A survey of the topological properties of symplectomorphism groups

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Abstract

The special structures that arise in symplectic topology (particularly Gromov–Witten invariants and quantum homology) place as yet rather poorly understood restrictions on the topological properties of symplectomorphism groups. This article surveys some recent work by Abreu, Lalonde, McDuff, Polterovich and Seidel, concentrating particularly on the homotopy properties of the action of the group of Hamiltonian symplectomorphisms on the underlying manifold $M$. It sketches the proof that the evaluation map $\pi_1(\text{Ham}(M)) \to \pi_1(M)$ given by $\{\phi_t\} \mapsto \{\phi_t(x_0)\}$ is trivial, as well as explaining similar vanishing results for the action of the homology of $\text{Ham}(M)$ on the homology of $M$. Applications to Hamiltonian stability are discussed.

1 Overview

The special structures that arise in symplectic topology (particularly Gromov–Witten invariants and quantum homology) place as yet rather poorly understood restrictions on the topological properties of symplectomorphism groups. This article surveys some recent work on this subject. Throughout $(M,\omega)$ will be a closed (ie compact and without boundary), smooth symplectic manifold of dimension $2n$, unless it is explicitly mentioned otherwise. Background information and more references can be found in [24, 23, 27].

The symplectomorphism group $\text{Symp}(M,\omega)$ consists of all diffeomorphisms $\phi : M \to M$ such that $\phi^*(\omega) = \omega$, and is equipped with the $C^\infty$-topology, the topology of uniform convergence of all derivatives. We will sometimes contrast this with the $C^0$ (i.e. compact-open) topology. The (path) connected component containing the identity is denoted $\text{Symp}_0(M,\omega)$. (Note that $\text{Symp}$ is locally path connected.) This group $\text{Symp}_0$ contains an important normal subgroup called the Hamiltonian group $\text{Ham}(M,\omega)$ whose
elements are the time-1 maps of Hamiltonian flows. These are the flows $\phi_t^H$, $t \in [0, 1]$, that at each time $t$ are tangent to the symplectic gradient $X^H_t$ of the function $H_t : M \to \mathbb{R}$, i.e.

$$\dot{\phi}_t^H = X^H_t, \quad \omega(X^H_t, \cdot) = -dH_t.$$ When $H^1(M, \mathbb{R}) = 0$ the groups Ham and Symp coincide. In general, there is a sequence of groups and inclusions

$$\text{Ham}(M, \omega) \hookrightarrow \text{Symp}_0(M, \omega) \hookrightarrow \text{Symp}(M, \omega) \hookrightarrow \text{Diff}^+(M),$$

where Diff$^+$ denotes the orientation preserving diffeomorphisms. Our aim is to understand and contrast the properties of these groups.

We first give an overview of basic results on the group Symp$^0$. Then we describe results on the Hamiltonian group, showing how a vanishing theorem for its action on $H_* (M)$ implies various stability results. Finally, we sketch the proof of this vanishing theorem. It relies on properties of the Gromov–Witten invariants for sections of Hamiltonian fiber bundles over $S^2$, that can be summarized in the statement (essentially due to Floer and Seidel) that there is a representation of $\pi_1(\text{Ham}(M, \omega))$ into the automorphism group of the quantum homology ring of $M$. The proof of the vanishing of the evaluation map $\pi_1(\text{Ham}(M)) \to \pi_1(M)$ is easier: it relies on a “stretching the neck” argument, see Lemma 3.2 below. A different but also relatively easy proof of this fact may be found in [23].

**Basic facts**

We begin by listing some fundamentals.

- **Dependence on the cohomology class of $\omega$.**

  The groups Symp($M, \omega$) and Ham($M, \omega$) depend only on the diffeomorphism class of the form $\omega$. In particular, since Moser’s argument implies that any path $\omega_t$, $t \in [0, 1]$, of cohomologous forms is induced by an isotopy $\psi_t : M \to M$ of the underlying manifold (i.e. $\psi_t^* (\omega_t) = \omega_0$, $\psi_0 = \text{id}$), the groups do not change their topological or algebraic properties when $\omega_t$ varies along such a path. However, as first notices by Gromov (see Proposition 1.3 below), changes in the cohomology class $[\omega]$ can cause significant changes in the homotopy type of these groups.

- **Stability properties of Symp($M$) and Symp$^0$($M$).**

  By this we mean that if $G$ denotes either of these groups, there is a $C^1$-neighbourhood $N(G)$ of $G$ in Diff($M$) that deformation retracts onto $G$. This follows from the Moser isotopy argument mentioned above. In the case $G = \text{Symp}(M)$, take

$$N(\text{Symp}) = \{ \phi \in \text{Diff}(M) : (1-t)\phi^* (\omega) + t\omega \text{ is nondegenerate for } t \in [0,1] \}.$$ By Moser, one can define for each $\phi$ a unique isotopy $\psi_t$ (that depends smoothly on $\phi^*(\omega)$) such that $\psi_t^* ((1-t)\omega + t\omega) = \omega$ for all $t$. Hence $\phi \circ \psi_1 \in \text{Symp}(M)$. Similarly, when $G = \text{Symp}_0(M)$ one can take $N(G)$ to be the identity component of $N(\text{Symp})$. Note also that these neighborhoods are uniform with respect to $\omega$. For example, given any compact subset $K$ of $\text{Symp}_0(M, \omega)$ there is a $C^\infty$-neighbourhood $N(\omega)$ of $\omega$ in the space of all symplectic forms such that $K$ may be isotoped into $\text{Symp}_0(M, \omega')$ for all $\omega' \in N(\omega)$. These statements, that we sum up in the rubric symplectic stability, exhibit the flabbiness, or lack of local invariants, of symplectic geometry.

The above two properties are “soft”, i.e. they depend only on the Moser argument. By way of contrast, the next result is “hard” and can be proved only by using some deep ideas, either from variational
• The group $\text{Symp}(M, \omega)$ is $C^0$-closed in $\text{Diff}(M)$.

This celebrated result of Eliashberg and Ekeland–Hofer is known as **symplectic rigidity** and is the basis of symplectic topology. The proof shows that even though one uses the first derivatives of $\phi$ in saying that a diffeomorphism $\phi$ preserves $\omega$, there is an invariant $c(U)$ (called a **symplectic capacity**) of an open subset of a symplectic manifold that is continuous with respect to the Hausdorff metric on sets and that is preserved by a diffeomorphism if and only if $\phi^*(\omega) = \omega$. (When $n$ is even, one must slightly modify the previous statement to rule out the case $\phi^*(\omega) = -\omega$.) There are several ways to define a suitable invariant $c$. Perhaps the easiest is to take Gromov’s width:

$$c(U) = \sup\{\pi r^2 : B^{2n}(r) \text{ embeds symplectically in } U\}.$$ 

Here $B^{2n}(r)$ is the standard ball of radius $r$ in Euclidean space $\mathbb{R}^{2n}$ with the usual symplectic form $\omega_0 = \sum_i dx_{2i-1} \wedge dx_{2i}$.

It is unknown whether the identity component $\text{Symp}_0(M)$ is $C^0$-closed in $\text{Diff}(M)$. In fact this may well not hold. For example, it is quite possible that the group $\text{Symp}^c(\mathbb{R}^{2n})$ of compactly supported symplectomorphisms of Euclidean space is disconnected when $n > 2$. (When $n = 2$ this group is contractible by Gromov [8].) Hence for some closed manifold $M$ there might be an element in $\text{Symp}(M) \setminus \text{Symp}_0(M)$ that is supported in a Darboux neighbourhood $U$ (i.e. an open set symplectomorphic to an open ball in Euclidean space). Such an element would be in the $C^0$-closure of $\text{Symp}_0(M)$ since by conformal rescaling in $U$ one could isotop it to have support in an arbitrarily small neighbourhood of a point in $U$.

We discuss related questions for the group $\text{Ham}(M)$ in Section 2 below. Though less is known about the above questions, some very interesting new features appear. Before doing that we shall give a brief summary of what is known about the homotopy groups of $\text{Symp}(M)$.

**The homotopy type of $\text{Symp}(M)$**

In dimension 2 it follows from Moser’s argument that $\text{Symp}(M, \omega)$ is homotopy equivalent to $\text{Diff}^+$. Thus $\text{Symp}(S^2)$ is homotopy equivalent to the rotation group $\text{SO}(3)$; $\text{Symp}_0(T^2)$ is homotopy equivalent to an extension of $\text{SL}(2, \mathbb{Z})$ by $T^2$; and for higher genus the symplectomorphism group is homotopy equivalent to the mapping class group. In dimensions 4 and above, almost nothing is known about the homotopy type of $\text{Diff}^+$. On the other hand, there are some very special 4-manifolds for which the (rational) homotopy type of $\text{Symp}$ is fully understood. The following results are due to Gromov [8]. Here $\sigma_Y$ denotes (the pullback to the product of) an area form on the Riemann surface $Y$ with total area 1.

**Proposition 1.1 (Gromov)**

(i) $\text{Symp}^c(\mathbb{R}^4, \omega_0)$ is contractible;

(ii) $\text{Symp}(S^2 \times S^2, \sigma_{S^2} + \sigma_{S^2})$ is homotopy equivalent to the extension of $\text{SO}(3) \times \text{SO}(3)$ by $\mathbb{Z}/2\mathbb{Z}$ where this acts by interchanging the factors;

(iii) $\text{Symp}(\mathbb{C}P^2, \omega_{FS})$ is homotopy equivalent to $\text{PU}(3)$, where $\omega_{FS}$ is the Fubini–Study Kähler form.

It is no coincidence that these results occur in dimension 4. The proofs use $J$-holomorphic spheres, and these give much more information in dimension 4 because of positivity of intersections.

In Abreu [1] and Abreu–McDuff [5] these arguments are extended to other symplectic forms and (some) other ruled surfaces. Here are the main results, stated for convenience for the product manifold $\Sigma \times S^2$
(though there are similar results for the nontrivial $S^2$ bundle over $\Sigma$.) Consider the following family\footnote{Using results of Taubes and Li–Liu, Lalonde–McDuff show in [14] that these are the only symplectic forms on $\Sigma \times S^2$ up to diffeomorphism.} of symplectic forms on $M_g = \Sigma_g \times S^2$ (where $g$ is genus$(\Sigma)$):

$$\omega_\mu = \mu \sigma_\Sigma + \sigma_{S^2}, \quad \mu > 0.$$  

Denote by $G^0_\mu$ the subgroup

$$G^0_\mu := \text{Symp}(M_g, \omega_\mu) \cap \text{Diff}(M_g)$$

of the group of symplectomorphisms of $(M_g, \omega_\mu)$. When $g > 0$, $\mu$ ranges over all positive numbers. However, when $g = 0$ there is an extra symmetry --- interchanging the two spheres gives an isomorphism $G^0_\mu \cong G^0_{1/\mu}$ --- and so we take $\mu \geq 1$. Although it is not completely obvious, there is a natural homotopy class of maps from $G^0_\mu$ to $G^0_{\mu + \varepsilon}$ for all $\varepsilon > 0$. To see this, let

$$G^0_{[a,b]} = \bigcup_{\mu \in [a,b]} \{\mu\} \times G^0_\mu \subset \mathbb{R} \times \text{Diff}(M_g).$$

It is shown in [5] that the inclusion $G^0_b \to G^0_{[a,b]}$ is a homotopy equivalence. Therefore we can take the map $G^0_\mu \to G^0_{\mu + \varepsilon}$ to be the composite of the inclusion $G^0_\mu \to G^0_{[\mu, \mu + \varepsilon]}$ with a homotopy inverse $G^0_{[\mu, \mu + \varepsilon]} \to G^0_{\mu + \varepsilon}$. Another, more geometric definition of this map is given in [22].

**Proposition 1.2** As $\mu \to \infty$, the groups $G^0_\mu$ tend to a limit $G^0_\infty$ that has the homotopy type of the identity component $D^0_\infty$ of the group of fiberwise diffeomorphisms of $M_g = \Sigma_g \times S^2 \to \Sigma_g$.

**Proposition 1.3** When $\ell < \mu \leq \ell + 1$ for some integer $\ell \geq 1$,

$$H^*(G^0_\mu; \mathbb{Q}) = \Lambda(t, x, y) \otimes \mathbb{Q}[w_\ell],$$

where $\Lambda(t, x, y)$ is an exterior algebra over $\mathbb{Q}$ with generators $t$ of degree $1$, and $x, y$ of degree $3$ and $\mathbb{Q}[w_\ell]$ is the polynomial algebra on a generator $w_\ell$ of degree $4\ell$.

In the above statement, the generators $x, y$ come from $H^*(G^0_\mu) = H^*(\text{SO}(3) \times \text{SO}(3))$ and $t$ corresponds to an element in $\pi_1(G^0_\mu), \mu > 1$ found by Gromov in [8]. Thus the subalgebra $\Lambda(t, x, y)$ is the pullback of $H^*(D^0_\infty, \mathbb{Q})$ under the map $G^0_\mu \to D^0_\infty$. The other generator $w_\ell$ is fragile, in the sense that the corresponding element in homology disappears (i.e. becomes null homologous) when $\mu$ increases. It is dual to an element in $\pi_d t$ that is a higher order Samelson product and hence gives rise to a relation (rather than a new generator) in the cohomology of the classifying space. Indeed, when $\ell < \mu \leq \ell + 1$,

$$H^*(BG^0_\mu) \cong \frac{\mathbb{Q}[T, X, Y]}{\{T(X - Y) \ldots (\ell^2 X - Y) = 0\}},$$

where the classes $T, X, Y$ have dimensions $2, 4, 4$ respectively and are the deloopings of $t, x, y$.

Anjos [2] calculated the full homotopy type of $G^0_\mu$ for $1 < \mu \leq 2$. Her results has been sharpened in Anjos–Granja [3] where it is shown that this group has the homotopy type of the pushout of the following diagram in the category of topological groups:

$$\begin{array}{c}
\text{SO}(3) \xrightarrow{\text{diag}} \text{SO}(3) \times \text{SO}(3) \\
\downarrow \\
S^1 \times \text{SO}(3).
\end{array}$$
Thus $G_0^\mu$ is a amalgamated free product of two compact subgroups, $\text{SO}(3) \times \text{SO}(3)$, which is the automorphism group of the product almost complex structure, and $S^1 \times \text{SO}(3)$. The latter appears as the automorphism group of the product almost complex structure with Kähler form $\omega_\mu$, namely the Hirzebruch structure on $\mathcal{P}(L_2 \oplus \mathbb{C})$ where the line bundle $L_2 \to \mathbb{C}P^1$ has Chern number 2. As mentioned in [3], this description has interesting parallels with the structure of some Kac–Moody groups.

McDuff [22] proves that the homotopy type of $G_0^\mu$ is constant on all intervals $(\ell - 1, \ell]$, $\ell > 1$. However, their full homotopy type for $\mu > 2$ is not yet understood, and there are only partial results when $g > 0$. Apart from this there is rather little known about the homotopy type of $\text{Symp}(\mathcal{M})$. There are some results due to Pinsonnault [26] and Lalonde–Pinsonnault [19] on the one point blow up of $S^2 \times S^2$ showing that the homotopy type of this group also depends on the symplectic area of the exceptional divisor. Also Seidel [31, 30] has done some very interesting work on the symplectic mapping class group $\pi_0(\text{Symp}(\mathcal{M}))$ for certain 4-manifolds, and on the case $\mathcal{M} = \mathbb{C}P^m \times \mathbb{C}P^n$.

2 The Hamiltonian group

Now consider the Hamiltonian subgroup $\text{Ham}(\mathcal{M})$. It has many special properties: it is the commutator subgroup of $\text{Symp}_0(\mathcal{M})$ and is itself a simple group (Banyaga). It also supports a biinvariant metric, the Hofer metric, which gives rise to an interesting geometry. Its elements also have remarkable dynamical properties. For example, according to Arnold’s conjecture (finally proven by Fukaya–Ono and Liu–Tian based on work by Floer and Hofer–Salamon) the number of fixed points of $\phi \in \text{Ham}$ may be estimated as

$$\#\text{Fix} \phi \geq \sum_k \text{rank} \ H^k(M, \mathbb{Q})$$

provided that the fixed points are all nondegenerate, i.e. that the graph of $\phi$ is transverse to the diagonal.

Many features of this group are still not understood, and it may not even be $C^1$-closed in $\text{Symp}_0$. Nevertheless, we will see that there are some analogs of the stability properties discussed earlier for $\text{Symp}$. Also the action of $\text{Ham}(\mathcal{M})$ on $\mathcal{M}$ has special properties.

Hofer Geometry

Because the elements of the Hamiltonian group are generated by functions $H_t$, the group itself supports a variety of interesting functions. First of all there is the Hofer norm [10] that is usually defined as follows:

$$\| \phi \| := \inf_{\phi^H = \phi} \int_0^1 \left( \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt.$$ 

Since this is constant on conjugacy classes and symmetric (i.e. $\| \phi \| = \| \phi^{-1} \|$), it gives rise to a biinvariant metric $d(\phi, \psi) := \| \phi \phi^{-1} \|$ on $\text{Ham}(\mathcal{M}, \omega)$. There are still many open questions about this norm — for example, it is not yet known whether it is always unbounded: for a good introduction see Polterovich’s lovely book [27].

Recently, tools (based on Floer homology) have been developed that allow one to define functions on $\text{Ham}$ or its universal cover $\tilde{\text{Ham}}$ by picking out special elements of the action spectrum $\text{Spec}(\tilde{\phi})$ of $\tilde{\phi} \in \tilde{\text{Ham}}$. This spectrum is defined as follows. Choose a normalized time periodic Hamiltonian $H_t$ that generates $\tilde{\phi}$, i.e. so that the following conditions are satisfied:

$$\int H_t \omega^n = 0, \quad H_{t+1} = H_t, \quad \tilde{\phi} = \tilde{\phi}^H := (\phi^H_1, \{ \phi^H_t \}_{t \in [0,1]}).$$
Denote by $\tilde{\mathcal{L}}(M)$ the cover of the space $\mathcal{L}(M)$ of contractible loops $x$ in $M$ whose elements are pairs $(x, u)$, where $u : D^2 \to M$ restricts to $x$ on $\partial D^2 = S^1$. Then define the action functional $A_H : \tilde{\mathcal{L}}(M) \to \mathbb{R}$ by setting

$$A_H(x, u) = \int_0^1 H_t(x_t) \, dt - \int_{D^2} u^*(\omega).$$

The critical points of $A_H$ are precisely the pairs $(x, u)$ where $x$ is a contractible 1-periodic orbit of the flow $\phi_t^H$. Somewhat surprisingly, it turns out that the set of critical values of $A_H$ depends only on the element $\tilde{\phi}^H \in \tilde{\text{Ham}}$ defined by the flow $\{\phi_t^H\}_{t \in [0,1]}$; in other words, these values depend only on the homotopy class of the path $\phi_t^H$ rel endpoints. Thus we set:

$$\text{Spec}(\tilde{\phi}^H) := \{ \text{all critical values of } A_H \}.$$ 

There are variants of the Hofer norm that pick out certain special homologically visible elements from this spectrum: see for example Schwarz [28] and Oh [25].

Even more interesting is a recent construction by Entov–Polterovich [7] that uses these spectral invariants to define a nontrivial continuous and homogeneous quasimorphism $\mu$ on $\tilde{\text{Ham}}(M, \omega)$, when $M$ is a monotone manifold such as $\mathbb{C}P^n$ that has semisimple quantum homology ring. A quasimorphism on a group $G$ is a map $\mu : G \to \mathbb{R}$ that is a bounded distance away from being a homomorphism, i.e. there is a constant $c = c(\mu) > 0$ such that

$$|\mu(gh) - \mu(g) - \mu(h)| < c, \quad g, h \in G.$$

It is called homogeneous if $\mu(g^m) = m\mu(g)$ for all $m \in \mathbb{Z}$, in which case it restricts to a homomorphism on all abelian subgroups. Besides giving information about the bounded cohomology of $G$, quasimorphisms can be used to investigate the commutator lengths and dynamical properties of its elements. The example constructed by Entov–Polterovich extends the Calabi homomorphism defined on the subgroups $\tilde{\text{Ham}}_U$ of elements with support in sufficiently small open sets $U$. Moreover, in the case of $\mathbb{C}P^n$, it vanishes on $\pi_1(\text{Ham})$ and so descends to the Hamiltonian group $\text{Ham}$ (which incidentally equals $\text{Sym}_0$ since $H^1(\mathbb{C}P^n) = 0$.) It is not yet known whether $\tilde{\text{Ham}}(M)$ or $\text{Ham}(M)$ supports a nontrivial quasimorphism for every $M$. Note that these groups have no nontrivial homomorphisms to $\mathbb{R}$ because they are perfect.

**Relation between $\text{Ham}$ and $\text{Sym}_0$.**

The relation between $\text{Ham}$ and $\text{Sym}_0$ is best understood via the Flux homomorphism. Let $\tilde{\text{Sym}}_0(M)$ denote the universal cover of $\text{Sym}_0(M)$. Its elements $\tilde{\phi}$ are equivalence classes of paths $\{\phi_t\}_{t \in [0,1]}$ starting at the identity, where $\{\phi_t\} \sim \{\phi_t'\}$ iff $\phi_1 = \phi_1'$ and the paths are homotopic rel endpoints. We define

$$\text{Flux}(\tilde{\phi}) = \int_0^1 [\omega(\dot{\phi}_t, \cdot)] \in H^1(M, \mathbb{R}).$$

That this depends only on the homotopy class of the path $\phi_t$ (rel endpoints) is a consequence of the following alternative description: the value of the cohomology class $\text{Flux}(\tilde{\phi})$ on a 1-cycle $\gamma : S^1 \to M$ is given by the integral

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$$\text{Flux}(\tilde{\phi})(\gamma) = \int_{\tilde{\phi}_* (\gamma)} \omega, \quad (1)$$
where $\hat{\phi}_s(\gamma)$ is the 2-chain $I \times S^1 \to M : (t, s) \mapsto \phi_t(\gamma(s))$. Thus Flux is well defined. It is not hard to check that it is a homomorphism.

One of the first results in the theory is that the rows and columns in the following commutative diagram are short exact sequences of groups. (For a proof see [24, Chapter 10].)

\[
\begin{array}{ccc}
\pi_1(\text{Ham}(M)) & \longrightarrow & \pi_1(\text{Symp}_0(M)) \\
\downarrow & & \downarrow \\
\text{Ham}(M) & \longrightarrow & \text{Symp}_0(M) \\
\downarrow & & \downarrow \\
\text{Ham}(M) & \longrightarrow & \text{Symp}_0(M) \\
\end{array}
\xrightarrow{\text{Flux}}
\begin{array}{c}
\Gamma_\omega \\
\downarrow \\
H^1(M, \mathbb{R}) \\
\downarrow \\
H^1(M, \mathbb{R})/\Gamma_\omega.
\end{array}
\]

Here $\Gamma_\omega$ is the so-called flux group. It is the image of $\pi_1(\text{Symp}_0(M))$ under the flux homomorphism.

It is easy to see that $\text{Ham}(M)$ is $C^1$-closed in $\text{Symp}_0(M)$ if and only if $\Gamma_\omega$ is a discrete subgroup of $H^1(M, \mathbb{R})$.

**Question 2.1** Is the subgroup $\Gamma_\omega$ of $H^1(M, \mathbb{R})$ always discrete?

The hypothesis that $\Gamma_\omega$ is always discrete is known as the Flux conjecture. One might think it would always hold by analogy with symplectic rigidity. In fact it does hold in many cases, for example if $(M, \omega)$ is a short exact sequence of groups. (For a proof see [24, Chapter 10]; some sharper bounds are found in Kedra [11], but the argument does not rule out the possibility that $\Gamma_\omega$ is indiscrete for certain values of $[\omega]$. Thus, for the present one should think of $\text{Ham}(M)$ as a leaf in a foliation of $\text{Symp}_0(M)$ that has codimension equal to the first Betti number of $M$.

**Hamiltonian stability**

When $\Gamma_\omega$ is discrete, the stability principle extends: there is a $C^1$-neighbourhood of $\text{Ham}(M, \omega)$ in $\text{Diff}(M)$ that deformation retracts into $\text{Ham}(M, \omega)$. Moreover, if this discreteness were uniform with respect to $\omega$ (which would hold if $(M, \omega)$ were Kähler), then the groups $\text{Ham}(M, \omega)$ would have the same stability with respect to variations in $\omega$ as do $\text{Symp}_0$ and Symp.

To be more precise, suppose that for each $\omega$ and each $\varepsilon > 0$ there is a neighbourhood $N(\omega)$ such that when $\omega' \in N(\omega)$ $\Gamma_{\omega'}$ contains no nonzero element of norm $\leq \varepsilon$. Then for any compact subset $K$ of Ham$(M, \omega)$ there would be a neighbourhood $N(\omega)$ such that $K$ isotops into Ham$(M, \omega')$ for each $\omega' \in N(\omega)$. For example if $K = \{\phi_t\}$ is a loop (image of a circle) in Ham$(M, \omega)$ and $\omega_s, 0 \leq s \leq 1$, is any path, this would mean that any smooth extension $\{\phi'_t\}, s \geq 0$, of $\{\phi_t\}$ to a family of loops in Symp$(M, \omega_s)$ would be homotopic through $\omega_s$-symplectic loops to a loop in Ham$(M, \omega_s)$.

Even if this hypothesis on $\Gamma_\omega$ held, it would not rule out the possibility of global instability: a loop in Ham$(M, \omega)$ could be isotopic through (nonsymplectic) loops in Diff$(M)$ to a nonHamiltonian loop in some other far away symplectomorphism group Symp$(M, \omega')$. One of the main results in Lalonde–McDuff–Polterovich [18] is that this global instability never occurs; any $\omega'$-symplectic loop that is isotopic in Diff$(M)$ to an $\omega$-Hamiltonian loop must be homotopic in Symp$(M, \omega')$ to an $\omega'$-Hamiltonian loop regardless of the relation between $\omega$ and $\omega'$ and no matter whether any of the groups $\Gamma_{\omega'}$ are discrete. This is known as Hamiltonian rigidity and is a consequence of a vanishing theorem for the Flux homomorphism: see Corollary 2.3 below. As we now explain this extends to general results about the action of $\text{Ham}(M)$ on $M$.
**Action of $\text{Ham}(M)$ on $M$**

There are some suggestive but still incomplete results about the action of $\text{Ham}(M)$ on $M$. The first result below is folklore. It is a consequence of the proof of the Arnold conjecture, but as we show below (see Lemma 3.2) also follows from a geometric argument. The second part is due to Lalonde–McDuff [15]. Although the statements are topological in nature, both proofs are based on the existence of the Seidel representation, a deep fact that uses the properties of $J$-holomorphic curves.

**Proposition 2.2** (i) The evaluation map $\pi_1(\text{Ham}(M)) \to \pi_1(M)$ is zero.

(ii) The natural action of $H_*(\text{Ham}(M), \mathbb{Q})$ on $H_*(M, \mathbb{Q})$ is trivial.

Here the action $\text{tr}_\phi : H_*(M) \to H_{*+k}(M)$ of an element $\phi \in H_k(\text{Ham}(M))$ is defined as follows:

- If $\phi$ is represented by the cycle $t \mapsto \phi_t$ for $t \in V^k$ and $c \in H_*(M)$ is represented by $x \mapsto c(x)$ for $x \in C$ then $\text{tr}_\phi(c)$ is represented by the cycle $V^k \times C \to M : (t, x) \mapsto \phi_t(c(x))$.

It is just the action on homology induced by the map $\text{Ham}(M) \times M \to M$. It extends to the group $(M^M)_{id}$ of self-maps of $M$ that are homotopic to the identity, and hence depends only on the image of $\phi$ in $H_k(M^M)_{id}$. To say it is trivial means that $\text{tr}_\phi(c) = 0$ whenever $c \in H_i(M), i > 0$.

Note that this does not hold for the action of $H_1(\text{Symp}_0(M))$. Indeed by (1) the image under the Flux homomorphism of a loop $\lambda \in \pi_1(\text{Symp}_0(M))$ is simply

$$\text{Flux}_\omega(\lambda)(\gamma) = \langle \omega, \text{tr}_\lambda(\gamma) \rangle.$$  

(3)

The rigidity of Hamiltonian loops is an immediate consequence of Proposition 2.2.

**Corollary 2.3** Suppose that $\phi \in \pi_1(\text{Symp}(M, \omega))$ and $\phi' \in \pi_1(\text{Symp}(M, \omega'))$ represent the same element of $\pi_1((M^M)_{id})$. Then

$$\text{Flux}_\omega(\phi) = 0 \iff \text{Flux}_{\omega'}(\phi') = 0.$$  

Proof. If $\text{Flux}_\omega(\phi) = 0$ then $\phi$ is an $\omega$-Hamiltonian loop and Proposition 2.2(ii) implies that $\text{tr}_\phi : H_1(M) \to H_2(M)$ is the zero map. But, for each $\gamma \in H_1(M)$, (3) implies that

$$\text{Flux}_{\omega'}(\phi')(\gamma) = \text{Flux}_{\omega'}(\phi)(\gamma) = \langle \omega', \text{tr}_\phi(\gamma) \rangle = 0.$$  

This corollary is elementary when the loops are circle subgroups since then one can distinguish between Hamiltonian and nonHamiltonian loops by looking at the weights of the action at the fixed points: a circle action is Hamiltonian if and only if there is a point whose weights all have the same sign. One can also consider maps $K \to \text{Ham}(M, \omega)$ with arbitrary compact domain $K$. But their stability follows from the above result because $\pi_k(\text{Ham}(M)) = \pi_k(\text{Symp}_0(M))$ when $k > 1$ by diagram (2). For more details see [16].

Thus one can compare the homotopy types of the groups $\text{Ham}(M, \omega)$ (or of $\text{Symp}(M, \omega)$) as $[\omega]$ varies in $H^2(M, \mathbb{R})$. More precisely, as Buse points out in [6], any element $\alpha$ in $\pi_*(\text{Ham}(M, \omega))$ has a smooth
extension to a family $\alpha_t \in \pi_*(\Ham(M, \omega_t))$ where $[\omega_t]$ fills out a neighborhood of $[\omega_0] = [\omega]$ in $H^2(M, \mathbb{R})$. Moreover the germ of this extension at $\omega = \omega_0$ is unique. Thus one can distinguish between robust elements in the homology or homotopy of the spaces $\Ham(M, \omega_t)$ and $B\Ham(M, \omega_t)$ whose extensions are nonzero for all $t$ near 0 and fragile elements whose extensions vanish as $[\omega_t]$ moves in certain directions. For example, any class in $H^*(B\Ham(M, \omega_0))$ that is detected by Gromov–Witten invariants (i.e. does not vanish on a suitable space of $J$-holomorphic curves as in Le–Ono [20]) is robust, while the classes $w_f$ of Proposition 1.3 are fragile. For some interesting examples in this connection, see Kronheimer [13] and Buse [6].

**c-splitting for Hamiltonian bundles**

From now on, we assume that (co)homology has rational coefficients. Since the rational cohomology $H^*(G)$ of any $H$-space (or group) is freely generated by the dual of its rational homotopy, it is easy to see that part (ii) of Proposition 2.2 holds if and only if it holds for all spherical classes $\phi \in H_k(\Ham(M))$. Each such $\phi$ gives rise to a locally trivial fiber bundle $M \to P_{\phi} \to S^{k+1}$ with structural group $\Ham(M)$. Moreover, the differential in the corresponding Wang sequence is precisely $tr_{\phi}$. In other words, there is an exact sequence:

$$\ldots \to H_i(M) \xrightarrow{tr_{\phi}} H_{i+k}(M) \to H_{i+k}(P_{\phi}) \xrightarrow{r_{[M]}} H_{i-1}(M) \to \ldots$$  

Hence $tr_{\phi} = 0$ for $k > 0$ if and only if this long exact sequence breaks up into short exact sequences:

$$0 \to H_{i+k}(M) \to H_{i+k}(P_{\phi}) \xrightarrow{r_{[M]}} H_{i-1}(M) \to 0.$$

Thus Proposition 2.2(ii) is equivalent to the following statement.

**Proposition 2.4** For every Hamiltonian bundle $P \to S^{k+1}$, with fiber $(M, \omega)$ the rational homology $H_*(P)$ is isomorphic as a vector space to the tensor product $H_*(M) \otimes H_*(S^{k+1})$.

Observe that the corresponding isomorphism in cohomology need not preserve the ring structure. We say that a bundle $M \to P \to B$ is c-split if the rational cohomology $H^*(P)$ is isomorphic as a vector space to $H^*(M) \otimes H^*(B)$.

**Question 2.5** Is every fiber bundle $M \to P \to B$ with structural group $\Ham(M)$ c-split?

It is shown in [15] that the answer is affirmative if $B$ has dimension $\leq 3$ or is a product of spheres and projective spaces with fundamental group of rank $\leq 3$. By an old result of Blanchard, it is also affirmative if $(M, \omega)$ satisfies the hard Lefschetz condition, i.e. if

$$\wedge_{\omega}^k : H^{n-k}(M, \mathbb{R}) \to H^{n+k}(M, \mathbb{R})$$

is an isomorphism for all $0 < k < n$. (This argument has now been somewhat extended by Haller [9] using ideas of Mathieu about the harmonic cohomology of a symplectic manifold.) If the structural group of $P \to B$ reduces to a finite dimensional Lie group $G$, then c-splitting is equivalent to a result of Atiyah–Bott [4] about the structure of the equivariant cohomology ring $H^*_G(M)$. This is the cohomology of the universal Hamiltonian $G$-bundle with fiber $M$

$$M \to M_G = EG \times_G M \to BG,$$
and was shown in [4] to be isomorphic to $H^\ast(M) \otimes H^\ast(BG)$ as a $H^\ast(BG)$-module. Hence a positive answer to Question 2.5 in general would imply that this aspect of the homotopy theory of Hamiltonian actions is similar to the more rigid cases, when the group is finite dimensional or when the manifold is Kähler. For further discussion see [15, 16] and Kedra [12].

Note finally that all results on the action of $\operatorname{Ham}(M)$ on $M$ can be phrased in terms of the universal Hamiltonian bundle $M \to M_{\operatorname{Ham}} = E_{\operatorname{Ham}} \times_{\operatorname{Ham}} M \to B_{\operatorname{Ham}}(M)$.

For example, Proposition 2.2 part (i) states that this bundle has a section over its 2-skeleton. Such a formulation has the advantage that it immediately suggests further questions. For example, one might wonder if the bundle $M_{\operatorname{Ham}} \to B_{\operatorname{Ham}}$ always has a global section. However this fails when $M = S^2$ since the map $\pi_3(\operatorname{Ham}(S^2)) = \pi_3(SO(3)) \to \pi_3(S^2)$ is nonzero.

3 Symplectic geometry of bundles over $S^2$

The proofs of Propositions 2.2 and 2.4 above rely on properties of Hamiltonian bundles over $S^2$. We now show how the Seidel representation

$$\pi_1(\operatorname{Ham}(M, \omega)) \to (QH^{ev}(M))^\times$$

of $\pi_1(\operatorname{Ham}(M, \omega))$ into the group of even units in quantum homology gives information on the homotopy properties of Hamiltonian bundles. As preparation, we first discuss quantum homology.

The small quantum homology ring $QH_{s}(M)$

There are several slightly different ways of defining the small quantum homology ring. We adopt the conventions of [18, 21].

Set $c_1 = c_1(TM) \in H^2(M, \mathbb{Z})$. Let $\Lambda$ be the Novikov ring of the group $\mathcal{H} = H\mathcal{S}_3(M, \mathbb{R})/\sim$ with valuation $L_\omega$ where $B \sim B'$ if $\omega(B - B') = c_1(B - B') = 0$. Then $\Lambda$ is the completion of the rational group ring of $\mathcal{H}$ with elements of the form

$$\sum_{B \in \mathcal{H}} q_B e^B$$

where for each $\kappa$ there are only finitely many nonzero $q_B \in \mathbb{Q}$ with $\omega(B) > -\kappa$. Set

$$QH_s(M) = QH_s(M, \Lambda) = H_s(M) \otimes \Lambda.$$

We may define an $\mathbb{R}$ grading on $QH_s(M, \Lambda)$ by setting

$$\deg(a \otimes e^B) = \deg(a) + 2c_1(B),$$

and can also think of $QH_s(M, \Lambda)$ as $\mathbb{Z}/2\mathbb{Z}$-graded with

$$QH^{ev} = H^{ev}(M) \otimes \Lambda, \quad QH^{odd} = H^{odd}(M) \otimes \Lambda.$$

Recall that the quantum intersection product

$$a * b \in QH_{s+j-2n}(M), \quad \text{for} \quad a \in H_i(M), b \in H_j(M)$$
is defined as follows:

\[ a \ast b = \sum_{B \in \mathcal{M}} (a \ast b)_B \otimes e^{-B}, \]

where \((a \ast b)_B \in H_{i+j-2n+2c_1(B)}(M)\) is defined by the requirement that

\[ (a \ast b)_B \cdot c = GW_M(a, b, c; B) \quad \text{for all } c \in H_*(M). \]

Here \(GW_M(a, b, c; B)\) denotes the Gromov–Witten invariant that counts the number of \(B\)-spheres in \(M\) meeting the cycles \(a, b, c \in H_*(M)\), and we have written \(\ast\) for the usual intersection pairing on \(H_*(M) = H_*(M, \mathbb{Q})\). Thus \(a \ast b = 0\) unless \(\dim(a) + \dim(b) = 2n\) in which case it is the algebraic number of intersection points of the cycles.

Alternatively, one can define \(a \ast b\) as follows: if \(\{e_i\}\) is a basis for \(H_*(M)\) with dual basis \(\{e_i^*\}\), then

\[ a \ast b = \sum_i GW_M(a, b, e_i; B) \, e_i^* \otimes e^{-B}. \]

The product \(\ast\) is extended to \(QH_*(M)\) by linearity over \(\Lambda\), and is associative. Moreover, it preserves the \(\mathbb{R}\)-grading in the homological sense, i.e. it obeys the same grading rules as does the intersection product.

This product \(\ast\) gives \(QH_*(M)\) the structure of a graded commutative ring with unit \(1 = [M]\). Further, the invertible elements in \(QH_{ev}(M)\) form a commutative group \((QH_{ev}(M, \Lambda))^\times\) that acts on \(QH_*(M)\) by quantum multiplication. By Poincaré duality one can transfer this product to cohomology. Although this is very frequently done, it is often easier to work with homology when one wants to understand the relation to geometry.

**The Seidel representation \(\Psi\)**

Consider a smooth bundle \(\pi : P \to S^2\) with fiber \(M\). Here we consider \(S^2\) to be the union \(D_+ \cup D_-\) of two copies of \(D\), with the orientation of \(D_+\). We denote the equator \(D_+ \cap D_-\) by \(\partial\), oriented as the boundary of \(D_+\), and choose some point \(\ast\) on \(\partial\) as the base point of \(S^2\). We assume also that the fiber \(M_\ast\) over \(\ast\) has a chosen identification with \(M\).

Since every smooth bundle over a disc can be trivialized, we can build any smooth bundle \(P \to S^2\) by taking two product manifolds \(D_+ \times M\) and gluing them along the boundary \(\partial \times M\) by a based loop \(\lambda = \{\lambda_t\}\) in \(\text{Diff}(M)\). Thus

\[ P = (D_+ \times M) \cup (D_- \times M)/\sim, \quad (e^{2\pi i t}, x)_- \equiv (e^{2\pi i t}, \lambda_t(x))_+. \]

A symplectic bundle is built from a based loop in \(\text{Symp}(M)\) and a Hamiltonian bundle from one in \(\text{Ham}(M)\). Thus the smooth bundle \(P \to S^2\) is symplectic if and only if there is a smooth family of cohomologous symplectic forms \(\omega_b\) on the fibers \(M_b\). It is shown in \([29, 24, 15]\) that a symplectic bundle \(P \to S^2\) is Hamiltonian if and only if the fiberwise forms \(\omega_b\) have a closed extension \(\Omega\). (Such forms \(\Omega\) are called \(\omega\)-compatible.) Note that in any of these categories two bundles are equivalent if and only if their defining loops are homotopic.

From now on, we restrict to Hamiltonian bundles, and denote by \(P_\lambda \to S^2\) the bundle constructed from a loop \(\lambda \in \pi_1(\text{Ham}(M))\). By adding the pullback of a suitable area form on the base we can choose the closed extension \(\Omega\) to be symplectic. The manifold \(P_\lambda\) carries two canonical cohomology classes, the first Chern class of the vertical tangent bundle

\[ c_{vert} = c_1(TP_\lambda^{vert}) \in H^2(P_\lambda, \mathbb{Z}), \]

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and the coupling class $u_\lambda$, i.e. the unique class in $H^2(P_\lambda, \mathbb{R})$ such that 
\[ \iota^*(u_\lambda) = [\omega], \quad u_\lambda^{n+1} = 0, \]
where $i : M \to P_\lambda$ is the inclusion of a fiber.

The next step is to choose a canonical (generalized) section class in $\sigma_\lambda \in H_2(P_\lambda, \mathbb{R})/\sim$. By definition this should project onto the positive generator of $H_2(S^2, \mathbb{Z})$. In the general case, when $c_1$ and $[\omega]$ induce linearly independent homomorphisms $H_2^S(M) \to \mathbb{R}$, $\sigma_\lambda$ is defined by the requirement that
\[ c_{\text{vert}}(\sigma_\lambda) = u_\lambda(\sigma_\lambda) = 0, \tag{7} \]
which has a unique solution modulo the given equivalence. If either $[\omega]$ or $c_1$ vanishes on $H_2^S(M)$ then such a class $\sigma_\lambda$ still exists.\(^\dagger\) In the remaining case (the monotone case), when $c_1$ is some nonzero multiple of $[\omega] = 0$ on $H_2^S(M)$, we choose $\sigma_\lambda$ so that $c_{\text{vert}}(\sigma_\lambda) = 0$.

We then set
\[ \Psi(\lambda) = \sum_{B \in H} a_B \otimes e^B \tag{8} \]
where, for all $c \in H_*(M)$,
\[ a_B : M \to \ GW_{P_\lambda}([M], [M], c; \sigma_\lambda - B). \tag{9} \]

Note that $\Psi(\lambda)$ belongs to the strictly commutative part $QH_{ev}$ of $QH_*(M)$. Moreover $\deg(\Psi(\lambda)) = 2n$ because $c_{\text{vert}}(\sigma_\lambda) = 0$. Since all $\omega$-compatible forms are deformation equivalent, $\Psi$ is independent of the choice of $\Omega$.

Here is the main result.

**Proposition 3.1** For all $\lambda_1, \lambda_2 \in \pi_1(\text{Ham}(M))$
\[ \Psi(\lambda_1 + \lambda_2) = \Psi(\lambda_1) \ast \Psi(\lambda_2), \quad \Psi(0) = 1, \]
where $0$ denotes the constant loop. Hence $\Psi(\lambda)$ is invertible for all $\lambda$ and $\Psi$ defines a group homomorphism
\[ \Psi : \pi_1(\text{Ham}(M, \omega)) \to (QH_{ev}(M, \lambda))^\times. \]

In the case when $(M, \omega)$ satisfies a suitable positivity condition, this is a variant of the main result in Seidel [29]. The general proof is due to McDuff [21] using ideas from Lalonde–McDuff–Polterovich [18]. It uses a refined version of the ideas in the proof of Lemma 3.2 below.

**Homotopy theoretic consequences of the existence of $\Psi$**

First of all, note that because $\Psi(\lambda) \neq 0$ there must always be $J$-holomorphic sections of $P_\lambda \to S^2$ to count. Thus every Hamiltonian bundle $\pi : P \to S^2$ must have a section $S^2 \to P$. If we trivialize $P$ over the two hemispheres $D_\pm$ of $S^2$ and homotop the section to be constant over one of the discs, it becomes clear that there is a section if and only if the defining loop $\lambda$ of $P$ has trivial image under the evaluation map $\pi_1(\text{Ham}(M)) \to \pi_1(M)$. This proves part (i) of Proposition 2.2.

In fact one does not need the full force of Proposition 3.1 in order to arrive at this conclusion, since we only have to produce one section.

\(^\dagger\)See [21, Remark 3.1] for the case when $[\omega] = 0$ on $H_2^S(M)$. If $c_1 = 0$ on $H_2^S(M)$ but $[\omega] \neq 0$ then we can choose $\sigma_\lambda$ so that $u_\lambda(\sigma_\lambda) = 0$. Since $c_{\text{vert}}$ is constant on section classes, we must show that it always vanishes. But the existence of the Seidel representation implies every Hamiltonian fibration $P \to S^2$ has some section $\sigma_P$ with $n \leq c_{\text{vert}}(\sigma_P) \leq 0$ (since it only counts such sections), and the value must be 0 because $c_{\text{vert}}(\sigma_{P_{\lambda'}}) = c_{\text{vert}}(\sigma_{P_\lambda}) + c_{\text{vert}}(\sigma_{P_{\lambda'}})$: see [21, Lemma 2.2].
Lemma 3.2 Every Hamiltonian bundle $P \to S^2$ has a section.

**Sketch of Proof** Let $\lambda = \{\lambda_t\}$ be a Hamiltonian loop and consider the family of trivial bundles $P_{\lambda,R} \to S^2$ given by

$$P_{\lambda,R} = (D_+ \times M) \cup (S^1 \times [-R,R] \times M) \cup (D_- \times M)$$

with attaching maps

$$(e^{2\pi it}, \lambda_t(x))_+ \equiv (e^{2\pi it}, -R, x), \quad (e^{2\pi it}, R, x) \equiv (e^{2\pi it}, \lambda_t(x))_-.$$

Thus, $P_{\lambda,R}$ can be thought of as the fiberwise union (or Gompf sum) of $P_{\lambda}$ with $P_{-\lambda}$ over a neck of length $R$. It is possible to define a family $\Omega_R$ of $\omega$-compatible symplectic forms on $P_{\lambda,R}$ in such a way that the manifolds $(P_{\lambda,R}, \Omega_R)$ converge in a well defined sense as $R \to \infty$. The limit is a singular manifold $P_{\lambda} \cup P_{-\lambda} \to S_\infty$ that is a locally trivial fiber bundle over the nodal curve consisting of the one point union of two 2-spheres. To do this, one first models the convergence of the 2-spheres in the base by a 1-parameter family $S_R$ of disjoint holomorphic spheres in the one point blow up of $S^2 \times S^2$ that converge to the pair $S_\infty = \Sigma_+ \cup \Sigma_-$ of exceptional divisors at the blow up point. Then one builds a suitable smooth Hamiltonian bundle

$$\pi_X : (\mathcal{X}, \tilde{\Omega}) \to S$$

with fiber $(M, \omega)$, where $S$ is a neighbourhood of $\Sigma_+ \cup \Sigma_-$ in the blow up that contains the union of the spheres $S_R, R \geq R_0$; see [21] §2.3.2. The almost complex structures $\tilde{J}$ that one puts on $\mathcal{X}$ should be chosen so that the projection to $S$ is holomorphic. Then each submanifold $P_{\lambda,R} := \pi_X^{-1}(S_R)$ is $\tilde{J}$-holomorphic.

The bundles $(P_{\lambda,R}, \Omega_R) \to S^2$ are all trivial, and hence there is one $J$-holomorphic curve in the class $[\sigma_0 = [S^2 \times pt]$ through each point $q_R \in P_{\lambda,R}$. (It is more correct to say that the corresponding Gromov–Witten invariant $GW_{P_{\lambda,R}}([M], [M], pt; \sigma_0)$ is one; i.e. one counts the curves with appropriate multiplicities.) Just as in gauge theory, these curves do not disappear when one stretches the neck, i.e. lets $R \to \infty$. Therefore as one moves the point $q_R$ to the singular fiber the family of $J$-holomorphic curves through $q_R$ converges to some cusp-curve (stable map) $C_\infty$ in the limit. Moreover, $C_\infty$ must lie entirely in the singular fiber $P_{\lambda} \cup P_{-\lambda}$ and projects to a holomorphic curve in $S$ in the class $[\Sigma_+] + [\Sigma_-]$. Hence it must have at least two components, one a section of $P_{\lambda} \to \Sigma_+$ and the other a section of $P_{-\lambda} \to \Sigma_-$. There might also be some bubbles in the $M$-fibers, but this is irrelevant.

The above argument is relatively easy, in that it only uses the compactness theorem for $J$-holomorphic curves and not the more subtle gluing arguments needed to prove things like the associativity of quantum multiplication. However the proof of the rest of Proposition 2.2 is based on the fact that each element $\Psi(\lambda)$ is a multiplicative unit in quantum homology. The only known way to prove this is via some sort of gluing argument. Hence in this case it seems that one does need the full force of the gluing arguments, whether one works as here with $J$-holomorphic spheres or as in Seidel [29] with Floer homology.

We now show how to deduce part (ii) of Proposition 2.2 from Proposition 3.1. So far, we have described $\Psi(\lambda)$ as a unit in $QH_*(M)$. This unit induces an automorphism of $QH_*(M)$ by quantum multiplication on the left:

$$b \mapsto \Psi(\lambda) \ast b, \quad b \in QH_*(M).$$

The next lemma shows that when $b \in H_*(M)$ then the element $\Psi(\lambda) \ast b$ can also be described by counting curves in $P_{\lambda}$ rather than in the fiber $M$.

**Lemma 3.3** If $\{e_i\}$ is a basis for $H_*(M)$ with dual basis $\{e^*_i\}$, then

$$\Psi(\lambda) \ast b = \sum_i GW_{P_{\lambda}}([M], b, e_i; \sigma - B) e_i^* \otimes e^B.$$
Sketch of Proof: To see this, one first shows that for any section class $\sigma$ the invariant $GW_{P_\lambda}([M], b, c; \sigma)$ may be calculated using a fibered $J$ (i.e. one for which the projection $\pi : P \to S^2$ is holomorphic) and with representing cycles for $b, c$ that each lie in a fiber. Then one is counting sections of $P \to S^2$. If the representing cycles for $b, c$ are moved into the same fiber, then the curves must degenerate. Generically the limiting stable map will have two components, a section in some class $\sigma - \lambda$ together with a $C$ curve that meets $b$ and $c$. Thus, using much the same arguments that prove the usual 4-point decomposition rule, one shows that

$$GW_{P_\lambda}([M], b, c; \sigma) = \sum_{A,i} GW_{P_\lambda}([M], [M], e_i; \sigma - A) \cdot GW_M(e_i^*, b, c; A).$$

(10)

But $\Psi(\lambda) = \sum q_i e_i^* \otimes e^B$ where

$$q_i = GW_{P_\lambda}([M], [M], e_i; \sigma - B) \in \mathbb{Q}.$$

Therefore

$$\Psi(\lambda) \ast b = \sum_{C,k} GW_M(\Psi(\lambda), b, e_k; C) e_k^* \otimes e^{-C} \quad \sum_{B,C,j,k} GW_{P_\lambda}([M], [M], e_j; \sigma - B) \cdot GW_M(e^*_j, b, e_k; C) e_k^* \otimes e^{B-C} \quad \sum_{A,k} GW_{P_\lambda}([M], b, e_k; \sigma - A) e_k^* \otimes e^A$$

where the first equality uses the definition of $\ast$, the second uses the definition of $\Psi(\lambda)$ and the third uses (10) with $\sigma = \sigma - (B - C)$. For more details, see [21, Prop 1.2].

Since $\Psi(\lambda)$ is a unit, the map $b \mapsto \Psi(\lambda) \ast b$ is injective. Hence for every $b \in H_*(M)$ there has to be some nonzero invariant $GW_{P_\lambda}([M], b, c; \sigma - B)$ in $P_\lambda$. In particular, the image $i_*(b)$ of the class $b$ in $H_*(P_\lambda)$ cannot vanish. Thus the map

$$i_* : H_*(M) \to H_*(P_\lambda)$$

of rational homology groups is injective. By (4), this implies that the homology of $P_\lambda$ is isomorphic to the tensor product $H_*(S^2) \otimes H_*(M)$. Equivalently, the map

$$\text{tr}_\lambda : H_*(M) \to H_{*+1}(M)$$

is identically zero. This proves Proposition 2.2 (ii) in the case of loops. The proof for the higher homology $H_*(\text{Ham})$ with $* > 1$ is purely topological. Since $H^*(\text{Ham})$ is generated multiplicatively by elements dual to the homotopy, one first reduces to the case when $\phi \in \pi_k(\text{Ham})$. Thus we need only see that all Hamiltonian bundles $M \to P \to B$ with base $B = S^{k+1}$ are c-split, i.e. that Proposition 2.4 holds. Now observe:

**Lemma 3.4** (i) Let $M \to P' \to B'$ be the pullback of $M \to P \to B$ by a map $B' \to B$ that induces a surjection on rational homology. Then if $M \to P' \to B'$ is c-split, so is $M \to P \to B$.

(ii) Let $F \to X \to B$ be a Hamiltonian bundle in which $B$ is simply connected. Then if all Hamiltonian bundles over $F$ and over $B$ are c-split, the same is true for Hamiltonian bundles over $X$.  

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(The proof is easy and is given in [15].) This lemma implies that in order to establish c-splitting when $B$ is an arbitrary sphere it suffices to consider the cases $B = \mathbb{C}P^n$, $B = \mathbb{C}$, the 1-point blow up $X_n$ of $\mathbb{C}P^n$, and $B = T^2 \times \mathbb{C}P^n$. But the first two cases can be proved by induction using the lemma above and the Hamiltonian bundle

$$\mathbb{C}P^1 \rightarrow X_n \rightarrow \mathbb{C}P^{n-1},$$

and the third follows by considering the trivial bundle

$$T^2 \rightarrow T^2 \times \mathbb{C}P^n \rightarrow \mathbb{C}P^n.$$

This completes the proof of Proposition 2.4. Though these arguments can be somewhat extended, they do not seem powerful enough to deal with all Hamiltonian bundles. For some further work in this direction, see Kedra [12].

References


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