Topological properties of Hamiltonian circle actions

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April 19, 2004

Abstract

We study conditions under which a circle action on a symplectic manifold gives rise to an essential (i.e. noncontractible) loop in the group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms. A fixed point $x$ of an $S^1$-action is called simple if the weights of the linearized action on $T_x M$ are 0 or $\pm 1$. We show that a circle subgroup that has a simple fixed point $x$ and that is inessential in a semisimple Lie subgroup $G$ of the diffeomorphism group is reversed by an element of $G$ that fixes the point $x$. We prove a homological analog of this result for Hamiltonian $S^1$-actions that are inessential in $\text{Symp}(M, \omega)$ and have no weights of absolute value $\geq 3$. We show that a Hamiltonian circle action is essential if the weights at the points at which the moment map is a maximum are sufficiently small. As a consequence, we show that if $(M, \omega)$ is a coadjoint orbit of $G$ then $\pi_1(G)$ injects into $\pi_1(\text{Ham}(M, \omega))$, thus answering a question raised by Weinstein. The results on Lie group actions are proved by a case by case study of the structure of semisimple Lie algebras, while the results on the Hamiltonian group use properties of the Seidel representation of $\pi_1(\text{Ham}(M, \omega))$ in the units of the quantum homology ring. We use our results about this representation to give a description of the quantum cohomology ring of an arbitrary toric manifold, that extends Givental’s calculation in the NEF case. There are two important technical ingredients; one relates the equivariant cohomology of $M$ to the Morse flow of the moment map, and the other is a version of the localization principle for calculating Gromov–Witten invariants on symplectic manifolds with $S^1$-actions.

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\textsuperscript{*}Partially supported by NSF grants DMS 0072512 and DMS 0305939.

\textsuperscript{†}Partially supported by NSF grant DMS 0204448
1 Introduction

This paper is an attempt to understand when a circle action on a symplectic manifold \((M,\omega)\) gives rise to a noncontractible loop in the symplectomorphism group \(\text{Symp}(M,\omega)\). A background motivation is to discover the extent to which \(\text{Symp}(M,\omega)\) behaves homotopically like a compact Lie group. Therefore, this paper also investigates circle subgroups of compact Lie group actions and tries to extend the results obtained to circle subgroups of \(\text{Symp}(M,\omega)\). (We shall always assume that our circles act nontrivially.)

We shall say that a circle subgroup \(\Lambda\) of a topological group \(H\) is essential in \(H\) if it represents a nonzero element in \(\pi_1(H)\) and inessential in \(H\) otherwise. The basic problem is therefore to find ways of recognizing essential circle subgroups. Our starting point is the following theorem of McDuff–Slimowitz [16].

Recall that a circle action is semifree if the stabilizer subgroup of each point \(x \in M\) is either \(S^1\) itself or the trivial group.

**Theorem 1.1** Any semifree circle action on a closed symplectic manifold is essential in \(\text{Symp}(M,\omega)\).

This is obvious if the action is not Hamiltonian since in this case the flux homomorphism

\[
\text{Flux} : \pi_1(\text{Symp}(M,\omega)) \longrightarrow H^1(M,\mathbb{R})
\]

does not vanish on \(\Lambda\). However, if the action is Hamiltonian with generating Hamiltonian \(K : M \longrightarrow \mathbb{R}\) then the result is not so easy: the proof in [16] involved studying the Hofer length of the corresponding paths \(\phi^t_k, t \in [0, T]\), in \(\text{Ham}(M,\omega)\). Another simpler proof due to Seidel is described in §2.3 below.

One immediate consequence of Theorem 1.1 is the following corollary that answers a question posed by Weinstein in [23].
Corollary 1.2 Let \((M,\omega)\) be a coadjoint orbit of a compact semisimple Lie group \(G\) on which \(G\) acts effectively. Then the inclusion \(G \rightarrow \text{Ham}(M,\omega)\) induces an injection on \(\pi_1\).

We prove this by showing that if \(G\) is a compact semisimple Lie group which acts effectively on a coadjoint orbit \((M,\omega)\), then every nonzero element of \(\pi_1(G)\) has a representative that acts semifreely on \(M\): see §5.3. In [22] Vina established a special case of this result by quite different methods.

The analog of Theorem 1.1 does not hold in the smooth (non-symplectic) category. For example, Claude LeBrun pointed out to us that the circle action on \(S^4\) induced by the diagonal action of \(S^1\) on \(\mathbb{C}^2 = \mathbb{R}^4 \subset \mathbb{R}^5\) is semifree but gives a nullhomotopic loop since \(\pi_1(\text{SO}(5)) = \mathbb{Z}/2\mathbb{Z}\). Nevertheless, the semifree condition does have consequences in the smooth category. In fact these consequences hold if there is a component of the fixed point set with a neighborhood on which the \(S^1\)-action is semifree. We shall say that such fixed point components are \textit{simple}. The proof of the following result is given in §6.

Theorem 1.3 Let a compact Lie group \(G\) act on a connected manifold \(M\) so that only a finite number of elements of \(G\) act trivially on \(M\). Let \(\Lambda \subset G\) be a circle subgroup that is inessential in \(G\). Then any simple component \(F\) of the fixed point set of \(\Lambda\) is symmetric, in the sense that there is an element \(g \in G\) whose action on \(M\) fixes \(F\) pointwise and which reverses \(\Lambda\).

Here, we say that \(g \in G\) \textit{reverses} a circle subgroup \(\Lambda \subset G\) if \(g t g^{-1} = t^{-1}\) for all \(t \in \Lambda\). Note that in many cases (e.g. \(G = \text{SO}(3)\)) any circle in \(G\) can be reversed by some \(g \in G\). In such a case the content of the above theorem is that the reversor \(g\) can be chosen to fix \(F\). Also it is important in the above theorem that \(\Lambda\) be inessential. Otherwise the result would apply to the semifree action of \(S^1 \subset \text{PU}(3) = G\) on \(\mathbb{C}P^2\) given by \([z_0 : z_1 : z_2] \mapsto [e^{2\pi i t} z_0 : z_1 : z_2]\). This has order 3 in \(\pi_1(G)\), while if it could be reversed by some \(g \in G\) it would have order 2.

1.1 Results on the Hamiltonian group

In the case of smooth circle actions, it is difficult to tell by looking at a fixed point component \(F\) whether or not there is a diffeomorphism that fixes \(F\) but reverses the action. However, there is more structure if the circle acts in a Hamiltonian way on a compact symplectic manifold \((M,\omega)\). We begin by explaining some basic facts about such actions.

Let \(K : M \rightarrow \mathbb{R}\) be the moment map, i.e. the generating vector field is \(X_K\) where \(\omega(X_K,\cdot) = -dK\). Unless explicit mention is made to the contrary, we assume that \(K\) is normalized so that the mean \(\int_M K\omega^n\) is zero (here \(2n = \dim M\)). Let \(J\) be an \(S^1\)-invariant \(\omega\)-compatible almost complex structure and \(\bar{F}\) be a component of the fixed point set \(M^{S^1}\). The linearized action on the tangent space \((T_xM, J)\) at a point \(x \in \bar{F}\) decomposes into a sum of 1-dimensional actions with weights \(m_1, \ldots, m_n\). These weights are independent of the choice of \(J\). Let \(m(F)\) denote their sum.

Theorem 1.3 has the following immediate consequence.

Corollary 1.4 Consider a Hamiltonian circle action \(\Lambda_K\) on a compact connected symplectic manifold \((M,\omega)\) with moment map \(K\). Let \(F\) be a simple component of the fixed point set. Let \(G \subset \text{Ham}(M,\omega)\) be a compact Lie group which contains \(\Lambda_K\). If \(\Lambda_K\) is inessential in \(G\), then \(K(F) = m(F) = 0\).

Proof: Theorem 1.3 implies that there is \(g \in G\) which fixes \(F\) but reverses \(\Lambda_K\). Then \(-K = K \circ g\), which immediately implies \(K(F) = m(F) = 0\).

It is natural to wonder whether this statement extends from such Lie subgroups to \(\text{Ham}(M)\) itself. Equivalently, one can ask the following question.
Proposition 1.7 Consider a Hamiltonian circle action $\Lambda_K$ with moment map $K$, and let $F$ be a simple component of the fixed point set. If $\Lambda_K$ is inessential in $\text{Ham}(M, \omega)$, then is $K(F) = m(F) = 0$? Are there other traces of a symplectomorphism which fixes $F$ and reverses $\Lambda_K$?

We have only managed to answer this question in a few special cases. Our most general result rules out extremal simple fixed point components.

Proposition 1.5 Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$. Assume that $\Lambda_K$ is inessential in $\text{Ham}(M, \omega)$. If $F$ is a simple component of the fixed point set, then $F$ cannot be the maximal or minimal fixed point component.

One special case where we can answer our question is the case of symplectic toric manifolds. Suppose that an $n$ dimensional torus $T$ acts on a $2n$ dimensional compact symplectic manifold $(M, \omega)$ with mean normalized moment map $\Phi : M \rightarrow \mathfrak{t}^*$. We call the triple $(M, \omega, \Phi)$ a symplectic toric manifold. The moment image is the polytope $\Delta = \Phi(M) \subset \mathfrak{t}^*$. Given a face $f \subset \Delta$ of dimension $k$, the preimage $\Phi^{-1}(f)$ represents a homology class on $M$ of degree $2k$. If $\Lambda$ is any circle in $T$, the moment map $K$ for $\Lambda$ is the composite of $\Phi$ with the associated projection.

Proposition 1.6 Fix a symplectic toric manifold $(M, \omega, \Phi)$ with moment image $\Delta$. Consider a circle subgroup $\Lambda_K \subset T$ which is inessential in $\text{Ham}(M, \omega)$. Let $F$ be a simple fixed point component for this circle action. Then $K(F) = m(F) = 0$. Moreover, let $f^+$ and $f^-$ be the largest faces of $\Delta$ whose minimum and maximum, respectively, are $\Phi(F)$. Then, $[\Phi^{-1}(f^+)] = [\Phi^{-1}(f^-)] \in H_* (M)$.

Our arguments can also deal with an intermediate simple fixed point component – and in fact give quite a bit more suggestive evidence – in the case when each point $x \in M$ that is not fixed by the action has isotropy (or stabilizer) subgroup of order at most two. In this case, we say that the action has at most twofold isotropy. This hypothesis implies that every point with nontrivial isotropy is contained in the symplectic submanifold $M^{2/2}$ of points fixed by $-1$.

As we shall see later (see Proposition 6.2) there is an analog of Theorem 1.3, which states that in the symplectic category if an inessential circle $\Lambda_K \subset G$ has at most twofold isotropy then there exists $g \in G$ which reverses $\Lambda_K$, even if there are no simple fixed points. This leads us naturally to the proposition below that describes some symmetry properties of the fixed point set.

Proposition 1.7 Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$ with moment map $K$. Assume that $\Lambda_K$ is inessential in $\text{Ham}(M, \omega)$ and that $\Lambda_K$ has at most twofold isotropy.

- For any connected component $N \subset M^{2/2}$, any integer $j \geq 0$, and any $\mu \in \mathbb{R}$,
  $$\bigoplus_{F \subset N^{2/1} \cap K^{-1}(\mu)} H_{j-\alpha_F}(F) = \bigoplus_{F' \subset N^{2/1} \cap K^{-1}(-\mu)} H_{j-\beta_{F'}}(F'),$$

  where the sum is over fixed components, $\alpha_F$ is the index of $F$ with respect to $K$, and $\beta_{F'}$ is the index of $F'$ with respect to $-K$.

- In particular, if $F$ is a simple fixed component, then $K(F) = 0$ and $m(F) = 0$.

- For all $\mu \in \mathbb{R}$, a homology class of $M$ can be represented in the sublevel set $K^{-1}((-\infty, \mu))$ if and only if it can be represented in the superlevel set $K^{-1}((-\mu, \infty))$.
Notice that the second claim follows immediately from the first, since every simple fixed component $F$ forms a connected component of $M^{2\mathbb{Z}(2)}$ with $m(F) = \beta_F - \alpha_F$.

These results generalize to fixed components $F$ which, although they are not simple, are almost simple in the following sense. Assume that the positive weights along $QH^*$ is a lift of $A$ and use them to deduce Propositions 1.5, 1.6, 1.7.

For a related generalization of Proposition 1.6, see Remark 1.12.

**Proposition 1.8** Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$ with moment map $K$. Assume that $\Lambda_K$ is inessential in $\text{Ham}(M, \omega)$. Let $F$ be an almost simple fixed component.

- If every point in the levels above $K(F)$ has at most twofold isotropy, then $K(F) = m(F) = 0$. In particular, $F$ cannot be the maximal or minimal fixed component.

- If the action has at most twofold isotropy on all of $M$ then $F$ is simple.

For a related generalization of Proposition 1.6, see Remark 1.12.

### 1.2 Results on the Seidel representation

The results in the previous section are based on a study of the Seidel representation for circle actions. In this section we state our main results on this representation and use them to deduce Propositions 1.5, 1.6, 1.7, and 1.8.

The Seidel representation is a quantum version of a classical homomorphism defined by Weinstein [23] using the action functional. Let $H^2(M) := H^2(M; \mathbb{Z})$ denote the spherical homology of $M$. Let $I_\omega, I_c : H^2(M) \longrightarrow \mathbb{R}$ denote the homomorphisms induced by evaluating the classes $[\omega]$ and $c_1 = c_1(TM) \in H^2(M, \mathbb{Z})$. Weinstein’s homomorphism $A_\omega : \pi_1(\text{Ham}(M)) \longrightarrow \mathbb{R}/(\text{im} I_\omega, I_c)$ takes the circle $\Lambda_K$ to the value $K(x)$ of the generating moment map at any critical point. As we show in §2.3 this extends to take the weights into account.

**Lemma 1.9** Let $(M, \omega)$ be a compact symplectic manifold. There is a homomorphism

$$A_{\omega, c} : \pi_1(\text{Ham}(M, \omega)) \longrightarrow \mathbb{R} \oplus \mathbb{Z}/\text{im}(I_\omega \oplus I_c)$$

whose value at a Hamiltonian circle action $\Lambda_K$ is $[K(x), -m(x)]$, where $x$ is any critical point of $K$.

The Seidel representation

$$S : \pi_1(\text{Ham}(M, \omega)) \longrightarrow \text{QH}_{\text{ev}}(M; \Lambda)^\times$$

is a lift of $A_{\omega, c}$ to the group of even units $\text{QH}_{\omega}(M; \Lambda)^\times$ of the quantum homology ring $\text{QH}_\omega(M) = \text{QH}_\omega(M; \Lambda) := H_\omega(M) \otimes \Lambda$ of $M$: see [20, 7, 11]. Here, following [14], we use coefficients $\Lambda := \Lambda_{\text{univ}}[q, q^{-1}]$ where $q$ is a variable of degree 2 and $\Lambda_{\text{univ}}$ is a generalized Laurent series ring in a variable $t$ of degree 0:

$$\Lambda_{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$
We shall order the elements $\sum_{d,\kappa} a_{d,\kappa} \otimes q^d t^\kappa$ in $\text{QH}_*(M; \Lambda)$ by the valuation\footnote{One must treat this ordering with some care. Although $v(a \ast b) \leq v(a) + v(b)$ for all $a, b \in \text{QH}_*(M)$ with equality only if the usual intersection product of the highest order terms is nonzero, in the case when this intersection product is zero the term of highest order in $a \ast b$ may not be equal to the product of the highest order terms in $a$ and $b$.}

$$v\left(\sum_{d,\kappa} a_{d,\kappa} \otimes q^d t^\kappa\right) = \max\{\kappa \mid \exists d : a_{d,\kappa} \neq 0\}.$$ 

For more details see §2.3.

The image $S(\Lambda) \in \text{QH}_*(M; \Lambda)^\times$ of the Hamiltonian loop $\Lambda$ is called the Seidel element of $\Lambda$. It has degree $\dim M$ and gives rise to a degree preserving automorphism of $\text{QH}_*(M)$ by quantum multiplication:

$$S(\Lambda)(a) := S(\Lambda) \ast a.$$ 

Thus $S(\Lambda) = S(\Lambda)(1)$ where $1$ denotes the unit $[M]$ in $\text{QH}_*(M)$. Our results are based on a partial calculation of $S(\Lambda_K)$. By analogy with the algebraic case, we say that the almost complex manifold $(M, J)$ is Fano (resp. NEF) if there are no $J$-holomorphic spheres in classes $B$ with $c_1(B) \leq 0$ (resp. $c_1(B) < 0$).

The following theorem is proved in section 3.2.

**Theorem 1.10** Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$ with normalized moment map $K$. Assume that the maximal fixed point component $F_{\text{max}}$ is simple, and define $K_{\text{max}} := K(F_{\text{max}})$ and $m_{\text{max}} := m(F_{\text{max}})$. Then:

(i) $$S(\Lambda_K) = [F_{\text{max}}] \otimes q^{-m_{\text{max}}} t^{K_{\text{max}}} + \sum_{B \in H_2^S(M), \omega(B) > 0} a_B \otimes q^{-m_{\text{max}} - c_1(B)} t^{K_{\text{max}} - \omega(B)},$$

where $a_B$ is the contribution from the section class $\sigma_{\text{max}} + B$. Moreover, if $a_B \neq 0$ then $\deg(a_B) = \dim F_{\text{max}} + 2c_1(B)$.

(ii) If $(M, J)$ is Fano (resp. NEF) for some $S^1$-invariant $\omega$-compatible almost complex structure $J$ then $a_B = 0$ unless $c_1(B) > 0$ (resp. $c_1(B) \geq 0$).

(iii) Assume that $(M, J)$ is NEF for some $S^1$-invariant $\omega$-compatible almost complex structure $J$. If $2c_1(B') \geq \text{codim } F_{\text{max}}$ for every $J$-holomorphic sphere $B'$ which intersects $F_{\text{max}}$ then all the lower order terms vanish. If the latter hypothesis holds except for spheres which lie in $F_{\text{max}}$ itself, then $a_B = 0$ unless $2c_1(B) < \text{codim } F_{\text{max}}$ and $B$ lies in the image of $H_2^S(F_{\text{max}})$.

The last sentence in part (i) of Theorem 1.10 expresses the fact that $S(\Lambda_K)$ preserves degree. If $a_B \neq 0$ then

$$\deg(a_B \otimes q^{-m_{\text{max}} - c_1(B)} t^{K_{\text{max}} - \omega(B)}) = \deg a_B - 2m(F_{\text{max}}) - 2c_1(B) = \dim M.$$ 

Moreover, $\dim F_{\text{max}} = \dim M + 2m(F_{\text{max}})$ because $F_{\text{max}}$ is simple. Therefore

$$\deg a_B = \dim F_{\text{max}} + 2c_1(B).$$ 

This theorem gives the most information when $\text{codim } F_{\text{max}} = 2$, for example in the case of a circle action on a toric variety that fixes one facet. If, in addition $(M, \omega, J)$ is Fano for some $S^1$-invariant $\omega$-compatible $J$, then, by part (iii), all the lower order terms vanish. That is, $S(\Lambda_K) = [F_{\text{max}}] \otimes q^{-m_{\text{max}}} t^{K_{\text{max}}}$. In §3.2 we shall give a more precise description of the lower order terms in $S(\Lambda)$. These remarks have consequences for the structure of the quantum cohomology of toric manifolds that are explained in §5.1.
Example 1.11 Consider the rotation of $S^2$ generated by the height function $K$ and let $A = [S^2]$. Then $\mathcal{S}(\Lambda_K) = [pt] \otimes qt^e(A)/2$.

Proof of Proposition 1.5. If $\Lambda_K$ is inessential in $\text{Ham}(M,\omega)$, then $\mathcal{S}(\Lambda_K) = \mathbb{I}$. Hence, Proposition 1.5 is an immediate consequence of the first claim of Theorem 1.10. $\square$

Proof of Proposition 1.6. Pick any $x \in F$. There is a neighborhood of $x$ and an isomorphism of $T$ with $(S^1)^n$ so that the action of $T$ is equivariantly symplectomorphic to the standard action of $(S^1)^n$ on $\mathbb{C}^n$. In these coordinates, the action of $\Lambda_K$ on $\mathbb{C}^n$ is given by $\lambda z = (\lambda^{m_1}z_1, \ldots, \lambda^{m_n}z_n)$, where $m_1, \ldots, m_n$ are the weights at $x$.

For $1 \leq i \leq n$, let $D_i$ be the facet of $\Delta$ which corresponds to $z_i = 0$. Let $\eta_i \in \ell$ denote the outward primitive normal vector to $D_i$, where $\ell \subset \mathfrak{l}$ is the integral lattice. Note that $K_i := (\eta_i, \Phi(i))$ is the moment map for a circle action $\Lambda_i$, and that $\Phi^{-1}(D_i)$ is a simple maximum for this action. By Theorem 1.10, $\mathcal{S}(\Lambda_i) = \Phi^{-1}(D_i) + \text{lower order terms}$. Since $\Lambda_K$ is inessential, $\mathcal{S}(\Lambda_K) = \mathbb{I}$. On the other hand, by looking at the action near the fixed point $x$ one sees that $\Lambda_K = \prod \Lambda_i^{-m_i}$. Therefore
\[
\prod \left( \left[ \Phi^{-1}(D_i) \right] \otimes q^{m_i} t^{-m_i} \right),
\]
where the product is taken in $\mathbb{Q}\text{H}_*(M;\Lambda)$.

Let $m_i = 1$ for $1 \leq i \leq r$, $m_i = -1$ for $r < i \leq r + s$, and $m_i = 0, i > r + s$. Then
\[
y_1 \cdots y_r \otimes q^{-r} t^{\eta_i(D_i) + \cdots + \eta_i(D_r)} = y_{r+1} \cdots y_{r+s} \otimes q^{-s} t^{\eta_{r+1}(D_{r+1}) + \cdots + \eta_{r+s}(D_{r+s})}.
\]
In particular, the highest order terms must agree. Since $D_1 \cap \cdots \cap D_r = f^+$ and $D_{1+r} \cap \cdots \cap D_{r+s} = f^-$, we have $[\Phi^{-1}(D_1)] \cap \cdots \cap [\Phi^{-1}(D_r)] = [\Phi^{-1}(f^+)]$ and $[\Phi^{-1}(D_{1+r})] \cap \cdots \cap [\Phi^{-1}(D_{r+s})] = [\Phi^{-1}(f^-)]$. Since these intersections are nontrivial, the highest order terms in the quantum product of the corresponding $y_i$ are given by these intersections. The result follows. $\square$

Remark 1.12 More generally, let $F$ be a fixed component of any inessential Hamiltonian loop $\Lambda_K \subset T$. Let $(m_1, \ldots, m_n)$ be the weights at $x \in F$ with corresponding facets $D_i$. Define homology classes in $M$ by $X^+ = \cap_{m_i > 0}[\Phi^{-1}(D_i)]^{m_i}$ and $X^- = \cap_{m_i < 0}[\Phi^{-1}(D_i)]^{-m_i}$. (Here, we are taking the ordinary coproduct product in homology.) If both $X^+$ and $X^-$ are nonzero, then, by an argument similar to the one above, $K(F) = m(F) = 0$ and $X^+ = X^-$.  

1.2.1 Semifree actions and canonical bases for homology
For a general action our methods do not give any information about $\mathcal{S}(\Lambda_K)(a) := \mathcal{S}(\Lambda_K) \ast a$ for $a \neq \mathbb{I}$. However, when the action is semifree, it is possible to describe the top order term in $\mathcal{S}(\Lambda_K)(a)$ for any $a \in \mathbb{H}_*(M)$. This formula is best written in terms of some canonical bases $\{c^-_n\}$ and $\{c^+_n\}$ for $\mathbb{H}_*(M)$.

Before explaining this, we introduce more notation. Given $\mu \in \mathbb{R}$, define
\[
M_\mu := K^{-1}(\mu, \infty), \quad M_{>\mu} := K^{-1}(\mu, \infty), \quad M^\mu := K^{-1}((-\infty, \mu]), \quad M^{<\mu} := K^{-1}((-\infty, \mu))
\]
The inclusions $M^\mu \hookrightarrow M$ and $M_{>\mu} \hookrightarrow M$ induce maps $\mathbb{H}_*(M^\mu) \rightarrow \mathbb{H}_*(M)$ and $\mathbb{H}_*(M_{>\mu}) \rightarrow \mathbb{H}_*(M)$ in rational homology. We call the images of these maps $F_\mu \mathbb{H}_*(M)$ and $F^{>\mu} \mathbb{H}_*(M)$, respectively.

We now give a brief review of equivariant cohomology: Let $S^1$ act on a space $N$. The equivariant cohomology $H^*_S(N)$ of $N$ is defined to be the cohomology of the total space
\[
N_{S^1} := S^\infty \times S^1 N
\]
of the universal \( N \)-bundle over the classifying space \( BS^1 = \mathbb{CP}^\infty \). Thus \( H^*_{S^1}(N) \) is a module over \( H^*_{S^1}(pt) \cong H^*(\mathbb{CP}^\infty) \), which is a polynomial ring with one generator \( u \) of degree 2. Moreover, there is a natural map from \( H^*_{S^1}(N) \) to \( H^*(N) \), given by restricting to any fiber.

If \( S^1 \) acts trivially on \( F \) there is a natural identification \( H^*_{S^1}(F) = H^*_F(pt) \otimes H^*(F) \). Given \( \tilde{Y} \in H^*_F(F) \), we say that the degree of \( \tilde{Y} \) in \( H^*_S \) is \( j \) if \( j \) is the smallest integer such that

\[
\tilde{Y} \in \bigoplus_{i=0}^j H^*_{S^1}(pt) \otimes H^*(F).
\]

We now explain a procedure for producing a natural set of generators for the equivariant cohomology of \( M \) given a set of generators for the cohomology of each fixed component.

**Lemma 1.13** Let \( S^1 \) act on a compact symplectic manifold \((M, \omega)\) with moment map \( K \).

Let \( F \subset M \) be any fixed component of index \( \alpha \); let \( e_F \in H^*_{S^1}(F) \) be the equivariant Euler class of the normal negative bundle to \( F \). Given any cohomology class \( Y \in H^*(F) \), there exists a unique cohomology class \( \tilde{Y}^+ \in H^*_{S^1}(M) \) so that

(a) The restriction of \( \tilde{Y}^+ \) to \( M^{<K(F)} \) vanishes,

(b) \( \tilde{Y}^+|_F = Y \cup e_F \), and

(c) the degree of \( \tilde{Y}^+|_F \) in \( H^*_{S^1}(pt) \) is less than the index \( \alpha_F \) of \( F' \) for all fixed components \( F' \neq F \).

Moreover, these classes generate \( H^*_{S^1}(M) \) as a \( H^*_{S^1}(pt) \) module.

We can use this lemma, which we prove in section 4.1.1, to create a set of generators for the homology of \( M \). Let \( F \) be a fixed component, and let \( \alpha_F \) and \( \beta_F \) denote the index of \( F \) with respect to \( K \) and \( -K \), respectively. Given a homology class \( c \in H_i(F) \), we define the **upwards extension** \( c^+ \in H_{i+\beta_F}(M) \) as follows:

- Let \( Y \in H^{\dim F-i}(F) \) be the Poincaré dual to \( c \).
- Let \( \tilde{Y}^+ \in H^{\dim F+\alpha_F-i}(M) \) be the unique equivariant cohomology class which satisfies the conditions of Lemma 1.13.
- Let \( Y^+ \in H^{\dim F+\alpha_F-i}(M) \) be the restriction of \( \tilde{Y}^+ \) to ordinary cohomology.
- Let \( c^+ \in H_{i+\dim F-i}(M) = H_{i+\beta_F}(M) \) be the Poincaré dual to \( Y^+ \).

Note that, by construction, \( c^+ \) lies in \( F_{K(F)}^{-}H_i(M) \). The **downwards extension** \( c^- \in H_{i+\alpha_F}(M) \), which lies in \( F^K(F)H_i(M) \) is defined analogously; simply replace \( K \) by \( -K \).

Since the classes \( \tilde{Y}^+ \) generate \( H^*_{S^1}(M) \) as a \( H^*_{S^1}(pt) \) module and the restriction \( H^*_{S^1}(M) \rightarrow H^*(M) \) is surjective, the classes \( Y^+ \) generate \( H^*(M) \) as a (rational) vector space. Hence, the classes \( c^+ \) (or, alternatively, the classes \( c^- \)) generate \( H_*(M) \) as a vector space.

When the action is semifree, the classes \( c^+ \) and \( c^- \) have a nice geometric description. Assume that \( c \) can be represented by an \( i \)-dimensional submanifold \( C \subset F \). By Lemma 4.6, if \( g_J \) is the metric associated to a generic \( S^1 \)-invariant \( \omega \)-compatible almost complex structure \( J \) and we choose \( C \) generically, the stable manifold \( W^s(C) \) is an \((i + \beta_F)\)-dimensional pseudocycle. (See section 4.1.2.) Hence, it represents a homology class \([W^s(C)] \in H_{i+\beta_F}(M) \). By Proposition 4.8, \([W^s(C)] = c^+ \). Similarly, \( c^- \) is represented by the unstable manifold \([W^u(C)] \).
Remark 1.14 We may define an automorphism $\mathcal{D}_K: H_*(M) \rightarrow H_*(M)$ by
\[ \mathcal{D}_K(c^+) = c^+, \quad c \in H_*(M^{S^1}). \]
For example, if $c = [F_{\text{max}}] \in H_*(F_{\text{max}})$ is the maximal fixed point set of $K$, then $c^- = \mathbb{I}$ while $c^+ = [F_{\text{max}}]$. Therefore
\[ \mathcal{D}_K(\mathbb{I}) = [F_{\text{max}}]. \]
If $K$ is Morse then $\mathcal{D}_K$ can be interpreted as a form of duality. If $\{c_i\}$ is given by the set of critical points of $K$, then the bases $\{c_i^\pm\} = \{\mathcal{D}_K(c_i^\pm)\}$ are dual with respect to the intersection pairing. Although it is tempting to think that $\mathcal{D}_K$ is an involution, in fact the correct relation is $\mathcal{D}_K \circ \mathcal{D}_K = \mathbb{I}$. □

The following theorem is proved in section 3.3.

**Theorem 1.15** Let $S^1$ act semifreely on a compact symplectic manifold $(M, \omega)$. Let $F$ be a component of the fixed point set, and choose a homology class $c \in H_*(F)$. Then
\[ S(\Lambda_K)(c^-) = c^+ \otimes q^{-m(F)} t^{K(F)} + \sum_{B \in H^2_*(M; \omega(B)>0} a_B \otimes q^{-m(F)-c_1(B)} t^{K(F)-\omega(B)}. \]
Moreover if $a_B \neq 0$ then $\deg a_B = \deg c^+ + 2c_1(B)$.

Since every element $a \in H_*(M)$ can be written as a linear combination of such $c^-$, this theorem gives the leading order term of $S(\Lambda_K)(a)$ for every $a \in H_*(M)$.

The last claim of the theorem follows from the fact that $S$ preserves degree. This implies that if $a_B \neq 0$, then
\[ \deg(a_B) - 2m(F) - 2c_1(B) = \deg(c^-). \]
Since the action is semifree, $m(F)$ is the number of positive weights minus the number of negative weights. But the degree of $c^+$ is the degree of $c$ plus twice the number of positive weights, and the degree of $c^-$ is the degree of $c$ plus twice the number of negative weights. Hence $\deg(a_B) = \deg(c^+) + 2c_1(B)$.

**Example 1.16** Think of $\mathbb{CP}^2$ as the manifold obtained from the closed unit ball in $\mathbb{C}^2$ by identifying its boundary to a complex line via the Hopf map, and consider the action
\[ (z_1, z_2) \mapsto (e^{-2\pi it} z_1, e^{-2\pi it} z_2). \]
Then $K(z_1, z_2) = \pi(c - |z_1|^2 - |z_2|^2)$ where $c = 2/3$, $F_{\text{max}} = \{pt\}$ and all critical points are simple. Since $c_1(L) = 3$ where $L = [\mathbb{CP}^2]$, there can be no lower order terms in the formula for $S(\Lambda_K)(a)$ since the dimensional condition can never be satisfied. Hence, since $\omega(L) := \pi$, we find:
\[ S(\Lambda_K)(\mathbb{I}) = [pt] \otimes q^2 t^{2\pi/3}, \quad S(\Lambda_K)(L) = \mathbb{I} \otimes q^{-1} t^{-\pi/3}, \quad S(\Lambda_K)([pt]) = L \otimes q^{-1} t^{-\pi/3}, \]
which is consistent with the formula $S(\Lambda_K)(a) = S(\Lambda_K) \ast a$. The above results also agree with the formulas found in [15] §4 for rotations of the one point blow up of $\mathbb{CP}^2$: see Example 5.6. In this example we shall also see that although Theorem 1.10 implies that there are no lower order terms in the Seidel element $S_{\Lambda}$ itself if $(M, \omega)$ is Fano and $F_{\text{max}}$ has codimension 2, there may be lower order terms in $S(\Lambda)(a)$ for such actions.
1.2.2 Actions with at most twofold isotropy

We can also obtain some information about $S(\Lambda_K)(c)$, though considerably less than before, when $\Lambda_K$ acts with at most twofold isotropy. Throughout the following discussion we denote by $\cdot Y$ the intersection pairing $H_k(Y) \times H_{m-k}(Y) \to \mathbb{Q}$ on the homology of an oriented $m$-dimensional manifold $Y$. For convenience we set $a \cdot Y b = 0$ whenever the dimensional condition $\deg(a) + \deg(b) = m$ is not satisfied.

The following theorem is proved in section 3.4.

**Theorem 1.17** Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$ with at most twofold isotropy. Let $F$ be a component of the fixed point set. Choose a homology class $c \in H_*(F)$, and write

$$S(\Lambda_K)(c^-) = \sum_{d, \kappa} c_{d, \kappa} \otimes q^d t^\kappa.$$

Then $c_{0,0} \in F_-K(F)H_*(M)$. Moreover, let $F'$ be a fixed component so that $K(F') = -K(F)$ but $F$ and $F'$ lie in different components of $M^Z/2)$. Given any $c' \in H_*(F')$, $c_{0,0} \cdot M (c')^- = 0$.

**Proof of Proposition 1.7.** Let $F$ be any fixed component whose indices with respect to $K$ and $-K$ are $\alpha F$ and $\beta F$, respectively. Let $c \in H_i(F)$ be any homology class, and let $c^- \in H_{i+\alpha F}$ be its downward extension. Since $\Lambda_K$ is inessential, $S(\Lambda_K)(c^-) = c^- \otimes 1 \mathbb{I}$. Hence, by Theorem 1.17 $c^- \in F_-K(F)H_i+\alpha F(M)$.

Moreover, let $F'$ be a fixed component so that $K(F') = -K(F)$ but $F$ and $F'$ lie in different components of $M^Z/2)$. Then, given any $c' \in H_*(F')$, $c' \cdot M (c')^- = 0$.

For any $\mu \in \mathbb{R}$, $F^\mu H_*(M)$ is generated by elements $c^-$, where $c \in H_*(F)$ and $F$ is a fixed component with $K(F) \leq \mu$. The above remarks imply that every such $c^-$ lies in $F_\mu^\mu H_*(M)$. Hence $F^\mu H_*(M) \subseteq F_\mu^\mu H_*(M)$. Similarly, applying the theorem to the moment map $-K$, $F^\mu H_*(M) \geq F_\mu^\mu H_*(M)$. Hence $F^\mu H_*(M) = F_\mu^\mu H_*(M)$. This proves the third statement.

Since both $K$ and $-K$ are perfect Morse functions, both $H_*(M^\mu) \to H_*(M)$ and $H_*(M^-\mu) \to H_*(M)$ are injections. Hence,

$$H_j(M^\mu, M^-\mu) = H_j(M^\mu, M^\mu).$$

By the Thom isomorphism theorem, this is equivalent to

$$\bigoplus_{F \subseteq M^Z \cap K^{-1}(\mu)} H_{j-\alpha F}(F) = \bigoplus_{F' \subseteq M^Z \cap K^{-1}(\mu)} H_{j-\beta F'}(F'),$$

where the sum is over connected components, $\alpha F$ is the index of $F$ with respect to $K$, and $\beta F'$ is the index of $F'$ with respect to $-K$.

Now let $F$ and $F'$ be fixed components which lie in different components of $M^Z/2)$ and satisfy $K(F) = -K(F') = \mu$. Consider $c \in H_*(F)$ and $c' \in H_*(F')$. We saw above that $c^- \cdot M (c')^- = 0$. Hence, the $F'$ component of the image of $c$ under the isomorphism above must be zero. Hence, the isomorphism above is still an isomorphism when restricted to any component of $M^Z/2)$.

Finally, let us consider the contribution of an almost simple component. The following theorem is also proved in §3.4.

**Theorem 1.18** Consider a Hamiltonian circle action $\Lambda_K$ on a compact symplectic manifold. Let $F$ be a fixed component. Assume that all the positive weights at $F$ are $+1$. Suppose further that every point in the levels above $K(F)$ has at most twofold isotropy. Choose a homology class $c \in H_*(F)$, and write

$$S(\Lambda_K)(c^-) = \sum_{d, \kappa} c_{d, \kappa} \otimes q^{-m(F)+d} \delta(K(F)+\kappa).$$
Then \( c_{0,0} \in F_{K(F)}H_*(M) \). Moreover, for any \( c' \in H_*(F) \),
\[
c_{0,0} \cdot_M (c')^\cdot = (e(E) \cap_F c) \cdot_F c'
\]
where \( e(E) \) denotes the Poincaré dual of the Euler class of the obstruction bundle \( E \to F \). (See equation (1).)

**Proof of Proposition 1.8.** Let \( F \) be an almost simple fixed component. Apply Theorem 1.18 with \( c = 1_F \in H_*(F) \) and \( c' \in H_*(F) \) chosen so that \( e(E) \cdot_F c' = k \neq 0 \). Then \( c_{0,0} \cdot_M (c')^\cdot = e(E) \cdot_F c' = k \neq 0 \). Therefore the coefficient \( c_{0,0} \) of \( q^{-m(F)} \) in \( S(\Lambda_F) \cdot (1_F)^\cdot \) is nonzero. Since \( \Lambda_F \) is inessential, \( S(\Lambda_F) \cdot (1_F)^\cdot = 0 \). Therefore \( c_{0,0} = 0 \) and \( K(F) = m(F) = 0 \). This proves the first claim. To prove the second claim, consider the component \( N \) of the isotropy submanifold \( M^{22} \) that contains \( F \). By Proposition 1.7, \( N \) is placed symmetrically with respect to \( K \), that is the maximum of \( K \) on \( N \) equals the negative of the minimum of \( K \) on \( N \). But if \( F \) is almost simple, all its positive weights are \( +1 \). Hence the maximum value of \( K \) on \( N \) is \( K(F) \). But we have just seen that \( K(F) = 0 \). It follows that \( N = F \), in other words, \( F \) is simple. \( \square \)

**Acknowledgements** The first author thanks Paul Seidel and Yong-Geun Oh for useful conversations and the Ellentuck Foundation and the Institute for Advanced Study for their generous support during Spring 2002.

## 2 Quantum homology and the Seidel representation

This section reviews the necessary background material. The main geometric idea behind our results, symplectic bundles over the two sphere, is explained in §2.1. We review (small) quantum homology in §2.2 to fix notational conventions, and then describe the Seidel representation in §2.3.

### 2.1 Symplectic bundles over the two sphere

Throughout we shall use the following notational/sign conventions. If \( H_t, 0 \leq t \leq 1 \), is a (time dependent) Hamiltonian then we define the corresponding vector field \( X_H \) by the identity
\[
\omega(X_H, \cdot) = -dH_t.
\]

Thus \( X_H = J(\text{grad} H_t) \), where \( J \) is an \( \omega \)-compatible almost complex structure and the gradient is taken with respect to the metric \( g_J \) given by \( g_J(x,y) = \omega(x, Jy) \). As an example, consider the unit sphere \( S^2 \) in \( \mathbb{R}^3 \), oriented via stereographic projection from the north pole.\(^2\) Then its area form is \( dx_3 \wedge d\theta \) and the vector field \( X_K \) generated by the normalized height function \( K = 2\pi x_3 \) is \( X_K = 2\pi \partial_{\theta} \). Thus the corresponding flow is the anticlockwise rotation of \( S^2 \) about the axis from the south to the north pole. Note that this flow is positive (i.e. anticlockwise) at the south pole \( s \) (the minimum of \( K \)) and negative at the north pole \( n \) (the maximum of \( K \)), which agrees with the usual conventions for defining the **moment map**.

Consider the locally trivial bundle \( P_\Lambda \to S^2 \) constructed by using \( \Lambda = \{ \phi_t \} \in \pi_1(\text{Ham}(M)) \) as a clutching function:
\[
P_\Lambda = (D_0 \times M) / \sim, \quad \text{where} \quad (e^{2\pi it}, \phi_t(x))_0 \sim (e^{2\pi it}, x)_{\infty}.
\]

\(^2\)This means that the vertical projection from the tangent space at the south pole to the \((x_1, x_2)\)-plane preserves orientation. Hence this orientation is the opposite of its orientation as the boundary of the unit ball.
Here we are thinking of $D_0$ as the closed unit disc centered at 0 in the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and of $D_\infty$ as another copy of this disc, embedded in $S^2 = \mathbb{C} \cup \{\infty\}$ via the orientation reversing map $re^{i\theta} \mapsto r^{-1}e^{i\theta}$. Correspondingly, we denote the fibers over $0, \infty$ by $M_0, M_\infty$. Note that our definition of $P_\Lambda$ agrees with that in [15] but differs in orientation from the convention used in [7, 11].

The fact that $\Lambda$ is Hamiltonian implies that there is a closed 2-form $\Omega$ on $P_\Lambda$ extending the fiberwise symplectic forms: see [20] or [13] Chapter 6 for example. Conversely, every pair consisting of a smooth bundle $\pi : P \longrightarrow S^2$ with fiber $M$ together with a closed 2-form $\Omega$ on $P$ that is nondegenerate on each fiber arises in this way from a loop in $\text{Ham}(M, \omega)$. By adding to $\Omega$ the pullback of a suitable area form on the base, we may assume that $\Omega$ is nondegenerate. Any such symplectic extension of the fiberwise forms will be called $\omega$-compatible. The set of these forms is contractible. Note that each such $\Omega$ gives rise to a connection on $P$ with Hamiltonian holonomy, whose horizontal distribution consists of the $\Omega$-orthonormals to the fibers.

Each such triple $(P, \pi, \Omega)$ admits a contractible family $\mathcal{J}(P, \pi, \Omega)$ of $\Omega$-compatible almost complex structures $\tilde{J}$ such that $\pi : (P, \tilde{J}) \longrightarrow (S^2, J_0)$ is holomorphic. Each $\tilde{J} \in \mathcal{J}(P, \pi, \Omega)$ preserves the tangent bundle to the fibers and hence also the horizontal distribution.

Now observe that the bundle $(P_\Lambda, \Omega) \longrightarrow S^2$ supports two canonical cohomology classes. The first is the first Chern class of the vertical tangent bundle

$$c_{\text{vert}} = c_1(TP_\Lambda^{\text{vert}}) \in H^2(P_\Lambda, \mathbb{Z}).$$

The second is the coupling class, which is the unique class $u_\Lambda \in H^2(P_\Lambda, \mathbb{R})$ such that

$$i^*(u_\Lambda) = [\omega], \quad u_\Lambda^{n+1} = 0,$$

where $i : M \longrightarrow P_\Lambda$ is the inclusion of a fiber.

Another important geometric fact about Hamiltonian bundles over $S^2$ is that they always have sections. A direct geometric argument shows that this is equivalent to saying that the map

$$\pi_1(\text{Ham}(M, \omega)) \longrightarrow \pi_1(M, x) : \{\phi_t\} \mapsto \{\phi_t(x)\}$$

given by evaluation at the base point $x$ is trivial. The latter statement follows from the proof of the Arnol’d conjecture or by the very existence of the Seidel representation: see [7]. Therefore, in particular, there always exists a section class, that is, a class $\sigma \in H^2_\text{sec}(P, \mathbb{R})$ that projects onto the positive generator of $H_2(S^2, \mathbb{Z})$. We shall denote by $H^2_\text{sec}(P, \mathbb{Z})$ the affine subspace of $H^2_\text{sec}(P, \mathbb{Z})$ consisting of such section classes.

Circle actions

We now assume that $\Lambda_K$ is a circle action with moment map $K$, and show how the ideas above simplify in this case. Let $S^1$ act on $S^3 \times M$ by the diagonal action

$$(z_1, z_2; \phi_t) \mapsto (e^{2\pi it}z_1, e^{2\pi it}z_2; \phi_t x).$$

We claim that $P_\Lambda$ can be identified with the quotient $S^3 \times S^1 / M$. To see this, write $[z_1, z_2; x]$ for the equivalence class containing the point $(z_1/r, z_2/r; x) \in S^3 \times M$, where $r^2 = |z_1|^2 + |z_2|^2$. In these coordinates,

$$D_0 \times M = \{[z, 1; x] : |z| \leq 1, \ x \in M\} \quad \text{and} \quad D_\infty \times M = \{[1, z; x] : |z| \leq 1, \ x \in M\}.$$

In particular, $M_0$ is the fiber at $[0 : 1] \in \mathbb{CP}^1 = S^3/S^1$, and $M_\infty$ is the fiber at $[1 : 0]$. Both have natural identifications with $M$. Since the orientation on $D_\infty$ was reversed, the gluing map is given by

$$[1, e^{-2\pi it}; x] \sim [e^{2\pi it}, 1; \phi_t x],$$

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as required.

Let \( \alpha \in \Omega^1(S^3) \) be the usual contact form on the unit sphere, normalized so that \( d\alpha = \chi^*(\tau) \) where \( \chi : S^3 \to S^2 \) is the Hopf map and \( \tau \) is the standard area form on \( S^2 \) with total area 1. Given any \( c \in \mathbb{R} \), the form \( \omega + c\,d\alpha - d(K\alpha) \in \Omega^2(S^3 \times M) \) is closed and basic, and hence descends under the projection \( \text{pr} : S^3 \times M \to S^3 \times_{\mathbb{S}^1} M \) to a closed two form \( \Omega_c \) on \( P_\Lambda \) which extends the fiberwise symplectic form. Thus,

\[
\Omega_c = \text{pr}_*(\omega + d((c - K)\alpha)) = \text{pr}_*(\omega + c\,d\alpha - d(K\alpha)).
\]

If \( c > \max K \), then \( \Omega_c \) is symplectic. The coupling class is simply \( [\Omega_0] \):

\[
u_\Lambda = [\Omega_0] = [\text{pr}_*(\omega - d(K\alpha))].\] (6)

**Remark 2.1** Note that, if \( c_1 \) and \( [\omega] \) are linearly dependent, then, since the \( S^1 \)-orbit of an arc going from the minimum to the maximum of \( K \) is a sphere on which both \( \omega \) and \( c_1 \) are positive, \( (M, \omega) \) must be monotone, that is, \( I_c = \mu I_\omega \) for some \( \mu > 0 \).

Each fixed point \( x \) of the \( S^1 \)-action gives rise to a section of \( P \)

\[
\sigma_x := S^3 \times_{\mathbb{S}^1} \{x\} = D_0 \times \{x\} \cup D_\infty \times \{x\}.
\]

We will sometimes write \( \sigma_c \) or \( \sigma_{\max} \) instead of \( \sigma_x \), when \( x \in F \) or \( x \in F_{\max} \), respectively. Here are some useful facts about these sections.

**Lemma 2.2** If \( x \) is a fixed point of a Hamiltonian circle action \( \Lambda_K \) on a symplectic manifold \( (M, \omega) \), then

\[
c_{\text{vert}}(\sigma_x) = m(x) \quad \text{and} \quad u_\Lambda(\sigma_x) = -K(x).
\]

Moreover, if \( B \) is the class of the sphere formed by the \( \Lambda \)-orbit of an arc from \( x \) to another fixed point \( y \), then \( B = \sigma_x - \sigma_y \).

**Proof:** The normal bundle of \( \sigma_x \) can be identified with a sum of holomorphic line bundles \( L_i \to \mathbb{CP}^1 \), one for each weight \( m_i \) at \( x \). Moreover, \( c_1(L_i) = m_i \). Thus \( c_{\text{vert}}(\sigma_x) = m(x) \). Further, by Equation (6) \( u_\Lambda(\sigma_x) = [\omega - d(K\alpha)](\sigma_x) = -K(x) \). This proves the first claim.

Using the sign conventions explained at the beginning of §2.1, one finds by an easy calculation that \( \omega(B) = K(y) - K(x) = u_\Lambda(\sigma_x - \sigma_y) = \omega(\sigma_x - \sigma_y) \). This identity holds for all \( \Lambda \)-invariant symplectic forms \( \omega' \) on \( M \). But, after averaging, any closed 2-form sufficiently close to \( \omega \) is a \( \Lambda \)-invariant symplectic form. Hence the classes \( [\omega'] \) fill out an open neighborhood of \( [\omega] \) in \( H^2(M) \), and so \( B = \sigma_x - \sigma_y \). \( \square \)

Let \( J \) be any \( S^1 \)-invariant almost complex structure on \( M \). The standard complex structure \( J_0 \) on \( \mathbb{C}P^2 \) is also \( S^1 \)-invariant (under the diagonal action), and its restriction to \( S^3 \) preserves the contact planes \( \ker \alpha \). Moreover, each vector \( \xi \in T_xP_\Lambda \) can be considered as an equivalence class of vectors on \( T(S^3 \times M) \); each such equivalence class has a unique representative in \( \ker \alpha \oplus TM \) at each point in the \( S^1 \)-orbit \( \text{pr}^{-1}(p) \). Therefore, the product complex structure \( J_0 \times J \) on \( \ker \alpha \oplus TM \) descends to an almost complex structure \( J \) on \( P_\Lambda \). By construction, if \( J \) is compatible with \( \omega \), then \( J \) is compatible with \( \Omega_c \) for all \( c > \max K \).

Moreover, \( J \) preserves the tangent spaces to the fibers, and the section \( \sigma_x \) is holomorphic for all fixed \( x \).

**Definition 2.3** We define \( \mathcal{J}_S(M) \) to be the set of all \( S^1 \)-invariant \( \omega \)-compatible almost complex structures on \( M \), and denote by \( \mathcal{J}_S(P) \) the space of almost complex structures on \( P \) constructed as above from the elements \( J \in \mathcal{J}_S(M) \). Note that \( \mathcal{J}_S(P) \subset \mathcal{J}(P, \pi, \Omega_c) \) for all \( c > \max K \).
2.2 Small quantum homology

We shall work with quantum homology with coefficients in the ring $\Lambda := \Lambda^{\text{univ}}[q, q^{-1}]$ where $q$ is a variable of degree 2 and $\Lambda^{\text{univ}}$ is a generalized Laurent series ring in a variable $t$ of degree 0:

$$\Lambda^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \ #\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

Correspondingly, quantum cohomology has coefficients in the dual ring $\tilde{\Lambda} := \Lambda^{\text{univ}}[q, q^{-1}]$ where $q$ is as before and

$$\tilde{\Lambda}^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \ #\{\kappa < c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

Thus we define

$$\text{QH}_*(M; \Lambda) = H_*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda, \quad \text{QH}^*(M; \tilde{\Lambda}) = H^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \tilde{\Lambda}.$$

These rings are $\mathbb{Z}$-graded in the obvious way:

$$\deg(a \otimes q^d t^c) = \deg(a) + 2d,$$

where $a \in H_*(M)$ or $H^*(M)$. They also have $\mathbb{Z}/2\mathbb{Z}$-gradings in which the even part is strictly commutative; for example,

$$\text{QH}_{\text{ev}} := H_{\text{ev}}(M) \otimes \Lambda, \quad \text{QH}_{\text{odd}} := H_{\text{odd}}(M) \otimes \Lambda.$$

Recall that the quantum intersection product

$$a \ast b \in \text{QH}_{i+j-\dim M}(M; \Lambda), \quad \text{for } a \in H_*(M), b \in H_j(M)$$

is defined as follows:

$$a \ast b = \sum_{B \in H^2(M, \mathbb{Z})} (a \ast b)_B \otimes q^{-c_1(B)} t^{-\omega(B)},$$

where $(a \ast b)_B \in H_{i+j-\dim M+2c_1(B)}(M)$ is defined by the requirement that

$$(a \ast b)_B \cdot M c = \text{GW}_{B, \beta}^M(a, b, c) \quad \text{for all } c \in H_*(M).$$

Here $\text{GW}_{B, \beta}^M(a, b, c) \in \mathbb{Q}$ denotes the Gromov–Witten invariant that counts the number of spheres in $M$ in the class $B$ that meet cycles representing the classes $a, b, c \in H_*(M)$. The product $\ast$ is extended to $\text{QH}_*(M)$ by linearity over $\Lambda$, and is associative. Moreover, it respects the $\mathbb{Z}$-grading.

This product $\ast$ gives $\text{QH}_*(M; \Lambda)$ the structure of a graded commutative ring with unit $1 = [M]$. Further, the invertible elements in $\text{QH}_{\text{ev}}(M; \Lambda)$ form a commutative group $\text{QH}_{\text{ev}}(M; \Lambda)^{\times}$ that acts on $\text{QH}_*(M; \Lambda)$ by quantum multiplication.

We shall work mostly with quantum homology since this is more geometric. However, some examples mention quantum cohomology. The multiplication (quantum cup product) is defined via Poincaré duality: given $\alpha, \beta \in H^*(M)$ with Poincaré duals $a = \text{PD}(\alpha), b = \text{PD}(\beta)$

$$\alpha \ast \beta = \text{PD}(a \ast b) = \sum_{B \in H_2(M, \mathbb{Z})} \text{PD}((a \ast b)_B) \otimes q^{c_1(B)} t^{\omega(B)},$$

Note that the coefficient is $q^{c_1(B)} t^{\omega(B)}$ rather than $q^{-c_1(B)} t^{-\omega(B)}$: in general the Poincaré duality map $\text{PD} : \text{QH}^*(M) \rightarrow \text{QH}_*(M)$ is given by $\text{PD}(\alpha \otimes q^d t^c) = \text{PD}(\alpha) \otimes q^{-d} t^{-\kappa}$. Thus in cohomology we must use the dual $\check{v}$ of the valuation $v$, namely

$$\check{v} \left( \sum_{d, \kappa} a_{d, \kappa} \otimes q^d t^\kappa \right) = \min \{ \kappa \mid \exists d : a_{d, \kappa} \neq 0 \}. \quad (8)$$
2.3 The Seidel representation

In this paper, we will study Hamiltonian loops \( \Lambda \) by examining the geometry of the symplectic bundle \( P_\Lambda \). On the classical level, this can be done by examining the (generalized) Weinstein homomorphism, which we mentioned in Lemma 1.9.

**Proof of Lemma 1.9.** Define the map

\[ A_{c,c} : \pi_1(\text{Ham}(M, \omega)) \to \mathbb{R} \oplus \mathbb{Z}/\text{im}(I_c) \]

by

\[ A_{c,c}(a) = -[u(a), c_{\text{vert}}(a)], \]

where \( c : S^2 \to P_\Lambda \) is any section. Note that this map is well defined. From the construction of the classes \( u(a) \) and \( c_{\text{vert}} \), and from the fact that \( P_{\Lambda_1 + \Lambda_2} \) is the fiber sum \( P_{\Lambda_1} \sharp P_{\Lambda_2} \), it is easy to see that \( A_{c,c} \)

is represented by sections.

Define the map

\[ a_\sigma : \mathbb{R} \to \mathbb{R} \]

by

\[ a_\sigma = \sum_{\sigma \in H^2_{\text{sec}}(P)} a_\sigma \otimes q^{-c_{\text{vert}}(\sigma)} t^{-u_\sigma(\sigma)}, \]

where \( a_\sigma \cdot M e = GW^{P_\Lambda}_\sigma(c) \) for all \( c \in H_*(M) \). Here \( H^2_{\text{sec}}(P) \) denotes the affine subspace of \( H^2(P; \mathbb{Z}) \) that is represented by sections.

Intuitively, \( a_\sigma \)

is represented by the class

\[ e_\text{ev} = (\mathcal{M}_{0,1}(P_\Lambda, \tilde{J}; \sigma)) \cap [M] \]

where \( \mathcal{M}_{0,1}(P_\Lambda, \tilde{J}; \sigma) \) is the moduli space of all \( \tilde{J} \)-holomorphic sections in class \( \sigma \) with one marked point, \( e_\text{ev} \) is the obvious evaluation map to \( P_\Lambda \) and \([M] \) denotes the homology class represented by a fiber; see [11]. This moduli space has formal dimension \( \dim M + 2c_{\text{vert}}(\sigma) + 2 \). We find that \( a_\sigma = 0 \) unless

\[ \deg(a_\sigma \otimes q^{-c_{\text{vert}}(\sigma)}) = \deg(a_\sigma) - 2c_{\text{vert}}(\sigma) = \dim M. \]

Because all dimensions are even, \( S(\Lambda) \)

belongs to the strictly commutative part \( \mathbb{Q}^2_{\text{ev}} \) of \( \mathbb{Q}^2(M) \). Moreover, \( S(\Lambda) \)

is independent of the choice of symplectic extension form \( \Omega \) since all of these are deformation equivalent. It is shown in [11] (using ideas from [20, 7]) that \( S(\Lambda) \) lies in \( \mathbb{Q}^2_{\text{ev}}(M; \Lambda)^{m} \), the group of multiplicative units in the ring \( \mathbb{Q}^2_{\text{ev}}(M) \), and that the correspondence \( S \) induces a group homomorphism

\[ S : \pi_1(\text{Ham}(M, \omega)) \to \mathbb{Q}^2_{\text{ev}}(M; \Lambda)^{m}. \]

It is immediate from the definition that it lifts the Weinstein homomorphism.

It is often useful to identify \( \mathbb{Q}^2_{\text{ev}}(M; \Lambda)^{m} \) with \( \text{Aut}(\mathbb{Q}^2(M; \Lambda)) \), the group of automorphisms of \( \mathbb{Q}^2(M; \Lambda) \) as a right \( \mathbb{Q}^2(M; \Lambda) \)-module, since every such automorphism is determined by its value at \( 1 \). Correspondingly we define

\[ S(\Lambda)(a) := S(\Lambda) \ast a \quad \forall a \in \mathbb{Q}^2_{\text{ev}}(M; \Lambda). \]

Since the Seidel element has degree \( \dim M \), this endomorphism preserves degree.
Definition 2.5 The Seidel representation is the group homomorphism

\[ S : \pi_1(\text{Ham}) \to \text{QH}_{\text{pers}}(M; \Lambda)^{\times} = \text{Aut}(\text{QH}_{\text{pers}}(M; \Lambda)). \]

It is shown in [11] (see also [20, 7, 14]) that

\[ S(\Lambda)(a) = \sum_{\sigma \in H^{\Lambda}_{\text{vert}}(P)} b_\sigma \otimes q^{-c_{\text{vert}}(\sigma)} t^{-u_\Lambda(\sigma)} \]  

(10)

where \( b_\sigma \cdot M \ c = GW^P_{\sigma}(a, c) \) for all \( c \in \text{QH}_1(M) \). Here one should think of \( a \) as represented by a cycle in the fiber \( M_0 \) over the center of the disc \( D_0 \), and \( b_\sigma \) and \( c \) as represented by cycles in the fiber \( M_\infty \) over the center of \( D_\infty \). Then the element \( S(\Lambda) \) induces a ring isomorphism from \( \text{QH}_*(M_0) \) to \( \text{QH}_*(M_\infty) \). Intuitively, the class \( S(\Lambda)(a) \) is represented by the intersection of \( M_\infty \) and the space of all \( J \)-holomorphic sections of \( P_\Lambda \) that meet the cycle in \( M_0 \) which represents \( a \). Since the connection in the bundle \( (P, \Omega) \to S^2 \) provides an identification of \( M_0 \) with \( M_\infty \) that is well defined up to symplectic isotopy, \( S(\Lambda) \) gives rise to a well defined element of \( \text{Aut}(\text{QH}_*(M; \Lambda)) \) as claimed.

Example 2.6 Consider the rotation of the unit sphere \( S^2 \) with \( K = 2\pi x_3 \). Then the fibration \( P_\Lambda \) can be identified with the nontrivial fibration from the one point blow up \( M_\infty \) of \( \mathbb{CP}^2 \) to \( S^2 \). By Lemma 2.2 the section \( \sigma_{\text{max}} \) corresponding to the maximum (the north pole) has normal bundle of Chern number \( m(\mu) = -1 \), and so is the exceptional divisor, while the section \( \sigma_{\text{min}} \) corresponding to the minimum (the south pole) has Chern number 1, and so lies in the class of a line. Since the the Seidel element \( S(\Lambda_K) \) has degree \( \dim M = 4 \), a section \( \sigma \) can only contribute to it if \( 0 \geq 2c_{\text{vert}}(\sigma) = 2c_1(X) \geq -4 \). Therefore, \( \sigma_{\text{max}} \) is the only holomorphic section of \( P_\Lambda \) that can contribute to the Seidel element. It follows easily that \( S(\Lambda_K) = [pt] \otimes q^{t^{2(\Lambda)}} \), as claimed in Example 1.11.

3 Computing the Seidel element

This section contains the main proofs. We begin by calculating the contribution to the Seidel element \( S(\Lambda) \) of the sections \( \sigma_{\text{max}} \) through points on the maximal fixed set \( F_{\text{max}} \). In Proposition 3.3 we show that this is nonzero precisely when \( F_{\text{max}} \) is almost simple. These arguments use easy results on the behavior of \( J \)-holomorphic spheres. To go further, we need a version of the localization theorem: there is a \( T^2 \)-action on the moduli spaces of stable maps and only the invariant elements contribute to \( S(\Lambda) \). This theorem is stated in §3.2. We defer the proof to §4.2, devoting the rest of this section to an investigation of the invariant elements. We first prove Theorem 1.10. Then, in §3.3, we consider the semifree case, and prove Theorem 1.15. Finally, in §3.4, we consider the case where the isotropy is at most twofold, and prove Theorems 1.17 and 1.18.

3.1 The contribution of the maximal fixed set

We begin with some preliminary remarks about \( \tilde{J} \)-holomorphic sections of \( P := P_\Lambda \). Throughout we assume \( J \in \mathcal{J}_S(P_\Lambda) \), the space of almost complex structures on \( P_\Lambda \) that are constructed from \( S^1 \)-invariant almost complex structures on \( M \) using the identification of \( P_\Lambda \) with a quotient of \( S^3 \times M \). See Definition 2.3.

Let \( M_{0,k}(P, \tilde{J}; \sigma) \) denote the space of equivalence classes \([u, z]\) of \( \tilde{J} \)-holomorphic maps \( u : S^2 \to P \) in class \( \sigma \) with \( k \) pairwise distinct marked points \( z := \{z_1, \ldots, z_k\} \). Here, two such pairs \((u, z)\) and \((u', z')\) are equivalent if there is \( \psi \in \text{PSL}(2, \mathbb{C}) \) such that

\[ u' = u \circ \psi, \quad \psi(z'_i) = z_i, \quad i = 1, \ldots, k. \]

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The compactification \( \overline{M}_{0,k}(P, \tilde{J}; \sigma) \) consists of equivalence classes \( \tau = [\Sigma(u), u, z] \) of \( \tilde{J} \)-holomorphic stable maps \( u : \Sigma(u) \to P \) with \( k \) marked points. Here \( \Sigma(u) \) is a union of copies of \( S^2 \) attached via a tree graph, and the equivalence relation is given by all reparametrizations that respect the special points, i.e. the attaching (or nodal) points and the marked points. If \( \tau \) is a section class, each element \( \tau \) in \( \overline{M}_{0,k}(P, \tilde{J}; \sigma) \) projects via \( \pi : P \to S^2 \) to an equivalence class of holomorphic maps \( \pi \circ u : \Sigma(u) \to S^2 \) of total degree 1. Such a map has just one component of degree 1; on all the other components \( \pi \circ u \) is constant. Thus \( \tau \) has a distinguished component that is a section, called the **root**. The other components are mapped into fibers.

We shall be specially interested in the case when \( k = 2 \) and the first marked point is mapped to \( M_0 \), the other to \( M_{\infty} \). In this case, there is a unique chain of spheres joining the component that contains the first marked point \( z_0 \) to the component that contains the second marked point \( z_\infty \); we call the components of this chain the **principle components**. The other spheres are called **bubbles**. The root is always a principle component.

For most of the results in this paper, we will need to look at invariant chains, as described in the next subsection. However, the following observation, due to Seidel\(^4\), allows us to give a simpler argument when we are studying curves in a class \( \sigma_{\max} \) with \( \omega(B) \leq 0 \).

**Lemma 3.1** Consider a Hamiltonian circle action \( \Lambda_K \) on a compact symplectic manifold \((M, \omega)\), and let \( \tilde{J} \in \mathcal{J}_S(P_{\Lambda}) \) be constructed from \( J \in \mathcal{J}_S(M) \). Fix \( B \in H^2_*(M; \mathbb{Z}) \), and consider the moduli space

\[
\overline{M}_{0,0}(P_{\Lambda}, \tilde{J}; \sigma_{\max} + B).
\]

- **If** \( B \neq 0 \) and \( \omega(B) \leq 0 \), **the moduli space is empty**.
- **If** \( B = 0 \), **the moduli space is compact and can be identified with** \( F_{\max} \) **itself**.

**Proof:** The symplectic form \( \Omega_c \) defined in (5) is compatible with \( \tilde{J} \) for any \( c > \max K \). Fix \([z, w] \in P_{\Lambda} = S^3 \times_{S^1} M\). Recall that any non-zero tangent vector \( \xi \in T_{[z,x]} P_{\Lambda} \) can be uniquely represented by a vector \( h + v \in T_{(z,x)} (S^3 \times M) \), where \( h \in \ker \alpha \subset T_z S^3 \) and \( v \in T_x M \). Now

\[
\Omega_c (\xi, \tilde{J}\xi) = (\omega - dK \wedge \alpha + (c - K) d\alpha)(h + v, J_0 h + Jv) = \omega(v, Jv) + (c - K) d\alpha(h, J_0 h) \geq (c - K_{\max}) d\alpha(h, J_0 h) = (c - K_{\max}) \chi^* \tau(\xi, \tilde{J}\xi)
\]

with equality impossible unless \( v = 0 \) and \( K(x) = \max K \). Since \( \chi^* \tau \) is the pullback by the Hopf map of the area form on \( S^2 \) with area 1, it follows that for any \( \tilde{J} \)-holomorphic section \( \sigma \)

\[
\Omega_c(\sigma) \geq c - K_{\max} = \Omega_c(\sigma_{\max}),
\]

with equality occurring exactly if \( \sigma \) is a constant section \( \sigma_x \) for some \( x \in F_{\max} \).

Since every stable map in a section class \( \sigma \) either consists of a section, or is the union of a section with other spheres \( A_i \) which lie in the fibers and satisfy \( \omega(A_i) > 0 \), the only stable maps that represent a section class \( \sigma \) with \( \Omega_c(\sigma) \leq c - \max K \) are the constant sections \( \sigma_x, x \in F_{\max} \). The result follows. \( \square \)

---

\(^4\)Private communication.
Lemma 3.2 Let \( x \) be any fixed point of the \( S^1 \)-action. For each \( \tilde{J} \in \mathcal{J}_S(P_A) \), the \( \tilde{J} \)-holomorphic curve \( \sigma_x \) is regular precisely when the negative weights at \( x \) are all equal to \(-1\).

**Proof:** Recall that \( \sigma_x \) is regular if and only if the linearization \( D_u \) of the corresponding Cauchy–Riemann operator is surjective. When \( J \in \mathcal{J}_S(P_A) \) the normal bundle of \( \sigma_x \) is holomorphic and splits into a sum of line bundles \( \oplus L_i \) that are preserved by \( D_u \). Moreover, \( D_u \) restricts on each \( L_i \) to the usual Dolbeault delbar operator. Thus \( D_u \) is surjective precisely when \( c_1(L_i) \geq -1 \) for all \( i \).

The next proposition generalizes part (i) of Theorem 1.10.

**Proposition 3.3** Consider a Hamiltonian circle action \( \Lambda_K \) on a compact symplectic manifold \((M, \omega)\) with normalized moment map \( K \). Let \( \iota(e(E_{\max})) \in H_*(F_{\max}) \) denote the Poincaré dual of the Euler class of the obstruction bundle at \( F_{\max} \) (see equation (1)), denote the inclusion \( H_*(F_{\max}) \rightarrow H_*(M) \) by \( \iota \), and set \( K_{\max} := K(F_{\max}) \) and \( m_{\max} := m(F_{\max}) \). Then:

\[
S(\Lambda_K) = \iota(e(E_{\max})) \otimes q^{-m_{\max}} \mu_{K_{\max}} + \sum_{B \in H^2(M; \mathbb{Z})_{\omega(B)>0}} a_B \otimes q^{-m_{\max} - c_1(B)} \mu_{K_{\max} - \omega(B)},
\]

where \( a_B \) is the contribution from the section class \( \sigma_{\max} + B \). Fix any \( c \in H_*(M) \). It is enough to show that \( a_0 \cdot_M c = \iota(e(E_{\max})) \cdot_M c \), and that \( a_B \cdot_M c = 0 \) for every nonzero \( B \in H_2^p(M; \mathbb{Z}) \) such that \( \omega(B) \leq 0 \). By definition, \( a_B \cdot_M c = GW_{\sigma_{\max} + B}(c) \). Choose an almost complex structure \( \tilde{J} \in \mathcal{J}_S(P) \). By Lemma 3.1, if \( \omega(B) \leq 0 \) and \( B \neq 0 \) then the moduli space \( M_{0,0}(P_A, \tilde{J}, \sigma_{\max} + B) \) is empty, and so \( a_B \cdot_M c = 0 \), as required. On the other hand, \( M_{0,1}(P_A, \tilde{J}, \sigma_{\max}) \) can be identified with the compact moduli space \( S^2 \times F_{\max} \).

(The \( S^2 \)-factor is the locus of the single marked point.) If \( F_{\max} \) is simple, then Lemma 3.2 implies that \( \sigma_x \) is regular for every \( x \in F_{\max} \). Hence the intersection of the evaluation pseudocycle \( ev : M_{0,1}(P, \tilde{J}, \sigma_{\max}) \rightarrow M_{\infty} \) with any class \( c \) in the fiber \( M_{\infty} \) is precisely \([F_{\max}] \cdot c\). Thus \( a_0 = [F_{\max}] \) in this case.

If any of the negative weights \( -k_i \) at \( F_{\max} \) is less than \(-1\), the elements of the compact manifold \( M := M_{0,1}(P, \tilde{J}, \sigma_{\max}) \) are not regular. Rather, for each \( i \), the cokernel of the restriction \( D_{u_i} : \mathcal{C}^\infty(S^2, E_i) \rightarrow \Omega^{0,1}(S^2, E_i) \) is a vector space of dimension \( \dim E_i \otimes \mathbb{C}^{k_i-1} \), and as \( x \) varies in \( F_{\max} \) these cokernels fit together to form the bundle \( E_i \otimes \mathbb{C}^{k_i-1} \) over \( M \). Thus the total obstruction bundle is the bundle \( \mathcal{E} \rightarrow M \) of equation (1). It follows from the standard theory (see for example [10, §5.3] or [14, Chapter 7.2]) that the regularized moduli space corresponds to the zero set of a generic section of \( \mathcal{E} := E_{\max} \). Therefore \( GW_{\sigma_{\max} + B}(c) = \iota(e(E)) \cdot_M c \) for each \( c \in H_*(M) \), and the result follows.

\[ \square \]

3.2 Invariant beads and chains

In order to understand the moduli spaces of sections in an arbitrary class \( \sigma \) we exploit the fact that \( T^2 \) acts on \((P_A, \Omega, J)\) when \( J \in \mathcal{J}_S(P_A) \). Here the first factor \( S^1 \times \{1\} \) acts on \( P_A \) by rotating the fibers via \( \phi_t \) while the second factor \( \{1\} \times S^1 \) acts by rotating the base as follows:

\[
\theta : [z, 1; x] = [e^{2\pi i\theta} z, 1; \phi_t x], \quad \theta : [1, z; x] = [1, e^{-2\pi i\theta} z; x] .
\]
Note that the only points of $P_\lambda$ fixed by the whole group are the points in $M_0$ and $M_\infty$ that are fixed by the original $S^1$-action $\phi_t$. Because the elements of $J_S(P_\lambda)$ are constructed from $S^1$-invariant almost complex structures on $M$ (see Definition 2.3), this action preserves $\tilde{J}$. Hence $T^2$ acts on the moduli spaces of stable maps via postcomposition.

The next result is a version of the localization principle for $T^2$-actions; it is well known in the algebraic case and is proved in the symplectic situation in §4.2. Given two (weighted) pseudocycles $f : Z \longrightarrow P_\lambda$ and $f' : Z' \longrightarrow P_\lambda$ (see Definition 4.1) and a section class $\sigma$, we define

$$\mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma; Z, Z') := ev^{-1}(f(Z) \times f'(Z'))$$

where $ev : \mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma) \longrightarrow P_\lambda \times P_\lambda$ is the evaluation map. The pseudocycles are said to be $S^1$-invariant if the images $f(Z)$ and $f'(Z')$ are closed under the action of $S^1$. In this case, if $\tilde{J} \in J_S(P_\lambda)$, then clearly there is an induced action of $T^2$ on this cutdown moduli space.

**Proposition 3.4** Suppose that $f : Z \longrightarrow M_0$ and $f' : Z' \longrightarrow M_\infty$ are $S^1$-invariant weighted pseudocycles which represent the classes $a$ and $a'$ in $H_*(M)$, respectively. Given $\tilde{J} \in J_S(P_\lambda)$, write

$$S(A)(a) = \sum_{\sigma \in H^2_{vert}(P)} a_\sigma \otimes q^{-c_{vert}(\sigma)} t^{-u_\lambda(\sigma)}.$$

Then $a_\sigma \cdot_M a' = 0$ unless the moduli space $\mathcal{M}^{cut} : = \mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma; Z, Z')$ contains a $T^2$-invariant element. Moreover, $a_\sigma \cdot_M a'$ is a sum of contributions, one from each connected component of the space $(\mathcal{M}^{cut})^{T^2}$ of invariant elements.

Note that most $T^2$-invariant elements in $\mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma; Z, Z')$ are not regular. Therefore it would be a nontrivial task to calculate their actual contributions to the invariant. In this paper we do not attempt such calculations.

The next task is to figure out the structure of the $T^2$-invariant elements in $\mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma)$. Note that each principal component has 2 special points joining it to the other principal components. We will place each principal component has 2 special points joining it to the other principal components. We will place

Note that most $T^2$-invariant elements in $\mathcal{M}_{0,2}(P_\lambda, \tilde{J}, \sigma; Z, Z')$ are not regular. Therefore it would be a nontrivial task to calculate their actual contributions to the invariant. In this paper we do not attempt such calculations.

Let $\tilde{J} \in J_S(P_\lambda)$ be constructed from $J \in J_S(M)$ and denote by $g_j$ the metric on $M$ defined by $J$ and $\omega$.

(i) If $A$ is a section class the only elements in $\mathcal{M}_{0,2}(P_\lambda, \tilde{J}, A)$ that are fixed by the $T^2$-action have the form $[u; 0, \infty]$ where $u : S^2 \longrightarrow P_\lambda$ is parametrized as a section and has as image some constant sphere $\sigma_x$ where $x \in M^{S^2}$.

(ii) If $A \in H_2(M)$ then the only elements in $\mathcal{M}_{0,k}(P_\lambda, \tilde{J}, A)$ that are fixed by the $T^2$-action lie in either $M_0$ or $M_\infty$. If such an element does not lie entirely in $M^{S^1}$, then $k \leq 2$ and there exists a parametrization $u : \mathbb{R} \times S^1 \longrightarrow M$ and a path $\gamma : \mathbb{R} \longrightarrow M$ which joins two fixed points $x$ and $y$ in $M$ so that the marked points lie in $u^{-1}(\{x, y\})$, and

$$u(s, t) = \phi_{pt/q}\gamma(s), \quad \text{and} \quad \gamma'(s) = \frac{p}{q} \text{grad}_{\gamma(s)} K,$$

where $p \neq 0$ and where $q > 0$ is the order of the isotropy group of the points in the image of $\gamma$. There is a unique choice of parametrization such that

$$\lim_{s \to -\infty} u(s, t) = x \quad \text{and} \quad \lim_{s \to \infty} u(s, t) = y.$$
Lemma 3.8 Let $A$ be the sum of the classes represented by the principal spheres and the bubbles. Additional ghost components at each of which a $T$-invariant chain from $x$ to $y$ in class $\sigma x + A$ is a sequence of critical points with root $z$. Given $u$ a holomorphic sphere, $\psi \in \text{PSL}(2, \mathbb{C})$ such that $\psi u = u \circ \psi$. It follows easily the assignment $\theta \mapsto \psi_{\theta}$ defines a homomorphism $S^1 \rightarrow \text{PSL}(2, \mathbb{C})$. Since the only circle subgroups of $\text{PSL}(2, \mathbb{C})$ consist of rotations about a fixed axis, there are two points in $S^2$, say $0, \infty$, that are mapped by $u$ into $M^{S^1}$. If $u \cap M^{S^2} = \{x, y\}$ we may suppose $u(0) = x$ and $u(\infty) = y$. If there are marked points on $u$ they must form a subset of $\{0, \infty\}$. Using coordinates $(s, t)$ on $S^2 \setminus \{0, \infty\} = \mathbb{R} \times S^1$ as above, we find that for some $q \neq 0$

\[ \psi_{\theta}(s, t) = (s, t + q\theta), \quad \phi_{\theta} \circ u(s, t) = u(s, t + q\theta). \]

Thus, in $u$ lies in the set of points with isotropy group $\mathbb{Z}/(q\theta)$. Denoting $\gamma(s) := u(s, 0)$, we have $u(s, \theta) = \phi_{\theta/q}\gamma(s)$. Moreover

\[ 0 = \partial_s u + J\partial_t u = (\phi_{\theta/q})_\gamma' + \frac{1}{q} J_X K(\gamma(s)), \]

where $X_K$ is the Hamiltonian flow induced by $K$. Thus $\gamma' = \frac{1}{2\pi} \text{grad} K$ because $-J_X K = \text{grad} K$. (Here we take the gradient with respect to the metric $g_{\gamma'}$.) Since every sphere is the $|p|$-fold cover of a simple sphere, this proves (ii). To get the stated result, we absorb any negative sign into $p$ rather than $q$.

Definition 3.6 Let $x$ and $y$ be two fixed points in $M$.

For $q > 0$ and $p \neq 0$, a bead from $x$ to $y$ of type $(p, q)$ is a map $u : \mathbb{R} \times S^1 \rightarrow M$ which satisfies equations (11) and (12).

For $q = 0$ and $p > 0$, a bead from $x$ to $y$ of type $(p, q)$ is a $p$-fold cover of a simple $J$-holomorphic sphere $u : \mathbb{R} \times S^1 \rightarrow M$ that lies entirely in one component of the fixed point set $M^{S^1}$ and which satisfies equation (12).

Definition 3.7 Given $x, y, z \in M^{S^1}$ an invariant principal chain from $x$ to $y$ in class $\sigma x + A$ and with root $z$ is a sequence of critical points $x = x_1, x_2, \ldots, x_k = y$ of $K$ joined by invariant $J$-holomorphic spheres with the following properties:

(a) there is $1 \leq i_0 \leq k$ such that $x_{i_0} = x_{i_0+1} = z$ and these points are joined by the section $\sigma z$;

(b) for each $1 \leq i < k$ where $i \neq i_0$, the points $x_i, x_{i+1}$ are joined by a $(p_i, q_i)$-bead in class $A_i$;

(c) $\sum_{i \neq i_0} A_i = A$.

Further an invariant chain from $x$ to $y$ in class $\sigma x + A$ and with root $z$ is a chain as above with additional ghost components at each of which a $T^2$-invariant tree of $(p, q)$ beads is attached. In this case, $A$ is the sum of the classes represented by the principal spheres and the bubbles.

The next lemma is an immediate consequence of Lemma 3.5 and the above definitions.

Lemma 3.8 Let $f : Z \rightarrow P_\Lambda$ and $f' : Z' \rightarrow P_\Lambda$ be $T^2$-invariant pseudocycles, $\sigma$ be a section class and choose $J \in \mathcal{J}_S(P_\Lambda)$. Then every $T^2$-invariant element of the cut down moduli space $\mathcal{M}_{0,2}(P_\Lambda, J, \sigma; Z, Z')$ is an invariant chain from a point $x \in Z$ to a point $y \in Z'$.

We will need the following useful facts about beads.
Lemma 3.9 Choose \( J \in J_S(M) \) and consider a (\( J \)-holomorphic) \((p,q)\)-bead from \( x \) to \( y \) in class \( A \). If \( q \neq 0 \), then \( A = p(\sigma_x - \sigma_y)/q \). Further:

(i) If \( K(y) > K(x) \) then \( p > 0 \), \( \omega(A) = p(K(x) - K(y))/q \) and \( c_1(A) = p(m(x) - m(y))/q \).

(ii) If \( K(y) < K(x) \) then \( p < 0 \), \( \omega(A) = |p||K(x) - K(y)|/q \) and \( c_1(A) = p(m(x) - m(y))/q \).

(iii) If \( K(y) = K(x) \) then \( \omega(A) > |K(y) - K(x)| \).

Proof: We saw in Lemma 2.2 that the homology class of the sphere formed by the \( \Lambda \)-orbit of an arc going from \( x \) to \( y \) is \( \sigma_x - \sigma_y \). Hence each \((p,q)\) bead from \( x \) to \( y \) lies in the class \( A = p(\sigma_x - \sigma_y)/q \) where \( \omega(A) = p(K(y) - K(x))/q \). Statements (i), (ii) and (iii) now follow from the fact that \( \omega(A) > 0 \) and \( c_1(A) = p(m(x) - m(y))/q \).

The proofs of our other results are based on a more careful study of the structure of the \( T^2 \)-invariant elements in \( \overline{M}_{0,2}(P_A, \bar{J}, \sigma_A + B) \). We begin by slightly strengthening the conclusion of Proposition 3.4.

Lemma 3.10 Consider a Hamiltonian circle action \( \Lambda_K \) on a compact symplectic manifold \((M,\omega)\) with normalized moment map \( K \). Let \( F_{max} \) be the maximal fixed component and choose \( J \in J_S(M) \). Given \( B \in H^2_\mathbb{Z}(M) \), let \( a_B \) denote the contribution of \( \sigma_{max} + B \) to \( S(\Lambda_K) \). Then \( a_B = 0 \) unless \( B \) can be represented by an invariant \( J \)-holomorphic stable map that intersects \( F_{max} \). More generally, if \( f' : Z' \to M \) is an invariant pseudocycle representing the class \( a' \), then \( a_B \cdot_M a' = 0 \) unless \( B \) can be represented by an invariant \( J \)-holomorphic stable map that intersects both \( F_{max} \) and \( f'(Z') \).

Proof: Assume \( a_B \cdot_M a' \neq 0 \). By Proposition 3.4, there must be a \( T^2 \)-invariant element in

\[ \overline{M}_{0,2}(P_A, \bar{J}, \sigma_{max} + B; M_0, f'(Z')) \],

where \( \bar{J} \in J_S(P_A) \) is constructed from \( J \) in the usual way. Hence, by Lemma 3.8, there is an invariant chain from \( x \in M_0 \) to \( y \in f'(Z') \) in the class \( \sigma_{max} + B \). Let \( z \) denote its root. Let \( A' \) be the sum of the homology classes represented by the subchain of spheres in \( M_0 \) from \( x \) to \( z \), and let \( A'' \) be the sum of the homology classes represented by the subchain of spheres in \( M_{\infty} \) from \( z \) to \( y \). Then \( A' + A'' + \sigma_z = \sigma_{max} + B \). Since the orbit of an upward gradient flow line from \( z \) to \( F_{max} \) is \( J \)-holomorphic, the class \( \sigma_z - \sigma_{max} \) is also represented by a \( J \)-holomorphic sphere.

Proof of Theorem 1.10. Part (i) is included in Proposition 3.3. Part (ii) follows immediately from Lemma 3.10.

Now assume that \((M, J)\) is \( \text{NEF} \) and that \( 2c_1(B') \geq \text{codim} F_{max} \) for all \( J \)-holomorphic spheres \( B' \) that do not lie entirely in \( F_{max} \). Assume also that \( a_B \neq 0 \). By Lemma 3.10, \( B \) can be represented by a \( J \)-holomorphic stable map which intersects \( F_{max} \). We must show that all components of this stable map lie in \( F_{max} \). Suppose the contrary. Then the assumptions imply that \( 2c_1(B) \geq \text{codim} F_{max} \). On the other hand, \( 0 \leq \text{deg}(a_B) = \text{dim} F_{max} + 2c_1(B) \leq \text{dim} M \). Therefore, \( 2c_1(B) = \text{codim} F_{max} \). Therefore, \( \text{deg}(a_B) = \text{dim} M \). Since \( a_B \) is not zero, this implies that it is a multiple of the generator of \( H_{\text{dim} M}(M) \); hence \( a_B \cdot [pt] \neq 0 \). Choose \( y \in F_{min} \). Then since \( a_B \cap [y] \neq 0 \), Lemma 3.10 implies \( B \) can be represented by a \( J \)-holomorphic stable map which intersects \( F_{max} \) and \( y \). Let \( B_1 \) be a sphere in the corresponding stable map which intersects \( F_{max} \) at \( x_1 \) but does not lie entirely in \( F_{max} \). Let \( x_2 \) denote the second marked point in \( B_1 \). Let \( B_2, \ldots, B_k \) be the remaining \( J \)-holomorphic spheres in \( B \). Then

\[ \text{codim} (F_{max}) = 2c_1(B) = 2 \sum_{i=1}^{k} c_i(B_i) \]
Since the assumptions imply that \( c_1(B_i) \geq 0 \) for all \( i \) and \( 2c_1(B_i) \geq \codim (F_{\text{max}}) \), we conclude that 
\[ 2c_1(B_1) = \codim (F_{\text{max}}) \] and \( c_1(B_i) = 0 \) for all \( i \neq 1 \). Since \( F_{\text{max}} \) is simple, \( B_1 \) is a bead of type \((p,q)\) with \( q = 1 \). By Lemma 3.9(ii), \( p < 0 \) and
\[
2c_1(B_1) = 2p \left( m(F_{\text{max}}) - m(x_2) \right) = -2p m(x_2) - p \codim (F_{\text{max}}).
\]
Since \( 2c_1(B_1) = \codim (F_{\text{max}}) \), \( m(x_2) \leq 0 \). Since \( c_1(B_i) = 0 \) for all \( i \neq 1 \), the next bead on the principal chain must connect \( x_2 \) to another point, \( x_3 \), which also satisfies \( m(x_3) \leq 0 \). Proceeding inductively, we see that \( m(y) \leq 0 \). But this is impossible, because \( m(y) > 0 \) for all \( y \in F_{\text{min}} \).

\[ \square \]

3.3 The semifree case

Suppose that the moment map \( K \) generates a semifree \( S^1 \)-action. By Lemma 4.5, for a generic almost complex structure \( J \in J_S(M) \), the pair \((K, g_J)\) is Morse regular where \( g_J \) denotes the metric associated to \( J \) (see Definition 4.2). Let \( F \) and \( F' \) be fixed components. By Lemma 4.6, for any generic submanifolds \( C \) of \( F \) and \( C' \) of \( F' \), the unstable manifolds \( W^u(C) \) and \( W^u(C') \) are pseudocycles. We shall denote them by \( W^u_j(C) \) and \( W^u_j(C') \) to emphasize that they depend on the choice of \( J \). By construction, these unstable manifolds are \( S^1 \)-invariant. Hence, to prove Theorem 1.15, we only need analyze the invariant chains in the moduli space \( \overline{M}_{0,2}(P_A, \tilde{J}, \sigma_F + B; W^u_j(C), W^u_j(C')) \), where \( \omega(B) \leq 0 \).

**Lemma 3.11** Consider a semifree Hamiltonian circle action \( \Lambda_K \) on a compact symplectic manifold \((M, \omega)\). Let \( \tilde{J} \) be a generic almost complex structure in \( J_S(P) \). Let \( F \) and \( F' \) be connected components of the fixed point set and let \( C \subset F \) and \( C' \subset F' \) be generic submanifolds. Fix \( B \in H^2_\mathbb{Z}(M) \) such that \( \omega(B) \leq 0 \), and consider the moduli space
\[
\overline{M}_{0,2}(P_A, \tilde{J}, \sigma_F + B; W^u_j(C), W^u_j(C')).
\]

(i) If \( B \neq 0 \), the moduli space contains no invariant chains.

(ii) If \( F \neq F' \), there are no invariant chains unless \( \dim(W^u_j(C)) + \dim(W^u_j(C')) > \dim M \).

(iii) If \( B = 0 \) and \( F = F' \), the only invariant chains are the constant sections \( \sigma_x \) for \( x \in C \cap C' \).

**Proof:** Assume that there is an invariant chain from \( x \in W^u_j(C) \) to \( y \in W^u_j(C') \) with root \( z \) in the class \( \sigma_F + B \). Note immediately that
\[
K(x) \leq K(F),
\]
with equality if and only if \( x \in C \). Let \( A' \) and \( A'' \) denote the classes represented by the invariant subchains from \( x \) to \( z \), and from \( z \) to \( x \), respectively. Then \( A' + A'' + \sigma_z = \sigma_F + B \), and so by Lemma 2.2
\[
\omega(A') + \omega(A'') - K(z) + K(F) = \omega(B) \leq 0.
\]
Because the action is semifree, every bead of type \((p,q)\) in the invariant chain from \( x \) to \( z \) has \( q = 1 \). Hence, by Lemma 3.9
\[
K(z) - K(x) \leq \omega(A')
\]
with equality impossible unless \( K(x) \leq K(z) \), \( A' \) is the class of a chain of \((1,1)\) beads from \( x \) to \( z \), and \( A' = \sigma_x - \sigma_z \). This implies both that there is a broken \( K \)-trajectory from \( z \) to \( x \), and that
\[
0 \leq \omega(A''),
\]
with equality if and only if $A'' = 0$. In this case, $z = y \in \overline{W^j_f(C')}$. Therefore, $K(z) \leq K(F')$.

Considering all four displayed inequalities together, it is clear that in fact they must all be equalities. This implies that $A' = \sigma_x - \sigma_z$ and $A'' = 0$, and also that $x \in C \subset F$, so $\sigma_x = \sigma_F$. Therefore $B = A' + A'' + \sigma_x - \sigma_F = 0$. This proves (i).

Next, since it implies both that $x \in C$ and that there is a broken $K$-trajectory from $z$ to $x$, there is a broken $K$-trajectory from $z$ to $C$. Additionally, since $z \in \overline{W^j_f(C')}$, by Lemma 4.4, there is a broken $K$-trajectory from $C'$ to $z$. Therefore, there is a broken $K$-trajectory from $C'$ to $C$. If $F \neq F'$, then by Lemma 4.3, this implies that $\dim W_j^j(C) + \dim W_j^j(C') > \dim M$. This proves (ii).

Finally, assume that $F = F'$. Then since $K(x) = K(F)$, $K(x) \leq K(z)$, and $K(z) \leq K(F')$, it follows that $K(x) = K(z)$. Thus $z \in F$ and $A' = \sigma_F - \sigma_z = 0$. Since also $A'' = 0$, the last claim follows. \(\Box\)

Before proving the rest of the theorems from the first section, we need to consider the contributions of fixed point sets other than $K$. To simplify the proof of Lemma 3.13 below, it is convenient to work with almost complex structures on $M$ that are well behaved near the fixed components. Each fixed component $F$ has a neighborhood $N_F$ that can be identified with a neighborhood of the zero section in a sum of Hermitian vector bundles $\pi_F : E_1 \oplus \cdots \oplus E_k \to F$ in such a way that the moment map $K$ is given by
\[
K(v_1, \ldots, v_k) = \sum_j \pi m_j ||v_j||^2, \quad m_j \in \mathbb{Z} \setminus \{0\}
\]
and $S^1$ acts in $E_j$ by rotation by $e^{2\pi i m_j}$. The symplectic connection with horizontal spaces $\text{Hor}_x$ equal to the $\omega$-orthogonals to the fibers is also $S^1$-invariant. Therefore, starting from any $\omega$-compatible $J_F$ on the components $F$, we may extend $J_F$ to an $S^1$-invariant $\omega$-compatible almost complex structure $J_M$ on $M$ whose restriction to each set $N_F$ agrees with the complex structure on the fibers of $\pi_F$, leaves the horizontal distribution invariant and is such that $\pi_F$ is holomorphic.

**Definition 3.12** Fix once and for all such an almost complex structure $J_M$ on $M$. Define $\mathcal{J}_S^S(M)$ to be the set of all $S^1$-invariant $\omega$-compatible almost complex structures on $M$ that equal $J_M$ near the fixed point components $F$. Let $\mathcal{J}_S^S(P_\Lambda)$ denote the subspace of $\mathcal{J}_S(P_\Lambda)$ constructed from $J \in \mathcal{J}_S^S(M)$ as in Definition 3.12.

Thus when $J \in \mathcal{J}_S^S(M)$ each fixed point component $F$ has a neighborhood $N_F$ that can be identified with a neighborhood of the zero section in the complex vector bundle $\pi_F : E^+ \oplus E^- \to F$, where $E^+$ (resp. $E^-$) is the subbundle of the normal bundle with positive (resp. negative) weights. Moreover, $\pi_F$ is $J$-holomorphic. Hence a neighborhood of the submanifold $S^2 \times F$ in $P_\Lambda$ can be identified with a neighborhood of the zero section in $\tilde{\pi}_F : \tilde{E}^+ \oplus \tilde{E}^- \to S^2 \times F$.

where the bundle $\tilde{E}^\pm \to S^2 \times F$ is induced in the obvious way from the $S^1$-action. Moreover, $\tilde{\pi}_F$ is $\tilde{J}$-holomorphic. Denote by $\mathcal{M}_{0,2}^{\infty}(P_\Lambda, \tilde{J}, \sigma_F)$ the moduli space of $J$-holomorphic maps $u : S^2 \to P$ in class $\sigma_F$ and parametrized as sections of $P_\Lambda \to S^2$, and by $\mathcal{F} \subset \mathcal{M}_{0,2}^{\infty}$ the subspace of constant sections.

**Lemma 3.13** Fix $\tilde{J} \in \mathcal{J}_S^S(P)$ and a fixed point component $F$. Then the evaluation map
\[
ev : \mathcal{M}_{0,2}^{\infty}(P_\Lambda, \tilde{J}, \sigma_F) \to M_0 \times M_\infty
\]
is transverse to $(N_F \cap E^-) \times (N_F \cap E^+) \subset M_0 \times M_\infty$ at all constant maps $u_x \in \mathcal{F}$. Moreover, if all the positive weights at $F$ are $+1$, $\nev$ is a local diffeomorphism onto $(N_F \cap E^+) \times (N_F \cap E^-)$. 23
Proof: The definitions imply that $F$ has a neighborhood $N(F)$ consisting of all holomorphic maps $\tilde{u} : S^2 \to \tilde{E}^+ \oplus \tilde{E}^-$ whose composite with the projection $\tilde{E}^+ \oplus \tilde{E}^- \to S^2 \times F$ is a holomorphic section of $\pi : S^2 \times F \to S^2$ in the class $[S^2 \times pt]$. Since $S^2 \times F \subset P_A$ has the product complex structure, $\tilde{u}$ must project to some sphere $S^2 \times \{x\}$ for $x \in F$. Therefore $\tilde{u}$ is a holomorphic section of the bundle $\tilde{E}^+ \oplus \tilde{E}^-|_{S^2 \times \{x\}}$. But this bundle is a sum of line bundles whose Chern classes are the nonzero weights of the $S^1$ action at $F$. The line bundles with negative Chern classes have no sections, but those with positive Chern classes have plenty. Moreover if all the positive weights are $+1$ then there is precisely one section of $\tilde{E}^+|_{S^2 \times \{x\}}$ through any pair of points lying in distinct fibers. The result follows.

Proof of Theorem 1.15. By Lemma 2.2, $u_A(\sigma_F) = -K(F)$ and $e_{vert}(\sigma_F) = m_F$. Therefore we may write

$$S(\Lambda_K)(c^-) = \sum_{B \in \mathcal{H}_2^r(M)} a_B \otimes q^{-m(F) - \epsilon_1(B)} t^{K(F) - \omega(B)}$$

where $a_B$ is the contribution from the section $\sigma_F + B$. Let $F'$ be a fixed component, and consider $c' \in H_*(F')$. Fix $B \in \mathcal{H}_2^r(M)$ so that $\omega(B) \leq 0$. We want to show that $a_B = 0$ unless $B = 0$ and $F' = F$, in which case $a_B = c^+$. Since the classes $(c')^-$ form a basis for $H_*(M)$ it is enough to show that $a_B \cdot_M (c')^- = 0$ unless $B = 0$ and $F' = F$, in which case $a_B \cdot_M (c')^- = c \cdot_F c'$.

Choose a generic almost complex structure $J \in \mathcal{J}_s^P(M)$. By Lemma 4.5, the pair $(K, g_J)$ is Morse regular, where $g_J$ is the metric associated to $J$. We may assume without loss of generality that $c$ and $c'$ can be represented by generic submanifolds $C \subset F$ and $C' \subset F'$. By Lemma 4.6, the unstable manifolds $W^u_J(C)$ and $W^u_J(C')$ are pseudocycles. Moreover, by Proposition 4.8, $[W^u_J(C)] = c^-$ and $[W^u_J(C')] = (c')^-$. Let $\tilde{J} \in \mathcal{J}_s^P(P)$ be the associated almost complex structure on $P_A$.

Assume first that $B \neq 0$. By Lemma 3.11, the moduli space $\mathcal{M}_{0,2}(P_A, \tilde{J}, \sigma_F + B; W^u_J(C), W^u_J(C'))$ contains no invariant chains. By Proposition 3.4 and Lemma 3.8, this implies that $a_B \cdot_M (c')^- = 0$.

Now suppose that $B = 0$ but $F \neq F'$. If $a_0 \neq 0$, then deg$(a_0) = \deg(c^+)$ is an invariant chain. By Proposition 3.4 and Lemma 3.8, this implies that $a_B \cdot_M (c')^- = 0$.

Finally, assume that $B = 0$ and $F = F'$. By Lemma 3.11, the space $\mathcal{M}_{0,2}(P_A, \tilde{J}, \sigma_F; W^u_J(C), W^u_J(C'))$ contains no invariant chains except the constant sections $\sigma_x$ for $x \in C \cap C'$. By Proposition 3.4 and Lemma 3.8, this implies that only those elements contribute to $a_0 \cdot_M (c')^-$. Now consider the full moduli space $\mathcal{M}_{0,2}(P_A, \tilde{J}, \sigma_F)$. It follows from Lemma 3.13 that the evaluation map

$$ev : \mathcal{M}_{0,2}(P_A, \tilde{J}, \sigma_F) \to M_0 \times M_\infty$$

intersects $W^u_J(C) \times W^u_J(C')$ transversally in $c \cdot_F c'$ points. Hence $GW_{\sigma_F}(c, c') = c \cdot_F c'$, as required.

3.4 The case of at most twofold isotropy

When the action is not semifree, the unstable manifolds given by an $S^1$-invariant metric may not be pseudocycles. Therefore, we shall need to consider more general objects.

Definition 3.14 Let $C$ be a submanifold of a fixed component $F$ with index $\alpha_F$. A downwards pseudocycle from $C$ is an $S^1$-invariant weighted pseudocycle $f : Z_C \to M$, or $Z_C$ for short, of dimension $\dim C + \alpha_F$ such that $f(Z_C)$ lies in $M^K(F)$, $f(Z_C) \cap K^{-1}(K(F)) = C$, and $[Z_C] = [C^-]$. Here, $[C^-] \in H_*(M)$ is the downwards extension of $[C] \in H_*(F)$ constructed in Section 1.2.1.
We show in Lemma 4.7 and Proposition 4.8 that given any generic submanifold \( C \subset F \), there exists a downwards pseudocycle \( Z_C \).

As in the semifree case, we consider a section class \( \sigma \), choose \( \tilde{J} \in \mathcal{J}_S(P) \), and investigate \( T^2 \)-invariant elements of the moduli space 

\[
\overline{\mathcal{M}}^{\text{cut}} := \overline{\mathcal{M}}_{0.2}(P, \tilde{J}, \sigma; Z_C^-, Z_C^-, Z_C^-).
\]

**Lemma 3.15** Consider a Hamiltonian circle action \( \Lambda_K \) on a compact symplectic manifold \((M, \omega)\) with at most twofold isotropy. Let \( \tilde{J} \) be a generic almost complex structure in \( \mathcal{J}_S(P) \). Let \( F \) and \( F' \) be (not necessarily distinct) fixed point components, and let \( C \subset F \) and \( C' \subset F' \) be generic submanifolds. Fix \( B \in H^2_F(M) \) such that \( \omega(B) \leq 0 \), and consider the moduli space

\[
\overline{\mathcal{M}}_{0.2}(P, \tilde{J}, \sigma; \frac{1}{2} \sigma_F + \sigma_F') + B; Z_C^-, Z_C^-).
\]

The moduli space contains no invariant chains unless \( B = 0 \) and \( F \) and \( F' \) lie in the same component of \( \overline{M}^{Z/2(2)} \).

**Proof:** Assume that there is an invariant chain from \( x \in \mathbb{Z}_C \) to \( y \in \mathbb{Z}_C' \) with root \( z \) in the class \( \frac{1}{2} (\sigma_F + \sigma_F') + B \). We see immediately that

\[
K(x) \leq K(F) \quad \text{and} \quad K(y) \leq K(F'),
\]

with equality if and only if \( x \in C \subset F \) and \( y \in C' \subset F' \).

Let \( A' \) and \( A'' \) denote the classes represented by the invariant subchains from \( x \) to \( z \), and from \( z \) to \( y \), respectively. Since \( A' + A'' + \sigma_z - \frac{1}{2} (\sigma_F + \sigma_F') = B \), by Lemma 2.2

\[
\omega(A') + \omega(A'') - K(z) + \frac{1}{2} (K(F) + K(F')) = \omega(B) \leq 0.
\]

Since the action has at most twofold isotropy, every bead of type \((p, q)\) in the invariant chain from \( x \) to \( z \) has \( q \leq 2 \). Hence, by Lemma 3.9

\[
\frac{1}{2} (K(z) - K(x)) \leq \omega(A'),
\]

with equality impossible unless \( A' \) is the class of a chain of \((1, 2)\) beads from \( x \) to \( z \) and hence \( A' = \frac{1}{2} (\sigma_z - \sigma_z) \).

In particular, in this case \( x \) and \( z \) lie in the same component of \( \overline{M}^{Z/2(2)} \). By similar reasoning,

\[
\frac{1}{2} (K(z) - K(y)) \leq \omega(A''),
\]

with equality impossible unless \( A'' = \frac{1}{2} (\sigma_y - \sigma_z) \) and \( y \) and \( z \) lie in the same component of \( \overline{M}^{Z/2(2)} \).

Considering all five displayed inequalities together, it is clear that they must all be equalities. First, this means that \( A' = \frac{1}{2} (\sigma_z - \sigma_z), A'' = \frac{1}{2} (\sigma_y - \sigma_z), \sigma_x = \sigma_F, \sigma_y = \sigma_F \); hence, \( B = 0 \). Second, it implies that \( F \) and \( F' \) lie in the same component of \( \overline{M}^{Z/2(2)} \). \(\square\)

**Proof of Theorem 1.17.** Let \( F' \) be a fixed component and consider \( c' \in H_4(F') \). Assume that either

- \( K(F') < -K(F) \), or
- \( K(F') = -K(F) \) and \( F \) and \( F' \) lie in different components of \( \overline{M}^{Z/2(2)} \).
It is enough to show that $a_{0,0} \cdot M(c')^- = 0$.

Choose a generic almost complex structure $J \in J_S(M)$. Let $\tilde{J} \in J_S(P_\Lambda)$ be the associated almost complex structure on $P_\Lambda$.

We may assume without loss of generality that $c$ and $c'$ can be represented by generic submanifolds $C \subset F$ and $C' \subset F'$, respectively. By Lemma 4.7 and Proposition 4.8 we can find downwards pseudocycles $Z_C$ and $Z_{C'}$ from $C$ and $C'$ as described above. By Proposition 3.4 and Lemma 3.8, it suffices to show that $\mathcal{M}_{0,2}(P_\Lambda, \tilde{J}, \sigma, Z_C, Z_{C'})$ contains no $T^2$-invariant chains when $\text{cータ}(\sigma) = u_\Lambda(\sigma) = 0$. Given such $\sigma$, let $B = \sigma - \frac{1}{2}(\sigma_F + \sigma_{F'})$. By Lemma 2.2, $\omega(B) = \frac{1}{2}(K(F) + K(F')) \leq 0$ with equality only if $K(F') = -K(F)$. Moreover, by assumption, in this case $F$ and $F'$ lie in different components of $M^{Z/(2)}$. Therefore, by Lemma 3.15, the moduli space contains no invariant chains. □

The previous result concerns the term $a_{0,0} \otimes 1$ in $\mathcal{S}(\Lambda)(a)$ and hence gives information only in cases when we know that $a_{0,0} \neq 0$, for example if $\Lambda$ is inessential. We next investigate the contribution from almost simple components $F$ for general $\Lambda$. Again our arguments work only if the isotropy at levels above $F$ is at most twofold.

**Lemma 3.16** Consider a circle action $\Lambda_K$ on a compact symplectic manifold $(M, \omega)$ with moment map $K : M \rightarrow \mathbb{R}$. Let $\tilde{J}$ be a generic almost complex structure in $J_S(P)$. Let $F$ and $F'$ be connected components of the fixed point set and let $C \subset F$ and $C' \subset F'$ be generic submanifolds. Assume $K(F') \leq K(F)$, that every positive weight at $F$ is $+1$, and also that the isotropy for points $w$ with $K(w) > K(F)$ is at most twofold. Consider the moduli space

$$\mathcal{M}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; Z_C, Z_{C'})$$

where $\omega(B) = 0$.

- If $B \neq 0$ or $F \neq F'$, there are no invariant chains in the moduli space.
- If $B = 0$ and $F = F'$, the only invariant chains are the constant sections $u_x$ for $x \in C \cap C'$.

**Proof:** Assume that there is an invariant chain in class $\sigma_F + B$ from $x \in Z_C$ to $y \in Z_{C'}$ with root $z$. We see immediately that

$$K(x) \leq K(F) \quad \text{and} \quad K(y) \leq K(F'),$$

with equality if and only if $x \in C \subset F$ and $y \in C \subset F'$.

Let $A'$ and $A''$ denote the classes represented by the invariant subchains from $x$ to $z$, and from $z$ to $y$, respectively. Since $A' + A'' + \sigma_z = \sigma_F + B$, by Lemma 2.2

$$\omega(A') + \omega(A'') - K(z) + K(F) = \omega(B) = 0.$$

Since the isotropy for points $w$ with $K(w) > K(F)$ is at most twofold, every $(p,q)$ bead in the part of the invariant chain from $x$ to $z$ which lies at least partially above $K(F)$ has $q \leq 2$. Hence, by Lemma 3.9,

$$\frac{1}{2} \left( K(z) - \max\{K(F), K(x)\} \right) \leq \omega(A'),$$

with equality impossible unless $K(z) \geq K(F)$. $A' = \frac{1}{2}(\sigma_x - \sigma_z)$, and $x$ and $z$ lie in the same component of $M^{Z/(2)}$. A similar reasoning applies to $A''$. Hence,

$$\frac{1}{2} \left( K(z) - \max\{K(F), K(y)\} \right) \leq \omega(A''),$$
with equality impossible unless \( K(z) \geq K(F), A'' = \frac{1}{2}(\sigma_y - \sigma_z) \), and \( y \) and \( z \) lie in the same component of \( M^\oplus/(2) \).

Considering all five displayed equations together with the hypothesis \( K(F') \leq K(F) \), it is clear that they must all be equalities.

This implies that \( x \in F \) and \( y \in F' \), that \( K(z) \geq K(F) \), that \( K(y) = K(F) \), and that \( x, y \) and \( z \) lie in the same connected component of \( M^\oplus/(2) \). Since all the positive weights at \( F \) are \(+1\), this implies that in fact \( z \in F \) and \( y \in F \), so that \( \sigma_x = \sigma_y = \sigma_z = \sigma_F \). Since \( A' = \frac{1}{2}(\sigma_x - \sigma_z) \) and \( A'' = \frac{1}{2}(\sigma_y - \sigma_z) \), this implies that \( B = 0 \), and the result follows.

**Proof of Theorem 1.18.** Let \( F' \) be a fixed component with \( K(F') \leq K(F) \) and consider \( c' \in H_*(F') \). Choose a generic almost complex structure \( J \in J^g(M) \). Let \( J \in J^g(P) \) be the associated almost complex structure on \( P \). We may assume without loss of generality that \( c \) and \( c' \) can be represented by generic transversally intersecting submanifolds \( C \subset F \) and \( C' \subset F' \), respectively. By Lemma 4.7 and Proposition 4.8, we can find downward pseudocycles \( Z_C \) and \( Z_{C'} \) as in Definition 3.14. These are constructed to coincide with the unstable manifolds \( W^u(C) \) and \( W^u(C') \) near \( F \). Since \( J \in J^g(M) \) is normalized near \( F \), these unstable manifolds agree with neighborhoods of the zero section in the restrictions of \( E \to F \) to \( C \) and \( C' \) respectively.

To show that \( c_{0,0} \in F_{K(F)}H_*(M) \), it is enough to show that if \( K(F') < K(F) \), then \( c_{0,0} \cdot_M (c')^- = 0 \) for all \( c' \). By Proposition 3.4 and Lemma 3.8, it suffices to show that \( \omega(B) = c_{0,0}(B) = 0 \) then the moduli space \( \overline{M}_{0,2}(P_A, J, \sigma_F + B; Z_C, Z_{C'}) \) contains no invariant chains. Since \( F' \neq F \), this is immediate from Lemma 3.16.

Now suppose \( F = F' \). Consider the moduli space \( \overline{M}^{\text{cut}} = \overline{M}_{0,2}(P_A, J, \sigma_F + B; Z_C, Z_{C'}) \). By Lemma 3.16, this is nonempty only if \( B = 0 \). Further, in this case the only invariant chains in \( \overline{M}^{\text{cut}} \) are the constant sections \( u_x \) for \( x \in C \cap C' \). These sections form one of the connected components of \( \overline{M}^{\text{cut}} \), and by Proposition 3.4 we may ignore any other components. Thus we may suppose that \( \overline{M}^{\text{cut}} \) reduces to the compact manifold \( C \cap C' \). If any of the negative weights along \( F \) are less than \(-1\) the elements of \( \overline{M}^{\text{cut}} \) are not regular. As in the proof of Proposition 3.3, their cokernels fit together to form the obstruction bundle \( E \to \overline{M}^{\text{cut}} \) of Equation (1). Because all positive weights along \( F \) are \(+1\) and because the sets \( Z_C, Z_{C'} \) coincide near \( F \) with the bundles \( E^-|_C, E^-|_{C'} \), it follows from Lemma 3.13 that the full moduli space intersects \( Z_C \times Z_{C'} \) transversally in \( \overline{M}^{\text{cut}} \). Hence standard theory implies that the regularized cut down moduli space represents the class \( e(E) \cdot_F [C \cap C'] = (e(E) \cap_F c) \cdot_F c' \) and that

\[
GW_{\sigma_F, F}(e^-, (c')^-) = (e(E) \cap_F c) \cdot_F c'.
\]

The result follows.

**4 Proofs of main technical lemmas**

We now establish the main technical results used in the paper.

**4.1 Invariant cycles in \( M \).**

This section establishes the properties of the canonical extension classes \( c^\pm \) used in Theorems 1.15, 1.17, and 1.18. We first prove Lemma 1.13 which is used to construct canonical downwards and upwards extensions of the homology classes of the fixed point set, and then show how to define representing cycles for these classes that have the properties claimed in Definition 3.14.
4.1.1 Canonical classes

Let $S^1$ act on a symplectic manifold $(M, \omega)$, with a moment map $K$ which is proper and bounded below. Then $K$ is Morse–Bott function with extraordinary properties. (For background information see [21].) First, $K$ is equivariantly perfect, that is, the restriction map $H^*_S(M) \to H^*_S(M^{<K})$ is surjective for all $\mu \in \mathbb{R}$, where $M^{<K} := K^{-1}(-\infty, \mu)$. The same proof shows that the restriction to the fixed point set is injective. More specifically, given any $\tilde{Y} \in H^*_S(M)$, then $\tilde{Y}|_{M^{<\mu}} = 0$ if and only if $\tilde{Y}|_{F^\prime} = 0$ for all fixed components $F^\prime$ with $K(F^\prime) < \mu$. The same argument also shows that $H^*_S(M)$ is equivariantly formal, that is, the restriction $H^*_S(M) \to H^*(M)$ is surjective.

Proof of Lemma 1.13. Let $S^1$ act on a compact symplectic manifold $(M, \omega)$ with moment map $K$. Let $F \subset M$ be any fixed component of index $\alpha$; and let $e_F \in H^*_{S^1}(F)$ be the equivariant Euler class of the negative normal bundle to $F$. Given any cohomology class $Y \in H^*(F)$, we must show that there exists a unique cohomology class $\tilde{Y}^+ \in H^*_{S^1}(M)$ so that

(a) the restriction of $\tilde{Y}^+$ to $M^{<K(F)}$ vanishes,

(b) $\tilde{Y}^+|_F = Y \cup e_F^*$, and

(c) the degree of $\tilde{Y}^+|_{F^\prime}$ in $H^*_S(pt)$ is less than the index $\alpha_{F^\prime}$ of $F^\prime$ for all fixed components $F^\prime \neq F$.

Moreover, we claim that these classes generate $H^*_S(M)$ as a $H^*_S(pt)$ module.

Since $K$ is equivariantly perfect, we can find $\tilde{Y}^+$ satisfying (a) and (b). In fact, in general there will be many such $\tilde{Y}^+$.

Enumerate the fixed sets other than $F$ by $F_1, \ldots, F_k$ so that $K(F_1) \leq K(F_{j+1})$ for all $j$. Assume that $\tilde{Y}^+$ satisfies (c) for all $F_j$ such that $j < i$. Let $\alpha_i$ denote the index of $F_i$, and let $m_{-}(F_i)$ denote the product of the negative weights at $F_i$; $m_{-}(F_i)$ is an element of $\mu\mathbb{R}$.

Therefore, $\tilde{Y}^+|_{F_i} \in H^*_S(F_i)$ can be written uniquely as a sum $\tilde{X} + \tilde{X}^{\prime}$, where $\tilde{X}$ is a multiple of $e_{F_i}$ and the degree of $\tilde{X}^{\prime}$ in $H^*_S(pt)$ is less than $\alpha_i$. Since $K$ is equivariantly perfect, there exists $\tilde{Y}^{\prime} \in H^*_S(M)$ so that $\tilde{Y}^{\prime}|_{F_i} = \tilde{X}$ and $\tilde{Y}^{\prime}_{F_j} = 0$ for all $j < i$. After subtracting $\tilde{Y}^{\prime}$, we find a new $\tilde{Y}^+$ that satisfies (c) for all $F_j$ such that $j \leq i$.

To see that $\tilde{Y}^+$ is unique, let $\tilde{Y}$ be the difference of two classes that satisfy (a), (b), and (c). Then the degree of $\tilde{Y}|_{F_i}$ in $H^*_S(pt)$ is less than the index of $F^\prime$ for every fixed component $F^\prime$ (and $\tilde{Y}|_F = 0$). Let $F_j$ be the smallest $j$ such that $\tilde{Y}|_{F_j} \neq 0$. Then, since the restriction to the fixed point set is injective, $\tilde{Y}$ vanishes when restricted to $H^*_S(M^{<K(F_j)})$. Hence, $\tilde{Y}|_{F_j}$ is a multiple of $e_{-}(F_j)$. But this is impossible, so $\tilde{Y}|_{F_j} = 0$ for all $i$. Hence, $\tilde{Y} = 0$.

Finally, for any $Y \in H^*(F)$, since $\tilde{Y}^+|_{M^{<K(F)}} = 0$ the restriction of $\tilde{Y}^+$ to $M_K(F)$ is an element of $H^*_S(M^{<K(F)})$. By injectivity, as $Y$ ranges over $H^*(F)$, these classes generate $H^*_S(M_K(F), M^{<K(F)})$ as a $H^*_S(pt)$ module. Hence, if we also let $F$ vary over all fixed components, then they generate $H^*_S(M)$.

\[\square\]

4.1.2 Morse cycles and equivariant cohomology

We shall work throughout with pseudocycles, and begin by recalling their definition from [12, 14]. A pseudocycle of dimension $d$ in a manifold $M$ is a smooth map $f : V \to M$ from an oriented smooth
$d$-dimensional manifold $V$ to $M$ whose $\Omega$-limit set

$$V^\infty := \{ x \in M : x = \lim_{i \to \infty} f(y_i), \text{ where } \{y_i\}_{i=1}^\infty \text{ has no limit point in } V \}$$

has codimension at least 2, i.e. it is in the image of a smooth map $g : W^{d-2} \to M$. Two pseudocycles $f_i : V_i \to M$ are bordant if they can be extended over a manifold $W$ with boundary $V_1 \cup -V_2$ by a map whose $\Omega$-limit set has dimension at most $d - 1$. Any $f : V^d \to M$ is bordant to a map that intersects a given codimension $d$ submanifold $X$ of $M$ transversally, i.e. $X \cap V^\infty = \emptyset$ and $f : V \to M$ meets $X$ transversally. Moreover, because the boundary has codimension at least 2, each bordism class $[f, V]$ of pseudocycles defines a unique rational homology class $c(f, V)$. (In fact, it defines a unique integral class: see Schwarz [19].) We say that two such cycles of complementary dimension meet transversally if $\lim_{\infty}$ of their images $f_i \to M$ are bordant if they can be extended over a manifold $W$ with boundary $V_1 \cup -V_2$ by a map whose $\Omega$-limit set has dimension at most $d - 1$. Any $f : V^d \to M$ is bordant to a map that intersects a given codimension $d$ submanifold $X$ of $M$ transversally, i.e. $X \cap V^\infty = \emptyset$ and $f : V \to M$ meets $X$ transversally. Moreover, because the boundary has codimension at least 2, each bordism class $[f, V]$ of pseudocycles defines a unique rational homology class $c(f, V)$. (In fact, it defines a unique integral class: see Schwarz [19].) We say that two such cycles of complementary dimension meet transversally if the closures of their images $f(V)$ and $f'(V')$ intersect only along their top strata $f(V)$ and $f'(V')$ and if these intersections are transverse in the usual way. It is shown in [12, 14] that the intersection number $c(f, V) \cdot c(f', V')$ can be calculated by perturbing $(f, V)$ to be transverse to $(f', V')$ and then counting the points of intersection of $f$ with $f'$ in the usual way.

In this paper it is convenient to work with rational combinations of such pseudocycles. Therefore we make the following definition.

**Definition 4.1** A **weighted pseudocycle** is a finite sum $\sum_i q_i f_i$, where $q_i \in \mathbb{Q}$ and $f_i : Z_i \to M$ is a pseudocycle as above. For short we sometimes forget the weights $q_i$ and denote this pseudocycle by $f : Z \to M$, where $Z := \cup_i Z_i$ and $f|Z_i := f_i$. We say that $(f, Z)$ is $S^1$-invariant iff the closure $\overline{f(Z)}$ of the image is invariant under the $S^1$-action. We also sometimes omit the map $f$ from the notation, denoting the cycle by $Z$ and the closure of its image by $\overline{Z}$.

All the cycles considered in this paper (except for the virtual moduli cycle) are weighted pseudocycles. Let $K$ be a Morse-Bott function, and let $g$ be any metric. Consider the **negative gradient flow**

$$\psi : \mathbb{R} \times M \to M$$

such that

$$\frac{\partial}{\partial t} \psi(t, x) = -\text{grad} f(\psi(t, x)) \quad \text{and} \quad \psi(0, x) = x \quad \text{for all } x \in M, t \in \mathbb{R}.$$

A **gradient trajectory** is a map $\gamma : \mathbb{R} \to M$ such that

$$\frac{d}{dt} \gamma(t) = -\text{grad} f(\gamma(t)) \quad \text{for all } t \in \mathbb{R}.$$

More generally, a **broken gradient trajectory** is a set of gradient trajectories $\gamma_1, \ldots, \gamma_n$ such that $\lim_{t \to -\infty} \gamma_i = \lim_{t \to -\infty} \gamma_{i+1}$ for all $i$. (By convention, we allow the case $n = 1$.)

Given any critical component $F$, we define the **stable manifold** and the **unstable manifold**, respectively, by

$$W^s(F) = \{ x \in M \mid \lim_{t \to \infty} \psi(t, x) \in F \}, \quad W^u(F) = \{ x \in M \mid \lim_{t \to -\infty} \psi(t, x) \in F \}.$$

Define maps

$$\pi_+ : W^s(F) \to F \quad \text{and} \quad \pi_- : W^u(F) \to F$$

by

$$\pi_+(x) = \lim_{t \to \infty} \psi(t, x) \quad \text{and} \quad \pi_-(x) = \lim_{t \to -\infty} \psi(t, x).$$
Both $\pi_+$ and $\pi_-$ are submersions; see [2].

Given a collection of distinct critical components $F_1, \ldots, F_k$, we define a natural map

$$f: W^u(F_1) \times \cdots \times W^u(F_{k-1}) \times W^s(F_2) \times \cdots \times W^s(F_k) \longrightarrow (M^{k-1} \times F_2 \times \cdots \times F_{k-1})^2$$

by

$$f(a_1, \ldots, a_{k-1}, b_2, \ldots, b_k) = (a_1, \ldots, a_{k-1}, \pi_-(a_2), \ldots, \pi_-(a_{k-1}), b_2, \ldots, b_k, \pi_+(b_2), \ldots, \pi_+(b_{k-1})). \quad (14)$$

Define $M(F_1, \ldots, F_k) = f^{-1}(\Delta)$. We can (and will) identify $M(F_1, \ldots, F_k)$ with tuples

$$(x_1, \ldots, x_{k-1}) \subset M^{k-1}$$

such that $x_i \in W^s(F_i) \cap W^u(F_{i+1})$ and $\pi^+(x_i) = \pi^-(x_{i+1})$ for all $i$. More geometrically, $M(F_1, \ldots, F_k)$ consists of all tuples $(x_1, \ldots, x_{k-1})$ for which there is a broken gradient trajectory $\gamma_1, \ldots, \gamma_{k-1}$ from $F_1$ to $F_k$ through $F_2, F_3, \ldots, F_{k-1}$ so that $\gamma_i$ contains the point $x_i$ for all $i$. Define maps

$$\pi_- : M(F_1, \ldots, F_k) \longrightarrow F_1 \quad \text{and} \quad \pi_+ : M(F_1, \ldots, F_k) \longrightarrow F_k$$

by

$$\pi_-(x_1, \ldots, x_{k-1}) = \pi_-(x_1) \quad \text{and} \quad \pi_+(x_1, \ldots, x_{k-1}) = \pi_+(x_{k-1}).$$

**Definition 4.2** We say that the pair $(K, g)$ is Morse regular if $f$ is transversal to the diagonal

$$\Delta \subset (M^{k-1} \times F_2 \times \cdots \times F_{k-1})^2$$

for every collection of critical components $F_1, \ldots, F_k$.

In general, this is stronger than assuming that $W^s(F)$ and $W^u(F')$ intersect transversally for all critical sets $F$ and $F'$, but it is equivalent if $K$ is a Morse function.

If $(K, g)$ is Morse regular, then by transversality, $M(F_1, \ldots, F_k)$ is a manifold of dimension $f_1 + \alpha_1 - \alpha_k$, where $f_i$ is the dimension of $F_i$ and $\alpha_i$ is the index of $F_i$. Note that the reparametrization group $\mathbb{R}^{k-1}$ acts on the elements $(\gamma_1, \ldots, \gamma_{k-1})$ in $M(F_1, \ldots, F_k)$ so that the set of points in $M$ that lie on a broken trajectory in $M(F_1, \ldots, F_k)$ has dimension $\leq f_1 + \alpha_1 - \alpha_k - (k - 1)$.

**Lemma 4.3** Let $M$ be a compact manifold. Let $K : M \longrightarrow \mathbb{R}$ be a Morse-Bott function and let $g$ be a metric so that the pair $(K, g)$ is Morse regular. Let $F$ and $F'$ be distinct critical components. If $C \subset F$ and $C' \subset F'$ are generic submanifolds, there is no broken gradient trajectory from $C'$ to $C$ unless

$$\dim W^s(C) + \dim W^u(C') > \dim M.$$

**Proof:** Assume that there is a broken trajectory from $F' = F_1$ to $F = F_k$ through critical components $F_2, \ldots, F_{k-1}$. By generality, we may assume that the maps $\pi_- : M(F_1, \ldots, F_k) \longrightarrow F_1$ and $\pi_+ : M(F_1, \ldots, F_k) \longrightarrow F_k$ are transverse to $C'$ and $C$, respectively. Therefore, the set $X = C' \times_{\pi_-} M(F_1, \ldots, F_k) \times_{\pi_+} C$ is a manifold of dimension $c' + \alpha' + c - \alpha - f$, where $c', c$, and $f$ denote the dimensions of $C', C$, and $F$, and $\alpha'$, $\alpha$ denote the index of $F', F$ respectively. There is a proper effective action of $\mathbb{R}$ on $M(F_1, \ldots, F_k)$ which moves $x_i$ along the gradient trajectory on which it lies; This induces an action on $X$. Hence, $X$ is empty unless $c' + \alpha' + c - \alpha - f > 0$. Since $\dim W^s(C) = c' + \alpha'$, and $\dim W^s(C) = c - f - \alpha$, $X$ is empty unless $\dim W^s(C) + \dim W^u(C') > \dim M$ as claimed. \qed

We will also need the following lemma, which can be easily proved by a slight variation of the proof for the analogous fact in the Morse case.
Lemma 4.4 Let $M$ be a compact manifold. Let $K : M \to \mathbb{R}$ be a Morse-Bott function and let $g$ be a metric so that the pair $(K, g)$ is Morse regular. Let $C$ be a submanifold of a critical component $F$. Every point in $\overline{W^u(C)}$, the closure of the unstable manifold of $C$, lies on a broken trajectory beginning in $C$.

Since we want the unstable manifold $W^u(C)$ to be $S^1$-invariant, we next investigate gradient flows with respect to invariant metrics. In general, due to the presence of isotropy spheres, there may be no $S^1$-invariant metric $g$ so that the pair $(K, g)$ is Morse regular, even if the moment map $K$ is Morse. For example, consider the action $[z_0 : z_1 : z_2] \mapsto [e^{2\pi i z_0} : z_1 : e^{-2\pi i z_2}]$ on $\mathbb{CP}^2$ and blow up the point $[0 : 1 : 0]$. The exceptional divisor $\Sigma$ has isotropy group $\mathbb{Z}/(2)$ and contains two critical points, both of index 2. Any $S^1$-invariant vector field must be tangent to $\Sigma$ since if $\phi$ denotes the generator of the isotropy subgroup $d\phi$ acts as $-1$ in the directions normal to $\Sigma$. In particular, the gradient grad $K$ of $K$ with respect to an invariant metric must be tangent to $\Sigma$ and hence have trajectories joining two critical points of equal index. The following lemma shows that this is the only obstruction to finding a Morse regular pair $(K, g)$.

Recall that $J_S(M)$ is the space of smooth invariant $\omega$-compatible almost complex structures on $M$ that are normalized near the fixed point components $F$ as described in Definition 2.3.

Lemma 4.5 Let $S^1$ act semifreely on a compact symplectic manifold $(M, \omega)$ with moment map $K$. For a generic almost complex structure $J \in J_S^0(M)$, the pair $(K, g_J)$ is Morse regular, where $g_J$ is the metric associated to $J$.

Proof: Salamon and Zehnder show in [18, Theorem 8.1] that the gradient flow of any Morse function $H$ on $(M, \omega)$ is Morse–Smale with respect to a generic metric of the form $g_J$, where $J$ ranges over the set of all $\omega$-compatible almost complex structures. We simply need to check that their argument continues to hold for Morse-Bott functions in the presence of a semifree $S^1$-action.

Inspection of the proof of [18, Theorem 8.1] shows that the map $f$ in equation (14) satisfies the required transversality condition provided that the tangent space $T_J(J_S)$ of the space $J_S := J_S(M)$ of allowable $J$ is large enough. (See also Austin–Braam [2, Proposition B.2].) This tangent space $T_J(J_S)$ is contained in the space of $S^1$-invariant sections of the bundle End of anti-$J$-holomorphic endomorphisms of $TM$ over $M$, and we need each gradient flow line $\gamma$ to go through a point $x \in M$ such that there are elements $Y \in T_J(J_S)$ whose value $Y(x)$ is an arbitrary element in $\text{End}_x$ and whose support intersects $\gamma$ in an arbitrarily small set. Since the isotropy group of $x$ is trivial for all points on $\gamma$ this is clearly the case; $\gamma$ is transverse to the level sets of the moment map $K$ and there are elements in $T_J(J_S)$ with support in $K^{-1}(a, a + \varepsilon)$ for arbitrarily small $\varepsilon$ and arbitrary value at $x$. If there were isotropy at $x$ then this argument would fail because $Y(x)$ would have to be fixed by $d\phi$ for all $\phi$ in the isotropy group at $x$.

The following lemma is adapted from Schwarz [19].

Lemma 4.6 Let $S^1$ act semifreely on a compact manifold $M$. Let $K : M \to \mathbb{R}$ be an $S^1$-invariant Morse-Bott function and let $g$ be an $S^1$-invariant metric so that the pair $(K, g)$ is Morse regular. Given a generic submanifold $C$ of a fixed component $F$, the unstable manifold $\overline{W^u(C)}$ is a pseudocycle.

Proof: The unstable manifold $W^u(C)$ is a submanifold of dimension $c + \alpha$, where $c$ is the dimension of $C$ and $\alpha$ is the index of $F$. Hence, we must show that $\overline{W^u(C)} \setminus W^u(C)$ has dimension at most $c + \alpha - 2$.

Because $C$ is generic, we may assume that $C$ is transverse to the map $\pi^{-1} : M(F_1, \ldots, F_n) \to F_1$ for every sequence of critical points $F = F_1, F_2, \ldots, F_n$. Therefore, $X = C \times_{\pi^{-1}} M(F_1, \ldots, F_n) \subset M(F_1, \ldots, F_n)$ is a submanifold of dimension $c + \alpha - \alpha_n$, where $\alpha_n$ is the index of $F_n$.

There is a smooth proper action of $\mathbb{R}$ on $X$, which moves the first coordinate $x_1$ along the gradient trajectory on which it lies. There is another smooth proper action of $S^1$ on $X$, which is given by the circle
action on $x_1$. If $n > 1$, the evaluation map $ev : X \rightarrow M$ defined by $ev(x_1, \ldots, x_n) = x_n$ is constant along the orbits of these actions. Hence, the image of the evaluation map has dimension at most $c + \alpha - 2$.

By Lemma 4.4, every point in the closure $\overline{W^u(C)}$ lies on a broken trajectory that begins on $C$, that is, it lies in the image of the evaluation map for $X = C \times_{\tau_\pm} M(F_1, \ldots, F_n) \subset M(F_1, \ldots, F_n)$ for some sequence of fixed points $F = F_1, \ldots, F_n$. Moreover, if the point does not lie $W^u(C)$ itself, then $n$ must be greater than one.

Lemma 4.7 Let $S^1$ act on a compact manifold $M$. Let $K : M \rightarrow \mathbb{R}$ be an $S^1$-invariant Morse-Bott function. Given a generic submanifold $C$ of a fixed component $F$ of index $\alpha_F$, there exists an $S^1$-invariant weighted pseudocycle $Z_C$ in $M^{K(F)}$ of dimension $\dim C + \alpha_F$ such that $Z_C \cap K^{-1}(K(F)) = C$.

Proof: In this case, as illustrated by the example after Lemma 4.4 there may be no $S^1$-invariant metric $g$ so that the pair $(K, g)$ is Morse regular. Instead, we begin with any $S^1$-invariant metric $g$, and then consider an $S^1$-invariant multivalued perturbation.

Briefly, the idea is this. Consider the space of all $S^1$-invariant smooth multivalued vector fields $Y$ on $M$. We will suppose for simplicity that $Y$ is single valued everywhere except on a finite number of disjoint slices of the form $M^u \backslash M^{u-\varepsilon}$ that contain no critical points of $K$, and that at each point $x$ in such a slice $Y(x)$ is a finite set that is invariant under the action of the isotropy group at $x$. The smoothness condition means that the graph $\{(x, v) : v \in Y(x)\}$ of $Y$ is a union of smoothly embedded open subsets of Euclidean space. For example, in the case of the blow up of CP$^2$ discussed at the beginning of this section, we allow $Y$ to take two values $\pm v(x)$ when $x \in M^u \backslash M^{u-\varepsilon}$ is near the isotropy submanifold. It is easy to check that there are enough perturbations of this kind so that for generic small $Y$ each solution $\gamma : \mathbb{R} \rightarrow M$ of the corresponding perturbed gradient flow relation

$$\frac{d}{ds} \gamma(s) \in \{- (\text{grad}_g K + Y)(\gamma(s))\}$$

is transverse to the level sets $K = \text{const}$ and regular in the sense of Salamon–Zehnder [18]. To keep the structure of the solution set as simple as possible we may assume that the number of elements in each set $Y(x)$ is constant and equal to $N$ for all $x$ lying in the interior of a slice, where $N$ is the l.c.m. of the orders of the stabilizer subgroups of the $S^1$-action. Then, a trajectory $\gamma$ that goes from a point $\gamma(-\infty) \in F$ to $F_{\text{min}}$ passes through some number $k$ of slices and hence satisfies one out of a set of $N^k$ possible equations. Moreover, because the set of trajectories that do not reach $F_{\text{min}}$ lie in a closed subset of codimension at least 2, there is a neighborhood $U$ of $\gamma(-\infty)$ in $F$ such that the set of trajectories that start in $U$ form a disjoint union of $N^k$ submanifolds. Thus for each generic submanifold $C$ in $F$ the set $W_C^{u,Y}$ of solutions to (15) that start at $C$ and end in $F_{\text{min}}$ is a manifold. As before, the transversality condition means that $W_C^{u,Y}$ intersects the corresponding stable manifolds transversally. (These are solutions to the relation $\frac{d}{ds} \gamma(s) \in \{(\text{grad}_g K + Y)(\gamma(s))\}$.) Hence the previous arguments apply to show that $W_C^{u,Y}$ is a pseudocycle. It is $S^1$-invariant by construction. Therefore we define $Z_C$ to be the weighted pseudocycle

$$Z_C := \frac{1}{N^k} W_C^{u,Y}.$$ 

This completes the construction.

Repeating the above construction for $-K$ we obtain upwards pseudocycles $Z_C^\pm$. It remains to prove that these extensions represent the canonical extensions $[C]^\pm$. 

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In this section we show that when calculating Gromov–Witten invariants on a manifold with $S^1$ of manifolds $X$ let $f$ be the dimension of $F$, $i$ be the dimension of $C$, and let $a$ be the index of $F$. Let $Y \in H^{i-1}(F)$ be the Poincaré dual to $C$. Let $Y^+ \in H^{i-1+\alpha}(M)$ denote the restriction to ordinary cohomology of the unique equivariant cohomology class $\tilde{Y}$ in $H^{i+\alpha}_S(M)$ described in Lemma 1.13. Recall from Section 1.2.1 that the upwards extension $[C]^+$ is defined to be the Poincaré dual of the restriction of $\tilde{Y}$ to $H^{i+\alpha}(M)$.

Proof: Let $Z^+$ denote either the pseudocycle $W^*(C)$ or the weighted pseudocycle $Z^+_C$, as appropriate. Let $f$ be the dimension of $F$, $i$ be the dimension of $C$, and let $a$ be the index of $F$. Let $Y \in H^{i+\alpha}(M)$ denote the restriction to ordinary cohomology of the unique equivariant cohomology class $\tilde{Y}$ in $H^{i+\alpha}_S(M)$ described in Lemma 1.13. Recall from Section 1.2.1 that the upwards extension $[C]^+$ is defined to be the Poincaré dual of the restriction of $\tilde{Y}$ to $H^{i+\alpha}(M)$.

Fix $N > d := f - i + \alpha$, and note that $\dim Z^+ = \dim M - d$. Since $Z^+$ is $S^1$-invariant, it can be extended to a cycle $(Z^+)_N := S^{2N+1} \times S^1$ $Z^+$ in the finite dimensional approximation $M_{S^1}^N := S^{2N+1} \times_{S^1} M$ to $M_{S^1}$. Denote by

$$\tilde{X}^N \in H^d(M_{S^1}^N)$$

the Poincaré dual of $(Z^+)_N$ in $M_{S^1}^N$. Clearly, the restriction of $\tilde{X}^N$ to $M$ is Poincaré dual to $[Z^+]$. Therefore, it is enough to show that $\tilde{X}^N$ is the restriction of $\tilde{Y}$ to $M_{S^1}$. By the injectivity of the restriction maps

$$H^d_{S^1}(M) \rightarrow H^d_{S^1}(M_{S^1}), \quad H^d_{S^1}(M) \rightarrow H^d_{S^1}(M_{S^1}),$$

it is enough to show that the restriction $\tilde{X}^N|_F$ of $\tilde{X}^N$ to $S^{2N+1} \times_{S^1} F$ satisfies the conditions (a), (b), and (c) of Lemma 1.13.

Because $Z^+$ has standard form near $C \subset F$, it is represented by the restriction over $C$ of the positive normal bundle of $F$. Therefore $\tilde{X}^N|_F = Y \cup e_F$ as required by property (b) of Lemma 1.13. Clearly, $\tilde{X}^N|_{F'} = 0$ for all fixed components $F'$ not equal to $F$, and let $\alpha'$ be the index of $F'$. We wish to show that the degree of $\tilde{X}^N|_{F'}$ in $H^*(BS^1)^N := H^*(\mathbb{C}P^N)$ is less than $\alpha'$, or equivalently that the degree of $\tilde{X}^N|_{F'}$ in $H^*(F')$ is greater than $d - \alpha'$. To prove this, it is enough to show that if $X \subset F'$ is a generic submanifold of dimension $d - \alpha'$, then $(S^{2N+1}/S^1) \times X \subset S^{2N+1} \times_{S^1} F$ does not meet $S^{2N+1} \times_{S^1} Z^\tau$. Hence it suffices to check that $X$ does not meet $Z^\tau$.

In the semifree case, $Z^+$ is the stable manifold $W^*(C)$ with respect to a generic metric $g_F$. By Lemma 4.4 every element in the closure $\overline{W}^*(C)$ lies on a broken geodesic ending at $C$. Therefore, by Lemma 4.3, $X \cap \overline{W}^*(C) \neq \emptyset$ only if $\dim X + \alpha' + \dim W^*(C) > \dim M$. Since by construction $\dim X + \alpha' + \dim W^*(C) = \dim M$, the intersection is empty.

In the general case, $Z^+$ is the sum of pseudocycles that are arbitrarily $C^0$-close to $\overline{W}^*(C)$. Therefore, the argument above shows that it can be constructed so that its closure $\overline{Z^\tau}$ is disjoint from any finite set of manifolds $X_i \subset F'$ that span the homology group $H_{d-\alpha'}(F')$. The result follows.

4.2 Localization

In this section we show that when calculating Gromov–Witten invariants on a manifold with $S^1$-action only the $S^1$-invariant stable maps contribute. Here is a formal statement of our results.

Let $(P, \omega)$ be a closed symplectic manifold, and, given classes $a_1, \ldots, a_k \in H_*(P)$, let $\alpha : Z \rightarrow P^k$ be a (possibly weighted) pseudocycle that represents their exterior product $a_1 \times \cdots \times a_k \in H_*(P^k)$. Define

$$\mathcal{N}_{a,k}(P, J, A; Z) := ev^{-1}(\overline{\alpha(Z)})$$
where $\text{ev} : \mathcal{M}(P, J, A) \to P^k$ is the evaluation map. First, we show that the calculation of the corresponding Gromov–Witten invariant can be localized in $P$ in the following sense.

**Lemma 4.9** The Gromov–Witten invariant $GW_P(a_1, \ldots, a_k; A)$ is a sum of contributions, one from each connected component of the cutdown moduli space $\mathcal{M}_{0,k}(P, J, A; Z)$.

Now consider the situation when $(P, \omega)$ carries an $S^1$-action. We assume that $J$ and the cycle $\alpha$ are $S^1$-invariant (what this means for cycles is explained in Definition 4.1) so that the cutdown space $\mathcal{M}_{0,k}(P, J, A; Z)$ also has an $S^1$-action.

**Proposition 4.10** Let $(P, \omega)$ be a closed symplectic manifold with an $S^1$-action $\{\phi_t\}_{t \in \mathbb{R}/\mathbb{Z}}$, and suppose that $J$ and $\alpha : Z \to P^k$ are $S^1$-invariant, where $\alpha$ represents $a_1 \times \cdots \times a_k$ as above. Then a connected component $C$ of $\mathcal{M}_{0,k}(P, J, A; Z)$ makes no contribution to $GW_P(a_1, \ldots, a_k; A)$ unless it contains an $S^1$-invariant element.

The next argument shows that this proposition is precisely what we need.

**Proof of Proposition 3.4.** Since the cycles $Z, Z'$ are $S^1$-invariant, the torus $T^2$ acts on the cutdown moduli space. Choose $N$ greater than the order of any of the isotropy subgroups of the $S^1$-action on $M$. Then the only sections of the bundle $P \to S^2$ that are invariant under the action of the subgroup $\{(Nt, t) : t \in \mathbb{R}^3\}$ of $T^2$ are the constant sections $\sigma_x$ at the fixed points $x \in M^{S^1}$. Therefore, it follows from Lemma 3.5 that the fixed points of this circle subgroup are the same as those for the action of the full torus. Hence by Lemma 4.9 and Proposition 4.10 the only components of the cutdown moduli space $\mathcal{M}^{cut} := \mathcal{M}_{0,k}(P, J, A; Z)$ that contribute to the GW invariant are those containing $T^2$-invariant elements.

The second statement in Proposition 3.4 goes one step further, and claims that the GW invariant is a sum of contributions one from each component of the space of invariant elements $(\mathcal{M}^{cut})^{T^2}$ in the cutdown moduli space. This is proved by applying the proof of Lemma 4.9 to the components of $(\mathcal{M}^{cut})^{T^2}$. The details are straightforward, and are left to the reader. \hfill $\square$

We now explain the idea of the proof of Proposition 3.4 assuming for simplicity that $\mathcal{M}_{0,k}(P, J, A; Z) := C$ is connected. If $C$ contains no $S^1$-invariant elements, $S^1$ acts with finite stabilizers on $C$ and hence also on some neighborhood $N(C)$ of $C$ in $\mathcal{M} := \mathcal{M}_{0,k}(P, J, A)$. We will see that we may give the quotient $N(C)/S^1$ an orbifold structure and hence construct the regularized moduli cycle $\text{ev}^{\nu} : \mathcal{M} \to P^k$ so that its subset

$$C^{\nu} := (\text{ev}^{\nu})^{-1}(\alpha(Z))$$

has a neighborhood $N(C^{\nu})$ that supports a free $S^1$-action. Moreover, $\text{ev}^{\nu} : N(C^{\nu}) \to P^k$ is $S^1$-equivariant. We will explain below the precise nature of the regularization $\mathcal{M}^{\nu}$, but suppose for now that it is a closed manifold. It then suffices to apply the following fact. Suppose that a closed oriented manifold $X$ supports a free $S^1$-action and that $f : X \to P$ is equivariant. Then $f : X \to P$ may be perturbed to an equivariant map whose image is disjoint from the closure of the image of any invariant pseudocycle $\alpha : Z \to P$ of complementary dimension. This holds because locally $X$ is the product of a transverse slice $Y$ with $S^1$, and it suffices to perturb the restriction $f|_{Y}$ so that it is disjoint from $\alpha(Z)$ and then extend by equivariance.

It is essential here that the action on $X$ is free; otherwise one could not extend an arbitrary perturbation of $f|_Y$ to $X$.

Similar arguments have been used by many authors, for example in connection with the calculation of the Floer homology of a time independent small function: cf. Fukaya–Ono [5] and Liu–Tian [8].
only difference is we are here dealing with an external $S^1$-action (i.e. one on the range of the stable maps) rather than a reparametrization action which lives on the domain.

To carry out the details of the proof we will first describe how to construct the regularized (or virtual) moduli cycle $ev^\nu : \overline{M}^\nu \to P^k$. We shall then prove Lemma 4.9, and finally the proposition. As in McDuff [10, 11], we will use the regularization process of Liu–Tian [8]; readers can substitute their preferred constructions.

4.2.1 Branched pseudocycles

To start, we describe what kind of object the virtual moduli cycle is. For more details see [8, 10].

First, it is a $d$-dimensional partially smooth space $\iota_X : X_{sm} \to X$. Here $X$ is a compact Hausdorff space (this is called the first topology), $\iota$ is a bijective continuous map, and $X_{sm}$ is a union of a finite number of disjoint smooth manifolds $X^i$ of dimensions $i \leq d$. The connected components of $X_{sm}$ are called strata. Maps from one partially smooth space to another are given by commutative diagrams

$$
\begin{align*}
X_{sm} \xrightarrow{\iota} X \\
\downarrow \quad \downarrow \\
Y_{sm} \xrightarrow{\iota} Y,
\end{align*}
$$

but for short they are often written $f : X \to Y$. Also any compact smooth manifold is a partially smooth space in which $P$ is given the usual topology and $P_{sm}$ has one stratum. Hence a partially smooth map $f : X \to P$ is continuous when thought of as a map from the Hausdorff space $X$ to the metric space $P$, and smooth when restricted to each stratum of $X_{sm}$. The virtual moduli cycle is a compact branched partially smooth labelled pseudocycle, or (compact) branched pseudocycle for short. This means it is a $d$-dimensional partially smooth space such that each $d$-dimensional stratum $X_j$ is oriented, has a rational label, and fits together with $(d-1)$-dimensional strata to form a branched manifold. More precisely, the closure $\overline{X_j}$ in $\iota(X^d \cup X^{d-1})$ of each component $X_j$ of $X^d$ can be given the structure of an oriented manifold with boundary; moreover, when one divides the top dimensional faces that meet each $(d-1)$-dimensional component into two sets according to their orientations, the sum of the labels in each of these sets must be equal. If $X$ has such structure then any map $f : X \to P$ represents a unique rational $d$-dimensional homology class. Just as with pseudocycles, there is an obvious notion of bordism.

A compact branched pseudocycle $X$ is said to have a free $S^1$-action with local slices if each point $x \in X$ has a neighborhood $U_{sm} \to U$ that is isomorphic to the product $Y_{sm} \times S^1 \to Y \times S^1$ with action $t \cdot (y, s) = (y, s + t)$.

The relevance of these definitions to the current problem is clear from the following lemma.

**Lemma 4.11** Suppose that the smooth manifold $P$ supports an $S^1$-action, that $X$ is a compact branched labelled pseudocycle with free $S^1$-action and that $f : X \to P$ is equivariant. Then $f$ can be perturbed to an equivariant map that is disjoint from the closure of the image of any $S^1$-invariant pseudocycle $\alpha : Z \to P$ of complementary dimension. Hence $f \cdot \alpha = 0$.

**Proof:** As before, local equivariant perturbations of $f$ may be constructed by perturbing $f$ on the local slices $Y$. The perturbation is constructed by induction over the strata $S$, starting with those of lowest dimension. The perturbations have the form $f|_{Y \cap S} \mapsto \phi \circ f|_{Y \cap S}$ where $\phi$ is a suitable small diffeomorphism of $P$, and hence, even though we have little control over the way the strata in $Y$ fit together, always extend from $Y \cap S$ to $Y$. Further details are left to the reader.  

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The above lemma does not extend to pseudocycles with free $S^1$-action, since the definition of pseudocycle does not give us enough control of the boundary. As an example, consider the standard $S^1$-action on $S^2$ and take $f : \mathbb{C} \setminus \{0\} \to S^2 = \mathbb{C} \cup \{\infty\}$ and $g : \{pt\} \to \{0\}$. Then $f$ is a pseudocycle representing the fundamental class, and it has a free $S^1$-action in the sense that $f$ is equivariant with respect to a free $S^1$-action on its domain. Nevertheless $f \cdot g \neq 0$. The reason is the following. By definition, $f \cdot g$ is calculated by first perturbing $f$ so that its boundary is disjoint from that of $g$ and then counting transverse intersection points: see [12, 14, Chapter 6]. In this example, the boundary of $f$ meets im $g$ in an essential way and we cannot prevent this by a hypothesis concerning only the $S^1$-action on the open set $\mathcal{Z}$; we must work with a closed domain.

4.2.2 Construction of the regularization

The basic idea in the construction of $\mathcal{M}_g^\nu$ is to perturb the compactification $\mathcal{M} := \mathcal{M}_g^\nu(P,J,A)$ of $\mathcal{M}_0^\nu(P,J,A)$ to a cycle of the correct dimension. The analytic input to the construction explained below is the standard gluing result, see [8, 5] or [14, Chapter 10] for example; the rest of the construction is purely topological. The most important step in the proof of Proposition 3.4 is to construct the local uniformizers of Step 1 below so that they support a free $S^1$-action.

The regularization process has four steps.

Step 1: Denote by $\tilde{B}$ the space of $k$-pointed stable maps $\tilde{\tau} = (\Sigma(u), u, z)$ where $\tilde{\tau}$, though not necessarily $J$-holomorphic, has the property that the group of self-maps $\Gamma_\tau = \{ \gamma : u \circ \gamma = u \}$ is finite. Let $B$ be the space of equivalence classes of such $\tilde{\tau}$. (More details are given in §4.2.3 below.) The elements of $B$ are organized into strata, depending on the topological types of their domains, and one can show that $B$ has the structure of an orbifold in the partially smooth category. (Objects in this category are spaces $B_{sm} \to B$ with two topologies, where the first is Hausdorff and the second is a finite union of disjoint Banach manifolds.) Thus each point $\tau \in B$ has a neighborhood $U$ with a uniformizer $(\tilde{U}, \pi, \Gamma)$ where $\pi : \tilde{U} \to \tilde{U}/\Gamma = U$ identifies $U$ with the quotient of $\tilde{U}$ by the action of the finite group $\Gamma := \Gamma_\tau$. Since the elements in $\tilde{U}$ are stable maps, constructing $\tilde{U}$ amounts to choosing a consistent set of parametrizations for the elements $\tau \in U$. More details are given below. Because $\mathcal{M} \subset B$ is compact, it is contained in the union $W$ of a finite number $U_1, \ldots, U_N$ of such locally uniformized sets $U$, each of which is a neighborhood of some point $\tau \in \mathcal{M}$. Throughout the construction one decreases the size of each $U_i$ (and hence increases their number) as appropriate. Our notational convention is that objects living on the uniformizers $U$ are decorated with tildes.

Step 2: We interpret the operator $\bar{J}_f$ as a section of an orbibundle $\mathcal{L} \to \mathcal{W}$. For each $U$ there is a locally trivial bundle $\tilde{\mathcal{L}}_U \to \tilde{U}$ on which the local isotropy group $\Gamma$ acts. The fiber of $\tilde{\mathcal{L}}_U$ at $\tilde{\tau} = (\Sigma(u), u, z) \in \tilde{U}$ is the space

$$\tilde{\mathcal{L}}_{\tilde{\tau}} := L^p(\Sigma(u), \Lambda^0_1 \otimes u^*(TM))$$

of anti-$J$-holomorphic 1-forms on $\Sigma(u)$ of class $L^p$ with values in $u^*(TM)$. The perturbations used to define $\mathcal{M}^\nu$ are built from sections of the local bundles $\tilde{\mathcal{L}}_U \to \tilde{U}$. In order to extend these local sections, we construct another object $\tilde{\mathcal{L}} \to \tilde{\mathcal{W}}$ from $\mathcal{L} \to \mathcal{W}$ that is called a multibundle. Here $\tilde{\mathcal{L}} \to \tilde{\mathcal{W}}$ is a collection of compatible maps $\tilde{\mathcal{L}}_I \to \tilde{V}_I$, where $I$ is a subset of the indexing set $\{1, \ldots, N\}$ for the $U_i$, $V_I$ is a suitable subset of $\cap_{i \in I} U_i$ and $\tilde{V}_I$ (resp. $\tilde{\mathcal{L}}_I$) is the fiber product of the $\tilde{U}_i$ (resp. $\tilde{\mathcal{L}}_I$) over $V_I$. The details of this construction are not important for what follows. All we need to know is that each section $\tilde{s}(\nu)$ of $\tilde{\mathcal{L}} \to \tilde{\mathcal{W}}$ (called a multisection) consists of a compatible collection $\{\tilde{s}(\nu)_I\}$ of multivalued sections of $\tilde{\mathcal{L}}_I \to \tilde{V}_I$. It turns out that each $\tilde{s}(\nu)_I$ is single valued over the top strata, but may well be multivalued over lower dimensional strata.

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Step 3: We construct a finite dimensional vector space $R$ and a map $\nu \mapsto \bar{s}(\nu)$ of $R$ into the space of multisections of $\bar{L} \rightarrow \bar{W}$ with the property that for generic small $\nu \in R$ the section $\bar{\partial}_J + \bar{s}(\nu)$ is transverse to the zero section. The vector space $R$ is a sum $\bigoplus_{i \in I} R_i$, where $\{U_i\}$ is an open covering of $W$ and for each $i$ $R_i$ is a suitable finite dimensional space of sections of $\bar{L}_{U_i} \rightarrow \bar{U}_i$. This space $R_i$ is formed from the local obstruction bundle. The essential requirement is that for each stable map $\bar{\tau} = (\Sigma(U_i), u, z) \in \bar{U}_i$ the subspace of $\bar{L}_{\bar{\tau}}$ formed by the values $\{\nu(\bar{\tau}) : \nu \in R_i\}$ projects onto the cokernel of the linearization $D_{\bar{\nu}}$ of $\bar{\partial}_J$ at $u$. The fact that suitable finite dimensional spaces $R_i$ exist is a consequence of the gluing construction and the compactness of $\bar{M}$. To see this, choose for each $\tau \in U \subset \bar{M}$:

(a) a lift $\tilde{\tau} = (\Sigma(U), u, z) \in \tilde{U}$ of $\tau$; and

(b) a subspace $R_{\tilde{\tau}} \subset \bar{L}_{\tilde{\tau}}$ that covers the cokernel of $D_{\bar{\nu}}$.

Then extend the elements $\nu \in R_{\tilde{\tau}}$ by parallel translation along small paths in $M$ to sections $\tilde{\nu}' \mapsto \nu(\tilde{\nu}')$ of $\bar{L}_{\tilde{\tau}}$ defined over some small neighborhood $N(\tilde{\tau})$ of $\tau$ in $\tilde{U}$. By the gluing construction, the subspace

$$R_{\tilde{\tau}}(\tilde{\nu}') = \{\nu(\tilde{\nu}') : \nu \in R_{\tilde{\tau}}\} \subset \bar{L}_{\tilde{\tau}},$$

projects onto coker$D_{\bar{\nu}}$ when $\tilde{\nu}'$ is sufficiently close to $\tilde{\tau}$. Moreover, we can choose this subset $\tilde{U}_{\tilde{\tau}}$ of $\tilde{U}$ to be invariant under the stabilizer group $\Gamma_{\tilde{\tau}}$ so that it has the form $\pi^{-1}(U_{\tilde{\tau}})$ for some neighborhood $U_{\tilde{\tau}}$ of $\tau$ in $\bar{M}$. Therefore, by compactness of $\bar{M}$, there is a finite set $\tau_1$ such that the corresponding pairs $(U_i, R_i) := (U_{\tau_i}, R_{\tilde{\tau}})$ have the required properties.

This defines the finite set of local pairs $(U_i, R_i), 1 \leq i \leq N$. One shows that each $\nu \in R_i$, when multiplied by a suitable cutoff function, gives rise to a multisection $\bar{s}(\nu)$ of $\bar{L} \rightarrow \bar{W}$. The most important point here is that the construction is local in $\bar{M}$, i.e. for each $\nu \in R_i$ the section $\bar{s}(\nu)_J = 0$ whenever the closure $\bar{U}_i$ is disjoint from all the sets $\bar{U}_j, j \in I$. Now set $R := \bigoplus_i R_i$. It follows from the construction that the local multisections $\bar{\partial}_J + \bar{s}(\nu)_J$ are transverse to the zero section for generic small $\nu \in R$. Hence the local zero sets $\bar{Z}_J \subset \bar{V}_J$ are submanifolds of the correct dimension $d$.

Step 4: We construct from the local zero sets $\bar{Z}_J$ of $\bar{\partial}_J + \bar{s}(\nu)$ a compact branched $d$ dimensional pseudomanifold $\bar{M}_{0,k}$. Its bordism class is independent of choices. There is a natural projection map

$$\text{proj} : \bar{M}_{0,k}(P, J, A) \rightarrow W$$

such that each element in the image lies in the zero set of the multivalued section $\bar{\partial}_J + \nu$, and the evaluation map factors through this projection. Moreover, the strata in $\bar{M}_{0,k}$ of dimensions $d, d - 1$ project to the top stratum of $W$, i.e. into stable maps whose domain has a single component. Therefore, when one evaluates the intersection number of $ev : \bar{M}_{0,k}(P, J, A) \rightarrow P^k$ with a cycle in $P^k$ one will be counting rationally weighted curves $u : S^2 \rightarrow P$ that satisfy a perturbed Cauchy–Riemann equation $\bar{\partial}_J u + \nu(u) = 0$.

Definition 4.12 The Gromov–Witten invariant $GW_P(a_1, \ldots, a_k; A)$ is the intersection number of the evaluation map $ev : \bar{M}_{0,k}(P, J, A) \rightarrow P^k$ with a generic representing pseudocycle $\alpha := Z \rightarrow P^k$ for the class $a_1 \times \cdots \times a_k$:

$$GW_P(a_1, \ldots, a_k; A) := ev \cdot \alpha.$$

It is zero by definition if the dimensional condition $\dim P + 2c_1(A) + 2k - 6 + \sum_i \dim a_i = k \dim P$ is not satisfied.
Lemma 4.9 claims that one would get the same answer by first cutting down the moduli space to \(\overline{M}(P, J; A; Z)\) and then regularizing each of its components separately.

**Proof of Lemma 4.9.** Let \(C_j, 1 \leq j \leq \ell\), be the connected components of \(\overline{M}_{0,k}(P, J; A; Z)\). By compactness there is \(\varepsilon > 0\) so that the set

\[
\mathcal{N}^{2\varepsilon} := \{z \in \overline{M} : d(\text{ev}(z), \alpha(Z)) \leq 2\varepsilon\},
\]

where \(d\) is the metric in \(P^k\), has \(\ell\) connected components \(\mathcal{N}_j^{2\varepsilon} \supset C_j\). Now choose pairs \((U_i, R_i)\) as in Step 3, where the open subsets \(U_i \subset B\) separate out the components \(C_j\) in the following sense: if \(U_i \cap (\cup_j \mathcal{N}_j^{2\varepsilon}) \neq \emptyset\) and \(U_i \cap (\overline{M} \setminus (\cup_j \mathcal{N}_j^{2\varepsilon})) \neq \emptyset\) then \(U_i\) and \(U_k\) are disjoint. Then construct a regularization \(\overline{\mathcal{M}}^\nu\) as described above. By the definition of \(\mathcal{N}_j^{2\varepsilon}\), the image under ev of the set \(\overline{\mathcal{M}}^\nu \setminus \text{proj}^{-1}(\cup_j \mathcal{N}_j^{2\varepsilon})\) has distance at least \(\varepsilon\) from \(\alpha(Z)\) and so does not contribute to the intersection \(\text{ev} \cdot \alpha\). On the other hand because the construction in Step 3 is local, the structure of \(\mathcal{N}(C_j^\nu) := \overline{\mathcal{M}}^\nu \cap \text{proj}^{-1}(\mathcal{N}_j^{2\varepsilon})\) depends only on the choices made for the open sets covering \(\mathcal{N}_j^{2\varepsilon}\). Hence if we define the intersection number of \(\text{ev} : \mathcal{N}(C_j^\nu) \rightarrow P^k\) with \(\alpha\) as the local contribution of \(C_j\) to the Gromov–Witten invariant \(\text{ev} \cdot \alpha\), this invariant is the sum of local and independent contributions as claimed. \(\square\)

Here is an easy corollary.

**Corollary 4.13** Let \(\alpha : Z \rightarrow P^k\) represent the class \(a_1 \times \cdots \times a_k\) in \(P^k\). If \(\overline{M}_{0,k}(P, J; A; Z) = \emptyset\) then \(GW_P(a_1, \ldots, a_k; A) = 0\).

### 4.2.3 The moduli space of stable maps as an orbifold

As preparation for the proof of Proposition 4.10, we describe the orbifold structure on the space of stable maps.

Let \((T, E)\) be a finite tree where \(T\) denotes the set of vertices and the relation \(E \subset T \times T\) describes the set of oriented edges. A genus zero stable map with \(k\) marked points modelled on \(T\) is a tuple

\[
\left(\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha \in E}, \{z_i, \alpha_i\}_{1 \leq i \leq k}\right)
\]

where \(u_\alpha : S^2 \rightarrow P\) is a map, \(z_{\alpha\beta} \in S^2\) denotes the point on the \(\alpha\)-th sphere that attaches to the \(\beta\)-th sphere, and \(z_i \in S^2\) is the \(i\)-th marked point lying on the \(\alpha_i\)-th sphere. Thus its domain \(\Sigma(\mathbf{u})\) is the quotient of \(S^2 \times T\) in which \((z_{\alpha\beta}, \alpha) \sim (z_{\beta\alpha}, \beta)\) whenever \(\alpha \in E\). We require that \(u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})\) whenever \(\alpha \in E\beta\) so that the \(u_\alpha\) induce a map \(\mathbf{u} : \Sigma(\mathbf{u}) \rightarrow P\). The special points \(\{z_{\alpha\beta} \mid \beta \in T\} \cup \{z_i \mid \alpha_i = \alpha\}\) on the \(\alpha\)-th sphere are assumed distinct. The stability condition states that every ghost component (i.e. component of \(\Sigma(\mathbf{u})\) on which \(\mathbf{u}\) is constant) has at least 3 special points.

Two such tuples \(\left(u_\alpha, z_{\alpha\beta}, (z_i, \alpha_i)\right)\) and \(\left(u_\alpha', z_{\alpha'\beta'}, (z_i', \alpha_i')\right)\) modelled on \(T, T'\) are equivalent if there is a tree isomorphism \(f : T \rightarrow T'\) and a collection \(\phi_\alpha \in PSL(2, \mathbb{C}), \alpha \in T, \) such that

\[
uf(\alpha) \circ \phi_\alpha = u_\alpha, \quad \phi_\alpha(z_{\alpha\beta}) = z_{f(\alpha)f(\beta)}, \quad f(\alpha_i), f(\alpha_i) = (z_i', \alpha_i').
\]

We shall call such tuples \(\mathbf{(u, z)}\) for short. The elements \(\tau = [\mathbf{u, z}]\) of the moduli space of stable maps \(\mathcal{M}\) are equivalence classes of such tuples. Each stratum consists of equivalence classes of stable maps modelled on a fixed tree \(T\) and has an obvious smooth topology. The Hausdorff topology on the whole space is discussed below.
We now describe Liu–Tian’s construction of the uniformizers \((\tilde{U}_\tau, \pi, \Gamma_\tau)\), where \(\tau = [u, z]\) is modelled on \(T\). Note that each uniformizer \(\tilde{U}_\tau\) is a subset of the ambient space \(\tilde{B}\). Let us first suppose that \(\Gamma_\tau = \{1\}\). Then the problem is to find a consistent way of parametrizing all the stable maps near \(\tau\). Choose a parametrization
\[
\tilde{\tau} := (u, z) := (u_\alpha, z_{\alpha\beta}, (z_i, \alpha_i))
\]
of \(\tau\). Add the minimum number of points \(w := (w_1, \alpha_{k+1}), \ldots, (w_t, \alpha_{k+t})\) to the set of labelled points in \(\tilde{\tau}\) to make its domain stable, i.e. so that each component has at least 3 special points. Pick out three of them for each component \(\alpha\) and denote by \(Y\) the resulting subset of the \(z_{\alpha\beta}, z_i\). The set \(w\) is chosen to be invariant under the action of any element in \(\Gamma_\tau\) that permutes the components of \(\Sigma(u)\), but so that in each component no two are on the same orbit of the stabilizer of this component in \(\Gamma_\tau\). Next choose for each \(j = 1, \ldots, t\), a small open disc \(H_j\) in \(P\) that is transverse to the image of \(u\) at the points \(u(w_j)\). (This is possible because the ghost components are already stable and hence never contain any of the added points \(w_j\).) If \(D_T\) denotes the stratum in \(\mathcal{B}\) containing \(\tau\) we define \(\tilde{U}_\tau \cap D_T\) to be a neighborhood of \(\tilde{\tau}\) in the slice
\[
\mathcal{S}_T := \left\{(u'_\alpha, z'_{\alpha\beta}, (z'_i, \alpha_i)) \mid (u'_{\alpha_{k+j}}, (w_j)) \in H_j, z'_{\alpha\beta} = z_{\alpha\beta} \text{ if } z_{\alpha\beta} \in Y, z'_i = z_i \text{ if } z_i \in Y \right\}.
\]
The domains of the stable maps \((u', z')\) near \(\tilde{\tau}\) are formed from the domain of \(\tilde{\tau}\) by gluing its components via the gluing parameters \(a_{\alpha\beta} \in T_{z_{\alpha\beta}}(\Sigma^2_{\alpha\beta}) \otimes T_{z_{\alpha\beta}}(\Sigma^2_{\beta\alpha})\). (Here for convenience we denote the \(\alpha\) component by \(\Sigma^2_{\alpha\beta}\).) Assuming \(r = |a_{\alpha\beta}|\) is sufficiently small we glue \(\Sigma^2_{\alpha\beta} \setminus B_r(z_{\alpha\beta})\) to \(\Sigma^2_{\beta\alpha} \setminus B_r(z_{\beta\alpha})\) along their boundaries by a rotation determined by \(\arg(a_{\alpha\beta})\). To describe this more precisely, let us suppose for simplicity that \(\tau\) does correspond to a unique tuple \((u, z)\) where the domain \(\Sigma^2_{\alpha\beta}\) has a unique identification \(\psi: \Sigma^2_{\alpha\beta} \to \Sigma^2_{\alpha'\alpha}\) at the points \(z_i\). Each such sphere \(\Sigma_{\alpha\beta}\) has a unique identification \(\psi: \Sigma_{\alpha\beta} \to \Sigma^2_{\alpha'\alpha}\) with its image in \(\Sigma^2_{\alpha\beta}\) in the obvious way. Note also that one can define the \(a\) so that \(c(a) = a\). Hence the tuple \((u; u', (z'_i, \alpha_i))\) can be written uniquely as a stable map \((u', z')\) \(\in \mathcal{M}_{0,k}(\Sigma^2; A, J)\) where \(u' : \Sigma^2 \to P\) is the composite \(u' \circ (\psi_{a_i})^{-1}\), and \(z'_{\alpha\beta} = \psi_{a_i}(z'_{\alpha\beta})\). Conversely, each stable map that is sufficiently close to \(\tilde{\tau}\) does correspond to a unique tuple \((u; u', (z'_i, \alpha_i))\) since the gluing parameter \(a\) is determined by the cross ratio of the four marked points \(z_1, z_2, z_3, z_4\). Therefore we may extend the slice \(\mathcal{S}_T\) by setting
\[
\mathcal{S} := \left\{(a; u', (z'_i, \alpha_i)) \mid u' : S_a \to P, u'(w_{k+j}) \in H_j, z'_i = z_i \text{ if } z_i = y_{am} \text{ for some } a, m \right\}.
\]
Finally we define \(\tilde{U}_\tau\) to be a neighborhood of \(\tilde{\tau}\) in \(\mathcal{S} \cup \mathcal{S}_T\). The projection to \(\mathcal{B}\) is given by dividing by the reparametrization group, i.e. by taking a stable map to its equivalence class. (We have not given a satisfactory description of the topology on \(\mathcal{B}\); for this see [8, 10].)
Now suppose that $\Gamma_\tau \neq \{\mathbb{I}\}$. We must extend the action of $\Gamma_\tau$ to $\bar{U}_\tau$. Suppose first that $\Gamma_\tau$ is a rotation group of order $n > 1$ with generator $\gamma \in \text{PSL}(2, \mathbb{C})$ that acts on a single component $\alpha_0$ of $\Sigma(u)$. This component can have at most two special points $y_1, y_2$. Let us suppose that it has precisely two, say $y_1, y_2$, and therefore one added point that we will call $w_1$. We may suppose $w_1$ chosen so that the set $w_{\alpha_0}(u_{\alpha_0}(w_1))$ contains $n$ distinct points at which $d\nu_{\alpha_0} \neq 0$. Choose disjoint little discs in $S_{\alpha_0}^1$ about these points that are permuted by $\gamma$. For any element $(u', z')$ that is close to $\bar{\tau}$ and in the same stratum, $(u'_{\alpha_0})^{-1}(H_1)$ is a collection of $n$ points, one in each of the little discs. Therefore there is unique point $w'$ in the little disc containing $\gamma(w_1)$ such that $u'(w') \in H_1$, and we define $\psi^\gamma_{u'} \in \text{PSL}(2, \mathbb{C})^{[T]}$ to be the unique element that acts as the identity in all components except for the $\alpha_0$-th and there fixes $y_1, y_2$ and takes $w_1$ to $w'$. Then set

$$
\gamma \cdot (u', z') = (u' \circ \psi^\gamma_{u'}, z') \in \bar{U}_\tau.
$$

It is not hard to check that this does define action of $\Gamma_\tau$ on a neighborhood of $\bar{\tau}$ in $\bar{U}_\tau \cap D_{\tau}$.

It extends over the full neighborhood $\bar{U}_\tau$ by acting on the gluing parameters $a$. We give a precise description in the case with $|T| = 2$ considered above. If $(a; u', w_1, (z'_i, \alpha_i)) \in S$ is sufficiently close to $\bar{\tau}$, then $(u')^{-1}H_1 \subset S_\alpha$ consists of $n$ points with precisely one, call it $w'$, in the little disc containing $\gamma(w_1)$. Because the map $a \mapsto c(a)$ is a diffeomorphism, there is a unique gluing parameter $\gamma(a)$ for which there is a biholomorphic map

$$
\psi_\gamma : (S_\gamma(a); y_0 = w_1, y_02, y_{\infty1}, y_{\infty2}) \longrightarrow (S_\alpha; y_0 = w', y_02, y_{\infty1}, y_{\infty2}).
$$

We define

$$
\gamma \cdot (a; u', (z'_i, \alpha_i)) := \left(\gamma(a); u' \circ \psi_\gamma, (\psi^{-1}_\gamma(z'_i), \alpha_i)\right) \in S.
$$

Alternatively, if we write the elements of $S$ in the form $(u'', z''_i)$ where $u'' : S^2 \longrightarrow P$ then

$$
\gamma \cdot (u'', z''_i) = (u'' \circ h, h^{-1}(z''_i)), \quad h := \psi_\alpha \circ \psi_\gamma \circ (\psi_\gamma(a))^{-1}.
$$

Again, one can check that this gives a well defined action of $\Gamma_\tau$ on $\bar{U}_\tau$. Its continuity (which is somewhat tricky) is proved in [11, §4.2]; see also [10].

The construction for other groups $\Gamma_\tau$ is similar. We need to consider the case when $\Gamma_\tau$ acts in a single component with one or no special points; then consider products of such actions; and finally consider an action that also permutes the components. These extensions are described in [8].

**Proof of Proposition 4.10** Let $C$ be a component of $\overline{M}_{0,k}(P, J, A, Z)$ on which the induced action of $S^1$ is locally free. We will show that its regularization $\mathcal{N}(C^u)$ can be constructed so as to support a free $S^1$-action. The result then follows from Lemma 4.11.

We show below that $C$ can be covered by $S^1$-invariant sets $W_j$ such that their uniformizers $\tilde{W}_j$, as well as the uniformizers of all sets they meet, support a free $S^1$-action with local slices $\tilde{Y}_j$. Granted this, the construction of the $(\bar{U}_i, R_i)$ in Step 3 can be made so that the sections in $R_i$ are $S^1$-invariant. To see this, choose for each $\tau \in W_j$ with lift $\tilde{\tau}$ a suitable finite dimensional space $R_{\tilde{\tau}}$ of the fiber $\tilde{L}_{\tilde{\tau}}$, extend its elements to a neighborhood $\tilde{Y}_{\tilde{\tau}}$ of $\tilde{\tau}$ in the slice $\tilde{Y}_j$ by parallel translation in $P$ and then extend over the product $\bar{U}_j := \tilde{Y}_j \times S^1$ using the $S^1$-action on $P$. Since this action preserves $J$ the transversality conditions continue to hold over the $S^1$-orbit.

Hence the local zero sets $Z^u_j$ all carry a free $S^1$-action with local slices. The local virtual cycle $\mathcal{N}(C^u)$ is made from these zero sets using partitions of unity, and one can check that its construction can carried in a way that respects the $S^1$-action. Moreover the induced $S^1$-action is free because each point in $\mathcal{N}(C^u)$
projects to one of the sets $\tilde{V}_i$ and hence to $\tilde{U}_i, i \in I$, where the action is free by construction: for details see Proposition 4.13 in [10].

Hence it remains to construct the $W_j$. To do this, we construct a different set of local uniformizers $(\tilde{W}_\tau, \pi, \tilde{\Gamma}_\tau)$ for a neighborhood $N(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{B}$ whose stabilizer subgroups $\tilde{\Gamma}_\tau$ incorporate not only the automorphism groups $\Gamma_\tau$ of the stable maps $\tau \in V$, but also the (finite) stabilizer subgroups $\text{Stab}(\tau) \subset S^1$ of the locally free $S^1$-action on $N(\mathcal{C})$. This amounts to defining an orbifold structure on the quotient $N(\mathcal{C})/S^1$ whose elements are equivalence classes $[\Sigma(u), u, z]_S$, where the equivalence relation $\sim_S$ is generated by the previous relation $\sim$ coming from the action of the reparametrization group together with the equivalence

$$(\Sigma(u), u, z) \sim_S (\Sigma(u), \phi_t \circ u, z), \quad t \in S^1,$$

where $\phi_t : P \to P$ denotes the action of $t \in S^1$.

The first task is to define the local group $\tilde{\Gamma}_\tau$ at $\tau \in \mathcal{C}$. Choose a parametrization $\tilde{\tau} = (u, z)$. Let $\Gamma_\tau := \{ \gamma \in \text{Aut}(\Sigma(u)) : u \circ \gamma = u \}$ denote its automorphism group, and denote by $N$ the order of the stabilizer subgroup $\text{Stab}(\tau)$ of $\tau$ in $S^1$. Then define

$$\tilde{\Gamma}_\tau := \{ (\gamma, k) \in \text{Aut}(\Sigma(u)) \times \text{Stab}(\tau) \mid u \circ \gamma = \phi_N \circ u \}. $$

There are exact sequences

$$\text{Stab}(\tau)' \hookrightarrow \text{Stab}(\tau) \to \text{Stab}(\tau)'' \quad \Gamma_\tau \times \text{Stab}(\tau)' \hookrightarrow \tilde{\Gamma}_\tau \to \text{Stab}(\tau)''$$

where $\text{Stab}(\tau)' := \{ t \in \mathbb{R}/\mathbb{Z} \mid \phi_t \circ u = u \}$ is the stabilizer subgroup of the image of $u$ in $P$.

We must show that every $\tau \in \mathcal{C}$ has a neighborhood $W_\tau$ with a uniformizer $(\tilde{W}_\tau, \pi, \tilde{\Gamma}_\tau)$ such that $(\tilde{W}_\tau, \tilde{\Gamma}_\tau)$ is equivariantly isomorphic to a product $\tilde{X} \times S^1$ with action induced by

$$(\gamma, k) \cdot (u', t) := (u' \circ \gamma, t - k/N).$$

Then the projection $\pi$ given by $\pi(u', z', t) := \phi_t \circ (u', z')$ is well defined and $S^1$-equivariant, and does quotient out by the action of $\tilde{\Gamma}_\tau$. For these formulas to make sense $\tilde{\Gamma}_\tau$ must be invariant under the action of $\Gamma_\tau \times \text{Stab}(\tau)'$. To find such a slice $\tilde{Y}_\tau$ we will use the fact that, by hypothesis, $\text{Stab}(\tau)$ is finite.

Suppose first that $\Gamma_\tau = \{1\}$. Choose the added points $w_j$ to be generic, i.e. so that $dw_j(w_j) \neq 0$ and the stabilizers $\text{Stab}(w_j(u))$ of the points $u(w_j) \in P$ are as small as possible, and then choose the slices $H_j \subset P$ to be $\text{Stab}(u(w_j))$-invariant. Note that $\text{Stab}(\tau)' \subseteq \text{Stab}(w_j(u))$ for all $j$. Suppose in addition that it is possible to choose one of the added points, say $w_2$, so that $\text{Stab}(u(w_2))$ is finite. Then there is a $\text{Stab}(u(w_2))$-invariant codimension 1 disc $X$ through $u(w_2)$ that is transverse both to the $S^1$-action and to $H_2$, and we set $H_2' := H_2 \cap X$. Then

$$\tilde{Y}_\tau := \{ \tilde{\tau}' \in \tilde{X} \mid u'(w_2) \in H_2' \}$$

is a slice for the induced local $S^1$-action on $\tilde{X}$; in particular it is $\text{Stab}(\tau)'$-invariant. Hence we may take

$$\tilde{W}_\tau := \tilde{Y}_\tau \times S^1, \quad W_\tau := \{ \phi_t \cdot \tau' \mid \tau' \in \pi(\tilde{Y}_\tau), t \in S^1 \}.$$ 

The projection $\tilde{W}_\tau \to W_\tau$ is given by $(\tilde{\tau}', t) \mapsto \phi_t \cdot \tau'$. Note that the uniformizer $\tilde{W}_\tau$ is no longer a subset of $\tilde{\mathcal{B}}$, but is defined so that it supports a free $S^1$-action.

Suppose now that we cannot choose $w_2$ as above. (For example, there may be no need to add any $w_j$ or the unstable components may all map into the fixed set.) Then, we choose any point $w_0 \in \Sigma(u)$ so that
Stab(\(u(w_0)\)) is finite. (This exists since Stab(\(\tau\)) is finite.) We choose the slice \(X\) through \(u(w_0)\) as before and define \(\bar{Y}_\tau \subset \bar{U}_\tau\) by the condition \(u'(w_0) \in X\). It is obvious what this means when \([u',z']\) is in the same stratum at \(\tau\). One extends to a neighboring strata as before. Note that in this case \(w_0\) lies on a component with at least 3 special points.

Finally suppose that \(\Gamma_\tau \neq \{1\}\). As before we treat the case when \(\Gamma_\tau\) is cyclic and acting on one component of \(\Sigma(u)\). Because this component contains at most 2 special points, \(w_0\) (if it has been defined) always lies on some other component. Thus the only case that needs special consideration is when \(w_2\) lies on the component on which \(\Gamma_\tau\) acts and so equals the point previously called \(w_1\). But then we may simply repeat the previous construction for the action of \(\Gamma_\tau\), replacing \(H_1\) by \(H_1' := H_1 \cap X\). This defines an action of \(\Gamma_\tau\) on \(\bar{Y}_\tau\) and hence completes the construction. \(\square\)

5 Applications and Examples

In the first section, we describe the small quantum cohomology of toric manifolds. Next, we work out \(S(\Lambda)\) in specific cases to illustrate the what may happen when the hypotheses of the main theorems do not hold. The last section concerns the action of a Lie group \(G\) on its coadjoint orbits and contains a proof of Corollary 1.2. It is independent of the rest of this paper.

5.1 The small quantum homology of smooth toric varieties

This section describes the general form of a set of generators and relations for the small quantum cohomology ring \(QH^*(M)\) of a toric manifold. In the case of a Fano variety the description is completely explicit, is determined by a simple algorithm from the moment polytope \(\Delta\), and agrees with Batyrev’s presentation [3]. In the NEF case we show that \(QH^*(M)\) is determined by a simple algorithm involving its moment polytope \(\Delta\) together with the Seidel elements of the circle actions corresponding to the primitive outward normals \(\eta_1, \ldots, \eta_N\) to the facets of \(\Delta\). In the general case, the relations involve some polynomials whose higher order terms are unknown. Our result elaborates a small part of Givental’s work on the mirror conjecture: see Cox–Katz [4, Examples 8.1.2.2, 11.2.5.2]. Throughout we work with quantum cohomology with the Novikov ring coefficients defined in §2.2 though one can extend the result to the full Novikov ring: see Remark 5.5. Batyrev used complex coefficients; for a discussion of the relation of these coefficient systems see [4, 8.1.3].

Before beginning our computation, let’s review a few facts about quantum cohomology. First, as in (8), define a valuation \(\bar{v}\) on \(\mathbb{Q}[x_1, \ldots, x_N] \otimes \bar{\Lambda}\) by

\[
\bar{v}\left(\sum_{d, \kappa} a_{d, \kappa} \otimes q^{d\kappa}\right) = \min\{\kappa \mid \exists d : a_{d, \kappa} \neq 0\}.
\]

Lemma 5.1 Let \((M, \omega)\) be a symplectic manifold. Fix \(x_1, \ldots, x_N \in H^*(M)\), and consider the natural homomorphisms of rings

\[
\theta : \mathbb{Q}[x_1, \ldots, x_N] \longrightarrow H^*(M), \quad \text{and}
\]

\[
\Theta : \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda \longrightarrow QH^*(M).
\]

(i) If \(\theta\) is surjective, then \(\Theta\) is also surjective. Further, given \(z \in QH^*(M)\), there is \(\tilde{z} \in \mathbb{Q}[x_1, \ldots, x_N] \otimes \bar{\Lambda}\) so that \(\Theta(\tilde{z}) = z\) and so that \(\bar{v}(\tilde{z}) \geq \bar{v}(z)\).
(ii) Let \( p_1, \ldots, p_m \in \mathbb{Q}[x_1, \ldots, x_N] \) generate the kernel of \( \theta \), and suppose \( q_1, \ldots, q_m \in \mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} \) are such that \( \Theta(q_i) = 0 \) and \( \check{v}(p_i - q_i) > 0 \) for all \( i \). Then \( q_1, \ldots, q_m \) generate the kernel of \( \Theta \).

**Proof:** Fix \( h > 0 \) such that it is less than the energy \( \omega(B) \) of every class \( B \neq 0 \) that contributes to the quantum multiplication, i.e. for which there is a nonzero three point Gromov–Witten invariant. Then 

\[
\check{v}(\alpha * \beta - \alpha \cup \beta) \geq h
\]

for all \( \alpha, \beta \in H^*(M) \), and hence

\[
\check{v}(\Theta(\check{z}) - \theta(\check{z})) \geq h, \quad \forall \check{z} \in \mathbb{Q}[x_1, \ldots, x_N].
\]

By possibly shrinking \( h \), we can also assume that \( v^*(p_i - q_i) > h \) for all \( i = 1, \ldots, m \).

Fix \( z \in \text{QH}^*(M) \). To prove (i) it is enough to find \( \check{z} \in \mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} \) so that

\[
\check{v}(z - \Theta(\check{z})) \geq \check{v}(z) + h, \quad \text{and} \quad \check{v}(\check{z}) \geq \check{v}(\check{y}),
\]

since then the argument can be completed by induction. Write

\[
z = \sum_{i=1}^{k} z_i \otimes q^d_i t^{\kappa_i} + r,
\]

where \( \check{v}(r) \geq \check{v}(z) + h, \ z_i \in H^*(M), \ d_i \in \mathbb{Z}, \) and \( \kappa_i \geq \check{v}(z) \). Since \( \theta \) is surjective, there exists \( \check{z}_i \in \mathbb{Q}[x_1, \ldots, x_N] \) so that \( \theta(\check{z}_i) = z_i \). Then \( \check{v}(z_i - \Theta(\check{z}_i)) \geq h \) by (16); so let \( \check{z} = \sum_{i=1}^{k} \check{z}_i \otimes q^d_i t^{\kappa_i} \).

Now fix \( \check{y} \in \ker \Theta \). To prove (ii), it is enough to find \( \check{z} \in \mathbb{Q}[q_1, \ldots, q_m] \) so that

\[
\check{v}(\check{z} - \check{y}) \geq \check{v}(\check{y}) + h, \quad \text{and} \quad \check{v}(\check{z}) \geq \check{v}(\check{y}),
\]

since then this argument can also be completed by induction. Write

\[
\check{y} = \sum_{i=1}^{k} \check{y}_i \otimes q^d_i t^{\kappa_i} + \check{r},
\]

where \( \check{v}(\check{r}) \geq \check{v}(\check{y}) + h, \ \check{y}_i \in \mathbb{Q}[x_1, \ldots, x_N], \ d_i \in \mathbb{Z}, \) and \( \check{v}(\check{y}) + h > \kappa_i \geq \check{v}(\check{y}) \). We may also assume that \( (d_i, \kappa_i) \neq (d_j, \kappa_j) \) if \( i \neq j \). Note that by (16)

\[
0 = \Theta(\check{y}) = \sum \Theta(\check{y}_i \otimes q^d_i t^{\kappa_i}) + \Theta(\check{r}) = \sum \theta(\check{y}_i) \otimes q^d_i t^{\kappa_i} + \check{r},
\]

where \( \check{v}(\check{r}) \geq \check{v}(\check{y}) + h \). Therefore, for all \( i, \ \theta(\check{y}_i) = 0 \), and hence \( \check{y}_i \) lies in the ideal generated by \( p_1, \ldots, p_m \). Hence, there exists \( \check{z}_i \in \langle q_1, \ldots, q_m \rangle \), so that \( \check{v}(\check{y}_i - \check{z}_i) \geq h \). Let \( \check{z} = \sum \check{z}_i \otimes q^d_i t^{\kappa_i} \).

We will now give a brief review of toric geometry. Good basic references are Cox–Katz [4, Ch 3] and Batyrev [3].

Consider a torus \( T \) with Lie algebra \( t \) and lattice \( \ell \). Let \( (M, \omega) \) be a smooth toric variety with moment map \( \Phi : M \rightarrow t^* \), chosen so that each of its components is mean normalized. Let \( \Delta \subset t^* \) be the image of the moment map. Let \( D_1, \ldots, D_N \) be the facets of \( \Delta \) (the codimension one faces), and let \( \eta_1, \ldots, \eta_N \in \ell \) denote the outward primitive integral normal vectors.\(^5\) Let \( \ell^* \subset t^* \) denote the lattice dual to \( \ell \). Let \( \Sigma \)

\(^5\)Choosing the \( \eta_i \in t \) to be the outward rather than the inward normal is more natural in our context. For then the corresponding circle action has \( \Phi^{-1}(D_i) \) as its maximal fixed point component, and it is this, rather than the minimal fixed point component, that is seen by the Seidel element: cf. Theorem 1.10. However, the authors of [3, 4] make the other choice, defining the polytope \( \Delta \) by equations of the form \( \{ v \in t^* : \langle \eta_i, v \rangle \geq -a_i \} \). If we take the inward normals then in the definition of \( SR_Y \) in (20) \( \beta_t \) should be replaced by \( -\beta_t \).
be the set of subsets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ so that $D_{i_1} \cap \cdots \cap D_{i_k} \neq \emptyset$. Define two ideals in $\mathbb{Q}[x_1, \ldots, x_N]$:

$$P(\Delta) = \left\langle \sum (\xi, \eta_i) x_i \mid \xi \in \ell^* \right\rangle,$$

and $SR(\Delta) = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \not\subseteq \Sigma \rangle$.

A subset $I \subseteq \{1, \ldots, N\}$ is called primitive if $I$ is not in $\Sigma$ but every proper subset is. Clearly,$$SR(\Delta) = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\} \text{ is primitive} \rangle.$$

The map which sends $x_i$ to the Poincaré dual of $\Phi^{-1}(D_i)$ (which we shall also denote by $x_i \in H^2(M)$) induces an isomorphism

$$\mathbb{Q}[x_1, \ldots, x_N]/(P(\Delta) + SR(\Delta)) \cong H^*(M, \mathbb{Q}).$$

Moreover, there is a natural isomorphism between $H_2(M; \mathbb{Z})$ and the set of tuples $(a_1, \ldots, a_N) \in \mathbb{Z}^N$ such that $\sum a_i \eta_i = 0$, under which the pairing between such an element of $H_2(M, \mathbb{Z})$ and $x_i$ is $a_i$. The linear functional $\eta_i$ is constant on $D_i$; let $\eta_i(D_i)$ denote its value. Under the isomorphism of (17) (extended to real coefficients)

$$[\omega] = \sum_i \eta_i(D_i)x_i, \quad \text{and} \quad c_1(M) = \sum_i x_i.$$  \hspace{1cm} (18)

We are now ready to examine the quantum cohomology of a toric variety. The Seidel representation in cohomology is the homomorphism

$$S^* : \pi_1(\text{Ham}(M, \omega)) \longrightarrow \text{QH}_{ev}(M; \Lambda)^\times, \quad \Lambda \mapsto \text{PD}(S(\Lambda)),$$

where $\text{QH}_{ev}(M; \Lambda)^\times$ is the group of even units in $\text{QH}^*(M)$ and $S$ is the representation in homology. For each $\eta_i$ define $\Phi^\eta : M \longrightarrow \mathbb{R}$ to be the composite of the moment map $\Phi : M \longrightarrow \ell^*$ with the linear functional $\eta_i \in \ell = \text{Hom}(\ell^*, \mathbb{R})$. Thus $\Phi^\eta$ is the moment map for the circle action $\Lambda_i$ with tangent vector $\eta_i \in \ell$ and with $F_{\text{max}} = \Phi^{-1}(D_i)$. Denote:

$$S^*(\Lambda_i) = y_i \otimes q^{-1}t^{-\eta_i(D_i)} \in \text{QH}_{ev}(M; \Lambda)^\times.$$

By Theorem 1.10 and the formula given for Poincaré duality in §2.2, $y_i = x_i + \text{higher order terms}$, where the terms are ordered by $\delta$.

Given any face of $\Delta$, let $D_{j_1}, \ldots, D_{j_k}$ be the facets that intersect to form this face. The dual cone is the set of elements in $\ell$ which can be written as a positive linear combination of $\eta_{j_1}, \ldots, \eta_{j_k}$. Every vector in $\ell$ lies in the dual cone of a unique face of $\Delta$. Therefore, given any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ there is a unique face of $\Delta$ so that $\eta_{i_1} + \cdots + \eta_{i_k}$ lies in its dual cone. Let $D_{j_1}, \ldots, D_{j_k}$ be the facets that intersect to form this unique face. Then there exist unique positive integers $c_1, \ldots, c_\ell$ so that

$$\eta_{i_1} + \cdots + \eta_{i_k} - c_1\eta_{j_1} - \cdots - c_\ell\eta_{j_\ell} = 0.$$

Batyrev showed that if $I$ is primitive the sets $I$ and $J = \{j_1, \ldots, j_\ell\}$ are disjoint. Let $\beta_i \in H_2(M, \mathbb{Z})$ be the class corresponding to the above relation. By (18), we see that

$$c_1(\beta_i) = k - c_1 - \cdots - c_\ell, \quad \text{and} \quad \omega(\beta_i) = \eta_{i_1}(D_{j_1}) + \cdots + \eta_{i_k}(D_{j_k}) - c_1\eta_{j_1}(D_{j_1}) - \cdots - c_\ell\eta_{j_\ell}(D_{j_\ell}).$$
Since \( \eta_1 + \cdots + \eta_k = c_1 \eta_1 + \cdots + c_t \eta_t \), the corresponding circle actions are also equal. Using the fact that the Seidel representation is in fact a homomorphism, we have
\[
y_{i_1} \cdots y_{i_k} \otimes q^{-k} t^{-\eta_1(D_{i_1}) - \cdots - \eta_k(D_{i_k})} = y_{i_1}^{c_1} \cdots y_{i_t}^{c_t} \otimes q^{-c_1 - \cdots - c_t} t^{-c_1 \eta_1(D_{i_1}) - \cdots - c_t \eta_t(D_{i_t})}.
\]

Therefore
\[
y_{i_1} \cdots y_{i_k} - y_{i_1}^{c_1} \cdots y_{i_t}^{c_t} \otimes q^{c_1(\eta_1)} t^{\omega(\eta_1)} = 0.
\]

Since \( x_1, \ldots, x_N \) generate \( H^*(M) \), by Lemma 5.1 the natural homomorphism
\[
\Theta : \mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} \longrightarrow \text{QH}^*(M)
\]
which takes \( x_i \) to the Poincaré dual of \( \Phi^{-1}(D_i) \) is surjective. To compute \( \text{QH}^*(M) \), we need to find the kernel of \( \Theta \). By Lemma 5.1, there exists
\[
Y_i = x_i + \text{higher order terms}
\]
such that \( \Theta(Y_i) = y_i \). Define an ideal \( \text{SR}_Y(\Delta) \subset \mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} \) by
\[
\text{SR}_Y(\Delta) = \left\{ Y_i, \ldots, Y_i - y_{i_1}^{c_1} \cdots y_{i_t}^{c_t} \otimes q^{c_1(\beta_1)} t^{\omega(\beta_1)} \mid I = \{ i_1, \ldots, i_t \} \text{ is primitive} \right\},
\]
where the \( Y_i \) are as in (19). Note that \( \text{SR}_Y(\Delta) \) depends on the \( Y_i \). Additionally, even if \( y_i \) is known, it is not in general possible to describe \( Y_i \) without prior knowledge of the ring structure on \( \text{QH}^*(M) \). On the other hand, \( \text{SR}_Y \) is clearly contained in the kernel of \( \Theta \). Moreover, Batyrev shows that \( \omega(\beta_i) > 0 \) for all primitive \( I \). Hence, applying Lemma 5.1, we obtain the following proposition:

**Proposition 5.2** Let \( \text{QH}^*(M) \) denote the small quantum cohomology of the toric manifold \((M, \omega)\). The map which sends \( x_i \) to the Poincaré dual of \( \Phi^{-1}(D_i) \) induces an isomorphism
\[
\mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} / (P(\Delta) + \text{SR}_Y(\Delta)) \cong \text{QH}^*(M).
\]

This is especially simple in the Fano case.

**Example 5.3 (Fano toric varieties)** Assume that \( M \) is Fano, i.e. that \( c_1(B) > 0 \) for every class \( B \in H_2(M) \) with a holomorphic representative. In this case the higher order terms in \( S(\Lambda_i) \) vanish by part (iii) of Theorem 1.10. Therefore \( y_i = x_i \) for all \( i \), so that we may set \( Y_i = x_i \). Hence
\[
\text{SR}_Y(\Delta) = \left\{ x_{i_1} \cdots x_{i_t} - x_{i_1}^{c_1} \cdots x_{i_t}^{c_t} \otimes e^{\beta_i} \mid I = \{ i_1, \ldots, i_t \} \text{ is primitive} \right\}.
\]

This gives exactly the formula for the small quantum cohomology of a Fano toric variety given by Batyrev and proved by Givental.

**Example 5.4 (NEF toric varieties)** Now assume that \( M \) is NEF, i.e. that \( c_1(B) \geq 0 \) for every class \( B \in H_2(M) \) with a holomorphic representative. Now there may be higher order terms in the Seidel elements \( y_i \). However, part (ii) of Theorem 1.10 implies that the higher order terms in \( S^*(\Lambda_i) \) have the form \( \alpha_B \otimes q^{-1} + c_1(B) \eta_i(D_i) + \omega(B) \) where \( B \in H_2(M) \) satisfies \( c_1(B) = 0 \) or 1. Since \( S^*(\Lambda_i) \) is homogeneous of degree 0, every nonzero \( \alpha_B \) must have degree 0 or 2. Therefore \( \alpha_B \) either lifts to the unit \( 1 \) in \( \mathbb{Q}[x_1, \ldots, x_N] \otimes \hat{\Lambda} \) or to some linear combination of the \( x_i \) that is unique modulo the additive relations \( P(\Delta) \). Hence we do not need to know the quantum multiplication in \( M \) in order to define the \( Y_i \). The rest of the information needed to define the relations \( P(\Delta) \) and \( \text{SR}_Y(\Delta) \) is contained explicitly in \( \Delta \).
Thus, in the NEF case, one knows the Seidel elements $S(\Lambda_i), i = 1, \ldots, N$, there is an easy formula based on the combinatorics of its moment polytope $\Delta$ for the quantum cohomology ring. This substitution of the $Y_i$ for the $x_i$ in the Stanley–Reisner ring $SR_Y$ is one way of looking at Givental’s change of variable formulae as discussed in [4, 11.2.5.2].

Remark 5.5 Often one wants to consider quantum cohomology with coefficients in a completion of the group ring of $H^2(M)$ rather than of a quotient of $H^2(M)$. Our methods give similar results in this case, but one must use a slightly different version of the Seidel representation. For more details see McDuff–Salamon [14, Chapter 11.4].

5.2 The Seidel representation: examples

The examples in this section show that even in the case of the simplest manifolds, namely rational ruled 5-dimensional $\mathbb{C}P^2$’s, the Seidel element can be quite complicated. The first example is Fano. We show how lower order terms may appear in the formula for symplectic 4-manifolds, the Seidel element can be quite complicated. The first example is Fano. We show how lower order terms may appear in the formula for $S(\Lambda)(a)$ and discuss a circle action with at most twofold isotropy. The second example illustrates the NEF case, in which, as already noted in Seidel [20], the expression for $S(\Lambda)$ can have infinitely many nonzero terms. We also show what can happen when the isotropy has order greater than two.

Example 5.6 (The one point blowup of $\mathbb{C}P^2$) Fix $\mu \in (0, 1)$. Identify the one point blow up $M_\star$ of $\mathbb{C}P^2$ with the region

$$\left\{(z_1, z_2) \in \mathbb{C}^2 \left| \frac{\mu^2}{\pi} \leq |z_1|^2 + |z_2|^2 \leq \frac{1}{\pi} \right. \right\}$$

with boundaries collapsed along the Hopf flow, and give it the corresponding symplectic form $\omega_\mu$. Let $E \in H_2(M_\star)$ denote the class of the exceptional divisor, let $L = [\mathbb{C}P^1]$, and let $B = L - E$ be the fiber class. Thus $\omega_\mu(L) = 1$. Let $p \in H_0(M)$ denote the homology class of a point, and let $\mathbb{I}$ be the generator of $H_4(M)$. The space $M_\star$ is a toric variety, where $T = S^1 \times S^1$ acts on $M_\star$ by $(\alpha_1, \alpha_2) \cdot (z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$. The standard complex structure $J$ on $M_\star$ is $T$-invariant and is compatible with $\omega_\mu$. The moment map $\Phi: M \longrightarrow \mathbb{R}^2$ is given by

$$\Phi(z_1, z_2) = (|z_1|^2 - \epsilon, |z_2|^2 - \epsilon), \quad \text{where} \quad \epsilon = \frac{1 - \mu^6}{3(1 - \mu^4)}.$$

The primitive outward normals are

$$\eta_1 = (-1, 0), \quad \eta_2 = (0, -1), \quad \eta_3 = (1, 1), \quad \text{and} \quad \eta_4 = (-1, -1).$$

Let $\Lambda_1$ denote the circle action corresponding to $\eta_1$. Since the moment map for $\Lambda_1$ takes its maximum on the set $z_1 = 0, \Lambda_1$ is the action $(z_1, z_2) \mapsto (e^{-2\pi i \mu} z_1, z_2)$. Similar arguments give explicit formula for the other $\Lambda_i$. Since $(M_\star, J)$ is Fano, part (iii) of Theorem 1.10 implies that

$$S(\Lambda_1) = S(\Lambda_2) = B \otimes q^t, \quad S(\Lambda_3) = L \otimes q t^{1-2\epsilon}, \quad \text{and} \quad S(\Lambda_4) = E \otimes q t^{2\epsilon-\mu^2}.$$

There are two primitive subsets, namely $\{3, 4\}$ and $\{1, 2\}$. Since $\eta_3 + \eta_4 = 0$,

$$\mathbb{I} = S(\Lambda_3) + S(\Lambda_4) = L + E \otimes q^2 t^{1-\mu^2}.$$
Since $\eta_1 + \eta_2 = \eta_4$, 
\[ E \otimes q^{2\varepsilon - \mu^2} = S(A_4) = S(A_1) \ast S(A_2) = B \ast B \otimes q^{2\varepsilon}. \]
Therefore 
\[ B \ast B = E \otimes q^{-1}\varepsilon - \mu^2 \quad \text{and} \quad L \ast E = \mathbb{I} \otimes q^{-2}\mu^2 - 1. \] (21)

The circle action $(A_1)^{-1}$ also has a simple maximum, namely the point $[(0, 1)] \in M_*$, the inverse image of the vertex $D_2 \cap D_3$. The holomorphic spheres $C$ through $F_{\max}$ all have $c_1(C) \geq 2$. Hence, again applying part (iii) of Theorem 1.10, we conclude 
\[ S(A_1^{-1}) = p \otimes q^{2\varepsilon}. \]
Since $-\varepsilon_1 = \varepsilon_3 + \varepsilon_2$, 
\[ p \otimes q^{2\varepsilon - \varepsilon} = S((A_1)^{-1}) = S(A_3)S(A_2) = B \ast L \otimes q^2\varepsilon^{1-\varepsilon}. \]
Therefore, 
\[ B \ast L = p. \] (22)

Note that equation (21) determines $\text{QH}_*(M_*)$ as a ring, but does not determine the product above. Together, equations (21) and (22) determine all possible products in $\text{QH}_*(M_*)$. In particular, using associativity, we find 
\[ p \ast p = L \otimes q^{-3}\varepsilon^{1-\varepsilon}, \quad E \ast p = B \otimes q^{-2}\mu^2 - 1, \quad \text{and} \quad p \ast B = \mathbb{I} \otimes q^{-3}\mu^2 - 1. \]
These products may also be derived directly from the 3-point Gromov–Witten invariants: it is not hard to check that the only nonzero invariants involving the classes $p, B$, and $E$ are 
\[ \text{GW}_{L,3}(p, p, B) = 1; \quad \text{GW}_{B,3}(p, E, E) = 1; \quad \text{and} \quad \text{GW}_{E,3}(A_1, A_2, A_3) = \pm 1 \quad \text{where} \quad A_1 = E \text{ or } B. \]

The natural action of $U(2)$ on $\mathbb{C}^2$ induces an action on $M_*$; this action contains the torus $T$. Since $\pi_1(U(2)) = \mathbb{Z}$, this shows that, as elements of $\pi_1(\text{Symp}(M_*, \omega))$, $A_1 = A_2$. Hence 
\[ A_3 = (A_4)^{-1} = A_1^{-2} = A_2^{-2} \]
It is a worthwhile exercise to check that $S(A_3) = S(A_4)^{-1} = S(A_1)^{-2} = S(A_2)^{-2}$.

Since $A_4$ is semifree, we can also apply Theorem 1.15 to this action. It has $F_{\max} = E$, the exceptional divisor. Let $r \in H_0(E)$ be the homology class of a point. Then the downwards extension $r^- = B \in H_2(M)$, and the upwards extension $r^+ = p \in H_0(M)$. Then 
\[ S(A_4)(r^-) = (E \otimes q^{2\varepsilon - \mu^2}) \ast B = r^+ \otimes q^{2\varepsilon - \mu^2} - E \otimes (q^{2\varepsilon - \mu^2})(q^{-1}\mu^2 - 1). \]
This agrees with Theorem 1.15, but also shows that lower order terms can appear, even in this simple example. This lower order term comes from an invariant chain consisting of the sphere $F_{\max}$ (in class $E$) together with a section $\sigma_z$ for $z \in F_{\max}$.

Now consider the circle action $\Lambda'$ corresponding to $\eta_1 + \eta_4 = (-2, -1)$. The corresponding moment map has a simple maximum, namely the point $[(0, \mu)] \in M_*$ that maps down to $D_1 \cap D_4 \in \Phi(M)$. Hence part (i) of Theorem 1.10 applies, but part (iii) does not because there is a holomorphic sphere $E$ through
the maximum with $2c_1(E) = 2 \leq \text{codim } F_{\text{max}} = 4$. Therefore our results do not rule out the presence of lower order terms in $S(\Lambda')$ and indeed these exist: since $(-2, -1) = \eta_1 + \eta_4$, \[ S(\Lambda') = S(\Lambda_1) * S(\Lambda_4) = B * E \otimes q^2 t^{3\epsilon - \mu^2} = p \otimes q^2 t^{3\epsilon - \mu^2} - E \otimes q t^{3\epsilon - 2\mu^2}. \]

Observe also that $\Lambda'$ has at most twofold isotropy, with isotropy submanifold $(M_*)\mathbb{Z}/(2)$ equal to $\Phi^{-1}(D_2)$. One can check this by writing $(-2, -1) = 2\eta_4 - \eta_2$: as explained in the proof of Proposition 1.6 in §2.2 the coefficient of $-\eta_4$ in this expression equals the weight on the transverse edge $D_2$. Therefore Theorem 1.18 applies to the fixed components $F_{13} := \Phi^{-1}(D_1 \cap D_3)$ and $F_{24} := \Phi^{-1}(D_2 \cap D_4)$, which are both isolated points. Since $F_{13}$ is simple, the Euler class $e(F_{13})$ is nonzero. On the other hand, $e(F_{24}) = 0$. Further, if $c_{ij} \in H_0(F_{ij})$ denotes a generator, we find $(c_{13})^- = L$, while $(c_{24})^- = E$. Therefore Theorem 1.18 implies that $S(\Lambda')(L)$ has a nontrivial summand $c_{0,0} \otimes t^{K}(F_{13})$ where $c_{0,0} \cdot L = 1$ and $K'$ denotes the moment map $\Phi^q + \eta^\ast$ of $\Lambda'$. (This is the contribution to $S(\Lambda')(L)$ of the constant section at the (almost) simple point $F_{13}$.) On the other hand, because $F_{24}$ is not almost simple, the constant section at $F_{24}$ makes no contribution to $S(\Lambda')(E)$ and so the coefficient of $q t^K(F_{24})$ in $S(\Lambda')(E)$ vanishes. This can be checked by direct calculation. For example $K'(F_{13}) = 3\epsilon - 1$ and $S(\Lambda')(L) = S(\Lambda')(E + B)$ contains one nonzero term of the form $a \otimes t^{-n}$, namely $B \otimes t^{3\epsilon - 1}$.

The manifold $M_*$ has many other toric structures; correspondingly there are many other elements of $\pi_1(\text{Ham}(M_*, \omega_\mu))$ that are represented by semifree circle actions. Indeed, whenever $\mu^2 > k/(k + 1)$, there is an $\omega_\mu$-compatible complex structure $J_k$ on $M_*$ such that the underlying complex manifold $(M_*, J_k)$ can be identified with the projectivization $\mathbb{P}(L_k \oplus \mathbb{C})$, where $L_k$ is the holomorphic line bundle over $\mathbb{C}P^1$ with Chern class $2k + 1$. The loop that rotates the fibers of $\mathbb{C}$ by $e^{2\pi i t}$ is semifree and represents the class $(4k + 2)\alpha$, where $\alpha = [-\Lambda_1] \in \pi_1(\text{Ham}(M_*, \omega_\mu))$. The classes $(2k + 1)\alpha$ are also represented by circle actions that preserve $J_k$ and rotate the base of the ruled surface $(M_*, J_k)$. When $k = 0$, $J_0$ is the standard complex structure discussed above, and the representative for $2\alpha$ is $\Lambda_1$ while the representative for $\alpha$ is $\Lambda_1^{-1}$. When $k > 0$ explicit formulas for these actions can be derived from the description of $(M_*, J_k)$ as a toric manifold given in [1] §2.3. In this case these actions have 4 isolated fixed points, two each in the fibers lying above the fixed points of the base rotation. However, these fixed points are not simple except when $k = 1$, in which case the fixed points in the fiber containing the overall minimum are simple. In the next example we shall discuss a similar action on $S^2 \times S^2$ in detail.

**Example 5.7 (Circle actions on $S^2 \times S^2$.)** Consider $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ with the symplectic form $\omega_\mu = \mu \pi_1^1(\sigma) + \pi_2^1(\sigma)$, where $\pi_i$ is projection onto the $i$th factor, $\sigma$ is the standard symplectic form on $\mathbb{C}P^1$ with total area 1. Assume that $\mu \geq 1$. Define $A$ and $B$ in $H_2(M)$ by $A = [\mathbb{C}P^1 \times \{q\}]$ and $B = [\{q\} \times \mathbb{C}P^1]$, where $q \in \mathbb{C}P^1$. Note that $\omega_\mu(A) = \mu$ and $\omega_\mu(B) = 1$. Let $p \in H_0(M)$ denote the holonomy class of a point, and let $\mathbb{L}$ denote the generator of $H_4(M)$.

The standard action of the torus $T = S^1 \times S^1$ on $(M, \omega_\mu)$ in which each $S^1$-factor rotates the corresponding sphere has moment map $\Phi : M \rightarrow \mathbb{R}^2$ given by

$$\Phi([x_1 : x_2], [y_1 : y_2]) = \left( \frac{\mu}{|x_1|^2 + |x_2|^2} |y_1|^2 - |y_2|^2, \frac{|x_1|^2 - |x_2|^2}{|x_1|^2 + |x_2|^2} |y_1|^2 + |y_2|^2 \right).$$

The primitive outward normals to the moment image $\Delta = \Phi(M)$ are

$$\eta_1 = (1, 0), \quad \eta_2 = (-1, 0), \quad \eta_3 = (0, 1), \quad \text{and} \quad \eta_4 = (0, -1).$$

Let $\Lambda_i$ be the circle action associated to $\eta_i$.

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6The formula in [1] Lemma 2.11(i) is slightly incorrect.
Since the standard complex structure on \(\mathbb{CP}^1 \times \mathbb{CP}^1\) is Fano and \(T\)-invariant, and since \(\Lambda_i\) acts semifreely for all \(i\), by Theorem 1.10
\[
S(\Lambda_1) = S(\Lambda_2) = B \otimes q^t t^\varphi, \quad \text{and} \quad S(\Lambda_3) = S(\Lambda_4) = A \otimes q^t t^\varphi.
\]
Since \(\eta_1 + \eta_2 = 0\) and \(\eta_3 + \eta_4 = 0\), \(S(\Lambda_1) * S(\Lambda_2) = \mathbb{I}\) and \(S(\Lambda_3) * S(\Lambda_4) = \mathbb{I}\). This implies that
\[
B * B = \mathbb{I} \otimes q^{-2} t^{-\mu} \quad \text{and} \quad A * A = \mathbb{I} \otimes q^{-2} t^{-\mu}.
\]
Let \(\Lambda' \subset S^1 \times S^1\) be the circle associated to \(\eta_1 + \eta_3 : = (1,1)\). Then \(\Lambda'\) acts by the diagonal action and so is semifree. Since \(c_1(C) \geq 2\) for every holomorphic sphere \(C\),
\[
S(\Lambda') = p \otimes q^t t^\frac{1+\mu}{2}.
\]
Because \(S(\Lambda') = S(\Lambda_1) * S(\Lambda_3)\) we find \(A * B = p\). As before, these products determine all the products in \(\text{QH}_* (M)\). In particular
\[
p * A = B \otimes q^{-2} t^{-1} \quad \text{and} \quad p * B = A \otimes q^{-2} t^{-\mu}.
\]
We now describe a second toric structure on \(M\). Let \(L_n\) denote the holomorphic bundle over \(\mathbb{CP}^1\) with Chern class \(n\). Let \(M'\) be the projectivization of the bundle \(L_2 \oplus \mathbb{C}\). Two commuting circles act naturally on \(M'\). First, the standard circle action on \(\mathbb{CP}^1\) lifts naturally to an action on \(T^\ast (\mathbb{CP}^1) = L_2\), and hence to \(M'\). Denote this circle action by \(\Gamma'\). Another circle, say \(\Gamma''\), acts by rotating each fiber. The standard complex structure \(J_2\) on \(M'\) is invariant under the resulting \(S^1 \times S^1\)-action. Moreover, if we assume that \(\mu > 1\), there exists a \(J_2\)-compatible invariant symplectic form \(\omega\) on \(M'\) so that \(M'\) is symplectomorphic to \((\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{\mu})\), which we consider to be fibered over \(\mathbb{CP}^1\) via projection to the first factor. In fact, we may assume that this symplectomorphism lifts the identity map on the base \(\mathbb{CP}^1\) and is equivariant with respect to the action of \(\Lambda'\) on \(M\) and \(\Gamma'\) on \(M'\): see for example [1]. Hence, we immediately conclude
\[
S(\Gamma') = S(\Lambda') = p \otimes q^t t^\frac{1+\mu}{2}.
\]
Here, and elsewhere, we identify \(p, A, B, \) and \(\mathbb{I}\) with their image in \(H_* (M')\).

As described above, \(M'\) is a smooth toric variety with moment map \(\Phi'\). The moment image \(\Delta' = \Phi' (M')\) is a quadrilateral with outward normals
\[
\gamma_1 = (0, 1), \quad \gamma_2 = (0, -1), \quad \gamma_3 = (1, -1), \quad \text{and} \quad \gamma_4 = (-1, -1).
\]
Further \((\Phi')^{-1} (D_1)\) is the diagonal in \(M' \equiv \mathbb{CP}^1 \times \mathbb{CP}^1\) and so contains points that we will call \(v_{ss}\) and \(v_{nn}\), where \(v_{ss}\) corresponds to \((0 : 1], [0 : 1])\), the pair (south pole, south pole), in \(\mathbb{CP}^1 \times \mathbb{CP}^1\) and \(v_{nn}\) corresponds to (north pole, north pole). Similarly, \((\Phi')^{-1} (D_2)\) is the antidiagonal and contains \(v_{sn}, v_{ns}\).

Indeed
\[
(\Phi')^{-1} (D_1 \cap D_3) = v_{nn}, \quad (\Phi')^{-1} (D_3 \cap D_2) = v_{ns}, \quad (\Phi')^{-1} (D_2 \cap D_4) = v_{sn}, \quad (\Phi')^{-1} (D_4 \cap D_1) = v_{ss}.
\]
The moment image itself is
\[
\Delta' = \{ \alpha \in \mathbb{R}^2 \mid (\alpha, \gamma_i) \leq c_i \}, \quad \text{where}
\]
\[
c_1 = \frac{1}{2} + \frac{\mu}{2} - \epsilon, \quad c_2 = \epsilon + \frac{1}{2} - \frac{\mu}{2}, \quad c_3 = c_4 = \epsilon, \quad \text{and} \quad \epsilon = \frac{\mu}{2} + \frac{1}{6\mu}.
\]
Let \(\Gamma_i \subset S^1 \times S^1\) be the circle associated to \(\gamma_i\). In our previous notation, \(\Gamma' = \Gamma_1 + \Gamma_3\) and \(\Gamma'' = \Gamma_1\).
The circle $\Gamma_1$ acts semifreely. Since every holomorphic sphere $C$ which intersects $F_{\text{max}}$ has $c_1(C) \geq 2$, it follows from part (iii) of Theorem 1.10 that there are no lower order terms in $S(\Gamma_1)$. Since $[F_{\text{max}}] = [(\Phi')^{-1}(D_1)] = A + B$, 

$$S(\Gamma_1) = (A + B) \otimes qt^{\frac{1}{2} + \frac{2}{9} + \epsilon}.$$ 

The circle $\Gamma_2$ also acts semifreely. In this case, $F_{\text{max}}$ itself is a holomorphic sphere in class $A - B$, so $c_1(F_{\text{max}}) = 0$. Therefore, part (iii) of Theorem 1.10 does not exclude lower order terms. On the other hand, every holomorphic sphere $C$ with $c_1(C) \leq 1$ lies entirely in $F_{\text{max}}$, so every term which contributes comes from a $C$ which lies in $F_{\text{max}}$. Indeed, since $\gamma_1 = -\gamma_2$,

$$S(\Gamma_2) = S(\Gamma_1)^{-1} = (A - B) \otimes \frac{qt^{\frac{1}{2} - \frac{2}{9} + \epsilon}}{1 - t^{1 - \mu}} = (A - B) \otimes qt^{\frac{1}{2} - \frac{2}{9} + \epsilon} \left(1 + t^{1 - \mu} + \cdots \right).$$

This calculation also appears in Remark 11.5 of [20].

Now consider $\Gamma_3$. Once again, Theorem 1.10 does not rule out lower order terms. Since $\gamma_3 = \gamma_2 + (1, 0)$,

$$S(\Gamma_3) = S(\Gamma_2) \ast S(\Gamma') = B \otimes q^{\epsilon} - (A - B) \otimes q^{\epsilon} \frac{t^{1 - \mu}}{1 - t^{1 - \mu}}.$$ 

A similar argument applies to $\Gamma_4$.

Let’s now pause for a moment to compare these results with the previous section. As above, let $D_i$ denote the facet that corresponds to $\gamma_i$; let $x_i$ denote the Poincare dual of $\Phi^{-1}(D_i)$; note that $[(\Phi')^{-1}(D_i)] = [(\Phi')^{-1}(D_1)] = A$, $[(\Phi')^{-1}(D_1)] = A + B$, and $[(\Phi')^{-1}(D_2)] = A - B$. Converting the equations above into cohomology, and using this notation, we find:

$$S^*(\Gamma_1) = x_1 \otimes q^{\epsilon} t^{-\frac{2}{9} + \epsilon},$$

$$S^*(\Gamma_2) = x_2 \otimes \frac{q^{\epsilon} t^{-\frac{2}{9} - \epsilon}}{1 - t^{1 - \mu}} ,$$

$$S^*(\Gamma_3) = \left(x_3 - x_2 \otimes \frac{t^{\mu - 1}}{1 - t^{1 - \mu}}\right) q^{-\epsilon} t^{-\epsilon} ,$$

$$S^*(\Gamma_4) = \left(x_4 - x_2 \otimes \frac{t^{\mu - 1}}{1 - t^{1 - \mu}}\right) q^{-\epsilon} t^{-\epsilon} .$$

Thus in equation (19) we may take

$$Y_1 = x_1, \quad Y_2 = x_2 \otimes \frac{1}{1 - t^{\mu - 1}}, \quad \text{and} \quad Y_4 = Y_3 = x_3 - x_2 \otimes \frac{t^{\mu - 1}}{1 - t^{1 - \mu}} .$$

We now look at $S(\tilde{\Gamma})$ for the circle action $\tilde{\Gamma}$ associated with $\gamma = (1, 2)$, which has threefold isotropy. In notation introduced earlier, we can describe the fixed set of $\tilde{\Gamma}$ as consisting of the points $v_{\alpha}$ (the maximum), the saddle points $v_{ss}$, $v_{ns}$ and the minimum $v_{nn}$. The maximum is not simple; in fact, because $(1, 2) = \eta_1 + 3\eta_2$, the diagonal $(\Phi')^{-1}D_1$ is stabilized by $Z/(3)$. Since the action does not have at most twofold isotropy, the arguments of Theorems 1.15 and 1.18 do not apply. We show that the conclusions of these theorems also fail. Since $(1, 2) = 2\gamma_1 + (1, 0)$,

$$S(\tilde{\Gamma}) = S(\Gamma_1)^2 \ast S(\Gamma') = \left(p + p \otimes t^{1 - \mu} + 2q^{-2} t^{-\mu}\right) \otimes q^{2} t^{\frac{1}{2} + \frac{2}{9} - 2\epsilon} .$$

50
First consider the minimum \( v_{sn} \) which is simple. Then, in the notation of Theorem 1.18, \((v_{sn})^- = p\) and \((v_{sn})^+ = 1\) and so one might expect the leading order term of \( S(\tilde{\Gamma})(p) \) to come from the section \( \sigma_{sn} \) and so have the form \( 1 \otimes q^t \mu \). But
\[
S(\tilde{\Gamma})(p) = (1 \otimes q^{-4}t^{-1-\mu} + 1 \otimes q^{-4}t^{-2\mu} + 2p \otimes q^{-2}t^{-\mu}) \otimes q^2t^{\frac{1}{2}+(\frac{2}{3})-2\epsilon}
\]
has the leading order term \( p \otimes t^{\frac{1}{2}+\frac{2}{3}-2\epsilon} \). It is not hard to check that this term comes from the invariant chain
\[
x = v_{sn} \xrightarrow{A-B} v_{ns} \xrightarrow{2B} v_{nn} \xrightarrow{\sigma_{nn}} y = v_{nn},
\]
where \( \sigma_{nn} \) is the constant section at \( v_{nn} \). This lies in class \( A + B + [\sigma_{nn}] = A + B + [\sigma_{nn}] = [\sigma_{sn}] - B \) since \( [\sigma_{sn}] - [\sigma_{nn}] = A + 2B \).

Next consider the simple saddle point \( v_{ss} \). Then \((v_{ss})^- = B \) and \((v_{ss})^+ = A + B \). Therefore, from Theorem 1.15 one would expect the leading order term in \( S(2\gamma + \tau_1 + \tau_2)(B) \) to be \((A + B) \otimes \pi^K(v_{ss})\), while in fact it is \((A + 2B) \otimes \pi^K(v_{ss})\). Since \((v_{ns})^+ = B\), one can get this extra term from an invariant chain going from \( x \in (v_{ss})^- \) to \( y \in (v_{ns})^- \) that lies in class \( [\sigma_{ss}] \). Since \( [\sigma_{nn}] = [\sigma_{ss}] - (A + B) \) such a chain is given by
\[
x = v_{sn} \in (v_{ss})^- \xrightarrow{E} v_{ns} \xrightarrow{B} v_{nn} \xrightarrow{\sigma_{nn}} v_{nn} \xrightarrow{B} y = v_{ns} \in (v_{ns})^-.
\]

\[ \square \]

## 5.3 Coadjoint orbits

Let \( G \) be a semisimple Lie group \( G \) with Lie algebra \( \mathfrak{g} \). Let \( M \subset \mathfrak{g}^* \) be a coadjoint orbit, together with the Kostant–Kirillov symplectic form \( \omega \). If the coadjoint action of \( G \) on \( M \) is effective, then \( G \) is naturally a subgroup of the group of Hamiltonian symplectomorphisms of \( M \). This inclusion induces a natural map from the fundamental group of \( G \) to the fundamental group of the group of Hamiltonian symplectomorphisms. In [23], Alan Weinstein asks when this map is injective. In this subsection, we show that this map is injective for all compact semisimple Lie groups. In fact we show that the composite
\[
\pi_1(G) \longrightarrow \pi_1(\text{Ham}(M, \omega)) \xrightarrow{\mathcal{S}} \text{QH}_{\text{ev}}(M; A)^\times
\]
is injective. Our argument is based on the following proposition.

**Proposition 5.8** Let \((M, \omega)\) be a coadjoint orbit of a compact semisimple Lie group \( G \). Assume that only a finite number of elements of \( G \) act trivially on \( M \). Then every nontrivial element in \( \pi_1(G) \) may be represented by a circle that acts semifreely on \( M \).

If \( K \) is the (finite) subgroup of \( G \) that acts trivially on \( M \), then the quotient \( G/K \) acts effectively on \( M \). Moreover, \( \pi_1(G) \) injects into \( \pi_1(G/K) \). Therefore, the proposition above follows from the special case where \( G \) acts effectively.

By Theorem 1.10, the Seidel element \( \mathcal{S}(\Lambda_K) \) is non trivial for every semifree Hamiltonian circle action \( \Lambda_K \); hence, we immediately deduce the following extension of Corollary 1.2.

**Corollary 5.9** Let \((M, \omega)\) be a coadjoint orbit of a compact semisimple Lie group \( G \). Assume that only a finite number of elements of \( G \) act trivially on \( M \). Then the Seidel representation \( \mathcal{S} : \pi_1(G) \longrightarrow \text{QH}_{\text{ev}}(M; A)^\times \) is faithful. Hence, the natural map \( \pi_1(G) \longrightarrow \pi_1(\text{Ham}(M, \omega)) \) is an injection.
Alternatively, the last statement follows from Theorem 1.1.

**Proof of Proposition 5.8.**

We begin with a brief review of a few facts about Lie groups.

Since each simply connected compact semisimple Lie group is a product of simple factors and its Lie algebra splits into a corresponding sum, the coadjoint orbits also are products of coadjoint orbits of simple groups. Moreover, the center of a simply connected compact Lie group is the product of the centers of its simple factors. Therefore, it suffices to assume that $G$ is simple.

Let $G$ be a simple compact Lie group. Let $\hat{G}$ denote the universal cover of $G$, and $\hat{G}$ denote the quotient of $G$ by its center. Let $\frak{g}$ denote the Lie algebra of $G$, and let $\frak{t}$ denote the Lie algebra of a maximal torus $T \subset G$. Let $\ell \subset \frak{t}$, $\ell \subset \frak{t}$, and $\ell \subset \frak{t}$ be the lattices consisting of vectors $\xi \in \frak{t}$ whose exponential is the identity in $G$, $\hat{G}$, and $\hat{G}$, respectively. There is a one-to-one correspondence between $\ell$ and circle subgroups of $G$, $\ell$ and circle subgroups of $\hat{G}$, and $\hat{\ell}$ and circle subgroups of $\hat{G}$, given by sending $\lambda$ to $t \mapsto \exp(t\lambda)$. Note that $\hat{\ell} \subset \ell \subset \hat{\ell}$. Because $\hat{G}$ is simply connected, 

$$\pi_1(G) \cong \ell/\ell \cong \hat{\ell}/\hat{\ell} \cong \pi_1(\hat{G}).$$

Let $t^*$ denote the dual to $t$, and let $\Delta \subset t^*$ denote the set of roots of $G$, i.e. the nonzero weights of the adjoint action $T$ on $t$, where $t$ is the complexification of $\frak{g}$. The lattice $\ell$ is dual to the lattice in $t^*$ generated by the roots, i.e. $\lambda \in \ell$ precisely when $\eta(\lambda) \in \mathbb{Z}$ for all $\eta \in \Delta$. If we use the Killing form $(\cdot, \cdot)$ to identify $t$ and $t^*$, then $\ell$ is generated by the set 

$$\left\{ \frac{2\eta}{(\eta, \eta)} \mid \eta \in \Delta \right\}.$$ 

Further the set of weights at any fixed point $p$ for the action of $T$ on $M$ is a nonempty subset of the set of roots. Therefore the result will follow if we find a representative $\lambda$ for each nontrivial class in $\ell/\ell$ such that $|\eta(\lambda)| \leq 1$ for every $\eta \in \Delta$.

We will check this on a case by case basis; in each case we will use the Killing form to identify $t$ and $t^*$. Let $(\cdot, \cdot)$ be the standard metric on $\mathbb{R}^k$ with the standard basis $e_1, \ldots, e_k$, and define 

$$\epsilon_i = e_i - \frac{1}{k} \sum_{j=1}^{k} e_j.$$ 

**I** For the group $A_n$, where $n \geq 1$, $t = t^* = \left\{ \lambda \in \mathbb{R}^{n+1} \mid \sum \lambda_i = 0 \right\}$ and the roots are $\epsilon_i - \epsilon_j = e_i - e_j$ for $i \neq j$. Hence 

$$\ell = \left\{ \lambda \in \mathbb{Z} \mid \lambda_i - \lambda_j \in \mathbb{Z} \right\},$$

and 

$$\hat{\ell} = \left\{ \lambda \in \mathbb{Z}^{n+1} \mid \sum \lambda_i = 0 \right\}.$$ 

As representatives for the quotient $\hat{\ell}/\ell \cong \mathbb{Z}/(n+1)$, we take $\lambda = \sum_{i=1}^{k} \epsilon_i$ for $0 \leq k \leq n$.

**II** For the group $B_n$, where $n \geq 2$, $t^* = \mathbb{R}^n$ and the roots are $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$. Hence 

$$\hat{\ell} = \mathbb{Z}^n,$$

and 

$$\hat{\ell} = \left\{ \lambda \in \mathbb{Z}^n \mid \sum \lambda_i = 0 \right\}.$$ 

As representatives of the quotient $\hat{\ell}/\ell \cong \mathbb{Z}/(2)$, we take 0 and $e_1$.

**III** For the group $C_n$, where $n \geq 3$, $t^* = \mathbb{R}^n$ and the roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i \neq j$. Hence 

$$\hat{\ell} = \{ \lambda \in \mathbb{R}^n \mid \lambda_i \pm \lambda_j \in \mathbb{Z}, \forall i,j \},$$

and 

$$\hat{\ell} = \mathbb{Z}^n.$$
As representatives of the quotient $\hat{\mathcal{E}}/\tilde{\mathcal{E}} \cong \mathbb{Z}/(2)$, we take $0$ and $\frac{1}{2} \sum_{i=1}^{n} e_i$.

(V, a) For the group $G$, where $n \geq 4$, $\mathfrak{t}^* = \mathbb{R}^n$ and the roots are $\pm e_i \pm e_j$ for $i \neq j$. Hence

$$\hat{\mathcal{E}} = \{\lambda \in \mathbb{R}^n \mid \lambda_i + \lambda_j \in \mathbb{Z}, \forall i, j\}, \text{ and } \tilde{\mathcal{E}} = \left\{\lambda \in \mathbb{Z}^n \mid \sum \lambda_i \in 2\mathbb{Z}\right\}.$$  

The quotient $\hat{\mathcal{E}}/\tilde{\mathcal{E}}$ is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ if $n$ is even, and to $\mathbb{Z}/(4)$ if $n$ is odd. Either way, as representatives for $\hat{\mathcal{E}}/\tilde{\mathcal{E}}$, we take $0, e_1, e_2, -e_1 - e_2,$ and $\varepsilon + e_1.$

(V, b) For the group $G$, $\mathfrak{t}^* = \mathbb{R}^6$ and the roots are $2 \varepsilon, e_i - e_j$, and $e_i + e_j + e_k \pm \varepsilon$ for $i, j, k$ distinct, where $\varepsilon = \frac{1}{2\sqrt{3}}(1, 1, 1, 1, 1)$. Hence,

$$\hat{\mathcal{E}} = \left\{n \varepsilon + (\xi_1, \ldots, \xi_6) \in \mathbb{R}^6 \mid \sum_{i=1}^{6} \xi_i = 0, n \in \mathbb{Z}, \frac{n}{2} + 3\xi_i \in \mathbb{Z} \text{ and } \xi_i - \xi_j \in \mathbb{Z} \forall i, j\right\}, \text{ and}$$

$$\tilde{\mathcal{E}} = \left\{n \varepsilon + (\xi_1, \ldots, \xi_6) \in \mathbb{R}^6 \mid \sum_{i=1}^{6} \xi_i = 0, n \in \mathbb{Z} \text{ and } \frac{n}{2} + \xi_i \in \mathbb{Z} \forall i\right\}.$$  

As representatives of the quotient $\hat{\mathcal{E}}/\tilde{\mathcal{E}} \cong \mathbb{Z}/(3)$, we take $0, e_1 + e_2,$ and $-e_1 - e_2.$

(V, c) For the group $G$, $\mathfrak{t}^* = \{\lambda \in \mathbb{R}^6 \mid \sum \lambda_i = 0\}$, and the roots are $e_i - e_j$, and $e_i + e_j + e_k + e_l$ for $i, j, k, l$ distinct. Hence

$$\hat{\mathcal{E}} = \{\lambda \in \mathfrak{t} \mid 4\lambda_i \in \mathbb{Z} \text{ and } \lambda_i - \lambda_j \in \mathbb{Z} \forall i, j\}, \text{ and } \tilde{\mathcal{E}} = \{\lambda \in \mathfrak{t} \mid \lambda_i \pm \lambda_j \in \mathbb{Z} \forall i, j\}.$$  

As representatives for the quotient $\hat{\mathcal{E}}/\tilde{\mathcal{E}} \cong \mathbb{Z}/(3)$, we take $0$ and $e_1 + e_2$.

Every group of type $G$, $F$, and $G_2$ is simply connected, so no further argument is necessary.

\section{Lie Group Actions}

This section contains a proof of Theorem 1.3, which we restate below for the convenience of the reader. This result is independent of the rest of the paper.

\textbf{Proposition 6.1} Let a compact Lie group $G$ act on a connected manifold $M$ so that only a finite number of elements of $G$ act trivially on $M$. Let $\Lambda \subset G$ be a circle subgroup that is inessential in $G$. Then any simple component $F$ of the fixed point set of $\Lambda$ is symmetric, in the sense that there is an element $g \in G$ whose action on $M$ fixes $F$ pointwise and which reverses $\Lambda$.

Recall, we say that $g \in G$ reverses a circle subgroup $\Lambda \subset G$ if $g \circ t \circ g^{-1} = t^{-1}$ for all $t \in \Lambda$.

We will also prove the following related proposition that does not assume there is a simple fixed component but adds a global isotropy assumption and assumes that the action is Hamiltonian.

\textbf{Proposition 6.2} Let a compact Lie group $G$ act in a Hamiltonian fashion on a connected symplectic manifold $(M, \omega)$ so that only a finite number of elements of $G$ act trivially on $M$. Let $\Lambda \subset G$ be a circle subgroup which is inessential in $G$. Assume that the action of $\Lambda$ has at most twofold isotropy. Then there exists $g \in G$ which reverses $\Lambda$. 

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Recall that we say that a circle action has at most twofold isotropy if every point which is not either fixed or free has stabilizer $\mathbb{Z}/(2)$. Further, a symplectic action of $G$ on $(M,\omega)$ is Hamiltonian if it is given by an equivariant moment map $\Phi : M \rightarrow g^\ast$.

Remark 6.3 (i) If $\Lambda \subset G$ is any circle subgroup of a simple group of type $B_n$, $C_n$, or $F_4$, then there exists a $g \in G$ which reverses $\Lambda$. In this case, Proposition 6.2 is trivial and the force of Proposition 6.1 is that we can choose $g$ so that it also fixes $p$.

(ii) If $G$ is a simple group, we do not need to assume that $(M,\omega)$ is symplectic in Proposition 6.2; we only need to assume that there exists a point $p$ which is fixed by a maximal torus containing $\Lambda$ but is not fixed by all of $G$.

(iii) In the proof of the propositions above, we pick a maximal torus $T$ which contains $\Lambda$. The reversor $g$ that we construct lies in $N(T)$ and has the property that $g^2$ lies in $T$. However, $g^2$ may not be equal to the identity.

Example 6.4 As an example, consider the action of $\text{SU}(2)$ on $\mathbb{CP}^2$ by the defining representation on the first two copies of $\mathbb{C}$. The standard maximal torus $S^1 \subset \text{SU}(2)$ acts by $\lambda \cdot [x : y : z] = [\lambda x : \lambda^{-1} y : z]$. Since $\text{SU}(2)$ is simply connected, this circle subgroup is inessential in $\text{SU}(2)$. It does have a simple fixed point, namely $[0,0,1]$, and has at most twofold isotropy. Therefore it has a reversor $g$ that fixes $[0,0,1]$. Note that $g^2 = -I$ for any such $g$.

On the other hand, the action of $S^1$ on $\mathbb{CP}^3$ given by $\lambda \cdot [x,y,z,w] = [\lambda^2 x,\lambda^{-1} y,\lambda^{-1} z, w]$ is inessential but has some isotropy of order 3. Since $F_{\text{max}}$ and $F_{\text{min}}$ have different structure, this action has no reversor.

To see that the symplectic hypothesis is needed in Proposition 6.2, consider the obvious action of $\text{SU}(3)$ on $S^3 := \mathbb{C}^3 \cup \{\infty\}$. The subgroup $\Lambda := \text{diag}(e^{i\theta},e^{-i\theta},e^{i\theta})$ acts with at most twofold isotropy but has no reversor. The proof fails because the only points fixed by $\Lambda$ are fixed by all of $G$.

Note that here, as elsewhere in this paper, by positive Weyl chamber we mean closed positive Weyl chamber.

Lemma 6.5 Let $G$ be a simply connected compact simple Lie group. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus $T \subset G$. Let $\ell$ be the integral lattice, let $\Delta$ denote the set of roots, and let $W$ denote the Weyl group. Use the Killing form to identify $\mathfrak{t}$ and $\mathfrak{t}^\ast$. Fix $\lambda \in \ell$. Choose a positive Weyl chamber which contains $\lambda$. Let $\delta \in \Delta$ denote the highest root. Then the following claims hold:

(a) If $(\lambda,\delta) \leq 2$, then there exist orthogonal roots $\eta_1, \ldots, \eta_k \in \Delta$ so that $\lambda = \sum a_i \eta_i$ and so that $(\lambda,\eta_i) = a_i(\eta_i,\eta_i) = 2$ for all $i$.

(b) Let $L \subset \Delta$ be a set of roots which contains every root $\eta \in \Delta$ such that $\delta + \eta$ or $\delta - \eta$ is also a root. Assume also that $L$ is closed under addition, that is, it contains every root which can be written as the sum of roots in $L$. Then $L$ contains all roots.

(c) If $(\lambda,\delta) > 2$ and $-\text{id} : \mathfrak{t} \rightarrow \mathfrak{t}$ is not an element of the Weyl group, then for every nonzero weight $\alpha \in \ell^\ast$ there exists $\sigma \in W$ so that $|\langle \sigma \cdot \alpha, \lambda \rangle| > 1$.

(d) If $-\text{id} : \mathfrak{t} \rightarrow \mathfrak{t}$ is not an element of the Weyl group, then $\delta$ is the only root which lies in the positive Weyl chamber.

Using this result, we can find find elements which reverse certain circle subgroups of simply connected compact simple Lie groups. Note that because $G$ is simply connected, every circle subgroup of $G$ is inessential in $G$. 

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Lemma 6.6 Let $\Lambda$ be a circle subgroup of a simply connected compact simple Lie group $G$.

(i) Let $\rho : G \to \text{GL}(V)$ be a nontrivial representation of $G$. If $\Lambda$ acts semifreely on $V$ then there exists $g \in G$ that reverses $\Lambda$.

(ii) Let $H \subsetneq G$ be a proper subgroup containing $\Lambda$. If the adjoint action of $\Lambda$ on $g/\mathfrak{h}$ is semifree, then there exists $h \in H$ that reverses $\Lambda$.

(iii) Let $H \subsetneq G$ be a proper subgroup containing a maximal torus which contains $\Lambda$. If the natural action of $\Lambda$ on $G/H$ has at most twofold isotropy, then there exists $g \in G$ that reverses $\Lambda$.

The assumption in (ii) above is a special case of (i) since the representation $V$ is restricted; however, the conclusion is stronger since it asserts that the reversor lies in $H$. Statement (ii) and (iii) are also related: the former makes a strong assumption about the action induced by $\Lambda$ on the tangent space to $G/H$ at the fixed point $eH$, the latter makes a weaker assumption about the action at all the fixed points on $G/H$.

We will now use the claims in Lemma 6.5 to prove Lemma 6.6. Let $T$ be a maximal torus which contains $\Lambda$. Let $\lambda \in \ell$ be the vector corresponding to $\Lambda$. Choose a positive Weyl chamber which contains $\lambda$. Let $\delta \in \Delta$ denote the highest root.

Recall that the Weyl group $W$ is the quotient $N(T)/T$, where $N(T)$ is the the normalizer of $T$ in $G$. Every root $\eta$ gives rise to an element $w_\eta \in W$ whose action on $\ell^*$ is given by $w_\eta(\beta) = \beta - \frac{2\langle \alpha, \delta \rangle}{\langle \alpha, \alpha \rangle} \eta$.

**Proof of Lemma 6.6 (i).**

Let $\rho : G \to \text{GL}(V)$ be a nontrivial representation of $G$. Assume that $\Lambda$ acts semifreely on $V$.

Suppose first that $(\lambda, \delta) \leq 2$. By claim (a), there exist orthogonal roots $\eta_1, \ldots, \eta_k \in \Delta$ so that $\lambda = \sum \alpha_i \eta_i$. Since the roots are orthogonal, for each $\eta_i$ the associated element of the Weyl group $w_{\eta_i}$ takes $\eta_i$ to $-\eta_i$ and leaves $\eta_j$ fixed for all $j \neq i$. Hence, their product $w = w_{\eta_1} \cdots w_{\eta_k}$ takes $\lambda$ to $-\lambda$, and so reverses $\Lambda$.

So assume instead that $(\lambda, \delta) > 2$. If $-\text{id}$ is in the Weyl group, then statement (i) is trivial. So we assume that it is not. Let $T$ act on $V$ via restriction, and pick any nonzero weight $\alpha \in \ell^*$ in the weight decomposition. By claim (c), we can find some $\sigma \in W$ such that $|\langle \sigma \cdot \alpha, \lambda \rangle| > 1$. Since $\sigma \cdot \alpha$ also appears in the weight decomposition, this contradicts the assumption that the action of $\Lambda$ on $V$ is semifree.

**Proof of Lemma 6.6 (ii).**

Let $H \subsetneq G$ be a proper subgroup which contains $\Lambda$. Assume that the adjoint action of $\Lambda$ on $g/\mathfrak{h}$ is semifree. Let $L$ be the set of roots $\eta \in \Delta$ so that the associated weight space $E_\eta \subset \mathfrak{g}_C$ lies in $\mathfrak{h}_C$. Clearly, if $|\langle \eta, \lambda \rangle| > 1$, then $\eta \in L$.

Suppose first that $(\lambda, \delta) \leq 2$. By claim (a), there exist orthogonal roots $\eta_1, \ldots, \eta_k \in \Delta$ so that $\lambda = \sum \alpha_i \eta_i$ and so that $(\eta_i, \eta_j) = 2$ for every $i$. Since $(\eta_i, \lambda) = 2$, $\eta_i$ lies in $L$ for all $i$. Hence, the associated element of the Weyl group $w_{\eta_i}$ lies in $H$ for all $i$. Thus $w = w_{\eta_1} \cdots w_{\eta_k}$ must lie in $H$.

So assume instead that $(\lambda, \delta) > 2$. We see immediately that $\delta$ and $-\delta$ lie in $L$. If $\eta, \eta'$ and $\eta + \eta'$ are all roots, then $[E_{\eta}, E_{\eta'}^\ast] = E_{\eta + \eta'}$. Hence, since $\mathfrak{h}_C$ is closed under Lie bracket, if $\eta$ and $\eta'$ are in $L$ then $\eta + \eta'$ lies in $L$ also, that is, $L$ is closed under addition. Additionally, if $\eta$ and $\eta'$ are roots such that $\delta = \eta + \eta'$, then either $(\lambda, \eta) > 1$ or $(\lambda, \eta') > 1$. If the former holds, then $\eta$ and $-\eta$ lie in $L$. Since $L$ is closed under addition, so do $\eta'$ and $-\eta'$. The other case is identical. Thus, claim (b) implies that every root lies in $L$. This contradicts the claim that $H$ is a proper subgroup.

**Proof of Lemma 6.6 (iii).**

Let $H \subsetneq G$ be a proper subgroup which contains the maximal torus $T$, and assume that the natural action of $\Lambda \subset T$ on $G/H$ has at most twofold isotropy.
If $(\lambda, \delta) \leq 2$, then part (iii) follows by the argument used to prove part (i). So assume that $(\lambda, \delta) > 2$. We may also assume that $-\text{id}$ does not lie in the Weyl group, because otherwise the claim is trivial. Since $H \subset G$ is proper, there exists at least one root $\eta$ so that the associated weight space $E_{\eta}$ is not contained in $\mathfrak{h}_C$. Then there is $\sigma \in W$ so that the root $\sigma \cdot \eta$ lies in the positive Weyl chamber. Hence by (d) $\sigma \cdot \eta = \delta$, and so $|(\sigma \cdot \eta, \lambda)| > 2$. Choose $\tilde{\sigma} \in N(T)$ which descends to $\sigma$. Then $\tilde{\sigma}H$ is a fixed point for $T$, and $\sigma \cdot \eta$ is one of the weights for $\Lambda$ at this fixed point. This contradicts the fact that the action has at most twofold isotropy.

We are now ready to deduce Propositions 6.1 and 6.2. Let $\tilde{G}$ denote the universal cover of $G$. Then $\tilde{G}$ is the direct product of a compact simply connected semisimple Lie group and a vector space. Since $\Lambda$ is inessential, it lifts to a circle subgroup of $\tilde{G}$. Since this lift must lie in the compact part of $\tilde{G}$, we may assume without loss of generality that $\tilde{G}$ is a compact simply connected semisimple Lie group. Thus $\Lambda$ is a compact simply connected semisimple Lie group. In fact, since it is enough to prove these claims for the universal cover of $G$, we may (and will) assume that $G$ itself is a compact simply connected semisimple Lie group. Thus $G$ is the product of compact simple and simply connected groups $G_1 \times \cdots \times G_n$. Let $\Lambda_i$ be the projection of $\Lambda$ to $G_i$. Without loss of generality, we may assume that $\Lambda_i \neq \{\text{id}\}$ for all $i$.

**Proof of Proposition 6.1.**

Let $G = G_1 \times \cdots \times G_n$ as above. Choose $p \in F$ and let $H \subset G$ be the stabilizer of $p$. Then $\Lambda \subset H$. There exists a representation $V$ of $H$, called the *isotropy representation*, so that a neighborhood of the $G$-orbit through $p$ is equivariantly diffeomorphic to a neighborhood of the zero section of $G \times_H V$. Fix some simple factor $G_i$, and let $H_i = H \cap G_i$.

Assume first that $H_i$ is a proper subgroup. Note that $\mathfrak{g}_i$ is invariant under the action of $\Lambda$. Thus, since $\Lambda$ acts semisfreely on $\mathfrak{g}/\mathfrak{h}$ via the adjoint action, $\Lambda_i$ acts semifreely on $\mathfrak{g}_i/\mathfrak{h}_i$. Thus, by Lemma 6.6 (ii) there exists an element $h_i \in H_i$ that reverses $\Lambda_i$.

So assume on the contrary that $H_i = G_i$. Let $N'$ be the projection of $\Lambda$ onto the product of all the simple factors except $G_i$. Since $\Lambda_i \subset G_i \subset H$ and $\Lambda \subset H$, we must have $N' \subset H$. Hence $N'$ acts on $V$. For any integer $k$, let $V_k$ denote the subspace of $V$ on which $N'$ acts with weight $k$. Since $N'$ commutes with $G_i$, $V_k$ is a representation of $G_i$. Since only a finite number of elements of $G$ act trivially on $M$, $G_i$ must act nontrivially on $G \times_H V$, and hence also on $V$. Therefore, there is some $k$ so that the representation $G_i$ on $V_k$ is nontrivial. Because $G_i$ is simple, $\Lambda_i$ must act with both positive and negative weights on $V_k$. But the weights for the action of $\Lambda$ on $V_k$ are the weights for the action of $\Lambda_i$ shifted by $k$. Hence, because $F$ is a fixed point for the action of $\Lambda_i$ on $V_k$, it is itself semifree. Therefore, by Lemma 6.6 (i) there exists $h_i \in G_i = H_i$ that reverses $\Lambda_i$.

Since $h_i$ reverses $\Lambda_i$ for each $i$, $g = (h_1, \ldots, h_n)$ reverses $\Lambda$, as required. Moreover, since $H_1 \times \cdots \times H_n \subset H$ (in general they are not equal), $g$ lies in $H$, and hence fixes $p$.

**Proof of Proposition 6.2.**

Fix some simple factor $G_i$. Let $W$ be the Weyl group of $G_i$. Let $T \subset G_i$ be a maximal torus of $G_i$ containing $\Lambda_i$. Let $\Phi : M \rightarrow t^*$ be the moment map for the $T$-action. Pick any $\xi \in t$ so that the one parameter subgroup generated by $\xi$ is dense in $T$. Let $p$ be any point which maps to the minimum value of $\Phi^T$, the component of $\Phi$ in the direction $\xi$. By construction, $p$ is a fixed point for $T$. Assume first that $\Phi^T(p) = 0$, that is, the function $\Phi^T$ is nonnegative on $M$. Since the moment polytope $\Phi(M)$ is invariant under the Weyl group $W$, this implies that $\Phi^T \cdot \xi$ is also nonnegative on $M$ for all $\sigma \in W$. Because $G_i$ is simple and $\xi$ is a generic point of $t$, for any nonzero $\sigma \in W$ there exists an element $\sigma \in W$ such that $(\sigma \cdot \xi, x) < 0$. Applying this to $x \in \Phi(M) \setminus \{0\}$, we see that $\Phi(M)$ must be the single point ${0}$, which is impossible, because the action is effective. Therefore, $\Phi(p) \neq 0$. 56
Now let us reconsider the action of $G$ on $M$. Let $H$ be the stabilizer of $p$ in $G$, and let $H_i = H \cap G_i$. Since $\Lambda$ acts with at most twofold isotropy on $G/H \subset M$, $\Lambda_i$ acts with at most twofold isotropy on $G_i/H_i$. Since $\Phi(p)$ is not zero, the stabilizer of $\Phi(p)$ in $G_i$ is a proper subgroup of $G_i$. Since $\Phi$ is equivariant, this implies that $H_i$ is a proper subgroup of $G_i$. By Lemma 6.6 (iii), this implies that there exists $g_i \in G_i$ which reverses $\Lambda_i$. Then $(g_1, \ldots, g_n)$ reverses $\Lambda$. \hfill $\square$

**Proof of Lemma 6.5.**

We now prove claims (a)-(d) on a case by case basis, using the classification of compact simple Lie groups. We will use the notation of §5.3. Note, however, that here $G = \tilde{G}$ since $G$ is simply connected.

**(I)** Recall that for the group $A_n$, where $n \geq 1$, $t = t^* = \{\xi \in \mathbb{R}^n \mid \sum \xi_i = 0\}$, the roots are $\epsilon_i - \epsilon_j = e_i - e_j$ for $i \neq j$, and the integral lattice is $\tilde{\ell} = \mathbb{Z}^{n+1} \cap t$. The positive Weyl chamber is $\{\xi \in t \mid \xi_1 \geq \cdots \geq \xi_{n+1}\}$.

The highest root is $\delta = \epsilon_1 - e_{n+1}$.

If $(\lambda, \delta) = \lambda_1 - \lambda_{n+1} \leq 2$, then $|\lambda_i| \leq 1$ for all $i$. Since $\sum \lambda_i = 0$ and $\lambda_i \in \mathbb{Z}$ for all $i$, there are an equal number of $+1$s and $-1$s, and the rest are $0$s. Hence, $\lambda$ is the sum of orthogonal roots of the form $\eta = \epsilon_i - \epsilon_j$. Since $(\eta, \eta) = 2$, this proves claim (a).

Since $\delta = (e_1 - e_k) + (e_k - e_{n+1})$, the roots $\pm(e_1 - e_k)$ and $\pm(e_k - e_{n+1})$ lie in $L$ for all $1 < k < n+1$. If neither $i$ nor $j$ is equal to 1, then $e_i - e_j = - (e_1 - e_i) + (e_1 - e_j)$ is also in $L$. This proves claim (b).

We now prove (c). The weight lattice is $\tilde{l} = \{\alpha \in t \mid \alpha_i - \alpha_j \in \mathbb{Z} \quad \forall \ i, j\}$. By permuting the coordinates of $\alpha$, we may assume $\alpha_1 \geq \cdots \geq \alpha_n$. Since $\alpha \neq 0$, there exists $k \in (1, \ldots, n)$ such that $\alpha_k - \alpha_{k+1} > 0$; since this difference lies in $\mathbb{Z}$, it must be at least 1. Since $\lambda_1 - \lambda_{n+1} = \lambda_1 + \sum_{i=1}^n \lambda_i > 2$ and $\lambda_i \geq \lambda_i + n - k$, $\sum_{i=1}^k \lambda_i + \sum_{i=1}^{n+1-k} \lambda_i > 2$. Therefore, either $\sum_{i=1}^k \lambda_i > 1$ or $\sum_{i=1}^{n+1-k} \lambda_i > 1$. In the former case,

$$(\alpha, \lambda) = \sum_{j=1}^{n} \left( (\alpha_j - \alpha_{j+1}) \sum_{i=1}^j \lambda_i \right) \geq (\alpha_k - \alpha_{k+1}) \sum_{i=1}^k \lambda_i > 1.$$ 

In the latter case, let $\alpha'$ be obtained from $\alpha$ by the permutation which reverses the coordinates, so that $\alpha'_i = \alpha_{n+2-i}$. Then

$$(\alpha', \lambda) = \sum_{j=1}^{n-1} \left( (\alpha_{n+2-j} - \alpha_{n+1-j}) \sum_{i=1}^j \lambda_i \right) \leq (\alpha_{k+1} - \alpha_k) \sum_{i=1}^{n+1-k} \lambda_i < -1.$$ 

The only facts we have used are that $t = \{\xi \in \mathbb{R}^n \mid \sum \xi_i = 0\}$, that the Weyl group contains the permutation group $S_n$, and that $\alpha_i - \alpha_j \in \mathbb{Z}$ for any $\alpha \in \tilde{l}$.

Finally, $\delta$ is the only root in the positive Weyl chamber.

**(II)** Recall that for the group $B_n$, where $n \geq 2$, $t = t^* = \mathbb{R}^n$, the roots are $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\ell = \{\xi \in \mathbb{Z}^n \mid \sum \xi_i \in 2\mathbb{Z}\}$. The positive Weyl chamber is $\{\xi \in t \mid \xi_1 \geq \cdots \geq \xi_n \geq 0\}$. The highest root is $\delta = e_1 + e_2$.

If $(\lambda, \delta) = \lambda_1 + \lambda_2 \leq 2$, then either $\lambda_1 = 2$ and $\lambda_i = 0$ for all $i \neq 1$, or $\lambda_i \leq 1$ for all $i$. Either way, since $\sum \lambda_i \in 2\mathbb{Z}$, we can write $\lambda$ as the sum of orthogonal roots $\eta_i$ such that $(\eta_i, \eta_i) = 2$.

\footnote{For uniformity, we shall always use the lexicographical order to choose the positive Weyl chamber.}
Since $\delta = (e_1 - e_k) + (e_2 + e_k) = (e_1 + e_k) + (e_2 - e_k)$, the roots $\pm e_1 \pm e_k$ and $\pm e_2 \pm e_k$ lie in $L$ for $k \neq 1$ or 2. Since $\delta = (e_1 + e_2)$, the roots $\pm e_1$ and $\pm e_2$ lie in $L$. Every root can be written as a sum of these roots.

Since $-\text{id}$ lies in the Weyl group, we are done.

(III) Recall that for the group $C_n$, where $n \geq 3$, $t = t^* = \mathbb{R}^n$, the roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\ell = \mathbb{Z}^n$. The positive Weyl chamber is $\{ \xi \in t \mid \xi_1 \geq \cdots \geq \xi_n \geq 0 \}$. The highest root is $\delta = 2e_1$.

If $(\lambda, \delta) = 2\lambda_1 \leq 2$, then $\lambda_i \leq 1$ for all $i$. Since $\lambda \in \mathbb{Z}^n$, we can write $\lambda$ as half the sum of orthogonal roots of the form $2e_i$. Note that $(\lambda, 2e_i) = 2$.

Since $\delta = (e_1 - e_k) + (e_1 + e_k)$, the roots $\pm e_1 \pm e_k$ lie in $L$ for $k \neq 1$. Every root can be written as a sum of these roots.

Since $-\text{id}$ lies in the Weyl group, we are done.

(IV) Recall that for the group $D_n$, where $n \geq 4$, $t = t^* = \mathbb{R}^n$, the roots are $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\ell = \mathbb{Z}^n$. The positive Weyl chamber is $\{ \xi \in t \mid \xi_1 \geq \cdots \geq \xi_{n-1} \geq |\xi_n| \}$. The highest root is $\delta = e_1 + e_2$.

If $(\lambda, \delta) = \lambda_i + \lambda_2 \leq 2$, then either $\lambda_i = 2$ and $\lambda_2 = 0$ for all $i \neq 1$, or $|\lambda_i| \leq 1$ for all $i$. Either way, since $\sum \lambda_i \in \mathbb{Z}^n$, we can write $\lambda$ as the sum of orthogonal roots $\eta_i$ such that $(\eta_i, \eta_i) = 2$.

Since $\delta = (e_1 - e_k) + (e_1 + e_k)$, the roots $\pm e_1 \pm e_k$ and $\pm e_2 \pm e_k$ lie in $L$ for $k \neq 1$ or 2. Every root can be written as a sum of these roots.

Now assume that $(\delta, \lambda) = \lambda_1 + \lambda_2 > 2$. Consider a nonzero weight $\alpha \in \tilde{\ell}^* = \{ \alpha \in \mathbb{R}^n \mid \alpha_i \pm \alpha_j \in \mathbb{Z} \forall i, j \}$. By applying the Weyl group, we may assume $\alpha$ lies in the positive Weyl chamber. Since $\lambda$ also lies in the positive Weyl chamber, $\alpha \lambda_i \geq 0$ for all $i \neq n$. Moreover, since $\alpha_{n-1} \geq |\alpha_n|$, and $\lambda_{n-1} \geq |\lambda_n|$, $\alpha_{n-1} \lambda_{n-1} + \alpha_n \lambda_n \geq 0$. Therefore, $\alpha_3 \lambda_3 + \cdots + \alpha_n \lambda_n \geq 0$. (Here, we have used that $n \geq 4$.) Since $\alpha$ is nonzero, either $\alpha_1 \geq 1$, or $\alpha_1 = \alpha_2 = \frac{1}{2}$. In either case, $\alpha_1 \lambda_1 + \alpha_2 \lambda_2 > 1$. (In the first case, we use the fact that $\lambda_1 + \lambda_2 > 2$ and $\lambda_1 \geq \lambda_2$ implies that $\lambda_1 > 1$.) Therefore, $(\alpha, \lambda) \geq \alpha_1 \lambda_1 + \alpha_2 \lambda_2 > 1$. This proves claim (c).

Finally, $\delta$ is the only root in the positive Weyl chamber.

(V, a) Recall that for the group $E_6$, $t = t^* = \mathbb{R}^6$ and the roots are $2\epsilon$, $\epsilon_i - \epsilon_j$, and $\epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon$ for $i, j$, and $k$ distinct, where $\epsilon = \frac{1}{2\sqrt{3}}(1,1,1,1,1,1)$. Therefore

$$\tilde{\ell} = \{ n \epsilon + (\xi_1, \ldots, \xi_6) \mid \sum_{i=1}^6 \xi_i = 0, n \in \mathbb{Z}, \frac{n}{2} + \xi_i \in \mathbb{Z} \forall i \}.$$ 

The positive Weyl chamber is

$$\{ n \epsilon + (\xi_1, \ldots, \xi_6) \in t \mid \sum_{i=1}^6 \xi_i = 0, \xi_2 \geq \cdots \geq \xi_6, \xi_1 + \xi_5 + \xi_6 \geq n/2 \geq 0 \}.$$ 

(Note that these conditions imply $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_6$.

Write $\lambda = n \epsilon + (\xi_1, \ldots, \xi_6)$, where $\sum \xi_i = 0$. Assume that $(\lambda, \delta) = \xi_1 - \xi_6 \leq 2$. Combining the inequalities $\xi_1 - \xi_6 \leq 2$, $\xi_2 \geq 2$, $\xi_4 \geq 2$, $\xi_5 \geq 2$, and $\xi_1 + \xi_5 + \xi_6 \geq 2$, we see that $\xi_4 \geq \frac{n-4}{6}$. Since also $\xi_2 + \xi_3 + \xi_4 \leq 0$, $\xi_2 \geq \xi_3$, and $\xi_3 \geq \xi_4$, we have $\xi_4 \leq 0$. Moreover, in both cases, if the final inequality in the sentence is an equality, so are all the preceding ones. Since $n \geq 0$, $0 \geq \xi_4 \geq -\frac{n-4}{6}$. Since $\lambda \in \tilde{\ell}$, $\xi_4 = 0$ or $\xi_4 = -\frac{1}{2}$. In the former case, $\xi_2 = \xi_3 = 0$, $\xi_1 + \xi_5 + \xi_6 = 0$, so $n = 0$. Hence, $\lambda = (\epsilon_1 - \epsilon_6)$. In the
latter case, $n$ is odd, so \( \xi_4 = -\frac{1}{2} \geq \frac{n-4}{6} \) implies that $n = 1$. In this case, $\lambda = (\epsilon_1 - \epsilon_0) + (\epsilon + \epsilon_1 + \epsilon_2 + \epsilon_0)$.

This proves claim (a).

Since $\delta = (\epsilon_1 - \epsilon_1) + (\epsilon_1 - \epsilon_0)$, the roots $\pm(\epsilon_1 - \epsilon_1)$ and $\pm(\epsilon_1 - \epsilon_0)$ lie in $L$ for all $1 < i < 6$. Moreover, $\delta = (\epsilon + \epsilon_1 + \epsilon_1 + \epsilon_1) - (\epsilon + \epsilon_1 + \epsilon_1 + \epsilon_0)$, so the roots $\pm(\epsilon_1 + \epsilon_1 + \epsilon_1 + \epsilon_0)$ lie in $L$ for all $1 < i < 6$. Since, for example, $\epsilon + \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon - \epsilon_4 - \epsilon_5 - \epsilon_6$, it follows easily that $L$ contains all roots.

Let $\alpha \in \tilde{\ell}^*$ be a nonzero weight. Write $\lambda = n\epsilon + \xi$ as before. By applying the Weyl group, we may assume that $\alpha = n\epsilon + (\zeta_1, \ldots, \zeta_6)$ is in the positive Weyl chamber. Since $\alpha = n\epsilon + (\zeta_1, \ldots, \zeta_6)$, it is enough to show that $\alpha = n\epsilon + (\zeta_1, \ldots, \zeta_6) > 1$. This fact now follows from the argument from $A_5$, since $(\delta, \lambda) = (\delta, \xi) > 2$, since the Weyl group contains the permutation group $S_5$, and since $\xi$ must satisfy $\zeta_1 - \zeta_j \in \mathbb{Z}$.

Finally, $\delta$ is the only root in the positive Weyl chamber.

(V, b) Recall that for the group $E_7$, $t = t^* = \{ \xi \in \mathbb{R}^8 | \sum \xi_i = 0 \}$; the roots are $\epsilon_i - \epsilon_j$ and $\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$ for $i, j, k$, and $l$ distinct, and the integral lattice is $\ell = \{ \xi \in t | \xi_i \pm \xi_j \in \mathbb{Z} \forall i, j \}$. The positive Weyl chamber is

$$\{ \xi \in t | \xi_2 \geq \cdots \geq \xi_8 \text{ and } \xi_1 + \xi_6 + \xi_7 + \xi_8 \geq 0 \}.$$ (Note that this automatically implies that $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_8$.

Assume that $\lambda = \lambda_1 - \lambda_8 \leq 2$. Combining the inequalities $\lambda_1 - \lambda_8 \leq 2$, $\lambda_1 + \lambda_8 + \lambda_7 + \lambda_8 \geq 0$, and $\lambda_5 \geq \lambda_1$ for $i = 6, 7$ and 8, we see that $\lambda_5 \geq -\frac{1}{2}$. Since also $\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \geq 0$ and $\lambda_1 \geq \lambda_5$ for $i = 2, 3$ and 4, $\lambda_5 \geq \lambda_5$. Moreover, in both cases, if the last inequality in the sentence is an equality, all the inequalities are equalities. Since $\lambda \notin \ell$, the only possibilities are $\lambda_5 = 0$ or $\lambda_5 = -\frac{1}{2}$. In the former case, we must have $\lambda = \epsilon_1 - \epsilon_8$. In the latter case, the only possibilites are $\lambda = (\epsilon_1 + \epsilon_3 + \epsilon_4) + (\epsilon_1 - \epsilon_4)$, or $\lambda = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + (\epsilon_1 - \epsilon_4) + (\epsilon_2 - \epsilon_3)$. The proves claim (a).

Since $\delta = (\epsilon_1 - \epsilon_1) + (\epsilon_1 - \epsilon_8)$, the roots $\pm(\epsilon_1 - \epsilon_1)$ and $\pm(\epsilon_1 - \epsilon_8)$ lie in $L$ for all $1 < i < 8$. Since $\delta = (\epsilon_1 + \epsilon_1 + \epsilon_1 + \epsilon_8) (\epsilon_1 + \epsilon_1 + \epsilon_1 + \epsilon_8)$, the roots $\pm(\epsilon_1 + \epsilon_1 + \epsilon_1 + \epsilon_8)$ and $\pm(\epsilon_1 + \epsilon_1 + \epsilon_1 + \epsilon_8)$ also lie in $L$ for all $1 < i < j < k < 8$. All roots can be written as a sum of these roots. This proves claim (b).

Since $\ell \subset \mathbb{Z}^9 \cap t$, $\alpha_i - \alpha_j \in \mathbb{Z}$ for every $\alpha \in \tilde{\ell}^* \subset t^*$. Hence, the argument for claim (c) follows from the argument for $A_7$.

Finally, $\delta$ is the only root in the positive Weyl chamber.

(V, c) For the group $E_8$, $t = t^* = \{ \xi \in \mathbb{R}^9 | \sum \xi_i = 0 \}$ and the roots are $\epsilon_i - \epsilon_j$, and $\pm(\epsilon_1 + \epsilon_2 + \epsilon_3)$ for $i, j$, and $k$ distinct. Hence the integral lattice is

$$\ell = \{ \xi \in t | 3\xi_i \in \mathbb{Z} \text{ and } \xi_i - \xi_j \in \mathbb{Z} \forall i, j \}.$$ The positive Weyl chamber is

$$\{ \xi \in t | \xi_2 \geq \cdots \geq \xi_9 \text{ and } \xi_2 + \xi_3 + \xi_4 \leq 0 \}.$$ (Note that these conditions imply that $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_9$.

Assume that $(\delta, \lambda) = \lambda_1 - \lambda_9 \leq 2$. Combining the inequalities

$$\lambda_1 - \lambda_9 \leq 2, \quad \lambda_1 + \lambda_9 = \lambda_7 + \lambda_8 + \lambda_9 \geq 0, \quad \lambda_1 \geq \lambda_4, i > 4,$$

we see that $\lambda_4 \geq -\frac{1}{4}$. Since $\lambda_2 + \lambda_3 + \lambda_4 \geq 0$ and $\lambda_2 \geq \lambda_3 \geq \lambda_4, \lambda_4 \geq 0$. Moreover, in both cases, if the last inequality in the sentence is an equality, all the inequalities are equalities. Since $\lambda \notin \ell$, the only possibilities are $\lambda_4 = 0$ or $\lambda_4 < -\frac{1}{4}$. In the former case, $\lambda = \epsilon_1 - \epsilon_9$. In the latter case, $\lambda = (\epsilon_1 - \epsilon_9) + (\epsilon_1 + \epsilon_2 + \epsilon_9)$. Claim (a) follows.
We now notice that $\delta = \epsilon_1 - \epsilon_9 = (\epsilon_1 - \epsilon_k) + (\epsilon_k - \epsilon_9) = (\epsilon_1 + \epsilon_i + \epsilon_j) - (\epsilon_i + \epsilon_j + \epsilon_9)$ for all $1 < k < 9$ and $1 < i < j < 9$. Therefore, the corresponding roots $\pm(\epsilon_1 - \epsilon_k), \pm(\epsilon_k - \epsilon_9), \pm(\epsilon_1 + \epsilon_i + \epsilon_j)$, and $\pm(\epsilon_i + \epsilon_j + \epsilon_9)$ all lie in $L$. Since every root can be written as a sum of these roots, claim (b) follows.

Finally, $\delta$ is the only root in the positive Weyl chamber.

(VI) For the group $F_4$, $t = t^* = \mathbb{R}^4$. The roots are $\pm \epsilon_i, \epsilon_i \pm \epsilon_j$ for $i \neq j$, and $\frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$. Hence the integral lattice is $\tilde{\ell} = \{ \xi \in \mathbb{Z}^4 \mid \sum \xi_i \in 2\mathbb{Z} \}$. The positive Weyl chamber is

$$\{ \xi \in t \mid \xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4 \geq 0 \text{ and } \xi_1 \geq \xi_2 + \xi_3 + \xi_4 \}.$$ 

(Note that automatically $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 + \epsilon_2$.

The argument for claim (a) carries over word for word from the argument for $B_4$.

Notice that if $k = 3$ or 4

$$\delta = e_1 + e_2 = (e_1 + e_2) = (e_1 - e_k) + (e_2 + e_k) = (e_1 + e_k) + (e_2 - e_k)$$

$$= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$$

$$= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + \frac{1}{2}(e_1 + e_2 + e_3 - e_4).$$

Hence, the corresponding roots all lie in $L$. Since every root can be written as the sum of these roots, this proves claim (b).

Since $-\text{id}$ lies in the Weyl group, we are done.

(VII) For the group $G_2$, $t = t^* = \{ \xi \in \mathbb{R}^3 \mid \sum \xi_i = 0 \}$. The roots are $\pm \epsilon_i$ and $\epsilon_i - \epsilon_j$ for $i$ and $j$ distinct. The positive Weyl chamber is $\{ \xi \in t^* \mid 0 \geq \xi_2 \geq \xi_3 \}$. (Note that automatically $\xi_1 \geq \xi_2$.) The integral lattice is $\tilde{\ell} = \mathbb{Z}^3 \cap t$. The highest root is $\delta = \epsilon_1 - \epsilon_3$.

The argument for claim (a) follows the argument for $A_3$ word for word.

Since $\delta = (\epsilon_1 - \epsilon_2) + (\epsilon_2 - \epsilon_3) = (\epsilon_1) + (-\epsilon_3)$, the roots $\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_2 - \epsilon_3), \pm \epsilon_1$ and $\pm \epsilon_3$ all lie in $L$. Since every root can be written as a sum of these roots, claim (b) follows.

Since $-\text{id}$ lies in the Weyl group, we are done.

References


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