2. The cotangent bundle of $S^2$

Let $T^*S^2 = \{(u, v) \mid u \cot v = 0, \|v\| = 1\}$ with the s. form $\omega = du \wedge dv$. The function $h(u, v) := \|u\|$ induces an $S^1$ action on $T^*(S^2) \setminus S^2$ given by

$$\sigma_t(u, v) = (R^u_{x\times v}(u), R^v_{-u\times v}(v)) = (\cos t u - \sin t \|u\| v, \cos t v + \sin t \frac{u}{\|u\|})$$

where $R^t_x$ denotes the rotation by angle $t$ about the axis $x/\|x\|$ and so is defined as long as $x \neq 0$. To see this, notice that this action is generated by the vector field $X := \partial|_{t=0}$ where

$$X = -\|u\|v \cdot \partial_u + \frac{u}{\|u\|} \cdot \partial_v.$$

(Here I am thinking of $\partial_u$ as a vector; thus the components of the tangent vector $u \cdot \partial_u$ are $(u_1, u_2, u_3)$.) Then $X$ is $\omega$-dual in $\mathbb{R}^6$ to the 1-form

$$-\iota(X)\omega = +\|u\|v \cdot dv + \frac{u}{\|u\|} \cdot du.$$

But $v \cdot dv = 0$ on $T^*S^2$. Hence when restricted to $T^*(S^2)$ this equals $dh = \frac{u}{\|u\|} \cdot u$.

This flow $(u, v) \mapsto \sigma_t(u, v)$ is the geodesic flow.

**Exercise 2.1.** (i) Consider the $S^1$ action

$$\sigma^s_t := R^t_{sv+u\times v}.$$

When $s \neq 0$ this is well defined on the whole of $T^*(S^2)$. Show that its generating vector field is

$$X^s = \frac{1}{h} (su \times v - \|u\|^2 v) \cdot \partial_u + u \cdot \partial_v,$$

where $h := h^s := \|sv + u \times v\| = \sqrt{s^2 + \|u\|^2}$.

(ii) Let $\omega^s := \omega + s\pi^*\beta$ where $\pi : T^*S^2 \to S^2$ is the obvious projection $(u, v) \mapsto v$ and $\beta$ is the area form on the base $S^2$ given by $\beta_t(X, Y) = v \cdot X \times Y$ for $X, Y \in T_v(S^2)$. (Thus $\beta$ is the restriction to $S^2$ of the closed form $x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2$ on $\mathbb{R}^3$.) Show that $\sigma^s$ is the Hamiltonian flow of $h^s$ with respect to $\omega^s$. **Hint:** I found this easiest to do using vector notation. eg write $\omega_s = du \cdot dv + \frac{1}{2}v \cdot (dv \times dv)$. Here, $dv$ denotes the 3-vector $(dx_1, dx_2, dx_3) \in \mathbb{R}^3$ (a 1-form with values in $\mathbb{R}^3$). So $dv \times dv$ is a 2-form with values in $\mathbb{R}^3$. Since $\times$ is a combination of the cross product on vectors and the wedge product on forms, $dx_1 \wedge dx_2 = dx_2 \times dx_1 \wedge dx_2$ has a single component in the 3rd direction. This notation works as expected. eg $v \cdot (u \times dv) = u \cdot (dv \times v)$. Note that $(u \times v) \cdot (u \times dv) = 0$ on $T^*S^2$. If you are confused by this notation, you might prefer to use that in Ex 9.7.5 in JHOL, which is equivalent, but written a little differently.

Note that the orbits of the geodesic flow $\sigma_t$ lie in the fibers of the projection

$$\rho : T^*(S^2) \setminus S^2 \to S^2, \quad (u, v) \mapsto \frac{u}{\|u\|} \times v.$$

Since these form the leaves of the characteristic foliation on the level set $\|u\| = const$, $\omega$ is nondegenerate on the fibers of $\rho$. 

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Exercise 2.2. Show that $\rho^*(-\beta) = \omega$ on the level set $\|u\| = 1$ in $T^*S^2$ (where $\beta$ is the standard area form on $S^2$). Hint: Because $\rho^*(\beta)$ vanishes on the orbits of the geodesic flow, which are the null directions of $\omega$ on $\|u\| = 1$, it must equal $\omega$ up to a positive or negative constant. So you just need to check this at one point.

Let $pr : L \to S^2$ be a symplectic line bundle with Chern number $-2$ with connection 1-form $\alpha$. Thus $d\alpha = -pr^*(\tau)$, where the curvature 2-form $\tau$ satisfies $\int_{S^2} \tau = -2$ the Chern number of $L$. Consider the form

$$\omega_L := d(r^2 \alpha) = 2rdr \land \alpha + r^2 d\alpha$$

where $r := \|u\|$ is the radial distance from the zero section.

Exercise 2.3. (i) Show that $\omega_L$ is nondegenerate away from the zero section $L_0$ of $L$.

(ii) Show that there is a symplectomorphism $\Psi$ that fits into the commutative diagram:

$$\begin{array}{ccc}
(T^*S^2 \setminus S^2, \omega) & \xrightarrow{\Psi} & (L \setminus L_0, \omega_L) \\
\rho \downarrow & & \downarrow pr \\
S^2 & = & S^2.
\end{array}$$

I am not sure if there is a nice explicit formula for $\Psi(u, v)$; in any case, to define this you would have to construct an explicit model for $L$. It might be better to argue more abstractly, defining $\Psi$ on the unit sphere bundles and then extending using the obvious Liouville vector fields. Note that $\Psi$ does not extend continuously over the zero section $S^2$ of $T^*S^2$.

Now consider the $\lambda$-disc bundle $(T^*_{\leq \lambda}S^2, \omega) = \{(u, v) \in T^2S^2 : \|u\| \leq 1\}$. Construct a symplectic manifold $(X, \omega_{\lambda,X})$ from the compact manifold $(T^*_{\leq \lambda}S^2, \omega)$ by identifying each null orbit of $\omega$ on the boundary $T^*_{= \lambda}S^2$ to a single point and smoothing in the $r$ direction. (Here it is good to think of $\omega$ as given near the boundary by formula (1), which is permissible by Exercise 2.3.) This is the symplectic cutting procedure of Lerman and the details are in his paper of that name.

Lemma 2.4. $(X, \omega_{\lambda,X})$ is symplectomorphic to $(S^2 \times S^2, \beta \oplus \beta)$ when $\lambda = \sqrt{2}$.

This follows from the classification of ruled surfaces: $(X, \omega_X)$ is diffeomorphic to $S^2 \times S^2$ and contains a symplectically embedded 2-sphere with self-intersection 2 (namely the image of the boundary $T^*_{= \lambda}S^2$) as well as a Lagrangian sphere $S$. Since $\int_{S^2} \beta = 4\pi$, we need $\omega_{\lambda,X}$ to integrate to $8\pi$ on $C$ and so need to take $\lambda = \sqrt{2}$. There is only one symplectic manifold of this kind.

One should be able to construct an explicit symplectomorphism $(X, \omega_X) \to (S^2 \times S^2, \beta \oplus \beta)$ that takes $C$ to the diagonal and the Lagrangian $S$ to the antidiagonal. A very similar problem appears in the proof of Prop 9.7.2 (ii) in JHOL (cf p 340) except that here it concerns the Hirzebruch surface corresponding to the form $\omega + s7^*\beta$. It seems to me that the same calculations should yield a fairly direct proof of Lemma ?? (ie. no need for $J$-holomorphic curves, just the Moser argument) but I haven’t checked the details.

Another possible way of getting explicit formula would be to look at the toric picture. Namely, consider the map

$$\Phi : (X, \omega_X) \to \mathbb{R}^2, \quad \Phi(u, v) = (e_1 \cdot (u \times v), \|u\|).$$
Its image is the triangle $T$ with vertices $(0,0), (-1,1), (1,1)$. Also, it is the moment map for a $T^2$ action on $\Phi^{-1}(T \setminus \{0,0\})$. In fact, $T$ corresponds to an orbifold, that is smooth except for the one singular point $T^{-1}(0,0)$. If $P_\varepsilon$ is the parallelogram obtained by cutting off the vertex $(0,0)$ of $T$ by the cut $y = \varepsilon$ (where $(x,y)$ are the coordinates in $\mathbb{R}^2$) then the corresponding toric manifold $M_\varepsilon$ is a Hirzebruch surface which is known to be symplectomorphic to $S^2 \times S^2$ with an appropriate symplectic form $\omega_\varepsilon$. This is the manifold obtained from $(T^*_\varepsilon S^2 \setminus T^*_\varepsilon S^2, \omega)$ by compactifying both its ends. It is easy enough to relate the limit of $M_\varepsilon$ as $\varepsilon \to 0$ with $X$. But to relate it to $S^2 \times S^2$ you still need an explicit symplectomorphism $M_\varepsilon \to (S^2 \times S^2, \omega_\varepsilon)$, which is essentially the same problem as before.

As you see, none of these approaches are very useful, i.e. at best they would involve you in significant computations. I think that it might be best to use $SO(3)$ actions: see Exercise 3.3

3. $SO(3)$ actions

The Lie algebra $so(3)$ has generators $\partial_i, i = 1, 2, 3$ where $\partial_i$ denotes an infinitesimal rotation about the $i$th axis. Thus

$$\partial_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \partial_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \partial_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie bracket is $[\partial_1, \partial_2] = \partial_3$ and cyclic permutations of this. Using these as basis, we can identify $so(3)$ with $\mathbb{R}^3$ and then identify the dual Lie algebra $so^*(3)$ with $so(3) = \mathbb{R}^3$ via the standard inner product $\langle \cdot, \cdot \rangle$.

Note that any dual Lie algebra $g^*$ is a **Poisson manifold**, i.e functions on $g^*$ have a Poisson bracket that satisfies the Jacobi identity. It is defined as follows, if $f, g : g^* \to \mathbb{R}$ then the value of the Poisson bracket at $x \in g^*$ is:

$$\{f, g\}(x) := \langle df(x), dg(x) \rangle, \quad x \in g^*.$$

To understand this, note that $df(x), dg(x) \in T^*_x g^* \cong g$, so that the Lie bracket $[df(x), dg(x)]$ makes sense and can be paired with $x \in g^*$.

To say that $SO(3)$ acts in a Hamiltonian way on $(M, \omega)$ means that there is a moment map $\Phi : M \to so^*(3)$ that is a Poisson map. Thus for any $\xi \in g$ the function

$$H_\xi : M \to \mathbb{R}, \quad p \mapsto \langle \xi, \Phi(p) \rangle,$$

has Hamiltonian vector field $X_\xi$, the tangent vector to the flow given by the action of $\exp(t\xi) \in SO(3)$ on $M$. (This is the usual property of a moment map.) But we also require that if $f, g : g^* \to \mathbb{R}$ are any two functions, then

$$\{f \circ \Phi, g \circ \Phi\}_M = \Phi \circ \{f, g\}_{g^*}.$$

This is the Poisson property.

NOTE: for consistency with Seidel’s signs, we require that $\iota(X_H)\omega = -dH$. Also, if $M$ is noncompact we require that $\Phi$ be proper, i.e. the inverse image of compact sets is compact.

The relation between the symplectic form and the Poisson bracket on any symplectic manifold is

$$\omega(X_F, X_G)(p) = \{F, G\}(p).$$
Hence if \( F = F_\xi \) (defined by \( F_\xi(p) = \Phi \circ \langle \xi, \Phi(p) \rangle \) for \( \xi \in g \)) and \( G = F_\eta \), then
\[
\omega(X_\xi, X_\eta)(p) = \langle [\xi, \eta], \Phi(p) \rangle.
\]
In other words, the symplectic form along the orbits of \( G = \text{SO}(3) \) is determined by the image of this orbit under the moment map. Since this is a coadjoint orbit of \( \text{SO}(3) \) the image is either the single point \( \{0\} \) (in which case the orbit is isotropic) or is a 2-sphere with nontrivial area form (in which case the orbit is a quotient of \( \text{SO}(3) \) by a finite subgroup with symplectic form pulled back from \( S^2 \)).

NOTE: this is consistent with the situation for Hamiltonian actions of the torus \( T^k \). In this case the Lie algebra is abelian, which means that all Poisson brackets vanish and the orbits are isotropic.

The next exercise shows that if \( \text{SO}(3) \) acts on \( M \) there is an additional \( S^1 \) action on most of \( M \). This means that the symplectic form on most of \( M \) is entirely determined by the moment image. Presumably, if \( M \) is closed \( \omega \) is determined everywhere. (There is a paper by Iglesias that classifies 4-dimensional manifolds with \( \text{SO}(3) \) action.) cf. Exercise 3.3.

**Exercise 3.1.** Check that the function \( r = \sqrt{\sum x_i^2} \) Poisson commutes with the coordinate functions on \( \text{so}^*(3) \equiv \mathbb{R}^3 \). Deduce that if \( \text{SO}(3) \) acts on a 4-manifold \( M \) with (proper) moment map \( \Phi \) then the function \( |\Phi| \) induces an \( S^1 \)-action on \( \Phi^{-1}(\mathbb{R}^3 \setminus \{0\}) \) that rotates the fibers of \( \Phi \). **Hint:** Recall that if two functions \( F, G : M \to \mathbb{R} \) Poisson commute then they generate commuting flows.

**Exercise 3.2.** (i) Consider the standard action of \( \text{SO}(3) \) on the unit sphere \( S^2 \) in \( \mathbb{R}^3 \). Show that the moment map \( \Phi : S^2 \to \mathbb{R}^3 \) for this action is \( -\iota \) where \( \iota \) is the obvious inclusion. **Hint:** This is equivalent to saying that the Hamiltonian function that generates the rotation about the \( x_3 \)-axis is the negative of the height function. Check this.

(ii) Check that the moment map for the diagonal action of \( \text{SO}(3) \) on \( (S^2 \times S^2, \beta \oplus \beta) \) is \( \mu : (x, y) \mapsto -x - y \). Points on the diagonal and antidiagonal are critical points. Show there no others. Note that the moment image is the ball centered at 0 and radius 2.

(iii) Show that the moment map for the obvious \( \text{SO}(3) \) action on \( (T^*S^2, \omega) \) is \( (u, v) \mapsto -u \times v \). **Hint:** By symmetry, you just need to check this for the rotation \( (u, v) \mapsto (R^t_e u, R^-t_e v) \), which you can do by direct calculation.

(iv) Deduce from (iii) that the moment image for the \( \text{SO}(3) \) action on \( (X, \omega_X) \) is the ball of radius 2. Again, what are the critical points?

**Exercise 3.3.** Consider the map
\[
\psi : S^2 \times S^2 \setminus \{x \neq \pm y\} \to T^*_{\leq 2} S^2, (x, y) \mapsto \left( \|x + y\| \frac{x \times y}{\|x \times y\|}, \frac{y - x}{\|y - x\|} \right).
\]
(i) Show that this map is well defined, i.e. does have image \( T^*_{\leq 2} S^2 \), and extends smoothly over the antidiagonal \( x + y = 0 \).

(ii) Show that it is \( \text{SO}(3) \)-equivariant.

(iii) Check that it is a symplectomorphism. Instead of doing this directly, it’s easiest to use the fact that the symplectic form on (most of) these spaces is determined by the moment map.

(iv) Prove Lemma 2.4.
3.1. **The pentagon space.** Consider the diagonal action of \( \text{SO}(3) \) on \((S^2)^5\). It has moment map \( \Phi((x_i)) = -\sum x_i \). Its critical points are those configurations \((x_i)\) in which all the points \(x_i\) lie in the same direction. Therefore if all the spheres have the same radius there are no critical points in \( \Phi^{-1}(\{0\}) \). The pentagon space is the corresponding reduced space

\[
P := (S^2)^5/\text{SO}(3) := \Phi^{-1}(\{0\})/\text{SO}(3).
\]

Each point in \( P \) can be thought of as a closed configuration of 5 rods of length 1 in \( \mathbb{R}^3 \), modulo the obvious action of \( \text{SO}(3) \). Varying the sizes of the spheres (or lengths of the rods) gives different spaces.