W \rightarrow M\) ranges over the set of all extensions of \(\alpha \vee \alpha \vee \alpha\) and \(u \in \pi_3(W)\) is a generator. Since \([\alpha, \alpha] = 0\) this set is nonempty. Moreover, because \(W\) consists of the wedge of three copies of \((S^2 \times S^2) \vee S^2\), \(f\) is determined by the way in which \(\alpha \vee \alpha\) is extended to \(S^2 \times S^2\), which can vary by an element \(\beta \in \pi_4(M)\). Therefore \([\alpha, \alpha, \alpha]\) is a
coset of the subgroup
\[ H := \{[\alpha, \beta] : \beta \in \pi_4(M)\} \subset \pi_5(M). \]

We say that \([\alpha, \alpha, \alpha]\) is nonzero if this coset does not contain the zero element. Hence \([\alpha, \alpha, \alpha] = 0\) if and only if some extension \(f : W \to M\) of \(\alpha \vee \alpha \vee \alpha\) extends further to \(S^2 \times S^2 \times S^2\).

In order to interpret the vanishing of \(\text{ev}_s(\rho) = 0\) in cohomological terms (rather than in terms of the minimal model) we need to make some further simplifying assumptions about the homotopy type of \(M\). We assume below that \(\pi_3(M) = 0\), so that \([\alpha, \alpha, \alpha] = 0\) a fortiori. Note that this hypothesis is satisfied by all generalized flag manifolds \(M(1, \ldots, m_k)\) with \(m_1 > 1\).

**Proposition 5.4.** Let \(\pi_1(M) = \pi_3(M) = 0\) and \(\lambda : S^1 \to G_p\) be a nontrivial action as in Proposition 5.3. Suppose that there is \(\alpha \in \pi_2(M)\) such that \(\partial \alpha = \lambda \in \pi_1(G_p)\). Consider \(c \in H^2(M)\) such that \(c(\alpha) \neq 0\). Then the higher Whitehead product \([\alpha, \alpha, \alpha]\) is nonzero if and only if \(c^3 \in I\), where \(I\) is as in Proposition 5.3.

**Proof.** One direction is trivial. Namely, if \([\alpha, \alpha, \alpha] = 0\) then \(\alpha \vee \alpha \vee \alpha\) extends to \(S^2 \times S^2 \times S^2\) and so \(c^3 \notin I\).

Now assume that \([\alpha, \alpha, \alpha] \neq 0\). Chose some extension of \(\alpha \vee \alpha \vee \alpha\) to \(S^2 \times S^2\) and using this define \(f : W \to M\) in a symmetric way. Denote \(\gamma_0 := f_*(u)\). By hypothesis the span in \(\pi_5(M)\) of the Whitehead products \([\alpha, \pi_4(M)]\) does not include \(\gamma_0\).

The minimal models of the spaces under consideration have the following forms:

- \(\mathcal{A}(W) = \Lambda(x_1, x_2, x_3; y_1, y_2, y_3; w; \ldots),\)
  where \(\deg x_i = 2, \deg y_i = 3, \deg w = 5\) and the nontrivial differentials are \(dy_i = x_i^2, dw = x_1x_2x_3\) and enough others in higher degrees to cancel out the cohomology. Note that this minimal model is infinitely generated.
- \(\mathcal{A}(M) = \Lambda(c_0, c_1, \ldots, c_k; v_1, \ldots, v_\ell; z; \ldots),\)
  where \(\deg c_i = 2, \deg v_i = 4, \deg z = 5\) and the first nontrivial differential is given by \(dz = \sum a_{ijk}c_ic_jc_k + \sum a_{\ell m}c_{\ell m}, \) for some \(a_{ijk}, a_{\ell m} \in \mathbb{Q}\). Moreover, \(c_0(\alpha) = 1\) and \(c_i(\alpha) = 0\) for \(i > 0\).

According to the above discussion about \([\alpha, \pi_4(M)]\) we can choose \(z \in A^3(M)\) so that \(z\) vanishes on all the elements \([\alpha, \beta]\) for \(\beta \in \pi_4(M)\) but \(z(\gamma_0) = 1\). From this we draw two conclusions. Firstly, because the quadratic part of \(dz\) is dual to the Whitehead product, \(dz\) includes no terms of the form \(c_0v_m\). In other words \(a_{0m} = 0\) for all \(m\) and \(dz = \sum a_{ijk}c_ic_jc_k\).

Secondly, because \(f^*(z) = w + \sum b_{ij}x_iy_j\) and \(f^*(c_i) = 0, i > 0,\)
\[
0 \neq x_1x_2x_3 + \sum b_{ij}x_ix_j^2 \\
= df^*(z) = f^*(dz) = a_{000}f^*(c_0^3).
\]

Therefore \(a_{000} \neq 0\), and the relation in cohomology given by setting \(dz\) to zero says that \([c_0]^3 \in I\). \(\square\)
Our final result concerns the question of whether \([\alpha, \alpha, \alpha] \cap \text{Im} \, ev_\ast \neq \emptyset\), that is whether the Whitehead product contains elements from the image of the evaluation map. Again, we need to strengthen the hypothesis that \([\alpha, \alpha] = 0\). The latter implies that \(\rho\) lifts to an element \(\rho_1 \in \pi_3(G_p)\), and we now assume that this lift can be chosen so that the Samelson product \(\langle \lambda, \rho_1 \rangle\) vanishes in \(G_p\). By Lemma 2.5, this will hold if, for example, \(\lambda\) is an essential circle in \(U(2) \subset G_p\) and the map \(U(2) \to G\) factors through \(SU(2)\). Again, this hypothesis holds for flag manifolds \(M(m_1, \ldots, m_k)\) with \(m_1 > 1\), and in particular for \(\mathbb{CP}^n\).

**Proposition 5.5.** Let \(\lambda\) be a nontrivial \(S^1\)-action on a simply connected \(c\)-symplectic manifold \((M, a)\), and let \(G := \text{Ham}(M, a)\). Suppose that there is \(\alpha \in \pi_2(M)\) such that \(\partial \alpha = \lambda \in \pi_1(G_p)\) and \([\alpha, \alpha] = 0\). Suppose further that \(\rho\) has a lift \(\rho_1\) to \(G_p\) such that \(\lambda \times \rho_1 = 0\) in \(G_p\). Then if \([\alpha, \alpha, \alpha] \neq 0\) there is a nonzero element in the intersection of \([\alpha, \alpha, \alpha]\) with the image of \(ev_\ast : \pi_5(G) \to \pi_5(M)\).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
S^2 \vee S^2 & \xrightarrow{\alpha \vee \alpha} & M \\
\downarrow & & \downarrow j \\
S^2 \times S^2 & \xrightarrow{\Lambda \times \Lambda} & BG_p.
\end{array}
\]

By assumption, the map \(\alpha \vee \alpha\) has some extension \(\phi\) to \(S^2 \times S^2\). However, because \(\pi \circ (\Lambda \times \Lambda)\) is nontrivial (where \(\pi : BG_p \to BG\)) we cannot choose \(\phi\) to make this diagram commute. On the other hand, because \(\pi \circ (\Lambda \times \Lambda)\) is the composite of the quotient map \(S^2 \times S^2 \to S^4\) with a representative of \(\overline{\eta}\), we can adjust \(\Lambda \times \Lambda\) on the top cell of \(S^2 \times S^2\) by a lift \(\tau\) of \(-\overline{\eta}\) to a map \(f := \Lambda \times \Lambda \# \tau : S^2 \times S^2 \to BG_p\) that does lift to \(M\). Thus we can arrange that \(f = j \circ \phi\) for suitable \(\phi\), i.e. that the following diagram commutes:

\[
\begin{array}{ccc}
S^2 \times S^2 & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow j \\
S^2 \times S^2 & \xrightarrow{f} & BG_p.
\end{array}
\]

Our assumptions imply that we may choose \(\tau\) so that in addition \([\Lambda, \tau] = 0\) in \(BG_p\). Consider the commutative diagram

\[
\begin{array}{ccc}
(S^2 \times S^2 \times S^2)_4 & \xrightarrow{\Phi} & M \\
\downarrow & & \downarrow j \\
(S^2 \times S^2 \times S^2)_4 & \xrightarrow{F} & BG_p,
\end{array}
\]

where \(\Phi\) is given by \(\phi\) and \(F\) is given by \(f\). Since \(\Lambda \vee \Lambda \vee \Lambda\) extends to the product map \(\Lambda \times \Lambda \times \Lambda\), the obstruction to extending \(F\) to the product \(S^2 \times S^2 \times S^2\) is 3\([\Lambda, \tau]\) and so vanishes. Therefore the obstruction to extending
\( \Phi \) is an element in \([\alpha, \alpha, \alpha]\) that has zero image in \(\pi_*(BG_p)\) and hence lies in the image of \(ev_*\).

If \([\alpha, \alpha, \alpha] = 0\) one could explore the third order product. However, to make sense of the results one would need further assumptions on \(M\) and the behavior of \(\lambda\).

References


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