

ON THE FLUX CONJECTURES

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ABSTRACT. The “Flux conjecture” for symplectic manifolds states that the group of Hamiltonian diffeomorphisms is C^1 -closed in the group of all symplectic diffeomorphisms. We prove the conjecture for *spherically rational* manifolds and for those whose minimal Chern number on 2-spheres either vanishes or is large enough. We also confirm a natural version of the Flux conjecture for symplectic torus actions. In some cases we can go further and prove that the group of Hamiltonian diffeomorphisms is C^0 -closed in the identity component of the group of all symplectic diffeomorphisms.

1. INTRODUCTION

Let (M, ω) be a compact symplectic manifold without boundary, and $G = \text{Symp}_0(M, \omega)$ be the identity component of the group of symplectic diffeomorphisms of M . There is an exact sequence

$$\tilde{H} \rightarrow \tilde{G} \xrightarrow{F} H^1(M; \mathbb{R})$$

where \tilde{G} is the universal cover of G , \tilde{H} is the subgroup of \tilde{G} formed by those paths which are homotopic in G with fixed endpoints to a Hamiltonian path, and F is the flux homomorphism defined by

$$F(\{\phi_t\}) = \int_0^1 \lambda_t dt.$$

Here the family of closed 1-forms $\{\lambda_t\}$ generates the isotopy $\{\phi_{t \in [0,1]}\}$ in the sense that

$$\omega\left(\frac{d\phi_t}{dt}, \cdot\right) = \lambda_t(\cdot) \quad \text{for all } t \in [0, 1].$$

Recall also that a path is Hamiltonian if its generating family $\{\lambda_t\}$ is exact at each time t . Thus the exact sequence simply expresses the fact that a path with vanishing (average) flux can be perturbed to have vanishing “instantaneous flux” for each t . Moreover, it is easy to check that $F(\{\phi_t\})$ is the class in $H^1(M; \mathbb{R})$ that assigns to each loop γ in M the integral of ω over the cylinder

$$C_\gamma : S^1 \times [0, 1] \rightarrow M$$

Date: February 2, 1997.

First author partially supported by NSERC grant OGP 0092913 and FCAR grant ER-1199, and second by NSF grant DMS 9401443. All three authors also supported by a NSERC Collaborative Project Grant CPG0163730.

defined by $C_\gamma(z, t) = \phi_t(\gamma(z))$. (Proofs of the above statements can be found in Banyaga [1] or McDuff–Salamon [13].) We will sometimes refer to the cylinder C_γ as the *trace* of $\{\phi_t\}$ on γ .

Let us denote by $\text{Ham}(M, \omega)$, or simply $\text{Ham}(M)$, the group of Hamiltonian diffeomorphisms, that is to say those elements of G which are endpoints of Hamiltonian paths.¹ Further, we define the *Flux subgroup* Γ of $H^1(M; \mathbb{R})$ to be the image by F of the closed loops in G :

$$\Gamma = \text{Image} \left(\pi_1(G) \xrightarrow{F} H^1(M; \mathbb{R}) \right).$$

If $\{\phi_{t \in [0,1]}\}$ is a path of symplectic diffeomorphisms then it is easy to see that its endpoint ϕ_1 belongs to $\text{Ham}(M)$ if and only if its flux $F(\{\phi_t\})$ belongs to Γ . In other words, there is an exact sequence

$$(*) \quad \text{Ham}(M, \omega) \rightarrow \text{Symp}_0(M, \omega) \rightarrow H^1(M, \mathbb{R})/\Gamma,$$

where the second map is induced by F . Thus, the group Γ contains crucial information about the manifold, being the key to whether or not a symplectic diffeomorphism is Hamiltonian. In particular it is important to know if Γ is discrete, and if it is, how to estimate the size of a neighborhood of $\{0\}$ in $H^1(M; \mathbb{R})$ that contains no element of Γ except $\{0\}$ itself. We begin with the following observation:

Proposition 1.1. *For any closed symplectic manifold, Γ is discrete if and only if the subgroup of Hamiltonian diffeomorphisms is C^1 -closed in the full group of (symplectic) diffeomorphisms of the manifold.*

This follows almost immediately from the existence of the exact sequence $(*)$ since all the maps involved are C^1 -continuous. A more detailed proof is given in §3 below. We can now state the main conjectures:

The C^1 -Flux Conjecture. *The flux subgroup is discrete for all symplectic manifolds.*

The C^0 -Flux Conjecture. *For any symplectic manifold, the group $\text{Ham}(M)$ is C^0 -closed in the identity component of the group of symplectic diffeomorphisms.*

The main problem posed by the Flux conjecture² can be presented in the following way. Consider a “long” Hamiltonian path $\{\phi_{t \in [0,1]}\}$ beginning at the identity whose graph at intermediate times escapes out of a Weinstein neighborhood U of the diagonal in $(M \times M, -\omega \oplus \omega)$, but which comes back inside that neighborhood at time $t = 1$. As is well-known, U can be identified with a neighborhood of the zero section in the cotangent bundle

¹When no specific mention is made, all our paths begin at the identity. Note also that the group \tilde{H} which appears in the above exact sequence is just the universal cover of $\text{Ham}(M, \omega)$.

²Classically, the “Flux conjecture” is what we call here the C^1 -Flux conjecture. For this reason, when we refer to the “Flux conjecture” with no qualification, we always have the C^1 -case in mind.

of M , and, when it is, a Hamiltonian isotopy that stays inside U is given by the graphs of a family of exact 1-forms. However, if the path goes outside U , then in general there is no obvious reason that the 1-form corresponding to its endpoint be exact.

Note also that the statement that $\text{Ham}(M)$ is C^k -closed for one $k \geq 1$ is obviously equivalent to the statement that it is C^k -closed for all $k \geq 1$. However, the question of whether it is C^0 -closed is a priori quite different, and seems to be very difficult to decide except in some particular cases like tori (see below). This is a fundamental issue. Indeed all known invariants (such as symplectic homology) are C^0 -invariants attached to Hamiltonian *paths*, but one is actually interested in the dependence on the Hamiltonian *endpoint*. Observe also that, although symplectic rigidity tells us that the group $\text{Symp}(M)$ of all symplectomorphisms is C^0 -closed in the group of all diffeomorphisms of M , it is not known whether or not the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$ is C^0 -closed in $\text{Symp}(M)$. To avoid this question, we look here at the C^0 -closure of $\text{Ham}(M)$ in $\text{Symp}_0(M)$.

As will become clear in this paper, the Flux conjectures lie at the very borderline between soft and hard symplectic topology. What we do here is to show how hard techniques and their consequences can be used to prove these conjectures in many cases where purely soft, topological methods seem to fail.

1.1. Some old results. The flux group Γ was first explicitly mentioned in Banyaga's foundational paper [1], where it was observed that Γ is discrete when M is Kähler, or, more generally, when M is Lefschetz.³ For the sake of completeness, we will briefly recall this and other early results on Γ .

(i) It is easy to check that there is a commutative diagram expressing the fact that the following two homomorphisms coincide up to a universal multiplicative constant depending only on the dimension of M :

$$\pi_1(G) \xrightarrow{ev} \pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \xrightarrow{PD} H^{2n-1}(M; \mathbb{Z})$$

and

$$\pi_1(G) \xrightarrow{F} H^1(M, \mathbb{R}) \xrightarrow{\cup[\omega]^{n-1}} H^{2n-1}(M; \mathbb{R}).$$

Here ev is the evaluation map at a point of M , PD is the Poincaré dual and F is the flux homomorphism. But the first map has discrete image. Thus, if a manifold is Lefschetz, the subgroup $\Gamma \subset H^1(M, \mathbb{R})$ is discrete too. Note also that in this case the flux homomorphism vanishes on all elements of $\pi_1(G)$ that evaluate trivially in the homology group $H_1(M, \mathbb{R})$.

(ii) Temporarily, let G denote the group of diffeomorphisms which preserve some given structure on M . Then the “ c -flux” homomorphism

$$\pi_1(G) \xrightarrow{F_c} H^1(M, \mathbb{R}),$$

³This means that $H^1(M, \mathbb{R}) \xrightarrow{\cup[\omega]^{n-1}} H^{2n-1}(M; \mathbb{R})$ is an isomorphism.

defined by assigning to the pair $(\phi_{t \in [0,1]}, \gamma) \in \pi_1(G) \times H_1(M; \mathbb{R})$ the integration of the class $c \in H^2(M)$ on the trace of $\{\phi_t\}$ on γ , vanishes identically if c is a characteristic class for the structure preserved by G . This was proved by McDuff in [11] by an argument based on Gottlieb theory. (A more transparent proof will be given in [7].) Restricting to the case when G is the group of symplectic diffeomorphisms and c is the first Chern class of the tangent bundle of the symplectic manifold, we see that the flux vanishes for monotone manifolds.⁴ (This application to monotone manifolds was not explicitly mentioned in [11], but appears in Lupton–Oprea [10].)

(iii) It is also proved in [11] that Γ is discrete when the class of the symplectic form is decomposable, that is to say when $[\omega]$ is the sum of products of elements of $H^1(M, \mathbb{R})$.

(iv) Finally it follows immediately from Ginzburg’s results in [5] that any symplectic torus action on a Lefschetz manifold has discrete flux: see §2 for more detail.

1.2. The evaluation map. In this section we discuss the relation of the Flux conjectures to the topological properties of the orbits $\{\phi_t(x)\}$ of a loop $\{\phi_t\}$. Our work is based on the following deep fact:

For every Hamiltonian flow $\{\phi_{t \in [0,1]}\}$ on a closed symplectic manifold, there is at least one fixed point $x \in M$ of ϕ_1 such that the loop $\{\phi_t(x)\}_{t \in [0,1]}$ is contractible. In particular, any loop in G with vanishing flux is sent by the evaluation map to $0 \in \pi_1(M)$.

This is proved along with the Arnold conjecture: indeed all proofs of the Arnold conjecture give a nonzero lower bound for the number of fixed points of ϕ_1 with *contractible* orbits $\{\phi_t(x)\}$: see Hofer–Salamon [6], Fukaya–Ono [4] and Liu–Tian [9].

As we will see in §3, the following result is an almost immediate consequence.

Proposition 1.2. *The flux homomorphism has discrete image on $\pi_1(G)$ if and only if it has discrete image on the subgroup K of $\pi_1(G)$ formed by loops with contractible orbits in M .*

Thus the Flux conjecture only depends on those symplectic loops which, like Hamiltonian ones, have contractible orbits. One sometimes says that the flux homomorphism of some symplectic manifold *factorizes* if it factorizes through the evaluation map $\pi_1(\text{Symp}(M)) \rightarrow \pi_1(M)$. In other words, it factorizes when every symplectic loop with contractible orbit has vanishing flux. Proposition 1.2 shows that factorization implies discreteness. We will discuss a result of Théret’s on factorization at the end of §1.4.

⁴We recall that a *monotone* manifold is a symplectic manifold satisfying $c = \kappa\omega$ for some $\kappa > 0$. It is *semi-monotone* if $\kappa \geq 0$.

Remark 1.3. Although both the above statements remain true for compactly supported Hamiltonian flows on noncompact manifolds, they do not extend to flows of arbitrary support, even if one considers the evaluation in *homology*. Indeed, the standard rotation of the annulus is Hamiltonian. On the other hand the fact that the evaluation in real homology vanishes for Hamiltonian loops on closed manifolds is very elementary and follows from the commutative diagram in §1.1. This illustrates a striking contrast between the homological (mod torsion) and homotopical points of view on the evaluation map: see Bialy-Polterovich [2] for further discussion. \square

In view of the above results, one might wonder if any symplectic loop with contractible orbits has to be Hamiltonian up to homotopy. In dimension 4 the answer to this question is not known in general, but it is false in general. Indeed, one has the following result:

Proposition 1.4. *Let (M, ω) be a closed symplectic manifold.*

(i) *In dimension 4 every symplectic S^1 -action with contractible orbits is Hamiltonian (and hence has fixed points.)*

(ii) *There is a nonHamiltonian symplectic S^1 -action on a 4-manifold with orbits that are homologous to zero but not contractible.*

(iii) *In dimension 6 there is a nonHamiltonian S^1 -action with fixed points and hence with contractible orbits.*

We will see in §2 that part (i) follows almost immediately from results in McDuff [12]. It is easy to construct an example for (ii) in which M is a nonKähler T^2 -bundle over T^2 , for example the Kodaira–Thurston manifold, while an example of type (iii) may be found in [12].

The above result shows that it is not in general possible to distinguish Hamiltonian loops from others by looking only at the topological properties of their orbits. However, the story becomes more interesting if one looks at the “evaluation map” $ev_\phi : H_k(M) \rightarrow H_{k+1}(M)$ of a loop $\{\phi_{t \in [0,1]}\}$ in dimensions $k > 0$. Here, ev_ϕ is the map which takes a k -cycle γ to its trace $C_\gamma = \cup_t \{\phi_t(\gamma)\}$. Thus $\{\phi_t\}$ is Hamiltonian up to homotopy if and only if $[\omega]$ vanishes on the image of $ev_\phi|_{H_1}$.

Recently we have shown that when M is monotone ev_ϕ vanishes identically for all k . The proof is based on Seidel’s description in [14] of the canonical action of $\pi_1(\text{Ham}(M))$ on the quantum cohomology ring of M . The vanishing of this map for $k = 1$ means that the flux for Hamiltonian loops

$$H_1(M) \xrightarrow{ev_\phi} H_2(M) \xrightarrow{\int \omega} \mathbb{R}$$

actually vanishes for *topological reasons*. Thus, if a loop $\beta = \{\phi_t\}$ in $\text{Diff}(M)$ is Hamiltonian with respect to ω and is homotopic to a loop β' which is symplectic with respect to some other symplectic form ω' , then β' is necessarily *Hamiltonian* (up to homotopy) with respect to ω' . This implies that if a loop is Hamiltonian with respect to ω , then, for any symplectic form ω'

sufficiently close to ω , it can be homotoped to a ω' -symplectic loop which is actually Hamiltonian. Note that this result can be interpreted as an obstruction in the following way. Let β be any ω -symplectic nonHamiltonian loop and denote by $A_{\beta,\omega} \neq \{0\}$ the image of the map $H_1(M, \mathbb{Z}) \rightarrow H_2(M, \mathbb{Z})$ induced by β . Let $a \in H^2(M, \mathbb{R})$ be any class that vanishes on $A_{\beta,\omega}$. Then a deformation of the symplectic structure from the class $[\omega]$ to a – if it exists – cannot be lifted to a deformation of the loop β . In other words, each space $A_{\beta,\omega}$ gives an obstruction either to the deformation of ω or to the deformation of the image of $\pi_1(\text{Symp}(M, \omega))$ inside $\pi_1(\text{Diff}(M))$. Because a symplectic structure ω and an ω -symplectic loop β determine a symplectic fibration V with fiber (M, ω) over the 2-sphere, this can also be interpreted in the following way. The existence of a ruled symplectic form on V compatible with a given symplectic fibration $V \rightarrow S^2$ depends only on the connected component of that fibration in the space of all symplectic fibrations. These results and various generalizations and corollaries will appear in our forthcoming paper [7].

1.3. The C^1 -Flux conjecture. We now present a list of manifolds for which we have been able to confirm the C^1 -Flux conjecture.

We start our discussion with the the case of symplectic torus actions. To motivate it, suppose that, for some symplectic manifold M , the group G retracts onto a finite dimensional Lie subgroup H . (This is known to be true for some “simple” symplectic manifolds, for instance for surfaces and some of their products.) Then all elements of the fundamental group of G are represented by elements of the fundamental group of a maximal torus inside H , and the flux conjecture reduces to the same conjecture about the fluxes of an autonomous action of the torus. Our first result confirms the Flux Conjecture in this case.

Theorem 1.5. *Let T^n act symplectically on a closed symplectic manifold. Then the restriction of the flux homomorphism F to $\pi_1(T^n)$ has discrete image in $H^1(M, \mathbb{R})$.*

The proof, which is given in §2, combines ideas from Ginzburg [5] with an analysis of the Morse-Bott singularities of the corresponding (generalized) moment map.

Let us go back to the general non-autonomous case. Note that the conjecture obviously holds if the integration morphism $\int \omega : H_2^T(M, \mathbb{Z}) \rightarrow \mathbb{R}$ has discrete image, where $H_2^T(M, \mathbb{Z})$ denotes the set of classes that can be represented by continuous maps of the 2-torus. The first statement below shows that it is enough to require that this integration morphism has discrete image on *spherical classes*

$$H_2^S(M, \mathbb{Z}) = \text{Im}(\pi_2(M) \rightarrow H_2(M, \mathbb{Z}))$$

alone. The second statement refers to the minimal spherical Chern number, which is by definition the nonnegative generator of the image of $H_2^S(M, \mathbb{Z})$ by the first Chern class $c = c_1(TM)$ of the tangent bundle of M .

Theorem 1.6. *The C^1 -Flux conjecture holds in the following cases:*

- (i): *The manifold is spherically rational, that is to say the image of the spherical 2-classes $H_2^S(M, \mathbb{Z})$ by the integration morphism $\int \omega$ is a discrete subgroup of \mathbb{R} (of the form $\lambda\mathbb{Z}$ for some nonnegative real number λ);*
- (ii): *The minimal spherical Chern number is either zero or is no less than $2n = \dim_{\mathbb{R}}(M)$;*
- (iii): *M has dimension 4 and $\pi_1(M)$ acts trivially on $\pi_2(M)$.*

The various parts of this theorem have different proofs. Parts (i) and (iii) follow from Proposition 1.2 by easy topological arguments. The main idea in the proof of part (ii) is the following. Suppose that some class $a \in H^1(M, \mathbb{R})$ belongs to Γ . Then the autonomous path $\{\psi_t\}$, generated by any representative λ_a of a has flux equal to a . But ψ_1 is also the endpoint of a Hamiltonian path $\{\phi_t\}$. Thus one can compare the Floer–Novikov homology of a generic perturbation of $\{\psi_t\}$ to the Floer homology of the path $\{\phi_t\}$. Morally, this should lead to strong constraints on the class a . In many cases, they are strong enough to give a complete description of Γ . If one is only interested in proving discreteness, one can limit this study to the comparison of these two paths when λ_a is so C^1 -small that its zeros are exactly the fixed points of ψ_1 . In this case, one compares the ordinary Morse–Novikov homology of the small 1-form λ_a to some Floer homology. In particular, if λ_a does not have enough zeroes to satisfy the constraints of the Arnold conjecture or if the zeroes do not have the appropriate indices, one concludes that it cannot belong to Γ .

Theorem 1.6 extends the results mentioned in §1.1 above. Indeed, an immediate corollary of (i) and (ii) is that the Flux conjecture holds for semi-monotone manifolds. Also (i) implies discreteness in the decomposable case, since any decomposable form vanishes on all spherical 2-classes.

1.4. The C^0 -Flux conjecture. We complete the introduction with a discussion of the C^0 -Flux conjecture. Again, set $G = \text{Symp}_0(M)$, and let G_i ($i = 0, 1$) be the C^i -closure of $\text{Ham}(M)$ in G . Let Γ_i be the image under the flux homomorphism of the lift of G_i to the universal cover of G . It is not hard to check that in this language Proposition 1.1 states that $\text{Closure}(\Gamma) = \Gamma_1$, while the Flux conjecture and the C^0 -conjecture are equivalent to the statements $\Gamma = \Gamma_1$ and $\Gamma = \Gamma_0$ respectively.

It turns out that when M is Lefschetz the group Γ_0 is contained in a group $\Gamma_{\text{top}} \subset H^1(M, \mathbb{R})$ which depends only on the topology of M and on the cohomology class of the symplectic form. Namely, consider the space of all smooth (or even continuous) maps $M \rightarrow M$, and denote by $\text{Map}_0(M)$ the connected component of the identity. The flux homomorphism F extends naturally to a homomorphism $\pi_1(\text{Map}_0(M)) \rightarrow H^1(M, \mathbb{R})$ by integrating the symplectic form on the evaluation of the path of maps on the 1-cycles of M . Denote by Γ_{top} its image $F(\pi_1(\text{Map}_0(M)))$. Clearly, $\Gamma \subset \Gamma_0$.

Theorem 1.7. *If M is Lefschetz, then $\Gamma_0 \subset \Gamma_{top}$.*

See §4 for the proof of this theorem. Since $\Gamma \subset \Gamma_0$ we immediately get the following useful consequence.

Corollary 1.8. *Assume that M is Lefschetz and that $\Gamma_{top} = \Gamma$. Then the C^0 -Flux conjecture holds for M .*

As an example, take M to be a closed Kähler manifold of nonpositive curvature such that its fundamental group has no center. For instance a product of surfaces of genus greater than 1 has this property. It is easy to see that in this case $\pi_1(\text{Map}_0(M))$ is trivial⁵ and hence $\Gamma_{top} = \{0\}$. Moreover, M is Lefschetz since it is Kähler. Thus the previous corollary implies that the C^0 -Flux conjecture holds on M .

Another immediate application of this corollary is that the C^0 -Flux conjecture holds for the $2n$ -dimensional torus with a translation invariant symplectic structure. Indeed, in this case M is Lefschetz so that the Flux homomorphism factors through $H_1(M, \mathbb{Z})$. Moreover translations generate a large enough subgroup of $\text{Symp}_0(M)$ for us to see that $\pi_1(\text{Symp}_0(M))$ maps onto $H_1(M, \mathbb{Z})$. Thus $\Gamma = \Gamma_{top}$, and so, by the above theorem, the conjecture holds. We refer the reader to McDuff–Salamon [13] for the explicit computation of Γ for tori.

Another proof of the C^0 -conjecture for these tori was first observed by Herman in 1983, just after Conley and Zehnder’s proof [3] of Arnold’s conjecture for the torus. His idea consists in the following basic observation: if a path $\{\psi_t\}$ has flux a and an endpoint ψ that is equal to the C^0 -limit of Hamiltonian diffeomorphisms, then, by composing everything by an appropriate Hamiltonian, we may assume that ψ is the translation of flux a . We must show that ψ is Hamiltonian. But, since all Hamiltonian diffeomorphisms have a fixed point by Arnold’s conjecture, so does the limit ψ . Hence ψ must be the identity, which, of course, is Hamiltonian.

A natural generalization of Herman’s idea is to replace the counting of fixed points by an analysis of the limiting behavior of Floer homologies associated to a sequence of Hamiltonian diffeomorphisms, which is just the approach we take in Theorem 1.6 (ii).

Let us mention finally that the questions which we discussed above can be posed in a more general context of *Lagrangian submanifolds* of a symplectic manifold. More precisely, let M be a symplectic manifold and $L \subset M$ be a closed Lagrangian submanifold. Denote by $H(L)$ (resp. $S(L)$) the space of all Lagrangian submanifolds which are obtained from L by a Hamiltonian (resp. Lagrangian) isotopy. The C^i -conjecture ($i = 0, 1$) in this case means that $H(L)$ is C^i -closed in $S(L)$. A result in this direction was obtained recently by Theret [15]. See also Bialy–Polterovich [2] for results on the evaluation in $\pi_1(M, L)$ in this situation.

⁵Note that $\pi_1(\text{Map}_0(M))$ always maps into the center of $\pi_1(M)$.

Throughout the paper we will assume that M is a smooth compact manifold without boundary. Many results apply when M is noncompact, provided that we consider symplectomorphisms of compact support. We leave such extensions to the reader.

2. TORUS ACTIONS

We begin by proving Theorem 1.5, and then discuss the examples mentioned in Proposition 1.4.

2.1. The discreteness of Γ for torus actions. Throughout this section we use the identification

$$\pi_1(T^d) = H_1(T^d, \mathbb{Z}) \subset H^1(T^d, \mathbb{R}).$$

We start with the following useful notion introduced by Viktor Ginzburg [5]. A symplectic action of T^d on M is called *cohomologically free* if the flux homomorphism $F : H_1(T^d, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ is injective. Clearly, for such an action the Flux subgroup $\Gamma = F(\pi_1(T^d))$ is discrete. It turns out that every torus action can be reduced to a cohomologically free torus action. More precisely, we say that a torus action on M is *reducible* if the following conditions (i)-(iii) hold:

(i) there exists a T^d -invariant symplectic submanifold $N \subset M$ such that the inclusion $H_1(N, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ is onto (we use the convention that a point is a symplectic submanifold);

(ii) there exists a symplectic T^r -action on N and a surjective homomorphism $j : T^d \rightarrow T^r$ such that $g|_N$ and $j(g)$ coincide as diffeomorphisms of N for all $g \in T^d$;

(iii) $r + \dim N < d + \dim M$.

This definition splits in two cases: either $\dim N < \dim M$ and in this case one can choose j as the identity, or $\dim N = \dim M$ which means that the whole action factorizes through a smaller torus.

As an example, the product of two standard circle actions on $T^2 \times S^2$ is reducible since we may take $T^r = S^1 \times \text{id}$ and N to be the product of an orbit on T^2 with a fixed point on S^2 .

Proposition 2.1. *Every irreducible torus action is cohomologically free.*

Proof. Assume that the action of T^d on M is not cohomologically free. Then for some non-zero element v in the Lie algebra of T^d the action generated by v is Hamiltonian. Denote by H the Hamiltonian function. Let $L \subset T^n$ be the subtorus (of positive rank!) defined as the closure of the 1-parametric subgroup V generated by v . Notice that every critical point of H is fixed by the action of L , since V is dense in L . Since the action of L is linearizable near each fixed point, we conclude that in some local complex coordinates (z_1, \dots, z_n) on M near a critical point one can write H (up to an additive constant) as a linear combination of the $|z_i|^2$. In particular, either H is *identically constant* or H is a *Morse-Bott function with even indices*.

In the first case, the action of L on M is trivial. Taking $N = M$, $T^r = T^n/L$, and defining j as the natural projection we find that the action is reducible.

In the second case, take N to be the minimum set of H . It follows from the previous discussion that N is a symplectic submanifold, and Morse-Bott theory implies that the inclusion $H_1(N, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ is an epimorphism. Moreover, since the action of L fixes N and commutes with the action of the whole group T^d , we conclude that N is T^d -invariant. Therefore, taking $T^r = T^d$ and defining j as the identity map we see that the action of T^d is reducible. This completes the proof. \square

Proof of Theorem 1.5.

Consider a T^d -action on M . Applying Proposition 2.1 repeatedly we end up with a cohomologically free reduction since the process terminates in view of (iii)! In other words there exists a T^d -invariant symplectic submanifold $N \subset M$ and a homomorphism $j : T^d \rightarrow T^r$ which satisfy conditions (i)-(iii) above and such that the T^r -action on N is cohomologically free.

Let $\Phi : H_1(T^r, \mathbb{R}) \rightarrow H^1(N, \mathbb{R})$ and $F : H_1(T^d, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ be the flux homomorphisms of the T^r -action on N and the T^d -action on M respectively. Let $j_* : H_1(T^d, \mathbb{R}) \rightarrow H_1(T^r, \mathbb{R})$ and $i^* : H^1(M, \mathbb{R}) \rightarrow H^1(N, \mathbb{R})$ be the natural maps. Clearly, $\Phi \circ j_* = i^* \circ F$, and thus

$$i^*(F(\pi_1(T^d))) = \Phi(j_*(\pi_1(T^d))).$$

The right hand side is discrete since j_* maps integer homology to integer homology and Φ is a monomorphism as the flux of a cohomologically free action. Moreover, i^* is a monomorphism in view of the condition (i). Hence the flux subgroup $\Gamma = F(\pi_1(T^d))$ is discrete. This completes the proof. \square

Corollary 2.2. *Every symplectic torus action on a closed manifold splits into a product of a Hamiltonian action and a cohomologically free action.*

Proof. This was proved by Ginzburg [5] for the case when either the symplectic manifold is Lefschetz, or the symplectic form represents an integer cohomology class. Combining Ginzburg's argument with our Theorem 1.5 one immediately gets this statement for general manifolds. \square

2.2. S^1 actions. It was shown in [12] that a symplectic S^1 -action on a closed 4-manifold (M, ω) is Hamiltonian if and only if it has fixed points. Therefore, part (i) of Proposition 1.4 follows from the next lemma.

Lemma 2.3. *In the 4-dimensional case every symplectic S^1 -action with contractible orbits has fixed points.*

Proof. Consider an action with no fixed points. This is generated by a nonvanishing vector field X . As in [12], one can slightly perturb ω to an invariant form which represents a rational cohomology class and then rescale ω , to reduce to the case when the flux of the action, which is represented by

the form $i_X(\omega)$, is integral. In this case there is a map $\mu : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$, called the generalised moment map, such that

$$i_X(\omega) = \mu^*(dt).$$

Since X has no fixed points μ is a fibration with fiber Q and we may think of M as made from $Q \times [0, 1]$ by identifying $Q \times \{0\}$ with $Q \times \{1\}$. Moreover, the action of S^1 on $Q \times [0, 1]$ is Hamiltonian and so has the usual structure of such actions. Thus there is a Seifert fibration $\pi : Q \rightarrow \Sigma$ and a family σ_t of area forms on Σ such that

$$\omega|_{Q \times \{t\}} = \pi^*(\sigma_t), \quad [\sigma_t] - [\sigma_s] = (t - s)c, \quad s, t \in [0, 1],$$

where $c \in H^2(\Sigma, \mathbb{Q})$ is the Euler class of π .

We claim that it is impossible for the orbits to be contractible in M . To see this, observe that the long exact homotopy sequence for $Q \rightarrow M \rightarrow S^1$ implies that $\pi_1(Q)$ injects into $\pi_1(M)$. Hence if the orbits are contractible in M , they contract in Q .⁶ But if they contract in Q , the Euler class c must be nontrivial. Therefore $[\sigma_0] \neq [\sigma_1]$, and it is impossible to find a suitable gluing map $Q \times \{0\} \rightarrow Q \times \{1\}$ since such a map would induce a symplectomorphism $(\Sigma, \sigma_0) \rightarrow (\Sigma, \sigma_1)$. \square

3. PROOF OF THE MAIN THEOREM IN THE C^1 -CASE

We will begin by proving the elementary results stated in Propositions 1.1 and 1.2. We then prove the various parts of Theorem 1.6, modulo a lemma needed for (ii) that is presented in §3.2.

3.1. The main arguments. Proof of Proposition 1.1.

We must show that the C^1 closure of $\text{Ham}(M)$ is equivalent to the discreteness of the flux group Γ .

Let ϕ_k be a sequence of Hamiltonian diffeomorphisms which C^1 -converges to some symplectic diffeomorphism ϕ . By composing the sequence and its limit with $\phi_{k_0}^{-1}$ for some fixed k_0 , we get a new sequence ψ_k of Hamiltonian diffeomorphisms converging to some ψ which is Hamiltonian iff ϕ is. By choosing k_0 large enough, the new sequence is C^1 -close to the identity for $k > k_0$, and therefore there are closed 1-forms λ, λ_k on M such that the autonomous symplectic path $\psi_{t \in [0,1]}^\lambda, \psi_{t \in [0,1]}^{\lambda_k}$ generated by the ω -duals of λ, λ_k are small paths whose endpoints are ψ, ψ_k and whose fluxes are equal to $[\lambda], [\lambda_k]$. For each $k \geq k_0$, we thus get a loop by composing (in the sense of paths, not pointwise) a Hamiltonian path $\{\phi_t^k\}$ from the identity to $\phi_1^k = \psi^k$ with the path $\{\psi_{1-t}^{\lambda_k}\}$. This loop has flux $-[\lambda_k]$, which is arbitrarily small for $k \geq k_0$. If Γ is discrete, these classes must vanish, and therefore their limit $[\lambda] = 0$. But this means that the path $\psi_{t \in [0,1]}^\lambda$ is Hamiltonian. Hence $\psi_1^\lambda = \psi$ is Hamiltonian, as required.

⁶This is the place where the argument fails in homology. It is possible for the orbit to represent a nonzero element of $H_1(Q)$ while being nullhomologous in M .

Conversely, if Γ is not discrete, there is an arbitrarily small class $[\lambda] \in H^1(M; \mathbb{R})$ which is not in Γ but is the limit of classes in Γ . Then the above construction gives a sequence of Hamiltonian diffeomorphisms which C^1 -converge to a nonHamiltonian one. \square

Proof of Proposition 1.2.

We must show that, if the flux homomorphism F has discrete image on the subgroup formed by symplectic loops with trivial evaluation in $\pi_1(M)$, its whole image Γ is discrete. To do this, it is clearly enough to show that there exists a neighborhood U of 0 in $H^1(M, \mathbb{R})$ with the following property:

every class $a \in \Gamma \cap U$ is the image under F of a symplectic loop with contractible orbits.

If U is sufficiently small, then every class $a \in U$ can be represented by a 1-form λ which generates an autonomous locally-Hamiltonian flow ψ_t^λ whose only closed orbits in the time interval $t \in [0; 1]$ are the zeros of λ . We claim that such U have the required property.

To see this suppose that $a \in \Gamma$. Then ψ_1^λ is the time-1 map of some Hamiltonian flow $\{\phi_t\}$. By the discussion after Proposition 1.2 there exists a fixed point x of $\phi_1 = \psi_1^\lambda$ such that the corresponding closed orbit of $\{\phi_t\}$ is contractible. As in the proof of Proposition 1.1, now consider the loop in $\text{Symp}_0(M)$ obtained by composing the paths $\{\phi_{1-t}\}$ and $\{\psi_t\}$. We see that its flux equals a . Moreover, if the only fixed points of ψ_1^λ are zeros of λ , its orbits are contractible. This completes the proof. \square

Proof of Theorem 1.6 (i)

Assume that the manifold M is spherically rational. We must show that Γ is discrete. In view of Proposition 1.2 it suffices to consider symplectic loops with contractible orbits only. Let $\{\phi_t\}$ be such a loop, and let $a \in H^1(M, \mathbb{R})$ be its flux. Then the value of a on each closed curve $\gamma \subset M$ is equal to the symplectic area of the torus C_γ . But this torus is the image of a sphere since one of its generating loops is contractible in M . Thus the homomorphism

$$H_1(M, \mathbb{R}) \xrightarrow{ev_\phi} H_2(M, \mathbb{R}) \xrightarrow{\int \omega} \mathbb{R}$$

factors through spheres

$$a : H_1(M, \mathbb{R}) \xrightarrow{ev_\phi} H_2^S(M, \mathbb{R}) \xrightarrow{\int \omega} \mathbb{R}.$$

But if the manifold is spherically rational, this map takes values in some discrete subgroup of \mathbb{R} which does not depend on a particular choice of a . This implies the discreteness of the flux subgroup Γ . \square

Proof of Theorem 1.6 (ii)

We must show that if the minimal Chern number of (M, ω) is zero or is $\geq 2n$, then the Flux conjecture holds. This is based on the comparison of the Maslov and Floer indices.

In view of Proposition 1.2 it is enough to restrict our attention to the subgroup K of $\pi_1(G)$ consisting of loops whose evaluation in π_1 vanishes. Fix a Riemannian metric g on M , and define a norm $\|a\|$ of a class $a \in H^1(M, \mathbb{R})$ as the C^1 - g -norm of the unique g -harmonic form in this class. The following lemma is proved in § .

Lemma 3.1. *There exists $\epsilon > 0$ such that every nonzero cohomology class a with $\|a\| < \epsilon$ is represented by a Morse form λ which has no critical point of index 0 and $2n$, and whose Hamiltonian flow has no nonconstant 1-periodic orbit.*

Granted this, assume that the image of $F(K)$ is not discrete, and choose ϵ as in Lemma 3.1. Then there exists $\beta \in K$ with $0 < \|F(\beta)\| < \epsilon$. Represent the class $F(\beta)$ by a form λ as in Lemma 3.1 and denote by ψ_1 the time-1 map of the corresponding symplectic flow $\{\psi_t^\lambda\}$. In view of our choice, the only 1-periodic orbits of $\{\psi_t^\lambda\}$ are the constant ones.

Since ψ_1 is a Hamiltonian diffeomorphism, we can join it to the identity by a Hamiltonian path $\{\phi_t\}$. Introduce the following notations. For a fixed point x of $\phi_1 = \psi_1$ denote by $i_\lambda(x)$ the Conley–Zehnder index coming from the flow of λ and by $i_H(x)$ the Conley–Zehnder index coming from the Hamiltonian path. If the minimal spherical Chern number N is 0 or is $\geq n$, then Hofer–Salamon proved in [6] that the Floer homology of a Hamiltonian path is isomorphic to the ordinary homology with coefficients in the Novikov ring of the group $H_2^S(M, \mathbb{Z})$. In particular there are fixed points x, y of ϕ with

$$i_H(x) - i_H(y) = 2n \pmod{2N}$$

(Recall that all relative indices of Floer homology are defined only modulo $2N$). On the other hand the loop which is the composition of $\{\phi_t\}$ with $\{\psi_{1-t}^\lambda\}$ has indices which are obviously equal to a constant $m \in \mathbb{Z}$ modulo $2N$. Thus, for any fixed point $z \in M$,

$$i_H(z) - i_\lambda(z) = m \pmod{2N}$$

and therefore

$$i_H(x) - i_H(y) = i_\lambda(x) - i_\lambda(y) \pmod{2N},$$

which implies that

$$i_\lambda(x) - i_\lambda(y) = 2n \pmod{2N}.$$

But our choice of λ implies that

$$-2n + 2 \leq i_\lambda(x) - i_\lambda(y) \leq 2n - 2$$

which means that $2 \leq 2n - (i_\lambda(x) - i_\lambda(y)) \leq 4n - 2$. The last equation states that $2n - (i_\lambda(x) - i_\lambda(y))$ is a multiple of $2N$. But this is impossible if N is 0 or $\geq 2n$. \square

Proof of Theorem 1.6(iii).

Suppose, by contradiction, that the flux is not discrete when π_1 acts trivially on π_2 and M has dimension 4. Because it is non-discrete, it is not factorizable. Thus there is a loop with trivial evaluation in $\pi_1(M)$ and non-zero flux $a \in H^1(M)$. By the commutativity of the diagramme in § 1.1 (i), we must have $a \cup [\omega^{n-1}] = 0$. Hence there is a non zero class such that:

- : (i) $a \cup [\omega] = 0$; and
- : (ii) there is a loop of symplectic diffeomorphisms $\{\phi_t \in [0,1]\}$ whose flux is a and which has contractible evaluation in $\pi_1(M)$.

Now choose an element $\gamma \in \pi_1 M$ such that $a(\gamma) = \kappa \neq 0$, and let $C = C_\gamma$, the trace of $\{\phi_t\}$ on γ by $\{\phi_t\}$. Then, if $p : \tilde{M} \rightarrow M$ is the universal cover, C lifts to a 2-sphere \tilde{C} in \tilde{M} , and

$$p^*([\omega])(\tilde{C}) = [\omega](C) = \kappa \neq 0.$$

Therefore it suffices to prove the following lemma.⁷

Lemma 3.2. *Let a be a nonzero class in $H^1(M; \mathbb{R})$ such that for some k*

$$a \cup [\omega]^k = 0, \quad p^*([\omega]^k) \neq 0.$$

Then π_1 acts nontrivially on $H_{2k}(M)$. Moreover, if $p^([\omega]^k)$ does not vanish on some element of $H_{2k}^S(\tilde{M})$, $\pi_1(M)$ acts nontrivially on $\pi_{2k}(M)$.*

Proof. Choose an element $\gamma \in \pi_1 M$ such that $a(\gamma) = \kappa \neq 0$, and let τ be the deck transformation associated to γ . Then, if the 1-form λ represents a , we have $p^*(\lambda) = dH$ where $\tau^*H - H = \kappa$. Further,

$$\lambda \wedge \omega^k = d\theta$$

for some $2k$ -form θ on M because $a \cup [\omega]^k = 0$. This implies

$$d(Hp^*\omega^k) = d(p^*\theta),$$

which implies in turn that

$$Hp^*\omega^k = p^*\theta + \sigma$$

for some closed $2k$ -form σ on \tilde{M} . Hence:

$$\tau^*\sigma - \sigma = (\tau^*H - H)p^*\omega^k = \kappa p^*\omega^k,$$

and so there is a $2k$ -cycle \tilde{C} in \tilde{M} such that

$$\int_{\tau(\tilde{C})} \sigma - \int_{\tilde{C}} \sigma = \int_{\tilde{C}} (\tau^*\sigma - \sigma) = \kappa \int_{\tilde{C}} p^*\omega^k \neq 0.$$

Therefore $\tau(\tilde{C})$ and \tilde{C} are not homologous in \tilde{M} , and the action of $\pi_1(M)$ on $H_{2k}(\tilde{M})$ is nontrivial. Finally, if $p^*([\omega]^k)$ does not vanish on some element of $H_{2k}^S(M)$, the same argument shows that the action of $\pi_1(M)$ on $\pi_{2k}(\tilde{M})$ is nontrivial too (note that this action is given by the deck transformations

⁷This lemma gives some results in all dimensions. However, it is only in dimension 4 that one gets a clean statement about the Flux conjecture.

is well-defined because $\pi_1(\tilde{M})$ vanishes). Because $\pi_{2k}(\tilde{M})$ is isomorphic to $\pi_{2k}(M)$, we conclude that $\pi_1 M$ acts nontrivially on $\pi_{2k}(M)$. \square

3.2. Proof of Lemma 3.1. It was shown by G. Levitt [8] that every non-zero cohomology class can be represented by a Morse 1-form without critical points of index 0 and $2n$. We show here that the lemma 3.1 follows easily from this statement. However one can also use a different approach based on the fact that a generic harmonic 1-form is Morse (see the Appendix).

Here is a short proof of Lemma 3.1 based on Levitt's theorem. On some small closed convex nbd V of $0 \in H^1(M, \mathbb{R})$ choose any smooth section σ of $Z^1(M) \rightarrow H^1(M, \mathbb{R})$ (where $Z^1(M)$ is the space of closed 1-forms) which is 0 at 0 (one can do this by choosing a Riemannian metric say). Then take any smooth convex hypersphere S in $H^1(M, \mathbb{R})$ containing 0 in its interior, and for each class $[\lambda]$ on S , choose a Levitt form λ (which means here a Morse form with no critical point of index 0 or $2n$) and define the nbd $U_\lambda \subset S$ as the intersection of S with the classes $[\lambda] + rV$, where r is a small real number. Each element in U_λ is represented by the form $\lambda + r\sigma(v)$. Thus if r is small enough, all elements of U_λ are represented by Levitt forms and this representation is continuous. Now there is a finite covering of S by such subsets. This implies that there is ε small enough so that the hypersphere εS as well as its interior is covered by Levitt forms that are such that the Hamiltonian flow has no non-constant 1-periodic orbit.

4. THE C^0 -CONJECTURE AND LEFSCHETZ MANIFOLDS

In this section we prove Theorem 1.7. We start with a discussion of the properties of the flux homomorphism in the more general context of paths of maps.

Let M be a closed manifold, and let $\alpha = \{\phi_t\}_{t \in [0;1]}$ be a path of smooth maps $M \rightarrow M$ such that $\phi_0 = \text{id}$ and ϕ_1 is a diffeomorphism. Let I_α^k be the ring of ϕ_1 -invariant differential forms on M of degree k . Define the flux map $\Phi : I_\alpha^k \rightarrow H^{k-1}(M, \mathbb{R})$ as follows. Given an invariant form $\sigma \in I_\alpha^k$ and a $(k-1)$ -cycle γ on M , the value $\Phi(\sigma)$ on γ is defined to be the integral of σ on the trace $\cup_{t \in [0,1]} \{\phi_t(\gamma)\}$ of γ under the path. It is not hard to see that this value depends only on the homology class of γ , so that Φ is well defined. Note that its domain I_α^k depends on ϕ_1 in general, and may well be empty when $\phi_1 \neq \text{id}$.

It turns out that Φ inherits the important *derivation* property of the usual flux associated to a closed path of maps which starts and ends at the identity. Namely, given two ϕ_1 -invariant forms σ_1, σ_2 (which for convenience we assume to have even degrees) we have

$$\Phi(\sigma_1 \wedge \sigma_2) = [\sigma_1] \cup \Phi(\sigma_2) + \Phi[\sigma_1] \cup [\sigma_2].$$

To see this, suppose first that the maps ϕ_t are diffeomorphisms, generated by the family of vector fields $X_t = \dot{\phi}_t$. Then it is easy to see that the class

$\Phi(\alpha)$ is represented by the form

$$\int_0^1 \phi_t^*(i_{X_t}\alpha) dt = \int_0^1 i_{Y_t}(\phi_t^* \alpha) dt,$$

where $Y_t = \phi_t^*(X_t)$. With this representation, the derivation property is obvious. In the general case it is not hard to make sense of the right hand side of this formula, even though the vector fields X_t do not exist.

To see this, consider the mapping torus N of the diffeomorphism ϕ_1 , that is the product $M \times [0, 1]$ with identifications $(x, 0) = (\phi_1(x), 1)$. Every ϕ_1 -invariant form σ on $M = M \times \{0\}$ admits an obvious canonical extension σ' to N , and the path α extends to a *closed* path of maps α' of N that takes $(x, 0)$ to $(\phi_t(x), t)$ for each t . Clearly the flux of σ' with respect to α' is a cohomology class of N whose restriction to $M \times \{0\}$ coincides with $\Phi(\sigma)$. Thus the derivation property for Φ follows from the derivation property of the flux for closed loops of maps.

In what follows we write $\Phi = \Phi_\alpha$ in order to emphasize the dependence of Φ on the path α .

Return now to our symplectic situation. Let (M, ω) be a closed symplectic manifold. Denote by \mathcal{X} the set of paths of maps $\alpha = \{\phi_{t \in [0;1]}\}$ with symplectic endpoints: $\phi_0 = \text{id}$ and $\phi_1 \in \text{Symp}_0(M)$. We will endow \mathcal{X} with the C^0 -topology associated to a Riemannian metric on M .

The proof of Theorem 1.7 is based on the following statement.

Lemma 4.1. *Assume that M is Lefschetz. Then the map*

$$\mathcal{X} \rightarrow H^1(M, \mathbb{R}), \quad \alpha \rightarrow \Phi_\alpha(\omega)$$

is continuous.

Proof. The derivation property implies that $\Phi_\alpha(\omega^n) = n[\omega]^{n-1} \cup \Phi_\alpha(\omega)$. The mapping $\alpha \rightarrow \Phi_\alpha(\omega^n)$ is continuous since the volume of a top-dimensional subset of M varies continuously under continuous deformations of the subset. The needed statement follows now from the Lefschetz property. \square

Proof of Theorem 1.7.

Suppose that the sequence $\psi_n \in \text{Ham}(M)$ converges C^0 to a symplectomorphism ψ in the identity component $\text{Symp}_0(M)$. We have to show that the flux $F(\psi)$ belongs to the quotient group $\Gamma_{\text{top}}/\Gamma$.

Fix n large enough and consider the map $\phi = \psi_n^{-1}\psi$. Define two paths of maps between the identity and ϕ . One of them, say α , is a short path of maps along geodesics in M . (If n is large then x and $\phi(x)$ can be joined by a unique geodesic in M for every point $x \in M$.) The second path, say α' , is a path of symplectic diffeomorphisms. Let β be the loop of maps formed by composing of α' with $-\alpha$, where $(-\alpha)_t = \alpha_{1-t}$. Clearly,

$$F(\beta) = F(\alpha') - \Phi_\alpha(\omega).$$

Note that $F(\beta)$ belongs to Γ_{top} , and that $F(\alpha') = F(\psi) \pmod{\Gamma}$. Further, Lemma 4.1 implies that $\Phi_\alpha(\omega)$ can be made arbitrarily small by choosing n sufficiently large. But both Γ and Γ_{top} are discrete since M is Lefschetz. Therefore we get that $F(\psi) = F(\beta) \pmod{\Gamma}$. This completes the proof. \square

5. APPENDIX

We present here another more geometric proof of Lemma 3.1, based on the representation of forms by harmonic forms. We first need the following folkloric statement, that was mentioned to us by M. Farber and M. Braverman.

Lemma 5.1. *After an arbitrarily small smooth perturbation of the metric, the harmonic form in a given nonzero cohomology class is Morse.*

This can be proved by a method similar to that in Uhlenbeck [16], and we give the argument for the sake of completeness.

Proof. Let \mathcal{R} be the space of all Riemannian metrics on M , and let Ω be the space of all closed 1-forms on M representing a given nontrivial cohomology class in $H^1(M, \mathbb{R})$. Define the subset $\Lambda \subset \mathcal{R} \times \Omega$ consisting of all pairs (g, λ) such that λ is g -harmonic.

The sets \mathcal{R} , Ω and Λ have a natural structure of Banach manifolds with respect to suitable Sobolev norms. Moreover, the natural projection $\Lambda \rightarrow \mathcal{R}$ is a smooth one-to-one map. Hence in view of Smale-Sard transversality theorem with parameters, it is enough to check that the evaluation map $F : \Lambda \times M \rightarrow T^*M$ defined by

$$F(g, \lambda, x) = (x, \lambda_x)$$

is transversal to the zero section.

Assume that λ is a g -harmonic form which vanishes at some point q . Note that $T_{(q,0)}T^*M = T_q^*M \oplus T_qM$. In order to prove the needed transversality result it suffices to show the following:

*for every $\xi \in T_q^*M$ there is a tangent vector*

$$(\dot{g}, \dot{\lambda}) \in T_{(g,\lambda)}\Lambda$$

such that $\dot{\lambda}_q = \xi$.

First of all we compute the tangent space $T_{(g,\lambda)}\Lambda$ that we consider as a linear subspace in the product $S^2(M) \times \Omega_0(M)$, where $S^2(M)$ is the space of all symmetric 2-forms on M and $\Omega_0(M)$ is the space of all exact 1-forms.

Denote by g^{-1} the natural operator associated with the metric g which transforms $(0, k)$ -tensors to $(1, k - 1)$ -tensors. A straightforward computation shows that a vector $(\dot{g}, \dot{\lambda}) \in S^2(M) \times \Omega_0(M)$ is tangent to Λ at (g, λ) if and only if:

$$\Delta_g \dot{\lambda} - d\Phi(\dot{g}) = 0,$$

where

$$\Phi(\dot{g}) = \operatorname{div}_g(-(g^{-1}\dot{g})g^{-1}\lambda + \frac{1}{2}\operatorname{tr}(g^{-1}\dot{g})g^{-1}\lambda).$$

In view of elliptic regularity, we can find an open set $B \subset M$ such that λ has no zero in B . For every smooth function f which is supported in B and has zero g -mean, one can find a symmetric form (also supported in B) \dot{g} in such a way that $\Phi(\dot{g}) = f$. Also, $\Delta_g \dot{\lambda} = d\Delta_g u$ where $du = \dot{\lambda}$, and hence $(\dot{g}, \dot{\lambda})$ is tangent to Λ at (g, λ) if $\Delta_g u = f$. In view of this discussion, the required transversality fact follows immediately from the next lemma. We write $\langle f \rangle$ for the g -mean value of a function f .

Lemma 5.2. *Let (M, g) be a closed Riemannian manifold, $B \subset M$ be an open subset and $q \in M$ be a point which is chosen away from the closure of B . Then for each $\xi \in T_q^*M$ there exists a smooth function f supported in B with $\langle f \rangle = 0$ such that the solution of the equation $\Delta_g u = f$ satisfies $d_q u = \xi$.*

Proof. Define the operator $E : C^\infty(M) \rightarrow C^\infty(M)$ such that $\langle Ef \rangle = 0$ and

$$\Delta_g Ef = E\Delta_g f = f - \langle f \rangle.$$

Its kernel (or *Green's function*) $e(x, y)$ is defined by

$$(Ef)(x) = \int_M e(x, y)f(y)dy,$$

and satisfies the equation $\Delta_g e(x, \cdot) = -1$ on $M - \{x\}$.

Introduce local coordinates x_1, \dots, x_n near q and identify tangent and cotangent spaces to M at q with \mathbb{R}^n .

Let V be the space of all smooth functions with zero mean which are supported in B , and define an operator $T : V \rightarrow \mathbb{R}^n$ by

$$T(f) = \frac{\partial(Ef)}{\partial x}(q).$$

We have to prove that T is surjective. Assume, on the contrary, that there exists a vector $w \in \mathbb{R}^n$ such that $(w, Tf) = 0$ for all $f \in V$ (here (\cdot, \cdot) is a scalar product). Then we immediately get that the function $b : M - \{q\} \rightarrow \mathbb{R}$,

$$b(y) = (w, \frac{\partial e}{\partial x}(q, y)),$$

is *constant* on B . Notice that b is harmonic, and in view of elliptic regularity it is therefore constant everywhere on $M - \{q\}$.

Let us show that this is impossible. Indeed choose a function h with $\langle h \rangle = 0$ which is harmonic near q and such that $(w, \frac{\partial h}{\partial x}(q)) \neq 0$. We write

$$h(x) = \int_M e(x, y)\Delta_g h(y)dy.$$

Since $\Delta_g h(y)$ vanishes near q , the derivative of the integral at q equals the integral of the derivative, and hence

$$\left(w, \frac{\partial h}{\partial x}(q)\right) = \int_M b(y) \Delta_g h(y) dy = 0$$

since b is constant. This contradiction proves the lemma. \square

Proof of Lemma 3.1

Fix $\epsilon > 0$ such that the Hamiltonian flow of every closed 1-form λ with C^1 - g -norm $< 2\epsilon$ has no nonconstant 1-periodic orbits. Fix any nontrivial class of norm $< \epsilon$. Take a very small perturbation g' of the metric g such that the g' -harmonic form λ in this class is Morse and has C^1 - g -norm less than 2ϵ . Notice that a harmonic Morse form has no critical points of index 0 and $2n$ by the maximum principle. This completes the proof. \square

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